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► **To cite this version:**

Gilles Pagès. Functional convex order for path-dependent American options using the Euler scheme of martingale jump diffusion process. 2012. hal-00767885v1

**HAL Id: hal-00767885**

**<https://hal.science/hal-00767885v1>**

Preprint submitted on 20 Dec 2012 (v1), last revised 23 Jul 2014 (v2)

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# Convex order for path-dependent American options using the Euler scheme of martingale jump diffusion process

GILLES PAGÈS \*

December 13, 2012

## Abstract

We explore the functional convex order of martingale diffusions and stochastic integrals with respect to their diffusion coefficient in both a Brownian and a jump framework. We finally extend this result to the Snell envelope of functionals of these process *i.e.* to American options with pathwise payoffs.

**Keywords** Convex order, local volatility models, Itô processes, Lévy-Itô processes, Laplace transform, Lévy processes, completely monotonic functions, pathwise European options, American options.

## 1 Introduction

We aim at elucidating the behaviour of the “réduites” in optimal stopping theory, or equivalently the American options, with respect to the convex order when viewed as functionals of the “volatility” of the “underlying” process supposed to be a martingale diffusion possibly with jumps. We propose an approach based on a transfer from discrete time dynamics to continuous time dynamics relying on the convergence properties of the Euler scheme. On our way to this target, we establish several convex order results for functionals of solutions of diffusions driven by Brownian motion or more general Lévy processes. We also consider the case of (martingale) stochastic integrals with respect to Lévy processes (when one of the two integrated processes is deterministic). We also obtain similar, although less general result since it only involves completely increasing functions on the real line, for stochastic integrals at a fixed time when both integrated processes at time  $t$  are functions of the driving processes at time  $t$ . We provide several counter-examples which show that our assumptions cannot be relaxed so easily and could possibly be sharp.

This kind of problem has been extensively investigated in the literature, often motivated by risk modeling in Finance. Among other, we can cite *e.g.* for American options several pioneering works like [3, 15, 11].

Our approach differs from that developed by Ruschendorf in a series of paper, especially as concerns the case of American options written on a càdlàg traded asset : the direct approximation by Snell envelopes of the Euler scheme does not work by elementary arguments like with martingale Brownian diffusions. So we propose an abstract approach using a filtration enlargement combined with a classical

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$H$ -assumption, without relying on the  $PDE$  aspects. The key is a continuity result of the Snell envelope established in an abstract framework in [10]). The case of Brownian diffusions remains closer in spirit with his results (see [15, 16]).

NOTATION: • For every  $T > 0$  and every integer  $n \geq 1$ , one denotes the uniform mesh of  $[0, T]$  by  $t_k^n = \frac{kT}{n}$ ,  $k = 0, \dots, n$ . Then for every  $t \in [\frac{kT}{n}, \frac{(k+1)T}{n})$ , we set  $\underline{t}_n = \frac{kT}{n}$  and  $\bar{t}_n = \frac{(k+1)T}{n}$  with the convention  $\underline{T}_n = T$ .

- For every  $u = (u_1, \dots, u_d)$ ,  $v = (v_1, \dots, v_d) \in \mathbb{R}^d$ ,  $(u|v) = \sum_{i=1}^d u_i v_i$ ,  $|u| = \sqrt{(u|u)}$  and  $x_{m:n} = (x_m, \dots, x_n)$  (where  $m \leq n$ ).
- $L_T^p = L^p([0, T], dt)$ .
- If  $\mathcal{F}$  denotes a filtration,  $\mathcal{T}_{[0, T]}^{\mathcal{F}} = \{\tau : \Omega \rightarrow [0, T], \mathcal{F}\text{-stopping time}\}$ .
- $\mathcal{F}^Y = (\mathcal{F}_t^Y)_{t \in [0, T]}$  is the smallest right continuous filtration  $(\mathcal{G}_t)_{t \in [0, T]}$  that makes  $Y$  a  $(\mathcal{G}_t)_{t \in [0, T]}$ -adapted process.
- $\mathcal{D}([0, T], \mathbb{R})$  denotes the set of right continuous left limited (or càdlàg following the French acronym) function defined on the interval  $[0, T]$ ,  $T > 0$ . It is usually endowed with the Skorokhod topology denoted  $Sk$  (see [8], chapter VI or [1], chapter 3, for an introduction to Skorokhod topology).

## 2 Functional convex order

### 2.1 Brownian martingale diffusion

**Proposition 2.1.** *Let  $\sigma, \theta : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$  be two continuous functions with linear growth in  $x$  uniformly in  $t \in [0, T]$ . Let  $X^{(\sigma)}$  and  $X^{(\theta)}$  be Brownian martingale diffusions, supposed to be the unique weak solutions starting from  $x$  at time 0 to the stochastic differential equations (with 0 drifts)*

$$dX_t^{(\sigma)} = \sigma(t, X_t^{(\sigma)})dW_t^{(\sigma)}, X_0^{(\sigma)} = x \quad \text{and} \quad dX_t^{(\theta)} = \theta(t, X_t^{(\theta)})dW_t^{(\theta)}, X_0^{(\theta)} = x$$

respectively where  $W^{(\sigma)}$  and  $W^{(\theta)}$  are standard one dimensional Brownian motions.

(a) *If there exists  $\kappa : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}_+$  a continuous function with linear growth in  $x$  uniformly in  $t \in [0, T]$ , such that  $\kappa(t, \cdot)$  is convex for every  $t \in [0, T]$  and the SDE*

$$dX_t^{(\kappa)} = \kappa(t, X_t^{(\kappa)})dW_t, X_0^{(\kappa)} = x$$

has a unique weak solution  $\kappa(t, \cdot)$  satisfying  $0 \leq \sigma \leq \kappa \leq \theta$ . Then, for every functional  $F : \mathcal{C}([0, T], \mathbb{R}) \rightarrow \mathbb{R}$ , convex and  $\|\cdot\|_{\text{sup}}$ -continuous functional with  $r$ -polynomial growth in the following sense

$$\forall \alpha \in \mathcal{C}([0, T], \mathbb{R}), \quad |F(\alpha)| \leq C(1 + \|\alpha\|_{\text{sup}}^r) \text{ for an } r \geq 1,$$

one has

$$\mathbb{E} F(X^{(\sigma)}) \leq \mathbb{E} F(X^{(\theta)}).$$

(a') *An equivalent form for claim (a) is: if  $\sigma(t, \cdot)$  is convex for every  $t \in [0, T]$  or  $\theta(t, \cdot)$  is convex for every  $t \in [0, T]$  and if  $0 \leq \sigma \leq \theta$ , then the conclusion of (a) still holds true.*

(b) *If  $|\sigma| \leq \theta$  and  $\theta$  is convex, then*

$$\mathbb{E} F(X^{(\sigma)}) \leq \mathbb{E} F(X^{(\theta)}).$$

**Lemma 2.1.** (*Extended Jensen's Lemma*) Let  $Z = (Z_1, \dots, Z_d) : (\Omega, \mathcal{A}, \mathbb{P}) \rightarrow \mathbb{R}^d$  be an integrable centered  $\mathbb{R}^d$ -valued random vector with independent components.

(a) Assume that  $Z \in L^r(\mathbb{P})$  for an  $r \geq 1$ . For every Borel function  $\varphi : \mathbb{R}^d \rightarrow \mathbb{R}$ , such that  $|\varphi(x)| \leq C(1 + |x|^r)$ ,  $x \in \mathbb{R}^d$ , we define

$$\forall u = (u_1, \dots, u_d) \in \mathbb{R}^d, \quad Q\varphi(u) = \mathbb{E} \varphi((u|Z)).$$

If  $\varphi$  is convex, then,  $Q\varphi$  is convex and, for every  $i \in \{1, \dots, d\}$ ,  $u_i \mapsto Q\varphi(u)$  non-decreasing on  $\mathbb{R}_+$ , non-increasing on  $\mathbb{R}_-$ .

(b) If  $Z$  has exponential moments in the sense that

$$\forall u \in \mathbb{R}^d, \quad \mathbb{E}(e^{(u|Z)}) < +\infty$$

(or equivalently  $\mathbb{E}(e^{a|Z|}) < +\infty$  for every  $a \geq 0$ ), then item (a) holds true for any convex function  $\varphi : \mathbb{R}^d \rightarrow \mathbb{R}$  satisfying the exponential growth condition  $|\varphi(x)| \leq Ce^{C|x|}$ ,  $x \in \mathbb{R}^d$ .

(c) If  $Z_i$  has a symmetric distribution for every  $i \in \{1, \dots, d\}$  and  $\varphi : \mathbb{R}^d \rightarrow \mathbb{R}$  is convex, then  $Q\varphi$  is an even function in each variable  $u_i$ , hence satisfying the following "local" maximum principle: for every  $a \in \mathbb{R}_+$ , or every  $i \in \{1, \dots, d\}$ ,

$$\forall (u_1, \dots, u_{i-1}, u_{i+1}, \dots, u_d) \in \mathbb{R}^{d-1}, \quad \sup_{|u_i| \leq a} Q\varphi(u_{1:d}) = Q\varphi(u_1, \dots, u_{i-1}, a, u_{i+1}, \dots, u_d).$$

**Proof.** (a)-(b) The existence and the convexity of  $Q\varphi$  are obvious. Temporarily assume that  $d = 1$ . The function  $Q\varphi$  is clearly finite on  $\mathbb{R}$  and convex. Furthermore, Jensen's Inequality implies that

$$Q\varphi(u) = \mathbb{E} \varphi((u|Z)) \geq \varphi(\mathbb{E}(u|Z)) = \varphi(0) = Q\varphi(0)$$

since  $Z$  is centered. Hence  $Q\varphi$  is convex and minimum at  $u = 0$  which implies that it is non-increasing on  $\mathbb{R}_-$  and non-decreasing on  $\mathbb{R}_+$ . The multi-dimensional extension follows from pre-conditioning by each one dimensional components  $Z_i$ .

