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Nonlinear damping and forced vibration analysis of laminated composite beams

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The purpose of the present work it to study the damping and forced vibrations of three-layered, symmetric laminated composite beams. In the analytical formulation, both normal and shear deformations are considered in the core by using the higher-order zig-zag theories. The harmonic balance method is coupled with a one mode Galerkin procedure for a simply supported beam. The geometrically nonlinear coupling leads to a nonlinear frequency amplitude equation governed by several complex coefficients. In the first part of the paper, linear and nonlinear damping parameters of laminated composite beams are obtained. In the second part, nonlinear forced vibration analysis is carried out for small and large vibration amplitudes. The frequency response curves are presented and discussed for various geometric and material properties.

1. Introduction

Fibre reinforced composites are widely used in the aerospace, automotive, nuclear, marine, biomedical and civil engineering. Composite materials can be tailored to offer a unique combination of material capabilities, which may include low density, high strength, high stiffness, high damping, chemical resistance, thermal shock resistance and others properties of interests. In order to control the resonant amplitudes of vibration and thus in extending service life of laminated composite beams under periodic load or impact, the damping in the core layer play an important role. At the constituent level, the energy dissipation in fibre-reinforced composites is induced by different processes such as the viscoelastic behaviour of the matrix, the damping at the fibre–matrix interface, the damping due to damage, etc. At the laminate level, damping is depending on the constituent layer properties as well as the layer orientations, interlaminar effects, stacking sequence, etc.

Most of the studies of laminated composite beams are devoted to linear vibration and damping analysis. Earlier works on this subject are done by Gibson and Plunkett [1] and Gibson and Wilson [2]. A good overview on the available literature dealing with the vibration behaviour in presence of viscoelastic material can be found in the survey articles by Nakra [3,4]. In the earlier works, some of the important contributions are the works of Heng et al. [5], He and Rao [6], Rikards [7] and Bhimaraddi [8]. In all these works, a complex modulus, which consists of a real part representing elastic stiffness and an imaginary part representing dissipation, has been widely used to model the behaviour of linear viscoelastic materials under harmonic vibrations. With respect to the introduction of geometrical nonlinearity for beams with viscoelastic cores, Kovac et al. [9] and Hyer et al. [10,11] studied the nonlinear vibration of a damped sandwich beam. This study is based on a multi-mode Galerkin procedure coupled with the harmonic balance method.

Sandwich and laminated composite beams have been analysed using the classical models developed for one-layer beams (solid beams). These models are based on a theory that neglects transverse shear and normal strains and leads to the classical laminate theory (CLT) [12,13]. Due to the drawbacks of the CLT, a first order shear deformation theory (FSDT) has been proposed to take into account the transverse shear deformation [14–16]. The effects of the transverse shear deformation are pronounced for composite beams because of the high ratio of the extensional modulus to the transverse shearing modulus. The FSDT is widely used, and assumes a constant transverse shear strain in the thickness direction [17]. Therefore, a shear correction factor is generally used to adjust the transverse shear stiffness in dynamic analyses of laminates [18–21]. To avoid the use of a shear correction factor, higher order shear deformation theories (HSDTs) have been developed [22–24]. These theories are more realistic, since they give zero transverse shear stress condition at the top and bottom surface boundaries of the structure. The HSDTs have been successfully and extensively applied to design of multi-layered structural components. The discontinuity of some mechanical properties in the thickness direction represents a flaw in these theories. Also, it should be emphasised that recent research [25,26] has shown that HSDTs considerably overestimate natural frequencies of

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soft-core sandwich plates. The HSDTs are therefore of limited values for analysing problems in which an accurate description of the transverse normal stress distribution and related consequences are of interest. To overcome such limitations, Kapuria et al. [27] have used zig-zag theories, satisfying the inter-laminar continuity of the transverse shear stresses, to predict the dynamic and buckling responses of laminated beams with arbitrary layouts.

The aim of this work is to develop a simple consistent theory for the nonlinear vibration analysis of laminated composite beams with large amplitudes. This theory couples the harmonic balance technique to Galerkin procedure. The nonlinear geometrical effect due to axial forces caused by axial restraints is modelled using higher order zig-zag theories, which incorporate various shear function models for the shear deformation in the core. The nonlinear amplitude–frequency and phase–frequency relationships are established. The nonlinear frequency value and, in turn, the system loss factor ratio are obtained for various amplitudes, for considering different geometric and material parameters.

2. Viscoelastic model for composite materials

The general form of linear theory of a viscoelastic body is given by [28]

$$\sigma(t) = \int_{-\infty}^{t} Q^{*}_{ijkl}(t-\tau)\,d\varepsilon_{kl}(\tau) \tag{1}$$

The moduli for a viscoelastic material are complex numbers. They have real and imaginary components and can be defined by

$$Q^{*}_{ijkl} = Q'_{ijkl} + iQ''_{ijkl} \tag{2}$$

where Q^{*}_{ijkl} are the complex moduli, Q'_{ijkl} are the storage moduli and Q''_{ijkl} are the loss moduli. The assumption of a time independent Poisson's ratio has been used, by many authors [29,30], to simplify the characterisation of glass and carbon fibre-reinforced composite, under structural loading conditions. The Poisson's ratio is considered as real and constant.

For this case, the complex moduli can be written in matrix form as

$$[Q^{*}] = [Q'] + i[Q''][\eta] \tag{3}$$

where η are the loss factors.

