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# On geometrically exact post-buckling of composite columns with interlayer slip—The partially composite *elastica*

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This paper is focused on the geometrically exact elastic stability analysis of two interacting kinematically constrained, flexible columns. Possible applications are to partially composite or sandwich columns. A partially composite column composed of two inextensible elastically connected sub-columns is considered. Each sub-column is modeled by the Euler–Bernoulli beam theory and connected to each other via a linear constitutive law for the interlayer slip. The paper discusses the validity of parallel and translational kinematics beam assumptions with respect to the interlayer constraint. Buckling and post-buckling behavior of this structural system are studied for cantilever columns (clamped-free boundary conditions). A variational formulation is presented in order to derive relevant boundary conditions in a geometrically exact framework. The exact post-buckling behavior of this partially composite beam-column is investigated analytically and numerically. The Euler *elastica* problem is obtained in the case of non-composite action. The “partially composite *elastica*” is then treated analytically and numerically, for various values of the interaction connection parameter. An asymptotic expansion is performed to evaluate the symmetrical pitchfork bifurcation, and comparisons are made with some exact numerical results based on the numerical treatment of the non-linear boundary value problem. A boundary layer phenomenon, similar to that also observed for the linearized bending analysis of partially composite beams, is observed for large values of the connection parameter. This boundary layer phenomenon is investigated with a straightforward asymptotic expansion, that also is valid for large rotations. Finally, the paper analyses the effect of some additional imperfection eccentricities in the loading mode, that lead to some pre-bending phenomena.

## 1. Introduction

Buckling and post-buckling of elastic columns is an old topic since the pioneering work of Euler [1], who first investigated the geometrically exact post-buckling behavior of axially loaded columns using elliptic integrals. Most of the published results on this stability topic concern the behavior of a single structural element that may include complex kinematics over the cross section (with higher-order shear models for instance), or various constitutive laws. However, the consideration of the non-linear behavior of multiple interacting columns has been probably less studied due to the difficulty associated with the geometrical constraints induced by the columns’ interaction [2]. This problem is of interest for a wide class of engineering applications in civil and aeronautic engineering, and also in a micromechanics perspective. For instance, understanding the complex interaction

between beams or columns can be very useful in the area of fibrous networks where the fibers are modeled as flexible beams. A possible mode of interaction in fibrous networks results from the presence of crosslinks that can be modeled as rigid connections between two beams at their point of intersection. This interaction can be seen as multiple concentrated kinematic constraints, forcing displacements to be the same at the cross-link locations. A specific distributed kinematic constraint which is of interest in this paper has also relevant applications in civil engineering, and also in many fields of natural sciences. The buckling of structural systems composed of parallel interacting columns belongs to this class of distributed kinematic constraint. A structural model, incorporating such constraints can be useful to understand the geological folding due to tectonic compression in multilayered sedimentary rocks (see for instance [3]). In the field of civil engineering, layered structural elements are often used and can be modeled by the same kind of parallel, kinematically constrained beams. Layered structural elements with interlayer slip are typically encountered in timber structures, where wooden beams are made up of layers assembled by means of nailing, bolting or gluing adhesives (with a soft shear modulus).

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Partially composite structures built up by sub-elements of different materials and connected by shear connectors to form an interacting unit, such as timber-concrete or steel-concrete elements, are widely used in building engineering. In the case of a flexible connection, the analysis procedure requires consideration of the interlayer slip between the sub-elements, leading to the partial interaction concept. For a detailed literature background on the partial composite theory, the readers are referred to Girhammar and Gopu [4] and Girhammar and Pan [5]. Möhler [6] obtained the buckling formulae of axially loaded partially-composite columns, using the Euler-Bernoulli model for each column. The lateral buckling problem of partially composite beam-columns subjected to both transverse and axial loading was investigated by Girhammar and Gopu [4] and Girhammar and Pan [5] and others for general boundary conditions. Xu and Wu [7] generalized the results of Girhammar and Gopu [4] by using Timoshenko's beam theory based on Engesser's theory for each beam-column and a uniform shear model distribution over the cross section. The work of Krawczyk et al. [8] should be mentioned for geometric non-linear layered beam theory including shear effects for each column, and based on a co-rotational Finite Element formulation. Cas et al. [9] investigated the buckling of layered wood columns neglecting the shear effect. Schnabl and Planinc [10] specifically studied the extensibility effect in the Euler-Bernoulli based solution of partial composite columns and demonstrated the importance of the choice of boundary conditions on the buckling loads. Ranzi et al. [11] investigated the non-linear geometrical behavior of the partially composite column and included some eccentric loading leading to pre-bending effects. More recently, Schnabl and Planinc [12] numerically characterized the buckling of a partially composite column with some non-uniform shear effect introduced from Reissner theory (which can be reformulated in terms of a generalized Haringx's model). Challamel and Girhammar [13] obtained the buckling load of the Engesser based and the Haringx based non-uniform shear model, and derived simple engineering shear formula for structural applications. Most of these studies are focusing on the linearized buckling analysis, even though the non-linear framework is formally presented in for instance the studies of Cas et al. [9], Krawczyk et al. [8] or Ranzi et al. [11].

Budd et al. [3] studied a similar problem adding an additional Winkler foundation, with application to parallel folding in layers of papers. The paper of Vinogradov and Derrick [14] can be also mentioned as they studied the post-buckling of extensible composite layered columns without the additional in-plane slip behavior for the connection between the sub-elements. Vinogradov and Derrick [14] also investigated the effect of additional eccentricities on the bending coupling of asymmetric composite layered columns. In the present paper, it is chosen to assume the inextensible property of each structural element for a simplification of the problem. The shear effect is concentrated in the layer connection between the two inextensible columns.

Another potential application of studies dealing with two interacting parallel columns is the modeling of the post-buckling behavior of sandwich columns. In fact, sandwich columns composed of three layers, two thin faces and a thick weak shear core are also concerned by this study [15–17]. There is a constitutive analogy between the uniform shear behavior of the soft core of the sandwich column, and the shear layer of the partially composite beam. The static in-plane behavior of sandwich beams is well established by Hoff [18]. The in-plane buckling problem of a three-layer sandwich column was studied by Hoff and Mautner [19] (see e.g. also [18,20,21] or [22]). The linear stability analysis of sandwich or partially composite columns has been well documented but the large displacement analysis of this problem still merits some complementary studies. More recently, Frostig [23] considered the

*elastica* problem of a sandwich column with axial extensibility. Symmetrically unstable post-bifurcation branches have been numerically observed for this problem. A related problem is the post-buckling behavior of solid elastic columns including shear effects. Such shear effects have been taken into account for buckling of solid beams in the studies of Huddleston [24], Sheinman and Adam [25], Goto et al. [26], Huang and Kardomateas [27] and Lee et al. [28] where stable post-bifurcation branches have been mainly noticed. However, imperfection sensitivity (symmetrically unstable bifurcation) can be eventually observed in presence of shear, as shown by Atanackovic and Spasic [29] or Beghini et al. [30]. For the rigorous *elastica* study of extensible homogeneous solid columns, reference is made to the works of Goto et al. [26] and Magnuson et al. [31].

It appears that the post-buckling of kinematically constrained parallel columns still merits some further investigation. The paper is focused on the geometrically exact elastic stability of two kinematically constrained flexible columns, with some possible applications to partially composite or sandwich columns. The partially composite column is composed of two inextensible, elastically connected sub-columns. Each sub-column is modeled by the Euler-Bernoulli beam theory and connected to each other via a linear constitutive law for the interlayer slip. The paper discusses the validity of parallel and translational beam assumptions with respect to the kinematic constraint.

## 2. Kinematics of the partially composite column

The state of the composite or sandwich column is specified by the in-plane cross-sectional rotation  $\theta(s)$  where  $s$  is the curvilinear abscissa. Each sub-domain is assumed to be composed of a column of width  $b_i$  with a depth  $h_i$ , as defined in Figs. 1 and 2. For

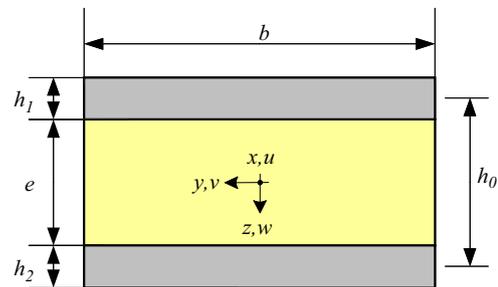


Fig. 1. Geometric parameters of a sandwich beam.