(c) is obvious.  $\square$

**Definition 2.1.** (a) For every integer  $n \geq 1$ , let  $i_n : \mathbb{R}^{n+1} \rightarrow \mathcal{C}([0, T], \mathbb{R})$  be the piecewise affine interpolation operator defined by

$$\forall x_{0:n} \in \mathbb{R}^{n+1}, \quad \forall k = 0, \dots, n-1, \quad \forall t \in [t_k^n, t_{k+1}^n], \quad i_n(x_{0:n})(t) = \frac{n}{T}((t_{k+1}^n - t)x_k + (t - t_k^n)x_{k+1}).$$

(b) For every integer  $n \geq 1$ , let  $I_n : \mathcal{F}([0, T], \mathbb{R}) \rightarrow \mathcal{C}([0, T], \mathbb{R})$  be the functional interpolation operator defined by

$$\forall \alpha \in \mathcal{F}([0, T], \mathbb{R}), \quad I_n(\alpha) = i_n(\alpha(t_0^n), \dots, \alpha(t_n^n)).$$

**Lemma 2.2.** Let  $X^n$ ,  $n \geq 1$  be a sequence of continuous processes weakly converging towards  $X$  for the  $\|\cdot\|_{\text{sup}}$ -norm. Then the continuously interpolated process  $\tilde{X}^n = I_n(X^n)$  of  $X^n$  is weakly converging toward  $X$  for the  $\|\cdot\|_{\text{sup}}$ -norm topology.

**Proof.** For every integer  $n \geq 1$  and every  $\alpha \in \mathcal{F}([0, T], \mathbb{R}^d)$ , the interpolation operators  $I_n(\alpha)$  reads

$$I_n(\alpha) = \frac{n}{T}((t_{k+1}^n - t)\alpha(t_k^n) + (t - t_k^n)\alpha(t_{k+1}^n)), \quad t \in [t_k^n, t_{k+1}^n], \quad k = 0, \dots, n-1.$$

Note that  $I_n$  maps  $\mathcal{C}([0, T], \mathbb{R}^d)$  into itself. One easily checks that  $\|I_n(\alpha) - \alpha\|_{\text{sup}} \leq w(\alpha, T/n)$  and  $\|I_n(\alpha) - I_n(\beta)\|_{\text{sup}} \leq \|\alpha - \beta\|_{\text{sup}}$ . We use the standard distance for weak convergence on Polish spaces defined by

$$d_w(\mathcal{L}(X), \mathcal{L}(Y)) = \sup \{ |\mathbb{E} F(X) - \mathbb{E} F(Y)|, [F]_{\text{Lip}} \leq 1, \|F\|_{\text{sup}} \leq 1 \}.$$

Then

$$\begin{aligned} d_w(\mathcal{L}(I_n(X^n)), \mathcal{L}(X)) &\leq d_w(\mathcal{L}(I_n(X^n)), \mathcal{L}(I_n(X))) + \text{dist}_w(\mathcal{L}(I_n(X)), \mathcal{L}(X)) \\ &\leq \text{dist}_w(\mathcal{L}(X^n), \mathcal{L}(X)) + \mathbb{E}(w(X, T/n) \wedge 2) \end{aligned}$$

which goes to 0 since  $X$  has continuous paths.  $\square$

**Proof of Proposition 2.1.** (a) STEP 1 (Discrete time). Let  $(Z_k)_{1 \leq k \leq n}$  be a sequence of independent, centered,  $\mathbb{R}$ -valued random vectors lying in  $L^r(\Omega, \mathcal{A}, \mathbb{P})$ ,  $r \geq 1$  and let  $(\mathcal{F}_k^Z)_{0 \leq k \leq n}$  denote its natural filtration. Let  $\Phi : \mathbb{R}^{n+1} \rightarrow \mathbb{R}$  be a convex function such that  $|\Phi(x)| \leq C(1 + |x|^r)$ . Let  $(X_k)_{0 \leq k \leq n}$  and  $(Y_k)_{0 \leq k \leq n}$  be two sequences of random vectors recursively defined by

$$X_{k+1} = X_k + \sigma_k(X_k)Z_{k+1}, \quad Y_{k+1} = Y_k + \theta_k(Y_k)Z_{k+1}, \quad 0 \leq k \leq n-1, \quad X_0 = Y_0 = x \quad (2.1)$$

where  $\sigma_k, \theta_k : \mathbb{R} \rightarrow \mathbb{R}$ ,  $k = 0, \dots, n-1$ , are Borel functions with linear growth. This implies by an easy induction that the r.v.  $X_k$  and  $Y_k$  all lie in  $L^r$ . Then, one defines the following martingales

$$M_k = \mathbb{E}(\Phi(X_{0:n}) | \mathcal{F}_k^Z) \quad \text{and} \quad N_k = \mathbb{E}(\Phi(Y_{0:n}) | \mathcal{F}_k^Z), \quad 0 \leq k \leq n.$$

Their existence follows from the growth assumptions on  $F$ ,  $\sigma_k$  and  $\theta_k$ ,  $k = 1, \dots, n$ . Now we define two sequences of functions  $\Phi_k, : \mathbb{R}^{k+1} \rightarrow \mathbb{R}$ ,  $k = 0, \dots, n$  by the backward inductions

$$\Phi_n = F \quad \text{and} \quad \Phi_k(x_{0:k}) = (Q\Phi_{k+1}(x_{0:k}, x_k + \cdot))(\sigma_k(x_k)), \quad x_{0:k} \in \mathbb{R}^{k+1}, \quad k = 0, \dots, n-1$$

on the one hand and, on the other hand,

$$\Psi_n = F \quad \text{and} \quad \Psi_k(x_{0:k}) = (Q\Phi_{k+1}(x_{0:k}, x_k + \cdot))(\theta_k(x_k)), \quad x_{0:k} \in \mathbb{R}^{k+1}, \quad k = 0, \dots, n-1.$$

It is clear by a backward induction and the definition of the kernel  $Q$  that for every  $k \in \{0, \dots, n\}$ ,

$$M_k = \Phi_k(X_{0:k}) \quad \text{and} \quad N_k = \Psi_k(X_{0:k}).$$

Let  $Q_k$  denote the operator attached to  $Z_k$  by Lemma 2.1. One derives from that for a convex function  $G : \mathbb{R}^{k+1} \rightarrow \mathbb{R}$  such that  $G(x) \leq C(1 + |x|^r)$ , the function  $(x_{0:k}, u) \mapsto (Q_{k+1}G(x_{0:k}, x_k + \cdot))(u)$  is convex, non-increasing on  $(-\infty, 0)$ , non-decreasing on  $(0, +\infty)$  as a function of  $u$ . This implies that, if  $\gamma : \mathbb{R} \rightarrow \mathbb{R}_+$  is convex (and non-negative), then  $x \mapsto Q_{k+1}G(x_{0:k}, x_k + \cdot)(\gamma(x))$  is convex.

As a consequence, one shows by a backward induction that, if the functions  $\sigma_k$ ,  $k = 0, \dots, n-1$ , are all non-negative and convex, then the functions  $\Phi_k$  are all convex.

Finally, we prove that  $\Phi_k \leq \Psi_k$  for every  $k = 0, \dots, n-1$ . We proceed again by a backward induction on  $k$ . First note that  $\Phi_n = \Psi_n = \Phi$ . If  $\Phi_{k+1} \leq \Psi_{k+1}$ , then

$$\begin{aligned} \Phi_k(x_{0:k}) &= (Q_{k+1}\Phi_{k+1}(x_{0:k}, x_k + \cdot))(\sigma_k(x_k)) \leq (Q_{k+1}\Psi_{k+1}(x_{0:k}, x_k + \cdot))(\theta_k(x_k)) \\ &\leq (Q_{k+1}\Psi_{k+1}(x_{0:k}, x_k + \cdot))(\theta_k(x_k)) = \Psi_k(x_{0:k}). \end{aligned}$$

In particular, when  $k = 0$  we get  $\Phi_0(x) \leq \Psi_0(x)$ , which also reads, taking advantage of the martingale property,  $\mathbb{E}\Phi(X_{0:n}) \leq \mathbb{E}\Psi(Y_{0:n})$ .

If all the functions  $\theta_k$ ,  $k = 0, \dots, n-1$  are convex, then all functions  $\Psi_k$ ,  $k = 0, \dots, n$ , are convex and  $\Phi_k \leq \Psi_k$  for every  $k = 0, \dots, n-1$  as well.

The proof for functionals with exponential growth and sequences  $Z_{1:n}$  having exponential moments of any order is the same.