If symmetry conditions of transversely isotropic materials are considered, only five storage modulus parameters and three damping coefficients are independent. The two matrices defined in Eq. (3) can be expressed in matrix form as [31]

$$[Q'] = \begin{bmatrix} Q'_{11} & Q'_{12} & Q'_{12} & 0 & 0 & 0 \\ Q'_{12} & Q'_{22} & Q'_{23} & 0 & 0 & 0 \\ Q'_{12} & Q'_{23} & Q'_{22} & 0 & 0 & 0 \\ 0 & 0 & 0 & Q'_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & Q'_{66} & 0 \\ 0 & 0 & 0 & 0 & 0 & Q'_{66} \end{bmatrix} \tag{4}$$

and

$$[\eta] = \begin{bmatrix} \eta_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \eta_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & \eta_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & \eta_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & \eta_6 & 0 \\ 0 & 0 & 0 & 0 & 0 & \eta_6 \end{bmatrix} \tag{5}$$

Therefore, according to this mathematical model, the viscoelastic constitutive relationships of transversely isotropic materials can be described in term of eight parameters; five independent dynamic stiffness parameters, $E'_1, E'_2, G'_{12}, v_{12}$ and v_{23}, plus three independent damping loss factors ($\tan\delta$), η_1, η_2 and η_6 written as follows:

$$Q'_{11} = \frac{E'_1}{1-\upsilon_{12}\upsilon_{12}} \tag{6a}$$

$$Q'_{22} = \frac{E'_2}{1-\upsilon_{12}\upsilon_{12}} \tag{6b}$$

$$Q'_{12} = \frac{E'_2}{1-\upsilon_{12}\upsilon_{12}} \tag{6c}$$

$$Q'_{55} = Q'_{66} = G'_{13} = G'_{12} \tag{6d}$$

$$Q'_{44} = \frac{E'_2}{2+2\upsilon_{12}} \tag{6e}$$

$$v_{21} = \frac{v_{12}E'_2}{E'_1} \tag{6f}$$

The normal and tangential stress σ_2 and τ, can be expressed as functions of strains by means of the stiffness matrix

$$\begin{Bmatrix} \sigma_2 \\ \tau_2 \end{Bmatrix} = \begin{bmatrix} \overline{Q}^{*}_{11} & 0 \\ 0 & \overline{Q}^{*}_{55} \end{bmatrix} \begin{Bmatrix} \varepsilon_2 \\ \varepsilon_{13} \end{Bmatrix} \tag{7}$$

in which

$$\begin{aligned}\bar{Q}^{*}_{11} = {} & Q'_{11}(1+i\eta_1)m^4 + (Q'_{12}(1+i\eta_1) + Q'_{12}(1+\eta_2) \\ & + 2Q'_{66}(1+\eta_6))m^2n^2 + Q'_{22}(1+\eta_2)n^4 \end{aligned} \tag{8a}$$

$$\bar{Q}^{*}_{55} = Q'_{44}(1+\eta_2)n^2 + Q'_{55}(1+\eta_6)m^2 \tag{8b}$$

where

$m = \cos\theta$ and $n = \sin\theta$

In which θ is the angle between the global axis and the local axis of the fibre in the composite material layer.

3. Formulation

3.1. Kinematics

The beam defined with dimensions and coordinate systems, is defined in Fig. 1. The laminated composite beam is considered with the coordinates x along the length, y along the width and z along

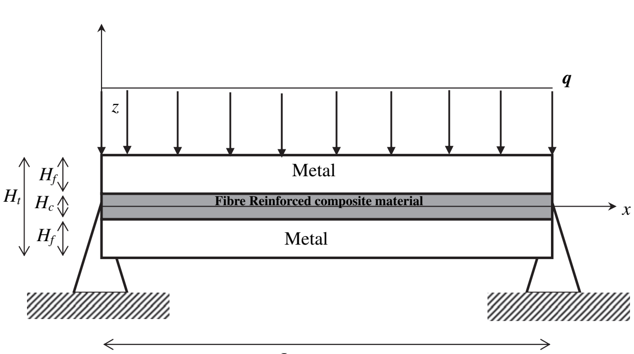

Fig. 1. Laminate beam with fibre Reinforced composite material core.

Table 1
Kinematics models considered in the study.

Model	Name	$f(z)$
1	Timoshenko [17]	$f(z) = z$
2	Reddy [22]	$f(z) = z - 4z^3/3H_c^2$
3	Touratier [23]	$f(z) = H_c \sin(\pi z/H_c)/\pi$
4	Afaq [24]	$f(z) = ze^{-2(z/H_c)^2}$

the thickness directions as shown in Fig. 1. The faces and core layer thickness are H_f and H_c, respectively. L is the length and H_t the total thickness of the beam. The derivation of the general governing equations is based on the following assumptions:

– No slipping occurs at the interfaces between the three layers of the beam.
– The kinematics of the beam is defined by the transverse displacement $w(x,t)$ and the independent rotation $\beta(x,t)$.