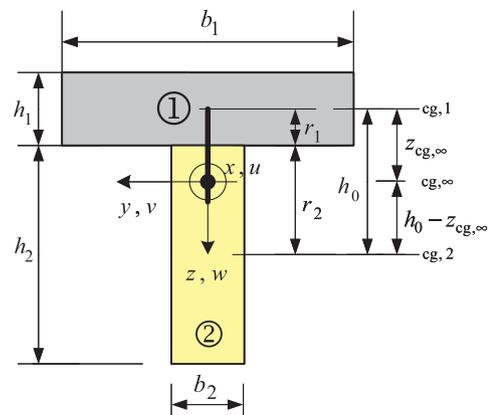


Fig. 2. Geometric parameters of a partially composite beam.

sandwich columns, it is often assumed that the width of each column is identical, i.e.  $b_i$  is equal to  $b$  (Fig. 1). The geometric parameters defining a typical composite beam with two sub-elements of different geometry and materials are shown in Fig. 2. The subscripts '1' and '2' refer to the top and bottom elements of the cross-section, respectively. Therefore, the kinematics of the composite beam is completely specified by the two rotation fields  $\theta_1(s)$  and  $\theta_2(s)$  in each beam,  $s \in [0; L]$  where  $L$  is the length of each beam. The bending stiffnesses of each column are denoted by  $EI_1$  and  $EI_2$ .

The stability of two shear connected Euler–Bernoulli inextensible columns is investigated. The engineering problem can be defined in Fig. 3. Clamped-free boundary conditions are specifically studied in this paper, as an archetypical structural problem. The composite column is loaded by two independent load parameters ( $P_1, P_2$ ) acting at the centroid of each column section at the free end. The length of each column is denoted by  $L$ . The sub-elements are connected together by means of some kind of discrete shear connectors or a weak shear layer, which are assumed to produce uniformly distributed slip forces or interlayer shear stresses. The shear connector, giving a shear layer force per unit length versus the slip, or shear displacement behavior, is linear elastic with a constant slip modulus  $k$ . Frictional effects and uplift at the shear interface are neglected. Full composite action (infinite slip modulus,  $k \rightarrow \infty$ ) and non-composite action (zero slip modulus,  $k \rightarrow 0$ ) represent upper and lower bounds for the partial composite action, respectively. The post-buckling configuration of this composite column is schematically illustrated in Fig. 3.

The centroidal axis of each column in the deformed configuration can be expressed with respect to its position angle (see Fig. 3) as follows:

$$\begin{cases} x_1(s) = \int_0^s \cos \theta_1(t) dt \\ y_1(s) = \int_0^s \sin \theta_1(t) dt \end{cases} \quad \text{and} \quad \begin{cases} x_2(s) = \int_0^s \cos \theta_2(t) dt \\ y_2(s) = \int_0^s \sin \theta_2(t) dt \end{cases} \quad (1)$$

where the variable  $t$  is the curvilinear abscissa (parametric curve).  $x_i(s)$  is the vertical coordinate along the  $x$ -axis whereas  $y_i(s)$  denotes the horizontal coordinate,  $i \in \{1; 2\}$ . It can be shown that the slip-displacement  $u(s)$  along the connected column interface

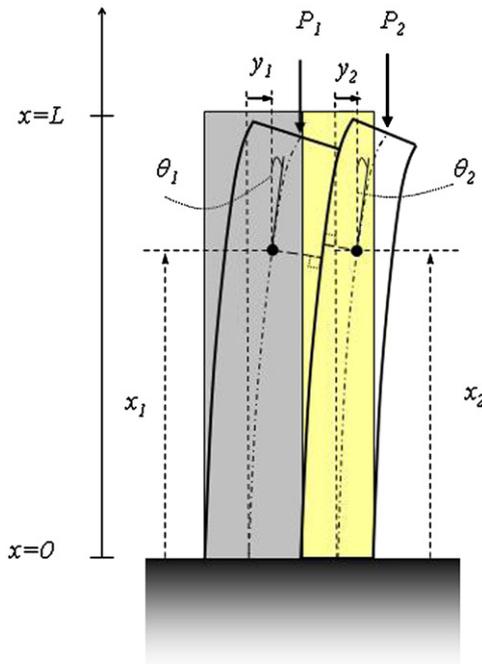


Fig. 3. Post-buckling behavior of a partially composite column.

can be expressed as a function of the two unknown rotations  $\theta_1(s)$  and  $\theta_2(s)$  as

$$u(s) = \int_0^s \left( \sqrt{[x_2^*(t)]^2 + [y_2^*(t)]^2} - \sqrt{[x_1^*(t)]^2 + [y_1^*(t)]^2} \right) dt \quad (2)$$

where the star notation ( $x_i^*(s), y_i^*(s)$ ) is relative to the projection of the point considered at the centroid of each column over the delimitation curve at the contact interface. These projections are calculated from simple geometrical considerations as

$$\begin{cases} x_1^*(s) = x_1(s) - \frac{h_1}{2} \sin \theta_1(s) \\ y_1^*(s) = y_1(s) + \frac{h_1}{2} \cos \theta_1(s) \end{cases} \quad \text{and} \quad \begin{cases} x_2^*(s) = x_2(s) + \frac{h_2}{2} \sin \theta_2(s) \\ y_2^*(s) = y_2(s) - \frac{h_2}{2} \cos \theta_2(s) \end{cases} \quad (3)$$

where  $h_i$  is the depth of each layer. Inserting the expression of Eq. (3) into the slip displacement of Eq. (2) leads to the simple formula:

$$u(s) = \frac{h_1}{2} \theta_1(s) + \frac{h_2}{2} \theta_2(s) \quad (4)$$

The shear interlayer force per unit length  $q(s)$  versus the slip or shear displacement  $u(s)$  is linear elastic with a constant slip modulus,  $k$ , expressed as

$$q(s) = ku(s) = \frac{k}{2} (h_1 \theta_1(s) + h_2 \theta_2(s)) \quad (5)$$

The connection between each sub-column leads to some kinematic constraints as the rotations of layers are constrained to each other via:

$$\exists (s_1, s_2) / \theta_1(s_1) = \theta_2(s_2) \quad (6)$$

The same methodology is presented in Kryżanowski et al. [32], with some different assumptions for the identification of the corresponding abscissa. In the geometrically exact framework, the columns are exactly parallel, but are not translational. The concept of parallel curves is well referenced in the book of Yates [33]. Since parallel curves have common normals, they have a common evolute. Leibniz [34] was the first to consider parallel curves at the end of the XVIIth century, prompted no doubt by the involutes of Huygens. The involute of a circle was discussed and utilized by Huygens in connection with his study of clocks without pendulums for service on ocean ships [35]. In the structural problem studied in this paper, each column is parallel to the each other with distance  $h_0$  between the centroidal axis of each sub-column. The relationship between the two curves is written as

$$\begin{cases} x_2 = x_1 - \frac{h_0}{\sqrt{x_1'^2 + y_1'^2}} y_1' \\ y_2 = y_1 + \frac{h_0}{\sqrt{x_1'^2 + y_1'^2}} x_1' \end{cases} \quad \text{with} \quad \begin{cases} x_1' = \cos \theta_1 \\ y_1' = \sin \theta_1 \end{cases} \quad (7)$$

leading to the geometrical constraint

$$\begin{cases} x_2 = x_1 - h_0 \sin \theta_1 \\ y_2 = y_1 + h_0 \cos \theta_1 \end{cases} \quad (8)$$

The rotations in each layer are equal at the following abscissas:

$$s_2 = \int_0^{s_1} \sqrt{x_2'^2(t) + y_2'^2(t)} dt \quad \text{with} \quad s_1 = \int_0^{s_1} \sqrt{x_1'^2(t) + y_1'^2(t)} dt \quad (9)$$

Developing Eq. (9), the rotation constrained can be expressed using the abscissa relationship:

$$s_2 = s_1 - h_0 \theta_1(s_1) \quad (10)$$

The constraint of the geometrically exact problem, based on the theory of parallel curves, finally gives

$$\theta_2(s - h_0 \theta_1(s)) = \theta_1(s) \quad (11)$$

The constraint Eq. (11) shows that  $\theta_2$  is a functional of  $\theta_1$ , leading to a theoretical one-single variable problem (expressed

for instance with  $\theta_1$  or  $\theta_2$ ). Such a kinematic constraint is mathematically complex and not often studied in the literature as it generates some distributed spatial delay between the two rotation functions.