STEP 2 (Continuous time). We consider now for both diffusion processes  $X^{(\sigma)}$  and  $X^{(\theta)}$  their continuous (also known as “genuine”) Euler schemes with step  $\frac{T}{n}$ , starting at  $x$ . To be more precise, this Euler scheme related to  $X^{(\sigma)}$  is defined by

$$\begin{aligned}\bar{X}_{t_{k+1}^n}^{(\sigma),n} &= \bar{X}_{t_k^n}^{(\sigma),n} + \sigma(t_k^n, \bar{X}_{t_k^n}^{(\sigma),n})(W_{t_{k+1}^n} - W_{t_k^n}), \quad k = 0, \dots, n-1, \quad \bar{X}_0^{(\sigma),n} = x \\ \bar{X}_t^{(\sigma),n} &= \bar{X}_{t_k^n}^{(\sigma),n} + \sigma(t_k^n, \bar{X}_{t_k^n}^{(\sigma),n})(W_t - W_{t_k^n}), \quad t \in [t_k^n, t_{k+1}^n),\end{aligned}$$

make up sequences are of the form (2.1) with the Brownian white noise sequence  $Z_k = (W_{t_k^n} - W_{t_{k-1}^n})$ ,  $k = 1, \dots, n$ . Furthermore, owing to the linear growth of  $\sigma$  and  $\theta$ , it is classical background on Euler scheme of Brownian diffusions that these sequences lie in  $L^p$  for any  $p \in (0, \infty)$ , uniformly in  $n$ , namely

$$\sup_{n \geq 1} \left\| \sup_{t \in [0, T]} |\bar{X}_t^{(\sigma),n}| \right\|_p + \sup_{n \geq 1} \left\| \sup_{t \in [0, T]} |\bar{X}_t^{(\theta),n}| \right\|_p < +\infty.$$

Furthermore,  $I_n(\bar{X}^{(\sigma),n}) = i_n((\bar{X}^{(\sigma),n})_{t_{0:n}^n})$  is but the piecewise affine interpolated Euler scheme (which coincide with  $\bar{X}^{(\sigma),n}$  at times  $t_k^n$ ). Note that the sup-norm of  $I_n(\bar{X}^{(\sigma),n})$  also has finite polynomial moments uniformly in  $n$  like the genuine Euler scheme.

Let  $F : \mathcal{C}([0, T], \mathbb{R}) \rightarrow \mathbb{R}$  be a convex functional with  $r$ -polynomial growth. For every integer  $n \geq 1$ , we define on  $\mathbb{R}^{n+1}$  the function  $F_n$  by

$$F_n(x_{0:n}) = F(i_n(x_{0:n})), \quad x_{0:n} \in \mathbb{R}^{n+1}. \quad (2.2)$$

It is straightforward that the convexity of  $F$  on  $\mathcal{C}([0, T], \mathbb{R})$  is transferred to the functions  $F_n$ ,  $n \geq 1$ . So does the polynomial growth property. It follows from Step 1 applied with  $\sigma_k = \sigma(t_k^n, \cdot)$  and  $\theta_k = \theta(t_k^n, \cdot)$  that, for every  $n \geq 1$ ,

$$\mathbb{E} F(I_n(\bar{X}^{(\sigma),n})) = \mathbb{E} F_n((\bar{X}_{t_k^n}^{(\sigma),n})_{k=0:n}) \leq \mathbb{E} F_n((\bar{X}_{t_k^n}^{(\theta),n})_{k=0:n}) = \mathbb{E} F(I_n(\bar{X}^{(\theta),n}))$$

On the other hand, it is a straightforward consequence from Theorem 3.39 in [8], p.551 that the *genuine (continuous) Euler schemes*  $\bar{X}^{(\sigma),n}$  weakly converges for the  $\|\cdot\|_{\text{sup}}$ -norm topology toward  $X^{(\sigma)}$ , unique weak solution of the *SDE*  $\equiv dX_t = \sigma(X_t)dW_t$ ,  $X_0 = x$ , as  $n \rightarrow +\infty$ ,

It follows from Lemma 2.2 and the  $L^p$ -boundedness of the sup-norm of the sequence  $(I_n(\bar{X}^{(\sigma),n}))_{n \geq 1}$  for  $p > r$  that

$$\mathbb{E} F(X^{(\sigma)}) = \lim_n \mathbb{E} F(I_n(\bar{X}^{(\sigma),n})) = \lim_n \mathbb{E} F_n((\bar{X}_{t_k^n}^{(\sigma),n})_{0 \leq k \leq n}).$$

The same holds true for the diffusion  $X^{(\theta)}$  and its Euler schemes. The conclusion follows.

(b) The proof follows the same lines by calling upon item (c) of the above Lemma in Step 1, having in mind that the distribution of standard Brownian increment is centered and has polynomial moments at any order (as Gaussian random vector).  $\square$

**Corollary 2.1.** *If  $\sigma, \theta : [0, T] \times I \rightarrow \mathbb{R}$ , where  $I$  is a nontrivial interval of  $\mathbb{R}$ , are continuous with polynomial growth and if the related SDE's satisfy a weak uniqueness assumption for every  $I$ -valued weak solution starting from  $(t, x)$ ,  $x \in I$ ,  $t \in [0, T]$ . Then the above Proposition remains true.*

**Proof.** (a) is obvious.  $\square$

**Proposition 2.2** (Extension to drifted diffusions). *Let  $b : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$  be a continuous function with linear growth in  $x$  uniformly in  $t$  and such that  $b(t, \cdot)$  is convex for every  $t \in [0, T]$ . Then, the conclusions of the above proposition remain true for the SDEs  $dY_t^{(\sigma)} = b(t, Y_t^{(\sigma)})dt + \sigma(t, Y_t^{(\sigma)})dW_t$  and  $dY_t^{(\theta)} = b(t, Y_t^{(\theta)})dt + \theta(t, Y_t^{(\theta)})dW_t$ , both starting from  $x$  and non-decreasing convex functions  $\varphi : \mathbb{R} \rightarrow \mathbb{R}$ .*

**Proof.** We have to introduce the operators  $Q_{b,\gamma}$ ,  $\gamma > 0$ , defined for every function  $\varphi : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$  (satisfying appropriate growth assumption in connection with the existing moments of  $Z$ ) by

$$Q_{b,\gamma}(\varphi)(x, u) = \mathbb{E} \varphi(x, x + \gamma b(x) + uZ).$$

One shows as in Lemma 2.1 above that  $Q_{b,\gamma}\varphi$  is convex in  $(x, u)$ , non-decreasing in  $u$  on  $\mathbb{R}_+$ , non-increasing in  $u \in \mathbb{R}_-$ .  $\square$

APPLICATIONS TO FUNCTIONAL PEACOCKS. We consider a local volatility model (on the discounted risky asset)

$$dS_t = S_t \sigma(t, S_t) dW_t, \quad S_0 = s_0 > 0, \quad (2.3)$$

where  $\sigma : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$  is a bounded continuous functions so that the above equation has at least a weak solution  $(S_t^{(\sigma)})_{t \in [0, T]}$  with distribution on a probability space  $(\Omega, \mathcal{A}, \mathbb{P})$  on which is defined a Brownian motion  $(B_t)_{t \in [0, T]}$  (with augmented filtration  $(\mathcal{F}_t^B)_{t \in [0, T]}$ ). Then,  $(S_t)_{t \in [0, T]}$  is a true  $(\mathcal{F}_t^B)_{t \in [0, T]}$ -martingale reading

$$S_t^{(\sigma)} = s_0 e^{\int_0^t \sigma(s, S_s^{(\sigma)}) dB_s - \frac{1}{2} \int_0^t \sigma^2(s, S_s^{(\sigma)}) ds}$$

(so that  $S_t^{(\sigma)} > 0$ ,  $t \in [0, T]$ ). One can introduce likewise the local volatility model related to the (bounded) volatility function  $\theta : [0, T] \times \mathbb{R}_+ \rightarrow \mathbb{R}$ , still starting from  $s_0 > 0$ . Then, if one of the following two additional conditions hold

(i)  $(x \mapsto x\kappa(t, x))_{x \in \mathbb{R}_+}$  has a non-negative convex extension on  $\mathbb{R}$ ,  $t \in [0, T]$

$$\& 0 \leq \sigma(t, \cdot) \leq \kappa(t, \cdot) \leq \theta(t, \cdot) \text{ on } \mathbb{R}_+, t \in [0, T],$$

(ii)  $(x \mapsto x\theta(t, x))_{x \in \mathbb{R}_+}$  has a non-negative convex extension on  $\mathbb{R}$ ,  $t \in [0, T]$  &  $|\sigma(t, \cdot)| \leq \theta(t, \cdot)$ ,  $t \in [0, T]$ ,

for every then, for every convex (hence continuous) function  $f : \mathbb{R} \rightarrow \mathbb{R}$  with at most polynomial growth

$$\mathbb{E} f \left( \int_0^T S_s^{(\sigma)} \mu(ds) \right) \leq \mathbb{E} f \left( \int_0^T S_s^{(\theta)} \mu(ds) \right)$$

where  $\mu$  is a signed (finite) measure on  $([0, T], \mathcal{B}or([0, T]))$ .

This claim is a straightforward consequence of the fact that for any  $\|\cdot\|_{\text{sup}}$ -continuous linear form  $L$  on  $\mathcal{C}([0, T], \mathbb{R})$ ,  $F \circ L$  is a  $\|\cdot\|_{\text{sup}}$ -continuous function. From a technical viewpoint, we apply Proposition 2.1 to the diffusion(s) processes (with 0 drift) and diffusion coefficients given by the above defined convex extensions of  $x\kappa(t, x)$  on the whole real line,  $\sigma$  and  $\theta$  in any appropriate way which preserves the non-negativity of  $x\sigma(t, x)$  a,d  $x\theta(t, x)$  on  $\mathbb{R}_-$  as well as the comparison with  $x\kappa(t, x)$ . In the end we take advantage of the *a posteriori* positivity of the solutions of these SDE when starting from a positive real number.