For the comparison of the various shear functions $f(z)$, the present work is limited to laminated composite and sandwich beams. In the higher order zig-zag theories, the displacement field is divided into three parts in order to satisfy displacement and transverse shear stresses continuity conditions at interfaces by introduction a shear function $f(z)$. The major drawback of these theories lies in the fact that the total number of the unknowns is dependent on the number of layers. The kinematics can be defined by

$$u_1(x,z,t) = u_1^0(x,t) - \left(z - \frac{H_c+H_f}{2}\right)\frac{\partial w(x,t)}{\partial x} \quad \frac{H_c}{2} < z \leqslant \frac{H_t}{2} \tag{9}$$

$$u_2(x,z,t) = u_2^0(x,t) - z\frac{\partial w(x,t)}{\partial x} + f(z)\beta(x,t), \quad -\frac{H_c}{2} \leqslant z \leqslant \frac{H_c}{2} \tag{10}$$

$$u_3(x,z,t) = u_3^0(x,t) - \left(z + \frac{H_c+H_f}{2}\right)\frac{\partial w(x,t)}{\partial x} \quad -\frac{H_t}{2} \leqslant z \leqslant -\frac{H_c}{2} \tag{11}$$

where $u_i(x,z,t)$ $(i = 1,\ldots,3)$ is the longitudinal displacement along the thickness of the layer i, $u_1^0(x,t)(i = 1,\ldots,3)$ is the axial displacement of the layer mid-plane. $w(x,t)$ is the common transverse displacement and $\beta(x,t)$ is the additional rotation of the normal to the mid-plane. By choosing the appropriate mathematical form for the shear function in the core, new kinematics refinements for the core modelling can be represented in Table 1. Considering the continuity of the displacements at the interfaces, Eqs. (9)–(11) the displacement fields $u_1^0(x,t)$ and $u_3^0(x,t)$ can be expressed as function of $u_2^0(x,t)$:

$$u_1^0(x,t) = u_2^0(x,t) - \left(\frac{H_c+H_f}{2}\right)\frac{\partial w(x,t)}{\partial x} + f\left(\frac{H_c}{2}\right)\beta(x,t), \tag{12}$$

$$u_3^0(x,t) = u_2^0(x,t) + \left(\frac{H_c+H_f}{2}\right)\frac{\partial w(x,t)}{\partial x} - f\left(\frac{H_c}{2}\right)\beta(x,t) \tag{13}$$

Substituting Eqs. (12) and (13) into Eqs. (9) and (11), respectively, leads to the axial displacement fields written as:

$$u_1 = (x,z,t) = u_2^0(x,t) - z\frac{\partial w(x,t)}{\partial x} + f\left(\frac{H_c}{2}\right)\beta(x,t), \tag{14}$$

$$u_2(x,z,t) = u_2^0(x,t) - z\frac{\partial w(x,t)}{\partial x} + f(z)\beta(x,t) \tag{15}$$

$$u_3(x,z,t) = u_2^0(x,t) - z\frac{\partial w(x,t)}{\partial x} - f\left(\frac{H_c}{2}\right)\beta(x,t) \tag{16}$$

According to the assumption of small strains and moderate rotations, the nonlinear strain–displacement relations for each layer can be expressed in the following form [32,33]:

$$\varepsilon_1(x,z,t) = \frac{\partial u_1(x,z,t)}{\partial x} + \frac{1}{2}\left(\frac{\partial w(x,t)}{\partial x}\right)^2 \tag{17a}$$

$$\varepsilon_2(x,z,t) = \frac{\partial u_2(x,t)}{\partial x} + \frac{1}{2}\left(\frac{\partial w(x,t)}{\partial x}\right)^2 \tag{17b}$$

$$\varepsilon_3(x,z,t) = \frac{\partial u_3(x,t)}{\partial x} + \frac{1}{2}\left(\frac{\partial w(x,t)}{\partial x}\right)^2 \tag{17c}$$

$$\gamma = \frac{\partial u_2(x,z,t)}{\partial z} + \frac{\partial w(x,t)}{\partial x} \tag{17d}$$

3.2. Formulation of forced vibration problem

The bending and membrane strains of the faces and the shear, bending and membrane strains of the composite interlayer are considered in the following formulation. In free vibration domain, the principal of virtual works is given by:

$$\delta P_{\text{int}} = \delta P_{ext} - \delta P_{acc} \tag{18}$$

δP_{ext} is the virtual works done by external uniform distributed load $q(t)$ and δP_{acc} represents the resulting of virtual works put into the system as acceleration. The internal virtual work δP_{int} is decomposed as

$$\delta P_{\text{int}} = \delta P_{\text{int}}^{(1)} + \delta P_{\text{int}}^{(2)} + \delta P_{\text{int}}^{(3)} \tag{19}$$

in which $\delta P_{\text{int}}^{(1)} + \delta P_{\text{int}}^{(2)} + \delta P_{\text{int}}^{(3)}$ are the virtual works of the upper metal, composite layer and lower metal respectively given by:

$$\delta P_{lint}^{(1)} = \int_0^L \left[N_1\left(\frac{\partial u(x,t)}{\partial x} + \frac{\partial w(x,t)}{\partial x}\frac{\partial \delta w(x,t)}{\partial x}\right) - \left(\frac{H_c+H_f}{2}\right) N_1\frac{\partial^2 \delta w(x,t)}{\partial x^2} + M_1\frac{\partial^2 \delta w(x,t)}{\partial x^2} + \left(f\left(\frac{H_c}{2}\right)N_1\right)\left(\frac{\partial \delta\beta(x,t)}{\partial x}\right)\right]dx \tag{20}$$