The zero order theory approximates the parallel curves with a translation as (translational assumption)

$$\theta_2(s) = \theta_1(s) \quad (12)$$

This zero order can be understood as the zero order asymptotic expansion of the non-linear constraint for sufficiently small rotation values. This assumption is generally adopted for generation of most numerical results presented in the literature on the buckling behavior of partially composite columns without shear effect (see for instance [4,5] or [9]). This assumption can be relaxed when taking into account the specific shear effect of the partially composite column (see for instance [12] or [13]). Continuing with the classification of the different approximated theories, a first-order theory would be expressed by

$$\theta_1(s) \ll 1 \Rightarrow \theta_1(s) = \frac{\theta_2(s)}{1+h_0\theta_2'(s)} \text{ or } \theta_1(s) = \theta_2(s)(1-h_0\theta_2'(s)) < \theta_2(s) \quad (13)$$

The total potential energy of this structural problem can be written as

$$U[\theta_1, \theta_2] = \int_0^L \frac{1}{2} El_1 \theta_1'^2 + \frac{1}{2} El_2 \theta_2'^2 + \frac{1}{2} k \left( \frac{h_1}{2} \theta_1 + \frac{h_2}{2} \theta_2 \right)^2 ds - P_1 \left[ L - \int_0^L \cos \theta_1(s) ds \right] - P_2 \left[ L - \int_0^L \cos \theta_2(s) ds \right] \quad (14)$$

with the geometrically exact constraints given by Eq. (11).

### 3. Theoretical model—fundamental equations

#### 3.1. Approximation of translational columns

The translational column theory ( $\theta_1(s) = \theta_2(s) = \theta(s)$ ) can be considered as the zero order approximation of the geometrically exact parallel column theory. In the case of translational column theory ( $\theta_1(s) = \theta_2(s) = \theta(s)$ ), the total potential energy of the structural problem can be written as

$$U[\theta] = \int_0^L \frac{1}{2} El_0 \theta'^2 + \frac{1}{2} k (h_0 \theta)^2 ds - P \left[ L - \int_0^L \cos \theta(s) ds \right] \text{ with} \quad (15)$$

$$El_0 = El_1 + El_2, \quad P = P_1 + P_2 \text{ and } h_0 = \frac{h_1 + h_2}{2}$$

The sum of the two axial forces  $P_1$  and  $P_2$  is equal to the total axial load denoted by  $P$ .  $El_0$  is the total bending stiffness, and  $h_0$  is the distance between the two theoretically parallel columns. With this simplified formulation, the slip  $u(s)$  is simply equal to  $h_0 \theta(s)$ , a result already anticipated by Budd et al. [3] for two identical layers. The main assumptions of this structural model are the axial and the shear inextensibility of each column. It is also assumed that the constraint between each column is a translational constraint. As pointed out in the above part, this strong assumption is a simplification of the exact problem. Note that the assumption of translational columns is generally assumed in the literature, even for large displacement analysis (see for instance [9] for the general geometrically non-linear framework, even if Cas et al. [9] only presented results for linearized buckling values). The governing equations of the problem are obtained from variational arguments (see the book of Buttazzo et al. [36] for general calculus of variation applied to one-dimensional media).

The stationary of the total potential energy is written as

$$\delta U[\theta] = \int_0^L El_0 \theta' \delta \theta' + kh_0^2 \theta \delta \theta - P \sin \theta \delta \theta ds = 0 \quad (16)$$

The non-linear differential equation is then obtained after an integration by part as

$$El_0 \theta'' - kh_0^2 \theta + P \sin \theta = 0 \quad (17)$$

with the essential and natural boundary conditions given by

$$[El_0 \theta' \delta \theta]_0^L = 0 \quad (18)$$

For the clamped-free case studied in this paper, the boundary conditions are

$$\theta(0) = \theta'(L) = 0 \quad (19)$$

The mathematical problem will now be expressed in dimensionless format with the following dimensionless parameters:

$$\bar{s} = \frac{s}{L}, \quad \beta = \frac{PL^2}{El_0} \text{ and } \kappa = \frac{kh_0^2 L}{El_0} \quad (20)$$

where  $\bar{s}$  is the dimensionless abscissa,  $\bar{s} \in [0; 1]$ ,  $\beta$  is the dimensionless load parameter and  $\kappa$  is the dimensionless connection parameter. Full composite action ( $\kappa \rightarrow \infty$ ) and non-composite action ( $\kappa \rightarrow 0$ ) represent upper and lower bounds for the partial composite action, respectively. The *elastica* problem of Euler [1] is covered in the case of non-composite action. For the partially composite column,  $\kappa$  is a dimensionless connection parameter that depends upon the shear connection parameter and the column characteristics. This is the same for the sandwich analogy where the dimensionless parameter  $\kappa$  directly depends on the shear stiffness of the soft core and characteristics of the faces.

The non-linear differential equation Eq. (17) is finally obtained in dimensionless form as

$$\frac{d^2 \theta}{d\bar{s}^2} = -\beta \sin \theta + \kappa \theta \quad (21)$$

with the boundary conditions

$$\theta(\bar{s} = 0) = \frac{d\theta}{d\bar{s}}(\bar{s} = 1) = 0 \quad (22)$$

#### 3.2. Buckling load—linearized problem

The initial buckling load can be calculated from the linearized equations given by

$$\frac{d^2 \theta}{d\bar{s}^2} + (\beta - \kappa) \theta = 0 \text{ with } \theta(\bar{s} = 0) = \frac{d\theta}{d\bar{s}}(\bar{s} = 1) = 0 \quad (23)$$

whose solution is expressed in the trigonometric format as

$$\theta(\bar{s}) = A \cos \sqrt{\beta - \kappa} \bar{s} + B \sin \sqrt{\beta - \kappa} \bar{s} \quad (24)$$

The buckling mode of the linearized system is then classically obtained from the boundary conditions as

$$\theta(\bar{s}) = \alpha \sin \left( \frac{\pi}{2} \bar{s} \right) \quad (25)$$

where  $\alpha$  is the rotation at the end of the column. The dimensionless buckling load  $\beta_c$  is then calculated as

$$\beta_c = \frac{\pi^2}{4} + \kappa \quad (26)$$

This dimensionless buckling load exactly corresponds to the solution of Hoff and Mautner [19] for sandwich column or Möhler [6] for inextensible partially composite columns (infinite axial stiffness). This is also the solution of Girhammar and Gopu [4] or Girhammar and Pan [5] in case of infinite axial stiffness (inextensibility assumption).