This result can be seen as an extension of the fact that  $\left( \int_0^T e^{\sigma B_t - \frac{\sigma^2 t}{2}} dt \right)_{\sigma \geq 0}$  is a peacock (see [2, 6]).

APPLICATIONS TO VOLATILITY COMPARISON RESULTS. It also appears as a functional extension of the comparison result in [3] since it shows that if

$$0 < \sigma_{\min}(t) \leq \sigma(t, \cdot) \leq \sigma_{\max}(t), \quad t \in [0, T]$$

then for every  $\|\cdot\|_{\text{sup}}$ -continuous convex functional  $F : \mathbb{R} \rightarrow \mathbb{R}$  with at most polynomial growth

$$\mathbb{E} F \left( S_s^{(\sigma_{\min})} \right) \leq \mathbb{E} F \left( S_s^{(\sigma)} \right) \leq \mathbb{E} F \left( S_s^{(\sigma_{\max})} \right) \quad (2.4)$$

where the left and right hand side are usually considered as quasi-closed forms since they correspond to Hull-White (or even Black-Scholes if  $\sigma_{\min}$ ,  $\sigma_{\max}$  are constant) models.

## 2.2 Counter-example (discrete time setting)

The above comparison results for the convex order can fail when the assumptions of the proposition 2.1 are not satisfied by the diffusion coefficient. In fact the counter-example below is developed in the discrete time framework corresponding to the first step of the proof. We consider the 2-period dynamics. Let  $X = X^{\sigma, x} = (X_{0:2}^{\sigma, x})$  satisfy

$$\begin{aligned} X_1 &= x + \sigma Z_1 \\ X_2 &= X_1 + \sqrt{2v(X_1)}Z_2 \end{aligned}$$

where  $Z_{1:2} \stackrel{\mathcal{L}}{\sim} \mathcal{N}(0; I_2)$ ,  $\sigma \geq 0$  and  $v : \mathbb{R} \rightarrow \mathbb{R}_+$  is a bounded  $\mathcal{C}^2$ -function such that  $v$  has a strict local maximum at  $x_0$  satisfying  $v'(x_0) = 0$  and  $v''(x_0) < -1$  (so is the case if  $v(x) = v(x_0) - \rho(x - x_0)^2 + o((x - x_0)^2)$ ,  $0 < \rho < \frac{1}{2}$ , in the neighbourhood of  $x_0$ ). Of course this implies that  $\sqrt{v}$  cannot be convex.

Let  $f(x) = e^x$ . It is clear that

$$\varphi(x, \sigma) := \mathbb{E}f(X_2) = e^x \mathbb{E}(e^{\sigma Z_1 + v(x + \sigma Z_1)})$$

Elementary computations show that

$$\begin{aligned} \varphi'_\sigma(x, \sigma) &= e^x \mathbb{E}(e^{\sigma Z_1 + v(x + \sigma Z_1)}(1 + v'(x + \sigma Z_1)Z_1)) \\ \varphi''_{\sigma^2}(x, \sigma) &= e^x \left( \mathbb{E}(e^{\sigma Z_1 + v(x + \sigma Z_1)}(1 + v'(x + \sigma Z_1)^2 Z_1^2)) + \mathbb{E}(e^{\sigma Z_1 + v(x + \sigma Z_1)}v''(x + \sigma Z_1)Z_1^2) \right). \end{aligned}$$

In particular

$$\varphi'_\sigma(x, 0) = e^{x+v(x)}(1 + v'(x))\mathbb{E}Z = 0$$

and

$$\varphi''_{\sigma^2}(x, 0) = e^{x+v(x)}((1 + v'(x))^2 + v''(x))$$

so that  $\varphi''_{\sigma^2}(x_0, 0) < 0$  which implies that  $\varphi'_\sigma(x_0, \sigma) < 0$  when  $\sigma \in (0, \sigma_0]$ ,  $\sigma_0 > 0$ . This clearly exhibits a counter-example to Proposition 2.1 when the convexity assumption does not hold on the functions  $(\sigma_k)$  or  $(\kappa_k)$ .

## 2.3 Lévy driven diffusions

Let  $Z = (Z_t)_{t \in [0, T]}$  be a Lévy process with Lévy measure  $\nu$  satisfying  $\int_{|z| \geq 1} |z|^p \nu(dz) < +\infty$ ,  $p \in [1, +\infty)$ . Then  $Z_t \in L^1(\mathbb{P})$  for every  $t \in [0, T]$ . If furthermore,  $\mathbb{E}Z_1 = 0$ , then  $(Z_t)_{t \in [0, T]}$  is an  $\mathcal{F}^Z$ -martingale.

**Proposition 2.3.** *Let  $Z = (Z_t)_{t \in [0, T]}$  be a martingale Lévy process with Lévy measure  $\nu$  satisfying  $\int_{|z| \geq 1} |z|^p \nu(dz) < +\infty$  for some  $p \in [1, +\infty)$  and let  $\kappa_i : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ ,  $i = 1, 2$ , be continuous functions, Lipschitz continuous in  $x$  uniformly in  $t$ . For  $i = 1, 2$ , let  $X^{(\kappa_i)} = (X_t^{(\kappa_i)})_{t \in [0, T]}$  be the unique diffusion process solution to*

$$dX_t^{(\kappa_i)} = \kappa_i(t, X_{t-}^{(\kappa_i)})dZ_t, \quad X_0^{(\kappa_i)} = x \in \mathbb{R}. \quad (2.5)$$

Let  $F : \mathcal{D}([0, T], \mathbb{R}) \rightarrow \mathbb{R}$  be a convex Sk-continuous functional

$$\forall \alpha \in \mathcal{C}([0, T], \mathbb{R}), \quad |F(\alpha)| \leq C(1 + \|\alpha\|_{\text{sup}}^r), \quad 0 < r < p.$$



(a) There exists  $\kappa : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}_+$  such that,  $\kappa(t, \cdot)$  is convex for every  $t \in [0, T]$  and  $0 \leq \kappa_1 \leq \kappa \leq \kappa_2$ , then

$$\mathbb{E} F(X^{(\kappa_1)}) \leq \mathbb{E} F(X^{(\kappa_2)}).$$

(a') An equivalent form for claim (a) is: if  $0 \leq \kappa_1 \leq \kappa_2$  and  $\kappa_1(t, \cdot)$  are convex for every  $t \in [0, T]$  or  $\kappa_2(t, \cdot)$  are convex for every  $t \in [0, T]$ . Then the conclusion of (a) still holds true.

(b) If  $|\sigma| \leq \theta$  and  $\theta$  is convex, then

$$\mathbb{E} F(X^{(\kappa_1)}) \leq \mathbb{E} F(X^{(\kappa_2)}).$$

**Lemma 2.3.** Let  $\alpha \in \mathcal{ID}([0, T], \mathbb{R})$ . The sequence of stepwiseconstant approximations defined by

$$\alpha_n(t) = \alpha(\underline{t}_n), \quad t \in [0, T],$$

converges toward  $\alpha$  for the Skorokhod topology.

**Proof.** See [8] Proposition 6.37 in Chapter VI (p.387, second edition). □

**Proof of Proposition 2.3.** STEP 1. Let  $(\bar{X}_t^n)_{t \in [0, T]}$  be the continuous (genuine) Euler scheme defined by

$$\bar{X}_t^n = x + \int_{(0, t]} \kappa(\underline{s}_n, \bar{X}_{\underline{s}_n}^n) dZ_s$$

where  $\kappa = \kappa_1$  or  $\kappa_2$ . Then, owing to the linear growth of  $\kappa$ ,

$$\left\| \sup_{t \in [0, T]} |X_t| \right\|_p + \sup_{n \geq 1} \left\| \sup_{t \in [0, T]} |\bar{X}_t^n| \right\|_p < +\infty.$$

On the other hand we know from [7] that the *stepwise constant Euler scheme*  $(\bar{X}_{\underline{t}_n}^n)_{t \in [0, T]}$ , satisfies

$$\text{dist}_{Sk}((\bar{X}_{\underline{t}_n}^n)_{t \in [0, T]}, (X_{\underline{t}_n})_{t \in [0, T]}) \rightarrow 0 \quad \text{as } n \rightarrow \infty \text{ in probability}$$

where  $\text{dist}_{Sk}$  is a metric for the Skorokhod topology on  $[0, T]$  (in fact the result established in [7] is a weak rate of convergence for the Skorokhod topology which is dramatically more sophisticated but what we need here is simply the convergence in probability). On the other hand, it follows from the above lemma 2.3 that  $\text{dist}_{Sk}((X_{\underline{t}_n})_{t \in [0, T]}, X) \rightarrow 0$  *a.s.* As a consequence

$$\text{dist}_{Sk}((\bar{X}_{\underline{t}_n}^n)_{t \in [0, T]}, X) \rightarrow 0 \quad \text{in probability.}$$

STEP 2. On the other hand, let  $F : \mathcal{ID}([0, T], \mathbb{R}) \rightarrow \mathbb{R}$  be a  $\mathbb{P}_X$ -Sk-continuous convex functional. For every integer  $n \geq 1$ , we still define the sequence of convex functions  $F_n : \mathbb{R}^{n+1} \rightarrow \mathbb{R}$  by  $F_n(x_{0:n}) = F\left(\sum_{i=0}^{n-1} x_i \mathbf{1}_{[t_k^n, t_{k+1}^n)} + x_n \mathbf{1}_{\{T\}}\right)$  so that  $F_n((\bar{X}_{t_k^n}^n)_{0:n}) = F((\bar{X}_{\underline{t}_n}^n)_{t \in [0, T]})$ .