$$\delta P_{lint}^{(2)} = \int_0^L \left\{N_2\left(\frac{\partial u(x,t)}{\partial x} + \frac{\partial w(x,t)}{\partial x}\frac{\partial \delta w(x,t)}{\partial x}\right) + (M_2)\ \frac{\partial^2 \delta w(x,t)}{\partial x^2}\right) - Q_{11}^{*} b\int_{-H_c/2}^{H_c/2}\left[(f(z)z\frac{\partial^2 w(x,t)}{\partial x^2} - f(z)^2\left(\frac{\partial \beta(x,t)}{\partial x}\right)\right]dz\ \frac{\partial \delta\beta(x,t)}{\partial x}\,dx + T\delta\beta(x,t)\right\}dx \tag{21}$$

$$\delta P_{lint}^{(3)} = \int_0^L \left[N_3\left(\frac{\partial u(x,t)}{\partial x} + \frac{\partial w(x,t)}{\partial x}\frac{\partial \delta w(x,t)}{\partial x}\right) + \left(\frac{H_c+H_f}{2}\right) N_3\frac{\partial^2 \delta w(x,t)}{\partial x^2} + M_3\frac{\partial^2 \delta w(x,t)}{\partial x^2} - \left(f\left(\frac{H_c}{2}\right)N_3\right)\left(\frac{\partial \delta\beta(x,t)}{\partial x}\right)\right]dx \tag{22}$$

Here, N_i and M_i $(i = 1,\ldots,3)$ are the axial forces and the bending moments in the laminated composite beam. T is the shear force in the core layer. They are defined by

$$N_1 = E_f S_f\ \ \frac{\partial u_1}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\right) \tag{23a}$$

$$M_1 = E_f I_f \frac{\partial^2 w}{\partial x^2} \tag{23b}$$

$$N_3 = E_f S_f \quad \frac{\partial u_3}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 \Bigg) \tag{23c}$$

$$M_3 = E_f I_f \frac{\partial^2 w}{\partial x^2} \tag{23d}$$

$$N_2 = S_c \bar{Q}^*_{11}\left(\frac{\partial u_2(x,t)}{\partial x} + \frac{1}{2}\left(\frac{\partial w(x,t}{\partial x}\right)^2\right) \tag{23e}$$

$$M_2 = I_c Q^*_{11} \quad \frac{\partial^2 w(x,t)}{\partial x^2}\Bigg) - \int_{-H_c/2}^{-H_c/2} Q^*_{11} z f(z) \frac{\partial \beta(x,t)}{\partial x} \tag{23f}$$

$$T = b\int_{-H_c/2}^{H_c/2} \bar{Q}^*_{55}\left(\frac{\partial f(z)}{\partial z}\right)^2 dz(\beta(x,t))) \tag{23g}$$

The geometrical quantities used in Eqs. (23a–g) are the cross sectional area S_f and S_c and the quadratic moments I_f and I_c of the faces and core layers. The virtual work expressions δP_{ext} and δP_{acc} are given respectively by:

$$\delta P_{ext} = \int_0^L q(t)\delta w(x,t)\,dx \tag{24}$$

$$\delta P_{acc} = (2\rho_f S_f + \rho S_c)\int_0^L \frac{\partial^2 w(x,t)}{\partial t^2}\delta w(x,t)dx \tag{25}$$

Using Eqs. (20)-(22), the variational relation Eq. (18) can be rewritten as:

$$\int_0^L \left[N_T\left(\frac{\partial u(x,t)}{\partial x} + \frac{\partial w(x,t)}{\partial x}\frac{\partial \delta w(x,t)}{\partial x}\right) + M_\beta \frac{\partial \delta\beta(x,t)}{\partial x}\right.$$
$$\left. + M_W \frac{\partial^2 \delta w(x,t)}{\partial x^2} + T(\delta\beta(x,t)\right]dx = \delta P_{ext} - \delta P_{acc} \tag{26}$$

in which

$$N_T(x,t) = (2E_f S_f + S_c \bar{Q}^*_{11})\varepsilon^0_2 \tag{27}$$

$$M_w = C_1 \frac{\partial^2 w(x,t)}{\partial x^2} - C_2 \frac{\partial \beta(x,t)}{\partial x} \tag{28}$$

$$M_\beta = -C_2 \frac{\partial^2 w(x,t)}{\partial x^2} + C_3 \frac{\partial \beta(x,t)}{\partial x} \tag{29}$$

$$T = C_4 \beta(x,t) \tag{30}$$

and

$$C_1 = 2E_f S_f\left(\frac{H_c + H_f}{2}\right)^2 + 2E_f I_f + \bar{Q}^*_{11} I_c \tag{31a}$$

$$C_2 = 2E_f S_f\left(\frac{H_c + H_f}{2}\right) f\left(\frac{H_c}{2}\right) + \bar{Q}^*_{11} b\int_{-H_c/2}^{H_c/2} z f(z)\,dz \tag{31b}$$

$$C_3 = 2E_f S_f f\left(\frac{H_c}{2}\right)^2 + \bar{Q}^*_{11} b\int_{-H_c/2}^{H_c/2} f(z)^2\,dz \tag{31c}$$

$$C_4 = \bar{Q}^*_{55} b\int_{-H_c/2}^{H_c/2}\left(\frac{\partial f(z)}{\partial z}\right)^2 dz \tag{31d}$$