### 3.3. Post-buckling behavior

The post-buckling behavior is now investigated from the geometrically exact equations obtained in the approximated framework of translational columns. The following methodology is identical to the one of the usual non-composite *elastica* investigated by Euler [1]—see also [37] or [38]. The non-linear differential equation Eq. (21) is first multiplied by the derivative of the rotation:

$$\frac{d^2\theta}{d\bar{s}^2} \frac{d\theta}{d\bar{s}} = -\beta \sin\theta \frac{d\theta}{d\bar{s}} + \kappa\theta \frac{d\theta}{d\bar{s}} \quad (27)$$

An integration of Eq. (27) gives the first-order differential equation

$$\frac{1}{2} \left( \frac{d\theta}{d\bar{s}} \right)^2 = \beta \cos\theta + \frac{\kappa}{2} \theta^2 + C \quad (28)$$

It is possible to express the integration constant with respect to the rotation of the column end, denoted by  $\alpha$

$$\theta(\bar{s}=1) = \alpha \Rightarrow C = -\beta \cos\alpha - \frac{\kappa}{2} \alpha^2 \quad (29)$$

Therefore, the first order differential equation can be also expressed as

$$\frac{d\theta}{d\bar{s}} = \sqrt{2\beta(\cos\theta - \cos\alpha) + \kappa(\theta^2 - \alpha^2)} \quad (30)$$

It can be checked that the square root is well defined for the post-buckling path characterized by a pitchfork bifurcation-type, where

$$\beta \geq \beta_c = \frac{\pi^2}{4} + \kappa \quad (31)$$

The term appearing in the square root of Eq. (30) is then developed in asymptotic expansion

$$\begin{aligned} 2\beta(\cos\theta - \cos\alpha) + \kappa(\theta^2 - \alpha^2) &\approx 2\beta \left( \frac{\alpha^2}{2} - \frac{\theta^2}{2} \right) + \kappa(\theta^2 - \alpha^2) \\ &\geq \frac{\pi^2}{4} (\alpha^2 - \theta^2) \geq 0 \end{aligned} \quad (32)$$

Therefore, the square root is well defined for the post-buckling path considered. The differential equation can be inverted in

$$\frac{d\bar{s}}{d\theta} = \frac{1}{\sqrt{2\beta(\cos\theta - \cos\alpha) + \kappa(\theta^2 - \alpha^2)}} \quad (33)$$

The post-buckling behavior can be finally expressed in the integral format:

$$1 = \int_0^\alpha \frac{1}{\sqrt{2\beta(\cos\theta - \cos\alpha) + \kappa(\theta^2 - \alpha^2)}} d\theta \quad (34)$$

Eq. (34) is also equivalent to the following formulae:

$$1 = \int_0^\alpha \frac{1}{\sqrt{4\beta(\sin^2(\alpha/2) - \sin^2(\theta/2)) + \kappa(\theta^2 - \alpha^2)}} d\theta \quad (35)$$

### 3.4. Approximation of translational columns, closed form solutions

In case of non-composite action, the dimensionless parameter  $\kappa$  is vanishing and the integral equation is simplified in

$$2\sqrt{\beta} = \int_0^\alpha \frac{1}{\sqrt{\sin^2(\alpha/2) - \sin^2(\theta/2)}} d\theta \quad (36)$$

The following change of variable can be considered to simplify the integration procedure:

$$p = \sin \frac{\alpha}{2} \quad \text{and} \quad p \sin \varphi = \sin \frac{\theta}{2} \quad (37)$$

**Table 1**

Numerical values of the symmetrically stable post-buckling path.

$\kappa$	0	10	100	1000
$\delta$	$\alpha$	$\alpha$	$\alpha$	$\alpha$
0.01	0.282	0.282	0.280	0.265
0.02	0.397	0.397	0.391	0.363
0.03	0.485	0.484	0.474	0.436
0.04	0.558	0.556	0.541	0.498
0.05	0.622	0.619	0.600	0.552
0.075	0.755	0.750	0.718	0.665
0.10	0.864	0.858	0.814	0.758
0.125	0.959	0.949	0.894	0.839
0.15	1.042	1.030	0.965	0.911

In case of non-composite action ( $\kappa=0$ ), the elastica problem is governed by a complete integral of the first kind, from

$$\sqrt{\beta} = \int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{1-p^2 \sin^2 \varphi}} d\varphi = K(p) \quad \text{with} \quad p = \sin \frac{\alpha}{2} \quad (38)$$

### 3.5. Numerical results

The computation of the non-linear boundary value problem is based on the MATLAB program *bvp4c*, a finite difference code that implements the three-stage Lobatto IIIa formula. This is a collocation formula and the collocation polynomial provides a C1-continuous solution that is fourth-order accurate, uniformly in the constant interval domain (Kierzenka and Shampine [39]). Mesh selection and error control are based on the residual of the continuous solution. Lobatto IIIa methods have been considered for boundary value problems due to their good stability properties [40]. Such a numerical method has been already used by Challamel [41] for initially post-buckling of an elastic column on a gradient foundation, within a geometrically exact framework. The non-linear second order differential equation Eq. (21) with the boundary conditions Eq. (22) (clamped-free boundary conditions) can be presented as a first-order differential equation:

$$\begin{aligned} \frac{d}{d\bar{s}} \begin{pmatrix} X^{(2)} \\ X^{(1)} \end{pmatrix} &= \begin{pmatrix} -\beta \sin X^{(1)} + \kappa X^{(1)} \\ X^{(2)} \end{pmatrix} \quad \text{with} \quad \begin{pmatrix} X^{(2)} \\ X^{(1)} \end{pmatrix} = \begin{pmatrix} \theta' \\ \theta \end{pmatrix} \quad \text{and} \\ X^{(1)}(0) = X^{(2)}(1) = 0; \quad X^{(1)}(1) = \alpha \end{aligned} \quad (39)$$

The pitchfork bifurcation is characterized by a dimensionless load parameter  $\beta$  greater than the dimensionless buckling load  $\beta_c$ . A dimensionless positive factor  $\beta$  can be introduced for the parametric numerical study

$$\beta = \beta_c(1 + \delta) \quad \text{with} \quad \delta \geq 0 \quad (40)$$

The numerical values are given in Table 1. The pitchfork bifurcation, composed of two stable symmetrical branches, is clearly shown in Figs. 4 and 5. The normalized post-buckling branches are seen to be clearly affected by the dimensionless connection parameter  $\kappa$ .

## 4. Asymptotic expansion of the post-buckling path

The post-buckling behavior of the partially composite column is investigated using an asymptotic expansion as given by

$$\begin{cases} \beta = \beta_0 + \varepsilon \beta_1 + \varepsilon^2 \beta_2 + \varepsilon^3 \beta_3 + \varepsilon^4 \beta_4 + \varepsilon^5 \beta_5 + \dots \\ \theta = \vartheta_0 + \varepsilon \vartheta_1 + \varepsilon^2 \vartheta_2 + \varepsilon^3 \vartheta_3 + \varepsilon^4 \vartheta_4 + \varepsilon^5 \vartheta_5 + \dots \end{cases} \quad (41)$$

where  $\varepsilon$  is a small parameter whose meaning will be developed later. The methodology is the same as the one presented by Thompson and Hunt [42] for the purely *elastica* problem (see also [43]). The non-linear differential equation is composed of

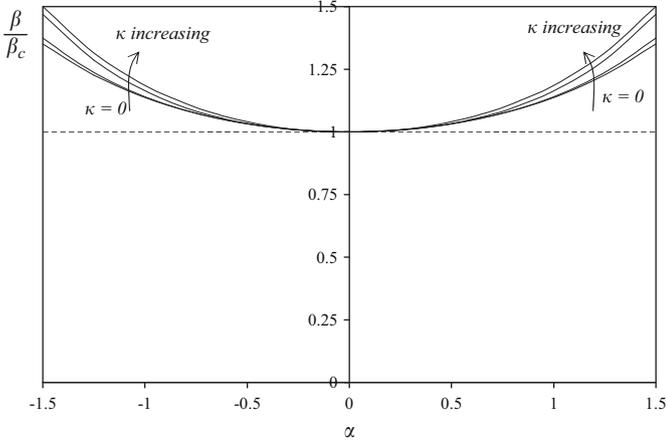


Fig. 4. Symmetrically stable post-buckling path; pitchfork bifurcation diagram;  $\kappa \in \{0; 10; 100; 1000\}$ .