Now, for every  $n \geq 1$ , the discrete time Euler schemes  $\bar{X}^{(\kappa_i), n}$ ,  $i = 1, 2$ , related to the jump diffusions with diffusion coefficients  $\kappa_1$  and  $\kappa_2$  are of the form (2.1) and  $|F_n(x_{0:n})| \leq C(1 + \|x_{0:n}\|^r)$ ,  $r \in (0, p)$ .

(a) Assume  $0 \leq \kappa_1 \leq \kappa_2$ . Then it follows from the first step of the proof of Proposition 2.1 that, for every  $n \geq 1$ ,  $\mathbb{E} F_n((\bar{X}_{t_k^n}^{(\kappa_1), n})_{0:n}) \leq \mathbb{E} F_n((\bar{X}_{t_k^n}^{(\kappa_2), n})_{0:n})$  *i.e.*  $\mathbb{E} F((\bar{X}_{\underline{t}_n}^{(\kappa_1), n})_{t \in [0, T]}) \leq \mathbb{E} F((\bar{X}_{\underline{t}_n}^{(\kappa_2), n})_{t \in [0, T]})$ . Letting  $n \rightarrow \infty$  completes the proof since  $F$  is  $\mathbb{P}_X$ -*a.s.* Sk-continuous. □

(b) is an easy consequence of the proof of Proposition 2.1(b) devoted to the discrete time setting (Step 1).

## 2.4 Bounds for non Markovian Itô and Doléans martingales

### 2.4.1 Itô martingales

This theorem provides an extension of Hajek's theorem (see [5]) and El Karoui-Jeanblanc-Shreve's Theorem (see [3]) which mainly deal with convex functions of the marginals of the processes.

We will extensively use the following classical result:

If  $(Y_n)_{n \geq 1}$  is a sequence of random variables taking values in a Banach space  $(E, |\cdot|_E)$ . If  $Y_n$  weakly converge toward  $Y_\infty$  and  $(\Phi(Y_n))_{n \geq 1}$  is uniformly integrable where  $\Phi : E \rightarrow \mathbb{R}$  is Borel function (going to infinity at infinity), then for every  $\mathbb{P}_{Y_\infty}$ -a.s. continuous Borel functional  $F$  defined on  $E$  such that  $|F(u)| \leq C(1 + \Phi(u))$  for every  $u \in E$ , one has  $\mathbb{E} F(Y_n) \rightarrow \mathbb{E} F(Y_\infty)$ .

**Proposition 2.4.** *Let  $F : \mathcal{C}([0, T], \mathbb{R}) \rightarrow \mathbb{R}$  be a convex  $\|\cdot\|_{\text{sup}}$ -continuous functional with exponential growth*

$$\forall \alpha \in \mathcal{C}([0, T], \mathbb{R}), \quad |F(\alpha)| \leq C(1 + \|\alpha\|_{\text{sup}}^r), \quad r \geq 0,$$

for a real constant  $C \geq 0$  and let  $(H_t)_{t \in [0, T]}$  be an  $(\mathcal{F}_t)$ -progressively measurable process and let  $h = (h_t)_{t \in [0, T]}$  be a bounded Borel function, dt-a.e. left continuous, such that Assume that

$$\mathbb{P}(d\omega)\text{-a.s. } t \mapsto H_t(\omega) \text{ is dt-a.e. left continuous.}$$

(a) If  $|H_t| \leq h_t$   $\mathbb{P}$ -a.s. for every  $t \in [0, T]$ , then

$$\mathbb{E} F \left( \int_0^\cdot H_s dW_s \right) \leq \mathbb{E} F \left( \int_0^\cdot h_s dW_s \right).$$

(b) If  $H_t \geq h_t \geq 0$   $\mathbb{P}$ -a.s. for every  $t \in [0, T]$  and  $\|H\|_{L^2([0, T], dt)} \in L^{r'}(\mathbb{P})$  for  $r' > r$ , then

$$\mathbb{E} F \left( \int_0^\cdot H_s dW_s \right) \geq \mathbb{E} F \left( \int_0^\cdot h_s dW_s \right).$$

**Remark.** The assumption  $h$  is satisfies as soon as  $h$  is left continuous over  $[0, T]$ .

**Proof.** STEP 1 (Discrete time). Let  $(Z_k)_{1 \leq k \leq n}$  be a sequence of independent centered  $\mathbb{R}$ -valued random vectors satisfying  $Z_k \in L^r(\Omega, \mathcal{A}, \mathbb{P})$ ,  $r \geq 1$ , and let  $(H_k)_{0 \leq k \leq n}$  be an  $(\mathcal{F}_k^Z)_{0 \leq k \leq n}$ -adapted sequence. Let  $\Phi : \mathbb{R}^{n+1} \rightarrow \mathbb{R}$  be a convex function such that  $|\Phi(x)| \leq C(1 + |x|^r)$ . Let  $X = (X_k)_{0 \leq k \leq n}$  and  $Y = (Y_k)_{0 \leq k \leq n}$  be two sequences of random vectors recursively defined by

$$X_{k+1} = X_k + H_k Z_{k+1}, \quad Y_{k+1} = Y_k + h_k Z_{k+1}, \quad 0 \leq k \leq n-1, \quad X_0 = Y_0 = x.$$

Then one proceeds like in the Step 1 of Proposition 2.1 to prove by backward induction that if  $|H_k| \leq h_k$ , then

$$\mathbb{E} \Phi(X) \leq \mathbb{E} \Phi(Y)$$

and if  $H_k \geq h_k \geq 0$ , then  $\mathbb{E} \Phi(X) \geq \mathbb{E} \Phi(Y)$ . We mainly rely on the fact that the sequence of functions recursively defined by

$$\Phi_n = \Phi, \quad \Phi_k = (Q\Phi_{k+1}(x_{0:k}, x_k + \cdot))(h_k)$$

are all convex and satisfy  $\Phi_k(Y_{0:k}) = \mathbb{E}(\Phi(Y_{0:n}) | \mathcal{F}_k^Z)$ . If  $e^{|Z_k|} \in L^r(\mathbb{P})$ , one shows likewise the result for functions  $\Phi$  such that  $|\Phi(x)| \leq C e^{r|x|}$ .

STEP 2. (a) The domination of  $H$  by  $h$  implies the existence of the stochastic integral since  $\int_0^T \mathbb{E} H_t^2 dt \leq \int_0^T h^2(t) dt < +\infty$ . Let  $n$  be a positive integer. It is clear that

$$\mathbb{E} \left\langle \int_0^\cdot H_s dW_s - \int_0^\cdot H_{s_n} dW_s \right\rangle_T = \int_0^T \mathbb{E} |H_t - H_{t_n}|^2 dt \rightarrow 0 \quad \text{as } n \rightarrow +\infty$$

so that, owing to Doob's Inequality,

$$\sup_{t \in [0, T]} \left| \int_0^t H_s dW_s - \int_0^t H_{\underline{s}_n} dW_s \right| \xrightarrow{L^2} 0 \quad \text{as } n \rightarrow +\infty.$$

On the other hand  $\sup_{t \in [0, T]} \left| \int_{\underline{s}_n}^t H_s dW_s \right| \rightarrow 0$  *a.s.* since paths of Brownian stochastic integrals are (uniformly) continuous over  $[0, T]$ . In particular,  $\left( \int_0^{\underline{s}_n} H_s dW_s \right)_{t \in [0, T]}$  weakly converges to  $\left( \int_0^{\cdot} H_s dW_s \right)_{t \in [0, T]}$  for the  $\|\cdot\|_{\text{sup}}$ -norm topology.

Let  $F_n$  be defined by (2.2). One derives from Step 1 that, for every  $n \geq 1$ ,

$$\begin{aligned} \mathbb{E} F \circ I_n \left( \int_0^{\cdot} H_{\underline{s}_n} dW_s \right) &= \mathbb{E} F_n \left( \int_0^{t_k^n} H_{\underline{s}_n} dW_s, k = 0, \dots, n \right) \\ &\leq \mathbb{E} F_n \left( \int_0^{t_k^n} h_{\underline{s}_n} dW_s, k = 0, \dots, n \right) = \mathbb{E} F \circ I_n \left( \int_0^{\cdot} h_{\underline{s}_n} dW_s \right) \end{aligned}$$

since  $F_n$  is convex. Combining the above functional weak convergence and the uniform integrability argument yields the expected inequality.