3.3. Nonlinear vibration mode and frequency–amplitude relationship

For the study of nonlinear harmonic vibrations, the response is assumed to be harmonic and proportional to the linear vibration mode. Based on the one mode Galerkin approximation, the nonlinear response is sought by the following form of deflection and rotation functions:

$$\begin{aligned} w(x,t) &= \frac{(Ae^{i\omega t} + \bar{A}e^{-i\omega t})}{2} w(x) \\ \beta(x,t) &= \frac{(Ae^{i\omega t} + \bar{A}e^{-i\omega t})}{2} B(x) \end{aligned} \tag{32}$$

where A is a complex unknown amplitude. The linear vibration mode of the undamped laminate beam, satisfies the following eigenvalue problem:

$$\int_0^L \left[M_\beta \frac{\partial \delta B(x)}{\partial x} + M_w \frac{\partial^2 w(x)}{\partial x^2} + T\delta B(x)\right]dx$$
$$= (2\rho_f S_f + \rho_c S_0)\omega^2 \int_0^L w\delta w(x)\,dx \tag{33}$$

From this equation and by integrating by part, one finds the following differential equations:

$$\begin{aligned} &\frac{\partial M_\beta}{\partial x} - T = 0 \\ &\frac{\partial^2 M_w}{\partial x^2} = (2\rho_f S_f + \rho_c S_c)\omega^2 w(x) \end{aligned} \tag{34a-b}$$

In the case of a simply supported beam, the boundary conditions are satisfied by the following expressions:

$$\begin{aligned} w(x) &= \sin(kx) \\ B(x) &= B\cos(kx) \end{aligned} \quad \text{where } k = \frac{n\pi}{L} \tag{35}$$

where the integer n is the mode number. Inserting Eqs. (29), (30), and (35) into (34a), one gets the expression for B as follows:

$$B = \frac{C_2 k^3}{C_3 k^2 + C_4} \tag{36}$$

Applying the variational principle to the displacements u, β and w, the following governing differential equations are obtained

$$\begin{aligned} &\frac{\partial N_T(x,t)}{\partial x} = 0 \\ &\frac{\partial M_\beta}{\partial x} - T = 0 \\ &- N_T(x,t)\frac{\partial^2 w(x,t)}{\partial x^2} + \frac{\partial^2 M_w}{\partial x^2} + (2\rho_f S_f + \rho_c S_c)\frac{\partial^2 w(x,t)}{\partial t^2} = q(t) \end{aligned} \tag{37a-c}$$

Eq. (37a) leads to a constant axial force $N_T(x,t) = N(t)$. Assuming that the ends are immovable, from Eq. (37a), one gets the following expression of the axial force:

$$N(t) = \frac{1}{L}\int_0^L N_T(x,t)\,dx = \frac{1}{2L}(2E_f S_f + \bar{Q}^*_{11} S_c)\int_0^L \left(\frac{\partial w(x,t)}{\partial x}\right)^2 dx \tag{38}$$

The substitution of Eqs. (28)–(30) into equations Eqs. (37b and 37c) and after some manipulation, one obtains a complex scalar differential equation similar to the ones established in [34–36] for linear calculation as given by

$$D_1 \frac{\partial^6 w(x,t)}{\partial x^6} - D_2 \frac{\partial^4 w(x,t)}{\partial x^4} + D_3 \frac{\partial^2 w(x,t)}{\partial x^2} + D_4 \frac{\partial^4 w(x,t)}{\partial x^2 \partial t^2}$$
$$- D_5 \frac{\partial^2 w(x,t)}{\partial t^2} = -D_6 q(t)$$

where

$$D_1 = \frac{C_1C_3}{C_2} - C_2 \tag{40a}$$

$$D_2 = \frac{C_3}{C_2}N(t) + \frac{C_1C_4}{C_2} \tag{40b}$$

$$D_3 = \frac{C_4}{C_2}N(t) \tag{40c}$$

$$D_4 = \frac{C_3}{C_2}(2\rho_f S_f + \rho_c S_c) \tag{40d}$$

$$D_5 = \frac{C_4}{C_2}(2\rho_f S_f + \rho_c S_c) \tag{40e}$$

$$D_6 = \frac{C_4}{C_2} \tag{40f}$$

In order to derive a simplified nonlinear amplitude–frequency equation, the expressions Eqs. (32) and (38) are inserted into Eq. (39). Using the harmonic balance method, a complex scalar nonlinear amplitude equation is then obtained as

$$-\omega^2 MA + KA + K_{nl}\bar{A}A^2 = Q \tag{41}$$

in which

$$M = \frac{D_5}{D_6}\int_0^L (w(x))^2\,dx - \frac{D_4}{D_6}\int_0^L \frac{\partial^2 w(x)}{\partial x^2}w(x)\,dx \tag{42}$$

$$K = \frac{C_1C_4}{D_6C_2}\int_0^L \frac{\partial^4 w(x)}{\partial x^4}w(x)\,dx - \frac{D_1}{D_6}\int_0^L \frac{\partial^6 w(x)}{\partial x^6}W(x)\,dx \tag{43}$$

$$K_{nl} = \frac{C_3}{D_6C_2}N(t)\int_0^L \frac{\partial^4 w(x)}{\partial x^4}w(x)\,dx - \frac{D_3}{D_6}\int_0^L \frac{\partial^2 w(x)}{\partial x^2}w(x)\,dx \tag{44}$$

and

$$Q = \int_0^l q(t)w(x)\,dx \tag{45}$$

3.4. Linear and nonlinear free vibration analysis

The linear version of the amplitude Eq. (41) (with $Q = 0$) leads to an approximate value of the complex eigenfrequency given by

$$\omega^2 = \frac{K}{M} = \Omega_{lin}^2(1 + i\eta_l) \tag{46}$$

where η_l is the loss factor and Ω_{lin} is the linear frequency. Free vibration analysis of Eq. (41) (with $Q = 0$) can be achieved by determining the relationship between the frequency ω^2 and the amplitude $a = |A|$.