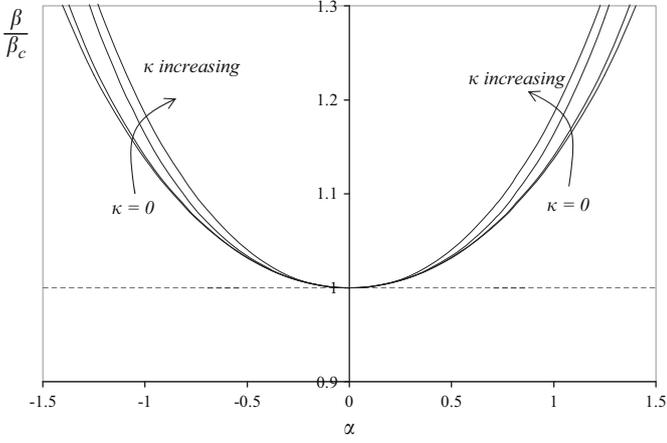


Fig. 5. Effect of the dimensionless connection parameter  $\kappa \in \{0; 10; 100; 1000\}$  on the symmetrically stable post-buckling path.

a sinusoidal term that can be developed using this asymptotic expression

$$\begin{aligned} \sin \theta = & \sin \vartheta_0 \left[ 1 - \frac{\vartheta_1^2}{2} \varepsilon^2 - \vartheta_1 \vartheta_2 \varepsilon^3 + \frac{\vartheta_1^4}{24} - \frac{\vartheta_2^2}{2} - \vartheta_1 \vartheta_3 \right] \varepsilon^4 \\ & + \frac{\vartheta_1^3 \vartheta_2}{6} - \vartheta_1 \vartheta_4 - \vartheta_2 \vartheta_3 \left] \varepsilon^5 \right] + \cos \vartheta_0 \left[ \vartheta_1 \varepsilon + \vartheta_2 \varepsilon^2 \right. \\ & + \left. \vartheta_3 - \frac{\vartheta_1^3}{6} \right] \varepsilon^3 + \left. \vartheta_4 - \frac{\vartheta_1^2 \vartheta_2}{2} \right] \varepsilon^4 \\ & + \left. \vartheta_5 - \frac{\vartheta_1 \vartheta_2^2}{2} - \frac{\vartheta_2^2 \vartheta_3}{2} + \frac{\vartheta_1^5}{120} \right] \varepsilon^5 + \dots \end{aligned} \quad (42)$$

The fundamental path is characterized by no prebuckling deformation

$$\vartheta_0 = 0 \quad (43)$$

This leads to the simplification of the trigonometric term as

$$\begin{aligned} \sin \theta = & \vartheta_1 \varepsilon + \vartheta_2 \varepsilon^2 + \left. \vartheta_3 - \frac{\vartheta_1^3}{6} \right] \varepsilon^3 + \left. \vartheta_4 - \frac{\vartheta_1^2 \vartheta_2}{2} \right] \varepsilon^4 \\ & + \left. \vartheta_5 - \frac{\vartheta_1 \vartheta_2^2}{2} - \frac{\vartheta_2^2 \vartheta_3}{2} + \frac{\vartheta_1^5}{120} \right] \varepsilon^5 + \dots \end{aligned} \quad (44)$$

The term  $\beta \sin \theta$  is then calculated as

$$\begin{aligned} \beta \sin \theta = & \beta_0 \vartheta_1 \varepsilon + (\beta_0 \vartheta_2 + \beta_1 \vartheta_1) \varepsilon^2 + \left. \beta_0 \vartheta_3 - \beta_0 \frac{\vartheta_1^3}{6} + \beta_1 \vartheta_2 + \beta_2 \vartheta_1 \right] \varepsilon^3 \\ & + \left. \beta_0 \vartheta_4 - \beta_0 \frac{\vartheta_1^2 \vartheta_2}{2} + \beta_1 \vartheta_3 - \beta_1 \frac{\vartheta_1^3}{6} + \beta_2 \vartheta_2 + \beta_3 \vartheta_1 \right] \varepsilon^4 \\ & + \left. \beta_0 \vartheta_5 - \beta_0 \frac{\vartheta_1 \vartheta_2^2}{2} - \beta_0 \frac{\vartheta_2^2 \vartheta_3}{2} + \beta_0 \frac{\vartheta_1^5}{120} + \beta_1 \vartheta_4 - \beta_1 \frac{\vartheta_1^2 \vartheta_2}{2} \right. \\ & \left. + \beta_2 \vartheta_3 - \beta_2 \frac{\vartheta_1^3}{6} + \beta_3 \vartheta_2 + \beta_4 \vartheta_1 \right] \varepsilon^5 + \dots \end{aligned} \quad (45)$$

Inserting this asymptotic expansion into the non-linear differential equation Eq. (21) and considering each power of the small parameter  $\varepsilon$  leads to the following system of five differential equations:

$$\begin{cases} \vartheta_1'' + (\beta_0 - \kappa) \vartheta_1 = 0 \\ \vartheta_2'' + (\beta_0 - \kappa) \vartheta_2 = -\beta_1 \vartheta_1 \\ \vartheta_3'' + (\beta_0 - \kappa) \vartheta_3 = -\beta_1 \vartheta_2 - \beta_2 \vartheta_1 + \beta_0 \frac{\vartheta_1^3}{6} \\ \vartheta_4'' + (\beta_0 - \kappa) \vartheta_4 = -\beta_1 \vartheta_3 - \beta_3 \vartheta_1 - \beta_2 \vartheta_2 + \beta_1 \frac{\vartheta_1^3}{6} + \beta_0 \frac{\vartheta_1^2 \vartheta_2}{2} \\ \vartheta_5'' + (\beta_0 - \kappa) \vartheta_5 = -\beta_1 \vartheta_4 - \beta_3 \vartheta_2 - \beta_2 \vartheta_3 - \beta_4 \vartheta_1 + \beta_2 \frac{\vartheta_1^3}{6} + \beta_0 \frac{\vartheta_1 \vartheta_2^2}{2} + \beta_0 \frac{\vartheta_2^2 \vartheta_3}{2} + \beta_1 \frac{\vartheta_1^2 \vartheta_2}{2} - \beta_0 \frac{\vartheta_1^5}{120} \end{cases} \quad (46)$$

associated with the boundary conditions for  $i \in \{1 \dots 5\}$ :

$$\vartheta_i(0) = 0 \quad \text{and} \quad \vartheta_i'(1) = 0 \quad (47)$$

with the normalization procedure (see also [42]) defined by

$$\vartheta_i(1) = 1 \quad \text{and} \quad \vartheta_i(1) = 0 \quad \text{for} \quad i \geq 2 \quad (48)$$

The first differential equation gives the linearized buckling mode

$$\vartheta_1(\bar{s}) = \sin\left(\frac{\pi}{2} \bar{s}\right) \quad \text{with} \quad \beta_0 = \beta_c = \frac{\pi^2}{4} + \kappa \quad (49)$$

For symmetrical reasons, it can be shown that some terms are vanishing in the asymptotic expansion as follows:

$$\text{for } k \geq 1, \quad \vartheta_{2k}(\bar{s}) = 0 \quad \text{and} \quad \beta_{2k-1} = 0 \quad (50)$$

The asymptotic characterization of the post-buckling path up to the fifth order is then reduced to the resolution of two differential equations

$$\begin{cases} \vartheta_3'' + \frac{\pi^2}{4} \vartheta_3 = -\beta_2 \vartheta_1 + \beta_0 \frac{\vartheta_1^3}{6} \\ \vartheta_5'' + \frac{\pi^2}{4} \vartheta_5 = -\beta_2 \vartheta_3 - \beta_4 \vartheta_1 + \beta_2 \frac{\vartheta_1^3}{6} + \beta_0 \frac{\vartheta_1^2 \vartheta_2}{2} - \beta_0 \frac{\vartheta_1^5}{120} \end{cases} \quad (51)$$

with  $\vartheta_1(\bar{s}) = \sin\left(\frac{\pi}{2} \bar{s}\right)$

The third-order term  $\vartheta_3$  is obtained from the resolution of a linear differential equation

$$\vartheta_3'' + \frac{\pi^2}{4} \vartheta_3 = \left(\frac{\beta_0}{8} - \beta_2\right) \sin\left(\frac{\pi}{2} \bar{s}\right) - \frac{\beta_0}{24} \sin\left(\frac{3\pi}{2} \bar{s}\right) \quad (52)$$

The general solution of such a differential equation is written as

$$\begin{aligned} \vartheta_3(\bar{s}) = & A \cos\left(\frac{\pi}{2} \bar{s}\right) + B \sin\left(\frac{\pi}{2} \bar{s}\right) + \frac{\bar{s}}{\pi} \left( \beta_2 - \frac{\beta_0}{8} \right) \cos\left(\frac{\pi}{2} \bar{s}\right) \\ & + \frac{\beta_0}{48\pi^2} \sin\left(\frac{3\pi}{2} \bar{s}\right) \end{aligned} \quad (53)$$