(b) Under the integrability assumption on  $\|H\|_{L^2([0, T], dt)}$  one still gets the weak of the “discretized” stochastic integrals for the  $\|\cdot\|_{\text{sup}}$ -norm topology. Furthermore it follows from the B.D.G. Inequality that, for any  $r' > r$ ,

$$\left\| \sup_{t \in [0, T]} \left| \int_0^t H_{\underline{s}_n} dW_s \right| \right\|_{r'} \leq c_{r'} \left\| \int_0^T H_{\underline{s}_n}^2 ds \right\|_{\frac{r'}{2}}^{\frac{1}{2}} \leq \left( \int_0^T h_{\underline{s}_n}^2 dt \right)^{\frac{1}{2}} \rightarrow \|h\|_{L_T^2} < +\infty$$

as  $n \rightarrow \infty$ . So we can apply Step 1 to get the lower bound for every  $n \geq 1$  and as a second step, let  $n$  go to infinity to conclude using a uniform integrability argument.  $\square$

## 2.4.2 Lévy-Itô martingales

**Proposition 2.5.** *Let  $Z = (Z_t)_{t \in [0, T]}$  be an integrable centered Lévy process with Lévy measure  $\nu$  satisfying  $\nu(|x|^p \mathbf{1}_{\{|x| \geq 1\}}) < +\infty$  for a real exponent  $p > 1$ . Let  $F : \mathcal{C}([0, T], \mathbb{R}) \rightarrow \mathbb{R}$  be a convex Skorokhod-continuous functional with  $r$ -polynomial growth,  $0 < r < p$ , namely*

$$\forall \alpha \in \mathcal{D}([0, T], \mathbb{R}), \quad |F(\alpha)| \leq C(1 + \|\alpha\|_{\text{sup}}^r)$$

for a real constant  $C \geq 0$ . Let  $(H_t)_{t \in [0, T]}$  be an  $(\mathcal{F}_t)$ -predictable process and let  $h = (h_t)_{t \in [0, T]}$  be a *dt*-a.e. left continuous function such that  $\|h\|_{L^2([0, T], dt)} < +\infty$ . Assume that

$$\mathbb{P}(d\omega)\text{-a.s. } t \mapsto H_t(\omega) \text{ is } dt\text{-a.e. left continuous.}$$

(a) If  $|H_t| \leq h_t$   $\mathbb{P}$ -a.s. for every  $t \in [0, T]$ , then

$$\mathbb{E} F \left( \int_0^{\cdot} H_s dZ_s \right) \leq \mathbb{E} F \left( \int_0^{\cdot} h_s dZ_s \right).$$

(b) If  $H_t \geq h_t \geq 0$   $\mathbb{P}$ -a.s. for every  $t \in [0, T]$  and  $\|H\|_{L^2([0, T], dt)} \in L^{r'}(\mathbb{P})$  for  $r < r' < p$ , then

$$\mathbb{E} F \left( \int_0^{\cdot} H_s dZ_s \right) \geq \mathbb{E} F \left( \int_0^{\cdot} h_s dZ_s \right).$$

**Proof.** (a) We will rely on the Step 1 of the proof for Brownian Itô processes. We first consider the Lévy-Khintchine decomposition of  $Z$

$$Z_t = cW_t + \tilde{Z}_t^\eta + Z_t^\eta, \quad c \geq 0,$$

where  $\tilde{Z}^\eta$  is martingale with jumps of size at most  $\eta$  and  $Z^\eta$  is compensated Poisson process with (finite) Lévy measure  $\nu(\cdot \cap \{|x| > \eta\})$ . Let  $n$  be a positive integer. Assume temporarily that  $p \in (0, 2]$ . It follows from B.D.G. Inequality that,

$$\begin{aligned} \left\| \sup_{t \in [0, T]} \left| \int_0^t H_s dZ_s - \int_0^t H_{\underline{s}_n} dZ_s \right| \right\|_p &\leq c\kappa_p \left\| \int_0^T (H_t - H_{\underline{t}_n})^2 dt \right\|_{\frac{p}{2}} \\ &\quad + \kappa_p \left\| \sum_{0 < s \leq T} (H_s - H_{\underline{s}_n})^2 (\Delta Z_s)^2 \mathbf{1}_{\{|\Delta Z_s| > \eta\}} \right\|_{\frac{p}{2}} \\ &\quad + \kappa_p \left\| \sum_{0 < s \leq T} (H_s - H_{\underline{s}_n})^2 (\Delta Z_s)^2 \mathbf{1}_{\{|\Delta Z_s| \leq \eta\}} \right\|_1 \end{aligned}$$

where we used in the last line that  $p \leq 2$  and the monotony of  $L^p$ -norm.

Owing to the compensation formula we have

$$\mathbb{E} \left| \sum_{0 < s \leq T} (H_s - H_{\underline{s}_n})^2 (\Delta Z_s)^2 \mathbf{1}_{\{|\Delta Z_s| > \eta\}} \right|^{\frac{p}{2}} \leq \mathbb{E} \int_0^T |H_t - H_{\underline{t}_n}|^p dt \nu(|x|^p \mathbf{1}_{\{|x| > \eta\}})$$

and

$$\mathbb{E} \left| \sum_{0 < s \leq T} (H_s - H_{\underline{s}_n})^2 (\Delta Z_s)^2 \mathbf{1}_{\{|\Delta Z_s| \leq \eta\}} \right| = \mathbb{E} \int_0^T (H_t - H_{\underline{t}_n})^2 dt \nu(|x|^2 \wedge \eta).$$

Since  $|H_t| \leq h(t)$  and the  $dt$ -a.e. continuity of  $t \mapsto H_t$ , we get that these above three terms go to 0 as  $n$  goes to infinity. It follows that

$$\sup_{t \in [0, T]} \left| \int_0^t H_s dZ_s - \int_0^t H_{\underline{s}_n} dZ_s \right| \xrightarrow{L^p} 0.$$

If  $p > 2$ , one proceeds by a standard ‘‘cascade’’ argument based on repeated applications of B.D.G. Inequality to the martingales

$$\sum_{0 \leq s \leq t} (\Delta Z_s)^{2^k} - t\nu(|x|^{2^k} \mathbf{1}_{\{|x| > \eta\}})$$

to switch from  $p$  to  $p/2$ ,  $p/2^2$  until we get  $p/2^k \leq 2$ .

At this stage we call upon Proposition 6.37 in [8] Chapter VI (p.387, second edition) which states that if  $\alpha_n \xrightarrow{Sk} \alpha$  then, the stepwise constant sequence  $\alpha_n(\cdot) \xrightarrow{Sk} \alpha$ . This shows as a straightforward consequence that

$$\int_0^{\cdot n} H_{\underline{s}_n} dZ_s \xrightarrow{Sk} \int_0^{\cdot} H_s dZ_s \quad \text{in probability.}$$

The same reasoning leads to the same convergence for the integrands related to  $h$ . Now one can conclude like in the proof of Proposition 2.5(a) by combining a functional  $Sk$ -weak convergence argument and a uniform integrability argument. Of course the interpolation operator is replaced by the stepwise constant approximation operator introduced in the proof of Proposition 2.3.

(b) is proved following the lines of Proposition 2.5.(b).  $\square$

### 2.4.3 Doléans (Brownian) martingales

A similar result holds true for Doléans exponential with obvious applications to the robustness of Black-Scholes formula for option pricing.

**Proposition 2.6.** *Let  $(H_t)_{t \in [0, T]}$  and  $h = (h_t)_{t \in [0, T]}$  be like in the previous proposition 2.5. Let  $F : \mathcal{C}([0, T], \mathbb{R}) \rightarrow \mathbb{R}$  be a convex  $\|\cdot\|_{\text{sup}}$ -continuous functional with exponential growth*

$$\forall \alpha \in \mathcal{C}([0, T], \mathbb{R}), \quad |F(\alpha)| \leq C(1 + \|\alpha\|_{\text{sup}}^r), \quad r > 0.$$

for a real constant  $C \geq 0$ .

(a) *If  $|H_t| \leq h_t$   $\mathbb{P}$ -a.s. for every  $t \in [0, T]$ , then*

$$\mathbb{E} F \left( \mathcal{E} \left( \int_0^\cdot H_s dW_s \right) \right) \leq \mathbb{E} F \left( \mathcal{E} \left( \int_0^\cdot h_s dW_s \right) \right).$$

(b) *If  $H_t \geq h_t \geq 0$   $\mathbb{P}$ -a.s. for every  $t \in [0, T]$  and  $\|H\|_{L^2([0, T], dt)} \in L^{r'}(\mathbb{P})$  for  $r' > r$ , then*

$$\mathbb{E} F \left( \mathcal{E} \left( \int_0^\cdot H_s dW_s \right) \right) \geq \mathbb{E} F \left( \mathcal{E} \left( \int_0^\cdot h_s dW_s \right) \right).$$

**Proof.** STEP 1: For a fixed integer  $n \geq 1$ , we consider the sequence of random variables defined by  $\Xi_0^n = 1$  and

$$\Xi_k^n = \Xi_{k-1}^n \exp \left( H_{t_{k-1}^n} \Delta W_{t_k^n} - \frac{T}{2n} H_{t_{k-1}^n}^2 \right), \quad k = 1, \dots, n$$

(where  $\Delta W_{t_k^n} = W_{t_k^n} - W_{t_{k-1}^n}$ ) and we also define a sequence  $(x_{k,n})_{0 \leq k \leq n}$  recursively defined in backward way by

$$\xi_{n,n} = \exp \left( h_{t_{n-1}^n} \Delta W_{t_n^n} - \frac{T}{2n} h_{t_{n-1}^n}^2 \right), \quad \xi_{k,n} = \xi_{k+1,n} \exp \left( h_{t_k^n} \Delta W_{t_k^n} - \frac{T}{2n} h_{t_k^n}^2 \right), \quad k = 0, \dots, n-1.$$

We introduce the family of continuously interpolated processes

$$X_t^{n,k} = i_n(\Xi_{0:k}^n, \xi_{k+1:n,n}^n)(t), \quad t \in [0, T].$$

We define the operator  $\tilde{Q}^{(n)}$  on Borel functions  $f : \mathbb{R} \rightarrow \mathbb{R}$  with polynomial growth by

$$\tilde{Q}^{(n)}(f)(x, h) = \mathbb{E} f \left( x \exp \left( h W_{T/n} - \frac{T}{2n} h^2 \right) \right)$$

It follows from general results on peacocks (see *e.g.* [2, 6, 13]) that, as soon as  $f$  is convex, the function

$$h \mapsto \tilde{Q}^{(n)}(f)(x, h) \text{ is (even and) non-decreasing on } \mathbb{R}_+.$$

We consider the discrete time filtration  $\mathcal{G}_k^n = \mathcal{F}_{t_k^n}$  and set  $\mathbb{E}_k = \mathbb{E}(\cdot | \mathcal{G}_k^n)$ .