The nonlinear frequency and loss factor are defined as follows:

$$\omega^2 = \Omega_{nl}^2(1 + i\eta_{nl}) \tag{47}$$

3.5. Solving the frequency–amplitude equation

The nonlinear Eq. (41) can be rewritten in a simple form, and solved when Q is not equal to zero. This Q is assumed to be real number. The complex numbers A, M, K and K_{nl} can be rewritten as:

$$\begin{aligned}
&-\omega^2 a|M|e^{i\xi} + |K|ae^{i\varphi} + a^3|K_{nl}|e^{i\psi} = Qe^{-i\chi}\\
&A = ae^{i\chi}\\
&: M = |M|e^{i\xi}\\
&K = |K|e^{i\varphi}\\
&K_{nl} = |K_{nl}|e^{i\psi}
\end{aligned} \tag{48}$$

The corresponding real and imaginary terms of the above equations are given by:

$$\begin{aligned}
&-\omega^2 a|M|\cos(\xi) + a|K|\cos(\varphi) + a^3|K_{nl}|\cos(\psi) = Q\cos(\chi)\\
&-\omega^2 a|M|\sin(\xi) + a|K|\sin(\varphi) + a^3|K_{nl}|\sin(\psi) = -Q\sin(\chi)
\end{aligned} \tag{49}$$

After some of manipulations, one obtains the following frequency–amplitude relation:

$$\omega^4|M|^2 - 2\alpha\omega^2 + \lambda = 0 \tag{50}$$

where

$$\begin{aligned}
&\alpha = |MK|\cos(\xi - \varphi) + a^2|MK_{nl}|\cos(\xi - \psi)\\
&\lambda = |K|^2 + a^4|K_{nl}|^2 + 2a^2|KK_{nl}|\cos(\varphi - \psi) - \frac{Q^2}{a^2}
\end{aligned} \tag{51a-b}$$

The solutions of Eq. (50) are as follows:

$$\omega(a) = \frac{\sqrt{\alpha \pm \sqrt{\alpha^2 - |M|^2\lambda}}}{|M|} \tag{52}$$

where $\omega(a)$ is an amplitude dependent frequency for nonlinear forced vibration problem.

4. Numerical results

To objectively assess the various higher order zig-zag theories associated with the shear function *f(z)* models listed in Table 1, linear and nonlinear free vibration analysis ($Q = 0$) are carried out. Laminated composite beams with simply supported edges are considered. The material and geometrical properties of the structure are given in Table 2. Results are obtained with MATLAB [37] developed for numerical application. The variation of the system loss factor ratio η_{nl}/η_l of the laminated composite beams is studied. Fig. 2 presents on how the various shear function models influence the ratio η_{nl}/η_l. It can be found from these figures that the models of Reddy [22], Touratier [23] and Affaq [24] are accurate enough to predict satisfactory loss factor ratio η_{nl}/η_l. However, in the case of thin core with $H_f/H_c = 7$, the Timoshenko model ($f(z) = z$) underestimates the loss factor ratio (η_{nl}/η_l), since this model cannot present exactly the transverse shear deformations. The influences of the fibre orientation on the loss factor ratio η_{nl}/η_l are addressed in Fig. 3. The obtained results show no great dependence between η_{nl}/η_l and the amplitude (a/H) for stiff core with $\theta = 0°$.Unlike, in the case of soft core ($\theta = 90°$), the dependence between η_{nl}/η_l and

Table 2
Materiel and geometrical parameters.

Face layers	Composite core	Whole beam
$E_f = 2.1.10^{11}$ Pa	$E_1 = 154.5$ GPa	$H = 0.01$ m
$\rho_f = 7800$ kg/m^3	$E_2 = 9.9$ GPa	$b = 4H$
	$E_3 = 9.9$ GPa	
	$G_{12} = 7.1$ GPa	
	$\upsilon_{23} = 0.49$	
	$\upsilon_{12} = 0.35$	
	$\eta_1 = 5.7 \times 10^{-3}$	
	$\eta_2 = 8.5 \times 10^{-3}$	
	$\eta_6 = 13.2 \times 10^{-3}$	
	$\rho_c = 1566$ kg/m^3	