Including the boundary conditions and the normalization condition leads to the following solution:

$$\vartheta_3(\bar{s}) = \frac{\beta_0}{48\pi^2} \left[ \sin\left(\frac{\pi}{2} \bar{s}\right) + \sin\left(\frac{3\pi}{2} \bar{s}\right) \right] \quad \text{and} \quad \beta_2 = \frac{\beta_0}{8} \quad (54)$$

The fifth-order term in the differential equation  $\vartheta_3$  is obtained from the resolution of a linear differential equation Eq. (51) which

can be written as

$$\vartheta_5'' + \frac{\pi^2}{4} \vartheta_5 = \frac{1}{496\pi^2} \left( \frac{\beta_0^2}{96} + \beta_0 - \beta_4 \right) \sin\left(\frac{\pi \bar{s}}{2}\right) - \frac{\beta_0}{320} \sin\left(\frac{3\pi \bar{s}}{2}\right) - \frac{1}{496\pi^2} \left( \frac{\beta_0^2}{16} + \frac{\beta_0}{120} \right) \sin\left(\frac{5\pi \bar{s}}{2}\right) \quad (55)$$

Inserting the boundary conditions leads to the fourth-order loading term:

$$\beta_4 = \frac{\beta_0}{96} \left( 1 + \frac{\beta_0}{4\pi^2} \right) \quad (56)$$

The normalization procedure has been based on

$$\alpha = \theta(1) = \varepsilon \quad (57)$$

Hence, the small parameter  $\varepsilon$  has the meaning of the tip rotation  $\alpha$ . The asymptotic expansion of the loading parameter is then written as

$$\frac{\beta}{\beta_c} = 1 + \frac{\alpha^2}{8} + \frac{\alpha^4}{96} \left( \frac{17}{16} + \frac{\kappa}{4\pi^2} \right) + \dots \quad (58)$$

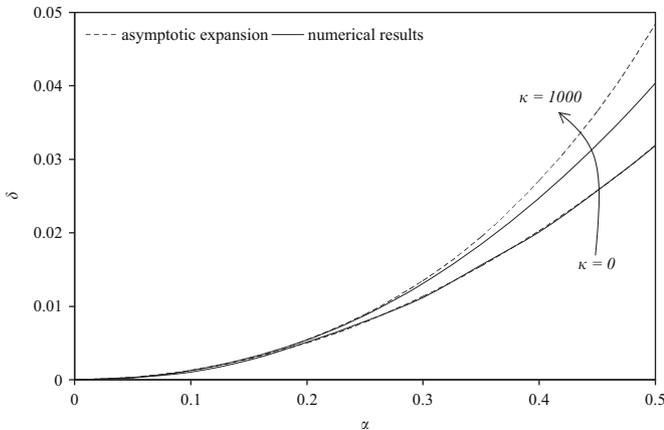
It can be seen that the second-order term related to the normalized dimensionless load is independent of the connection parameter  $\kappa$  that only appears on the fourth-order term. Of course, the buckling load  $\beta_c$  is an increasing function of the connection parameter  $\kappa$ . The post-buckling path of the non-composite column (*elastica problem*) is written as

$$\kappa = 0 \Rightarrow \frac{\beta}{\beta_c} = 1 + \frac{\alpha^2}{8} + \frac{17\alpha^4}{16 \cdot 96} + \dots \quad (59)$$

Finally, the post-buckling path can be also expressed using the parameter  $\delta$  as

$$\delta = \frac{\beta}{\beta_c} - 1 = \frac{\alpha^2}{8} + \frac{\alpha^4}{96} \left( \frac{17}{16} + \frac{\kappa}{4\pi^2} \right) + \dots \quad (60)$$

The validity of this asymptotic expansion with respect to the exact numerical solution is given in Fig. 6. The asymptotic expansion is better for small values of the connection parameter  $\kappa$ . It clearly appears that the pitchfork bifurcation has two stable post-bifurcation branches, whatever the connection parameter value. It is also remarkable that the dimensionless connection parameter only affects the fourth-order term in Eq. (60). Hence, for the post-buckling behavior of the inextensible partially composite elastica with this loading mode, only stable post-bifurcation branches are observed and associated with the imperfection insensitivity phenomenon. Interestingly, Budd et al. [3] suggest the possible use of negative connection stiffness, leading to some



**Fig. 6.** Asymptotic expansion of the post-buckling path, comparison with the exact numerical solution  $\kappa \in \{0; 1000\}$ .

possible softening phenomena in the post-bifurcation range for the present developed model.

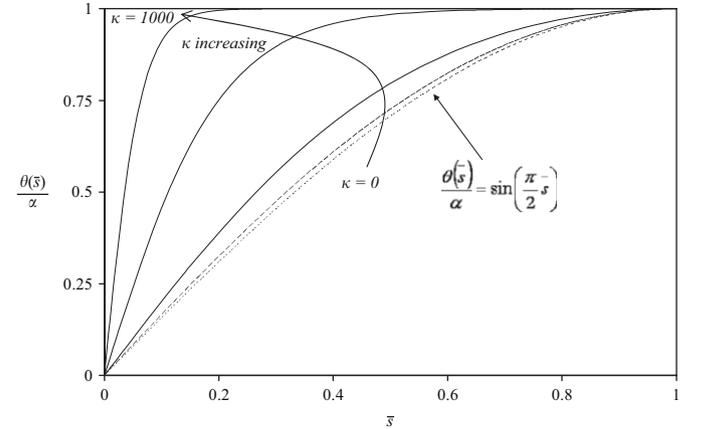
## 5. Boundary layer phenomenon

A boundary layer phenomenon is observed for large values of the dimensionless connection parameter  $\kappa$ , i.e. for full composite action (see Figs. 7 and 8). Boundary layer phenomenon for a stiff connection has already been observed for the linearized bending behavior of partially composite beams (Challamel and Girhammar [44]). In the present paper, the phenomenon appears during the post-buckling range of the partially composite column. Boundary layers are regions in which a rapid change occurs in the value of a variable, namely the rotation of the partially composite column near the clamped section. Mathematically, the occurrence of a boundary layer is associated with the presence of a small parameter multiplying the highest derivative ([45,46]). This is confirmed from the introduction of the small parameter  $\gamma$  for very stiff connection, defined by

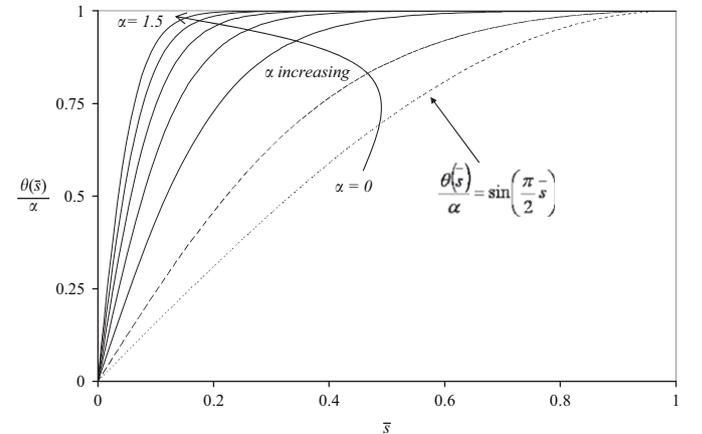
$$\gamma = \frac{1}{\sqrt{\kappa}} \quad (61)$$

The non-linear differential equation Eq. (21) can then be expressed as

$$\gamma^2 \frac{d^2 \theta}{d\bar{s}^2} + \beta \gamma^2 \sin \theta - \theta = 0 \quad (62)$$



**Fig. 7.** Boundary layer phenomenon for stiff connection;  $\kappa \in \{0; 10; 100; 1000\}$ ;  $\alpha = 1.5$ .



**Fig. 8.** Propagation of the boundary layer during the post-buckling range;  $\alpha \in \{0; 0.25; 0.5; 0.75; 1; 1.25; 1.5\}$ ;  $\kappa = 1000$ .