Now let  $F : \mathcal{C}([0, T]) \rightarrow \mathbb{R}$  be a (Borel) functional with linear growth with respect to the sup-norm and let  $F_n = F \circ i_n$ .

We will show by a backward induction on  $k \in \{0, \dots, n\}$  that

$$\mathbb{E}_k F_n(\Xi_{0:n}^n) \leq \mathbb{E}_k F_n(\Xi_{0:k}^n, \xi_{k+1:n,n}^n). \tag{2.6}$$

For  $k = n$  the equality is trivial. Assume it holds true for  $k \in \{0, \dots, n-1\}$ .

$$\begin{aligned}
\mathbb{E}_k F_n(\Xi_{0:k+1}, \xi_{k+2:n,n}) &= \mathbb{E}_k \left( F_n(\Xi_{0:k}, X_k^n \exp(H_{t_k^n} \Delta W_{t_{k+1}^n} - \frac{T}{2n} H_{t_k^n}^2), \xi_{k+2:n,n}) \right) \\
&= \left( \mathbb{E} \left( F_n(x_{0:k}, x_k \exp(\eta \Delta W_{T/n} - \frac{T}{2n} \eta^2), \xi_{k+2:n,n}) \right) \right)_{|x_{0:k} = X_{0:k}^n, \eta = H_{t_k^n}} \\
&= \left( \tilde{Q}^{(n)}(F_n(x_{0:k}, \cdot, \xi_{k+2:n,n})(X_k^n, H_{t_k^n})) \right)_{|x_{0:k} = X_{0:k}^n} \\
&\leq \left( \tilde{Q}^{(n)}(F_n(x_{0:k}, \cdot, \xi_{k+2:n,n})(X_k^n, h_{t_k^n})) \right)_{|x_{0:k} = X_{0:k}^n} \\
&= \mathbb{E}_k \left( F_n(\Xi_{0:k}, \xi_{k+1:n,n}) \right)
\end{aligned}$$

where we used the convex order property of the operator  $\tilde{Q}^{(n)}$  in the penultimate line. Plugging this inequality in (2.6) and using the chain rule for conditional expectation completes the induction.

Inequality (2.6) at  $k = 0$  reads

$$\mathbb{E} F(X^{n,n}) \leq \mathbb{E} F(X^{n,0})$$

STEP 2: Under the assumptions made on  $H$  and  $h$ , it is clear that  $\log X^{n,n}$  and  $\log X^{n,0}$  converge in  $L^2(\mathbb{P})$  for the sup norm, hence in distribution, toward  $\mathcal{E}(\int_0^\cdot H_s dW_s)$  and  $\mathcal{E}(\int_0^\cdot h_s dW_s)$  respectively. Elementary arguments about uniform integrability imply that

$$\lim_n \mathbb{E} F(X^{n,n}) = \mathbb{E} F \left( \mathcal{E} \left( \int_0^\cdot H_s dW_s \right) \right) \quad \text{and} \quad \lim_n \mathbb{E} F(X^{n,0}) = \mathbb{E} F \left( \mathcal{E} \left( \int_0^\cdot h_s dW_s \right) \right)$$

which completes the proof. Claim (b) follows likewise.  $\square$

#### 2.4.4 A counter-example

The counter-example below shows that Theorem 2.5 does not remain true if we relax the assumption that the dominating process  $(h_t)_{t \in [0, T]}$  is deterministic.

We consider the 2-period dynamics. Let  $X = X^\sigma = (X_{0:2}^\sigma)$  satisfy

$$\begin{aligned}
X_1 &= \sigma Z_1 \\
X_2 &= X_1 + \sqrt{2v(Z_1)} Z_2
\end{aligned}$$

where  $Z_{1:2} \stackrel{\mathcal{L}}{\sim} \mathcal{N}(0; I_2)$ ,  $\sigma \geq 0$  and  $v : \mathbb{R} \rightarrow \mathbb{R}_+$  is a bounded non-increasing sequence.

Let  $f(x) = e^x$ . It is clear that

$$\varphi(\sigma) := \mathbb{E} f(X_2) = \mathbb{E} \left( e^{\sigma Z_1 + v(Z_1)} \right).$$

Elementary computations show that

$$\varphi'(\sigma) = \mathbb{E} \left( e^{\sigma Z_1 + v(Z_1)} Z_1 \right)$$

so that

$$\varphi'(0) = \mathbb{E} \left( e^{v(Z_1)} Z_1 \right) < \mathbb{E} e^{v(Z_1)} \mathbb{E} Z_1 = 0$$

by a one-dimensional co-monotony argument:  $z \mapsto e^{v(z)}$  is non-increasing and  $z \mapsto z$  is non-decreasing and none of these functions are  $\mathbb{P}_{Z_1}$ -a.s. constant (which implies the strict inequality).

As a consequence,  $\varphi$  is decreasing on a right neighbourhood  $[0, \sigma_0]$  ( $\sigma_0 > 0$ ) of 0.

To include this into a Brownian stochastic integral framework, one proceeds as follows: let  $W$  be a standard Brownian motion and  $\sigma, \tilde{\sigma} \in (0, \sigma_0]$ ,  $\sigma < \tilde{\sigma}$ .

$$H_t = \sigma \mathbf{1}_{[0,1]}(t) + \sqrt{2v(W_1)} \mathbf{1}_{(1,2]}(t), \quad \tilde{H}_t = \tilde{\sigma} \mathbf{1}_{[0,1]}(t) + \sqrt{2v(W_1)} \mathbf{1}_{(1,2]}(t).$$

It is clear that  $0 \leq H_t \leq \tilde{H}_t$ ,  $t \in [0, 2]$  whereas

$$\mathbb{E} \left( e^{\int_0^2 H_s dW_s} \right) > \mathbb{E} \left( e^{\int_0^2 \tilde{H}_s dW_s} \right).$$

This makes up a counter-example to Proposition 2.5.

It has to be noted that if  $v$  is non-decreasing, then choosing  $f(x) = e^{-x}$  leads to a similar result since

$$\psi(\sigma) := \mathbb{E} f(X_2) = \mathbb{E} \left( e^{-\sigma Z_1 + v(Z_1)} \right)$$

satisfies  $\Psi'(\sigma) = -\mathbb{E} \left( e^{-\sigma Z_1 + v(Z_1)} \right)$ . In particular one still has by a co-monotony argument that  $\psi'(0) < 0$  if  $v$  is not constant.

#### 2.4.5 A comparison theorem for Laplace transform of Brownian stochastic integrals

As usual we start by a discrete time result which has its own interest for applications.

**Proposition 2.7.** *Let  $(Z_k)_{k \geq 1}$  be a sequence of  $\mathcal{N}(0; 1)$ -random variables. We set  $S_0 = 0$  and  $S_k = Z_1 \cdots + Z_k$ ,  $k = 1, \dots, n$  for convenience. We consider the two discrete time stochastic integrals*

$$X_k = \sum_{\ell=1}^k f_\ell(S_{\ell-1}) Z_\ell \quad \text{and} \quad Y_k = \sum_{\ell=1}^k g_\ell(S_{\ell-1}) Z_\ell, \quad k = 1, \dots, n$$

with  $X_0 = Y_0 = 0$  and  $f_k, g_k : \mathbb{R} \rightarrow \mathbb{R}_+$ ,  $k = 1, \dots, n$  are non-negative Borel functions satisfying:

*all the functions  $f_k$  are non-decreasing or all the functions  $g_k$  are non-decreasing.*

*Furthermore, if  $0 \leq f_k \leq g_k$  for all  $k = 1, \dots, n$ , then*

$$\forall \lambda \geq 0, \quad \mathbb{E} e^{\lambda X_n} \leq \mathbb{E} e^{\lambda Y_n}.$$

**Proof.** We start from the Cameron-Martin identity

$$\forall \sigma \in \mathbb{R}, \quad \mathbb{E} e^{\sigma Z + \varphi(Z)} = e^{\frac{\sigma^2}{2}} \mathbb{E} e^{\varphi(Z + \sigma)}.$$

which holds true in  $\overline{\mathbb{R}}_+$  for any Borel function  $\varphi : \mathbb{R} \rightarrow \mathbb{R}$ .

First we define in a backward way functions  $\tilde{f}_k$  and  $\tilde{g}_k$ ,  $k = 1, \dots, n+1$  as follows:  $\tilde{f}_{n+1} = \tilde{g}_{n+1} \equiv 0$ ,

$$\tilde{f}_k(x) = \frac{\lambda^2}{2} f_{k+1}^2(x) + \log \mathbb{E} \left( e^{\tilde{f}_{k+1}(x + \lambda f_k(x) + Z)} \right), \quad k = 1, \dots, n, \quad (2.7)$$

where  $Z \sim \mathcal{N}(0, 1)$ , the functions  $\tilde{g}_k$  being defined from the  $g_k$  the same way round.