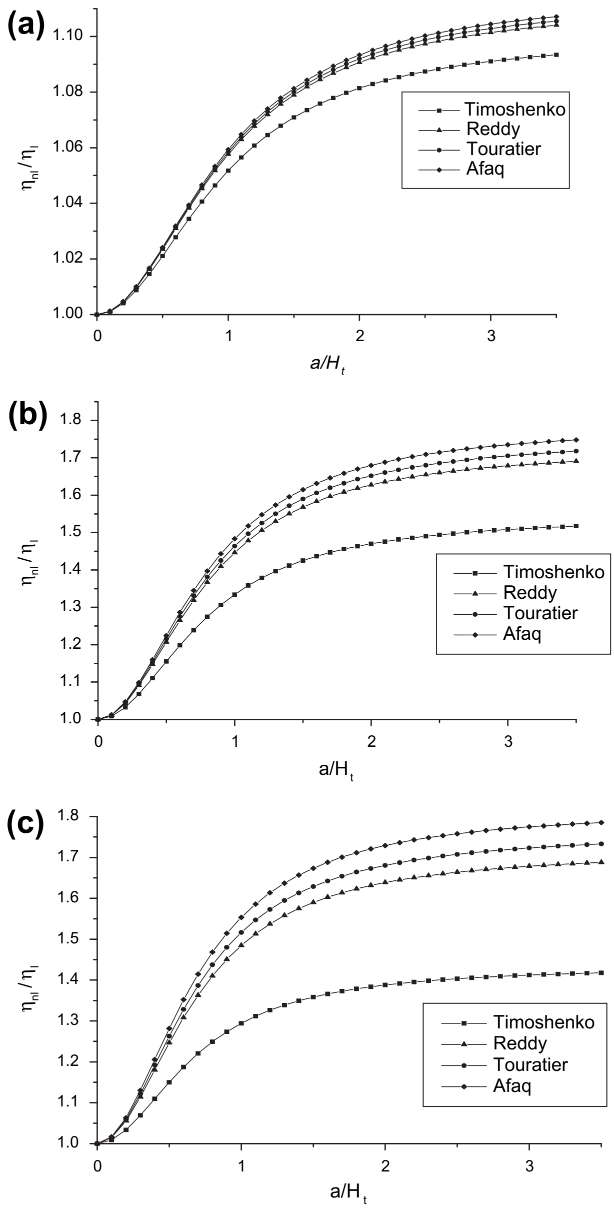

Fig. 2. Loss factor ratio η_{nl}/η_l as a function of the amplitude (a/H) for difference models with $\theta = 90°$ and $L/H = 100$. (a) $H_f/H_c = 1/7$; (b) $H_f/H_c = 1$ and (c) $H_f/H_c = 7$.

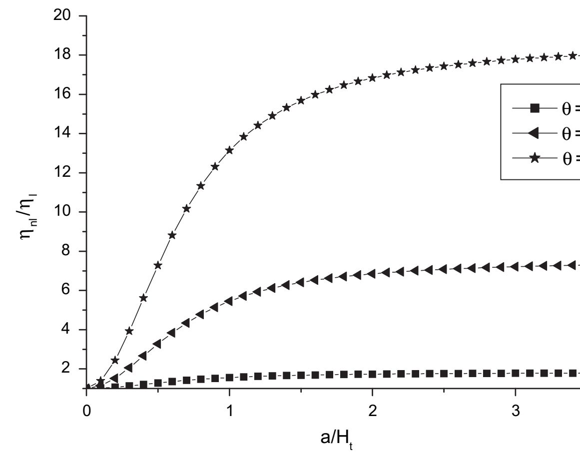

Fig. 3. Loss factor ratio η_{nl}/η_l as a function of the vibration amplitude (a/H_t) for various values of θ with Touratier model; $b = 1$; $H_f/H_c = 7$; $L/H_t = 100$.

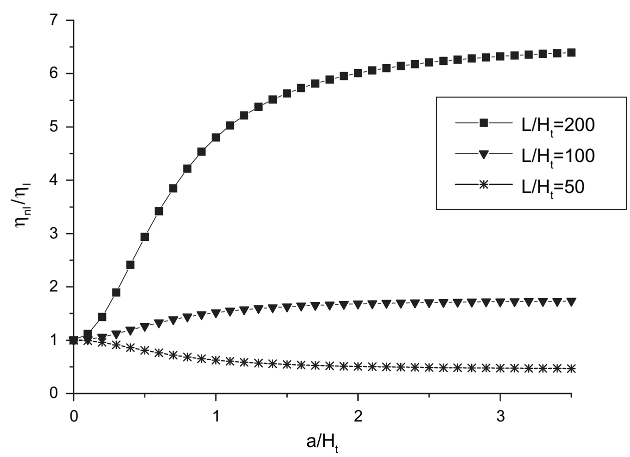

Fig. 4. Loss factor η_{nl}/η_l as a function of the vibration amplitude (a/H_t) for various values of L/H_t with Touratier model; $b = 1$; $H_f/H_c = 7$; $L/H_t = 100$ and $\theta = 90°$.

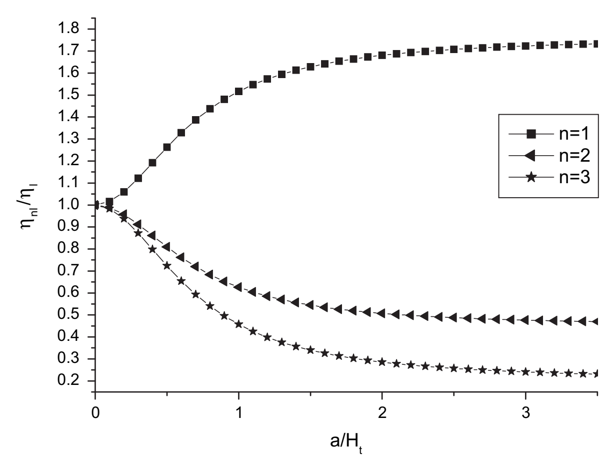

Fig. 5. Loss factor η_{nl}/η_l as a function of the vibration amplitude (a/H_t) for various modes of vibrations (n = 1–3) with Touratier model; $b = 1$; $H_f/H_c = 7$; $L/H_t = 100$ and $\theta = 90$ and; $L/H_t = 100$.

(a/H) is more strong at small amplitudes (a/H) $\in [0;2]$ and remain unchanged at large amplitudes.