The second derivative in the non-linear differential equation is affected by the small terms, leading to the boundary layer phenomenon. The dimensionless parameter  $\tau$  can be introduced as

$$\tau = \frac{\bar{s}}{\gamma} \quad \text{with} \quad \gamma = \frac{1}{\sqrt{\kappa}} \quad (63)$$

The buckling problem is then expressed by

$$\frac{d^2\theta}{d\tau^2} + \hat{\beta} \sin\theta - \theta = 0 \quad \text{with} \quad \hat{\beta} = \beta\gamma^2 \quad (64)$$

The post-buckling behavior of the "stiff" partially composite column is investigated using an asymptotic expansion

$$\begin{cases} \hat{\beta} = \hat{\beta}_0 + \gamma\hat{\beta}_1 + \dots \\ \theta = \theta_0 + \gamma\theta_1 + \dots \end{cases} \quad (65)$$

Note that  $\theta$  is not necessarily small, the small parameter  $\gamma$  being a structural parameter (linked to the connection between the two columns). Introducing the asymptotic expansion into the differential equation Eq. (64) leads to the following system of differential equations:

$$\begin{cases} \frac{d^2\theta_0}{d\tau^2} + \hat{\beta}_0 \sin\theta_0 - \theta_0 = 0 \\ \frac{d^2\theta_1}{d\tau^2} + (\hat{\beta}_0 \cos\theta_0 - 1)\theta_1 = -\hat{\beta}_1 \sin\theta_0 \end{cases} \quad (66)$$

The zero-order solution is the uniform solution (that does not verify the boundary conditions at the clamped section) and is defined by

$$\theta_0(\tau) = \alpha \Rightarrow \hat{\beta}_0 = \frac{\alpha}{\sin\alpha} \quad (67)$$

The first-order solution is then obtained from resolution of a linear differential equation

$$\frac{d^2\theta_1}{d\tau^2} - \left(1 - \frac{\alpha}{\tan\alpha}\right)\theta_1 = -\hat{\beta}_1 \sin\alpha \quad (68)$$

Thanks to the mathematical property

$$\alpha \leq \tan\alpha \quad \text{for} \quad \alpha \in \left[0; \frac{\pi}{2}\right] \quad (69)$$

the solution of Eq. (68) can be expressed in the exponential format given by

$$\theta_1(\tau) = C_1 \cosh\left(\sqrt{1 - \frac{\alpha}{\tan\alpha}}\tau\right) + C_2 \sinh\left(\sqrt{1 - \frac{\alpha}{\tan\alpha}}\tau\right) + \hat{\beta}_1 \frac{\sin\alpha}{1 - (\alpha/\tan\alpha)} \quad (70)$$

Prandtl's matching condition is written for this problem as (see for instance [44,45], or [46])

$$\lim_{\tau \rightarrow \infty} \theta_1(\tau) = 0 \Rightarrow C_2 = -C_1 \quad \text{and} \quad \hat{\beta}_1 = 0 \quad (71)$$

The boundary condition at the clamped section is used for the approximated solution in the boundary layer

$$\theta(0) = 0 \Rightarrow \theta_1(0) = -\frac{\alpha}{\gamma} \quad (72)$$

The asymptotic solution in the boundary layer is finally written as

$$\begin{aligned} \frac{\theta(t)}{\alpha} &= 1 - e^{-\sqrt{1 - (\alpha/\tan\alpha)}\tau} \quad \text{or equivalently} \\ \frac{\theta(\bar{s})}{\alpha} &= 1 - e^{-\sqrt{1 - (\alpha/\tan\alpha)}\sqrt{\kappa}\bar{s}} \end{aligned} \quad (73)$$

Fig. 9 shows the comparison between the approximate asymptotic solution in the boundary layer (based on Eq. (73)) with the exact numerical solution. The approximate solution is relevant for sufficiently large values of the dimensionless connection parameter  $\kappa$ , i.e. for sufficiently small values of the  $\gamma$  parameter.

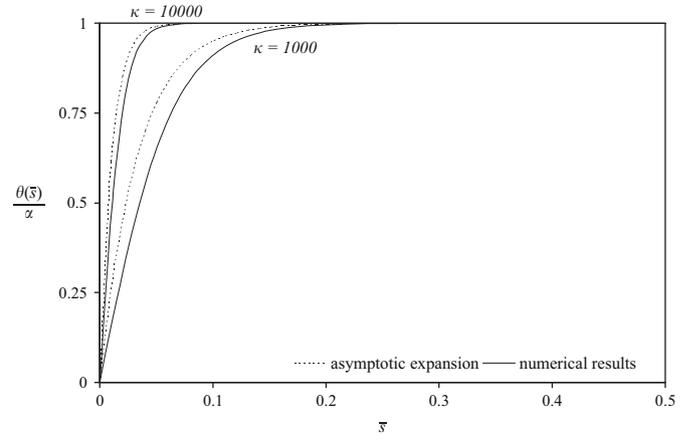


Fig. 9. Asymptotic expansion of the boundary layer;  $\kappa \in \{1000; 10000\}$ ;  $\alpha = 1.5$ .

## 6. Imperfection analysis with loading eccentricities

### 6.1. Pre-bending effects

Following the methodology of Challamel and Girhammar [13], the effect of some eccentricities in the loading mode is investigated in this part. In practical applications, it is not uncommon that the axial forces are introduced in the composite column with some eccentricities. Eccentric axial loadings are referred to load configurations when the individual axial loads are applied on the individual sub-elements outside their individual centroids. These eccentricities may also be thought as an imperfection of the loading mode.

Consider the load configuration in Fig. 10 (see also [13]). The individual axial loads ( $P_1$  and  $P_2$ ) are applied outside on the opposing sides of the centroid of each sub-element with some eccentricities denoted by  $e_1$  and  $e_2$ . This imperfect problem can be solved by adding a correction term in the total energy of the system given by Eq. (15) as

$$\begin{aligned} U[\theta] &= \int_0^L \frac{1}{2} EI_0 \theta'^2 + \frac{1}{2} k (h_0 \theta)^2 ds - P \left[ L - \int_0^L \cos\theta(s) ds \right] \\ &\quad + (P_1 e_1 - P_2 e_2) \sin\theta(L) \quad \text{with} \\ EI_0 &= EI_1 + EI_2, \quad P = P_1 + P_2 \quad \text{and} \quad h_0 = \frac{h_1 + h_2}{2} \end{aligned} \quad (74)$$

As shown by Challamel and Girhammar [13] for the linearized analysis, this problem is equivalent to the perfect one if the eccentricities are related by the following mathematical relationship:

$$\frac{e_1}{e_2} = \frac{P_2}{P_1} \quad (75)$$

The stationarity of the total potential energy  $\delta U = 0$  leads to the non-linear differential equation Eq. (17) with the following modified boundary conditions:

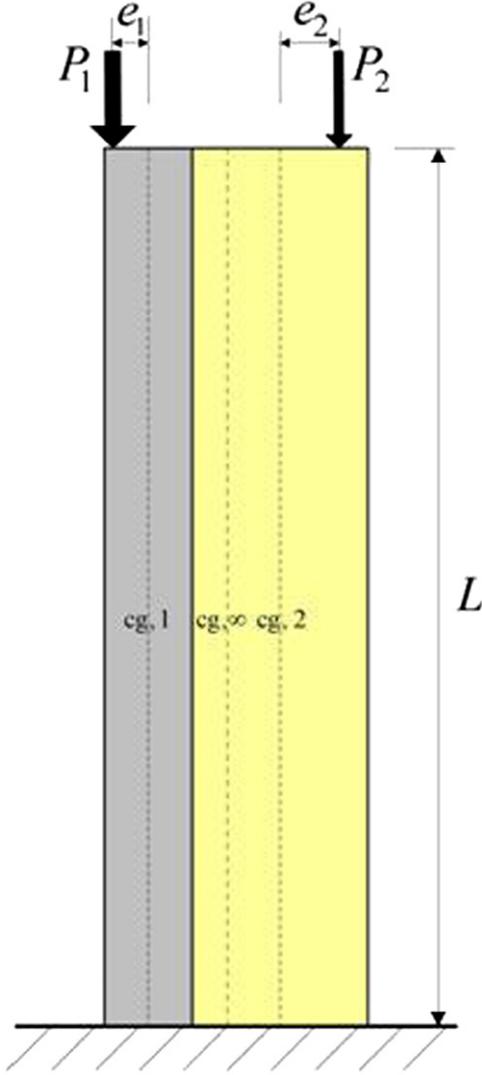
$$[EI_0 \theta' \delta\theta]_0^L + (P_1 e_1 - P_2 e_2) \cos\theta(L) \delta\theta(L) = 0 \quad (76)$$