The influences of the geometrical parameter L/H on η_{nl}/η_l , are addressed in Fig. 4. From this figure, one can note that an important reduction of η_{nl}/η_l is observed with respect to the amplitudes (a/H) for short beams ($L/H = 10$ and 50). In the case of slender beams with ($L/H = 200$), a great increasing of η_{nl}/η_l is observed at small amplitudes (a/H) $\in [0;2]$. For the values of $a/H = 2$ and greater, the ratio η_{nl}/η_l remains approximately constant. Fig. 5 shows the affect on the loss factor ratio of the first three vibration modes. As can be observed, the ratio η_{nl}/η_l due to the second and third ($n = 2$ and 3) modes of vibrations, initially decreases with a/H and increases for the first mode of vibration ($n = 1$).

Nonlinear forced vibration analysis is carried out in this study to analyse the hardening changes of laminated composite beams. As seen from Figs. 6 and 7, the geometric parameters H_f/H_c and L/H have strong effects on the nonlinear behaviour. In theses figures, some more pronounced hardening effects are obtained with thin cores ($H_f/H_c = 7$) and for slender beams ($L/H = 200$). Fig. 8 shows

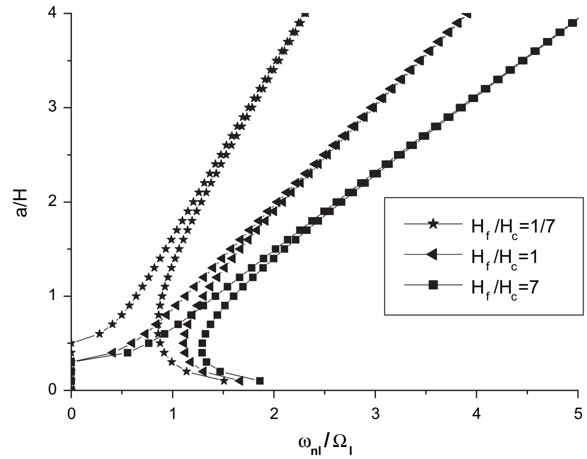

Fig. 6. Forced linear frequency amplitude response for various values of H_f/H_c with Touratier model; $\theta = 90°$; $L/H_t = 50$; $H_t = 0.01$. $b = 4H_t$ and $Q = 1000$.

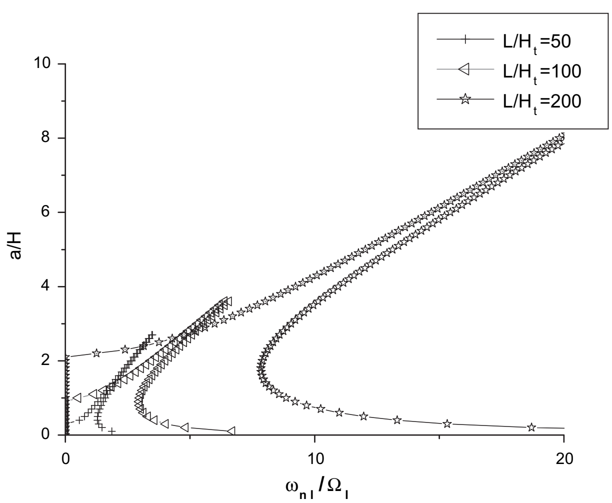

Fig. 7. Forced linear frequency amplitude response for various values of L/H_t with Touratier model; $\theta = 90°$; $H_f/H_c = 7$; $H_t = 0.01$. $b = 4H_t$ and $Q = 1000$.

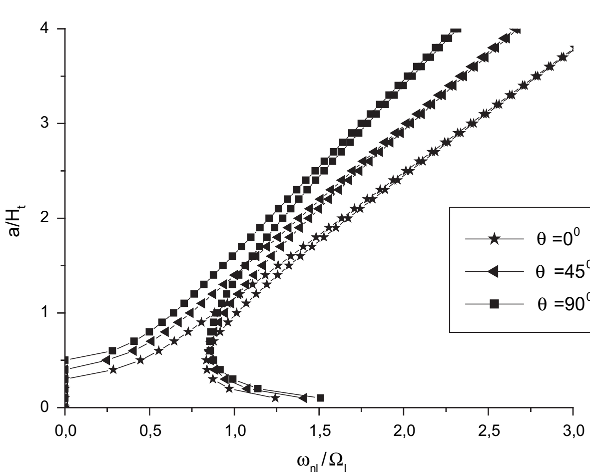

Fig. 8. Forced linear frequency amplitude response for various values of θ with Touratier model; $H_f/H_c = 7$; $L/H_t = 50$; $H_t = 0.01$. $b = 4H_t$ and $Q = 1000$.

that a softer effect can be achieved by using a soft core, in which, the fibre orientation angle θ is of 90°.

5. Conclusion

Nonlinear forced vibration analysis of laminated composite beams was investigated. Higher order zig-zag theories were used for the displacement field. Based on the harmonic balance method and Galerkin procedure, a scalar complex nonlinear amplitude–frequency relationship was established and a closed form analytical solution for this problem was determined.

Parametric studies indicated that the geometrical parameters H_f/H_c and L/H have strong effects on the loss factor ratio η_{nl}/η_l and the hardening changes for forced vibration analysis. The material properties of the reinforced fibre composite core have also a great influence on the loss ratio η_{nl}/η_l and the hardening changes. Therefore, the soft core with a fibre angle orientation of 90° is more suitable for passive vibration control of laminated composite beams, especially in the case of large amplitude excitations. This research provides a good foundation fore future investigations and contributes to the understanding of nonlinear damping properties of other type of structures under dynamic loads.

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