These boundary conditions can be expressed for the clamped-free column studied in this paper as

$$\theta(0) = 0 \quad \text{and} \quad EI_0 \theta'(L) = (P_2 e_2 - P_1 e_1) \cos\theta(L) \quad (77)$$

Such a mixed boundary condition has been already considered by Vinogradov and Derrick [14] with one single load (for instance  $P_1 = 0$ ). The equivalent eccentricity can be introduced from

$$e_0 = \frac{P_2 e_2 - P_1 e_1}{P_1 + P_2} \quad (78)$$

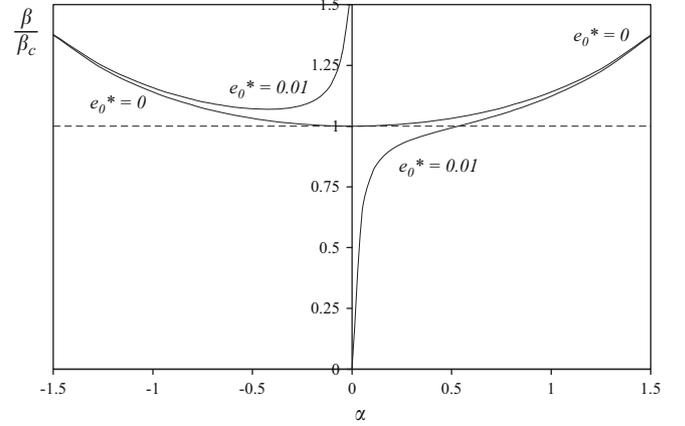


**Fig. 10.** A partially composite beam-column subjected to two axial loads applied at opposing sides of the centroid of the sub-elements inducing opposing eccentricity moments at the top.

Therefore, the boundary conditions can be expressed in a dimensionless format as

$$\theta(\bar{s}=0) = 0 \text{ and } \frac{d\theta}{d\bar{s}}(\bar{s}=1) = \beta e_0^* \cos \theta(\bar{s}=1) \text{ with } e_0^* = \frac{e_0}{L} \quad (79)$$

The dimensionless non-linear differential equation is still given by Eq. (21). Above,  $e_0^*$  is an imperfection parameter that controls the pre-bending behavior of the partially composite column. The influence of this imperfection parameter is shown in Fig. 11, obtained by numerical calculation of the non-linear boundary value problem for  $\kappa=10$ . The imperfection parameter breaks the internal symmetry of the problem. A significant difference can be noticed between the second-order analysis detailed in [13] (including the axial extensibility assumption) and the present geometrically exact theory (assuming the inextensibility assumption), where the post-buckling path of the imperfect problem in the present paper does not converge towards the initial constant buckling load, but asymptotically follows the hardening post-buckling path of the perfect system.



**Fig. 11.** Influence of a small eccentricity on the response of the partially composite column; imperfection analysis— $\kappa=10$ ;  $e_0^* \in \{0; 0.01\}$ .

## 6.2. Some remarks on the application of the equilibrium method

Following the methodology of Kryżanowski et al. [32], the direct equilibrium equations can be also used to obtain the differential equations of the problem. The two sub-columns can be considered separately and are loaded by some distributed tangential load, normal load and also a distributed torque. Hence, each sub-column is solicited by some follower loading, the contribution at the global scale being classified as a conservative problem. The equilibrium equations can be written as

$$\begin{cases} M_1' = R_x^1 \sin \theta_1 + R_y^1 \cos \theta_1 - m_1 \\ R_x^1 = -q_1 \cos \theta_1 - p_1 \sin \theta_1 \\ R_y^1 = q_1 \sin \theta_1 - p_1 \cos \theta_1 \\ M_2' = R_x^2 \sin \theta_2 + R_y^2 \cos \theta_2 - m_2 \\ R_x^2 = q_2 \cos \theta_2 + p_2 \sin \theta_2 \\ R_y^2 = -q_2 \sin \theta_2 + p_2 \cos \theta_2 \end{cases} \quad (80)$$

where the internal forces are defined at the centroid of each beam. Here,  $q_1$  (respectively  $q_2$ ) is the distributed follower tangential load,  $p_1$  (respectively  $p_2$ ) is the distributed follower normal load, and  $m_1$  (respectively  $m_2$ ) is the distributed torque.

The constitutive law of each beam and of the interlayer is given by

$$M_1 = EI_1 \theta_1', \quad M_2 = EI_2 \theta_2' \quad \text{and} \quad q_1 = k \left( \frac{h_1}{2} \theta_1 + \frac{h_2}{2} \theta_2 \right) \quad (81)$$

The distributed torque acting in each beam are obtained from equilibrium:

$$m_1 = q_1 \frac{h_1}{2} \quad \text{and} \quad m_2 = q_2 \frac{h_2}{2} \quad (82)$$

For the loading case of Fig. 3 with applied load at the centroid of each column, the loading boundary conditions at the top of the column are given by

$$R_x^1(L) = -P_1 \quad \text{and} \quad R_x^2(L) = -P_2 \quad (83)$$

A kinematic constraint should be added to identify the unknown normal reaction denoted by  $p_1$  or  $p_2$ . The reaction can be associated with a Lagrange multiplier of the system. The translated beam theory ( $\theta_1(s) = \theta_2(s) = \theta(s)$ ) can be considered as the zero order approximation of the geometrically exact parallel beams theory. In this case, we have

$$\theta_1 = \theta_2 \Rightarrow \begin{cases} q_2 = q_1 \\ p_2 = p_1 \end{cases} \quad (84)$$

The force equilibrium equations in Eq. (80) show that

$$\begin{cases} R_x^1 + R_x^2 = 0 \\ R_y^1 + R_y^2 = 0 \end{cases} \Rightarrow \begin{cases} R_x^1 + R_x^2 = -P_1 - P_2 = -P_0 \\ R_y^1 + R_y^2 = 0 \end{cases} \quad (85)$$

The sum of the moment equation leads to

$$M_1' + M_2' = (R_x^1 + R_x^2)\sin\theta + (R_y^1 + R_y^2)\cos\theta - m_1 - m_2 \quad (86)$$

Then, the differential equation of Eq. (17) is found again as

$$EI_0\theta'' = -P_0\sin\theta - m_0 \quad \text{with } m_0 = m_1 + m_2 = qh_0 = kh_0^2\theta \quad (87)$$

The direct equilibrium method is strictly equivalent to the energy-based equation introduced at the beginning of the paper.

## 7. Summary and conclusions

This paper is devoted to the elastic buckling and post-buckling of a partially composite column. Each sub-column is modeled with the Euler–Bernoulli beam theory and connected to each other via a linear constitutive law for the interlayer slip. A variational formulation is presented in order to derive relevant boundary conditions. The post-buckling behavior is analytically and numerically investigated. A boundary layer phenomenon is numerically observed for the post-buckling behavior in the case of stiff connections between the two sub-columns. This phenomenon is theoretically and asymptotically explained from a straightforward asymptotic expansion.

The present study is based on the inextensibility assumption for each sub-column. A natural extension of this research would consist in relaxing this last assumption to take into account the axial extensibility. The additional shear effect may also be theoretically modeled in each sub-column (see for instance [12,13]). Some more refined gradient or non-local connection law could theoretically be envisaged, especially in the presence of high gradient terms as observed for very stiff connections (see [47] for the use of gradient or non-local connection laws in buckling problems of two-connected columns). Furthermore, it would be interesting to investigate the effect of exact parallel beams theory for large rotations, instead of the translational beam theory theoretically investigated in the paper. It should be expected however that both theory are approximately equivalent for most engineering situations.

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