Then, we check by induction relying on the chaining rule for conditional expectations that that

$$\mathbb{E} e^{\lambda X_n} = \mathbb{E} e^{\lambda X_k + \tilde{f}_{k+1}(S_k)}, \quad k = 1, \dots, n.$$

so that finally  $\mathbb{E} e^{\lambda X_n} = e^{\tilde{f}_1(0)}$ . It follows from (2.7) using a backward induction that, for every  $k = 1, \dots, n$ ,  $\tilde{f}_k$  is non-decreasing if the functions  $f_k$  are so and idem for  $\tilde{g}_k$  with respect to the

functions  $g_k$ . Assume *e.g.* that the  $\tilde{f}_k$  are non-decreasing. Then, a backward induction shows that  $\tilde{f}_k \leq \tilde{g}_k$  since  $\tilde{f}_n \leq \tilde{g}_n$ , and if  $\tilde{f}_{k+1} \leq \tilde{g}_{k+1}$ , for every  $x \in \mathbb{R}$ ,

$$\tilde{f}_{k+1}(x + \lambda f_k(x) + Z) \leq \tilde{f}_{k+1}(x + \lambda g_k(x) + Z) \leq \tilde{g}_{k+1}(x + \lambda g_k(x) + Z).$$

Plugging this inequality in (2.7) with  $f_k^2 \leq g_k^2$ , one concludes that  $\tilde{f}_k \leq \tilde{g}_k$ . A similar reasoning can be carried out if the functions  $\tilde{g}_k$  are non-decreasing.  $\square$

A continuous time version of this result is the following

**Theorem 2.1.** *Let  $f, g : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}_+$  two bounded Borel functions such that*

$$\left\{ \begin{array}{l} (i) \quad f, g \text{ are } dt \otimes dx\text{-a.e. continuous,} \\ (ii) \quad 0 \leq f \leq g, \\ (iii) \quad \left( \forall t \in [0, T] f(t, \cdot) \text{ is non-decreasing} \right) \text{ or } \left( \forall t \in [0, T] g(t, \cdot) \text{ is non-decreasing} \right). \end{array} \right. \quad (2.8)$$

Then,

$$\forall \lambda \geq 0, \quad \mathbb{E} e^{\lambda \int_0^T f(s, W_s) dW_s} \leq \mathbb{E} e^{\lambda \int_0^T g(s, W_s) dW_s}$$

so that, for every completely increasing function  $\varphi : \mathbb{R} \rightarrow \mathbb{R}_+$

$$\mathbb{E} \varphi \left( \int_0^T f(s, W_s) dW_s \right) \leq \mathbb{E} \varphi \left( \int_0^T g(s, W_s) dW_s \right).$$

**Remarks.** • The finiteness of these integrals follows from Novikov's criterion.

• Completely increasing functions are defined as Laplace transform of non-negative measures supported by the non-negative real line, namely

$$\varphi(x) = \int_{\mathbb{R}_+} e^{\lambda x} \mu(d\lambda).$$

• One can reformulate (2.8) into

$$\left\{ \begin{array}{l} (i) \quad f, g \text{ are } dt \otimes dx\text{-a.e. continuous,} \\ (ii) \quad \exists h : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}_+ \text{ such that } \left\{ \begin{array}{l} (a) \quad 0 \leq f \leq h \leq g \text{ and} \\ (b) \quad \forall t \in [0, T] h(t, \cdot) \text{ is non-decreasing.} \end{array} \right. \end{array} \right. \quad (2.9)$$

**Proof.** Assume *e.g.* that  $f(t, \cdot)$  is non-decreasing for every  $t \in [0, T]$ . It follows from the above assumption (2.8)(i) that

$$\left\| \int_0^T f(s, W_s) dW_s - \int_0^T f(\underline{s}_n, W_{\underline{s}_n}) dW_s \right\|_2^2 = \int_0^T \mathbb{E} (f(s, W_s) - f(\underline{s}_n, W_{\underline{s}_n}))^2 ds \rightarrow 0$$

as  $n \rightarrow +\infty$ . It suffices to show that  $\mathbb{E} (f(s, W_s) - f(\underline{s}_n, W_{\underline{s}_n}))^2 \rightarrow 0$  *ds-a.e.* since  $f$  is bounded. Furthermore, *ds-a.e.*,  $x \mapsto f(s, x)$  is *dx-a.e.* continuous hence  $\mathbb{P}_{W_s}$ -*a.s.* continuous. the convergence of  $W_{\underline{s}_n}$  toward  $W_s$  completes the proof of the convergence.

Now, define for every  $k = 1, \dots, n$ ,

$$X_k = \int_0^T f(\underline{s}_n, W_{\underline{s}_n}) dW_s = \sum_{\ell=1}^k \sqrt{\frac{T}{n}} f(t_{\ell-1}^n, W_{t_{\ell-1}^n}) U_\ell^n$$



where  $U_\ell^n = \sqrt{\frac{n}{T}}(W_{t_\ell^n} - W_{t_{\ell-1}^n})$ ,  $\ell = 1, \dots, n$ . We define likewise  $(Y_k)_{k=0:n}$  with respect to the function  $g$ . It is clear that  $(X_k)$  and  $(Y_k)$  satisfy the assumptions of the above Proposition 2.7 so that

$$\forall \lambda \geq 0, \quad \mathbb{E} e^{\lambda \int_0^T f(\underline{s}_n, W_{\underline{s}_n}) dW_s} \leq \mathbb{E} e^{\lambda \int_0^T g(\underline{s}_n, W_{\underline{s}_n}) dW_s}.$$

One concludes by combining the above quadratic (hence weak) convergence and the uniform integrability argument since

$$\forall \lambda > 0, \quad \sup_n \mathbb{E} e^{\lambda \int_0^T f(\underline{s}_n, W_{\underline{s}_n}) dW_s} \leq e^{\frac{\lambda^2}{2} \|f\|_{\text{sup}} T} < +\infty. \quad \square$$

### 3 Application to the “réduite” and to path-dependent American options

#### 3.1 Bermuda options

We consider the same discrete time dynamics as for the “European” case. Let  $(Z_k)_{1 \leq k \leq n}$  be an i.i.d. sequence of  $\mathbb{R}^d$ -valued random vectors satisfying  $Z_k \in L^r(\Omega, \mathcal{A}, \mathbb{P})$ ,  $r \geq 1$ . Let  $(X_k)_{0 \leq k \leq n}$  and  $(Y_k)_{0 \leq k \leq n}$  be two sequences of random vectors recursively defined by

$$X_{k+1}^x = X_k^x + \sigma_k(X_k^x)Z_{k+1}, \quad Y_{k+1}^x = Y_k^x + \theta_k(Y_k^x)Z_{k+1}, \quad 0 \leq k \leq n-1, \quad X_0^x = Y_0^x = x$$

where  $\sigma_k, \theta_k$ ,  $k = 0, \dots, n$  are functions from  $\mathbb{R}$  to  $\mathbb{R}$ , all with linear growth. This implies by an easy induction that the r.v. s  $X_k^x$  and  $Y_k^x$  all lie in  $L^r$ . We define the “réduite” associated to these two dynamics

$$u_0(x) = \sup \{ \mathbb{E} F_\tau(X_{0:\tau}^x), \tau \mathcal{F}\text{-stopping time} \} \quad \text{and} \quad v_0(x) = \sup \{ \mathbb{E} (F_\tau(Y_{0:\tau}^x)), \tau \mathcal{F}\text{-stopping time} \}.$$

These quantities are closely related to the optimal stopping attached to these dynamics (see [12, 4] for details).

The proposition below is the counterpart of the first step of Proposition 2.1 (discrete time).

**Proposition 3.1.** *Let  $F_k : \mathbb{R}^{k+1} \rightarrow \mathbb{R}_+$ ,  $k = 0, \dots, n$  be a sequence of nonnegative convex functions satisfying a polynomial growth  $0 \leq F_k(x_{0:k}) \leq C(1 + |x_{0:k}|^r)$ ,  $k = 0, \dots, n$ ,  $r \geq 0$ . Let  $\mathcal{F} = (\mathcal{F}_k)_{0 \leq k \leq n}$  be a filtration such that  $Z_k$  is  $\mathcal{F}_k$ -adapted and  $Z_k$  is independent of  $\mathcal{F}_{k-1}$ ,  $k = 1, \dots, n$ .*

(a) *If, for every  $k \in \{0, \dots, n-1\}$ , there exists a convex function  $\kappa_k$  such that  $0 \leq \sigma_k \leq \kappa_k \leq \theta_k$ , then, for every  $x \in \mathbb{R}$ ,*

$$\sup \{ \mathbb{E} (F_\tau(X_{0:\tau}^x)), \tau \mathcal{F}\text{-stopping time} \} \leq \sup \{ \mathbb{E} (F_\tau(Y_{0:\tau}^x)), \tau \mathcal{F}\text{-stopping time} \}.$$

(b) *If the r.v.  $Z_k$  have symmetric distributions,  $\theta_k$ ,  $k = 1, \dots, n$  are convex and  $|\sigma_k| \leq \theta_k$ ,  $k = 1, \dots, n$  then the same result holds true.*

PROOF. We introduce  $U = (U_k)_{0 \leq k \leq n}$  and  $V = (V_k)_{0 \leq k \leq n}$  be the  $\mathcal{F}$ -Snell envelopes of  $(F_k(X_{0:k}))_{0 \leq k \leq n}$  and  $(F_k(Y_{0:k}))_{0 \leq k \leq n}$  respectively i.e.

$$U_k = \mathbb{P}\text{-supess} \left\{ \mathbb{E} (F_\tau(X_{0:\tau}) | \mathcal{F}_k), \tau \mathcal{F}\text{-stopping time}, \tau \geq k \right\}$$

and

$$V_k = \mathbb{P}\text{-supess} \left\{ \mathbb{E} (F_\tau(Y_{0:\tau}) | \mathcal{F}_k), \tau \mathcal{F}\text{-stopping time}, \tau \geq k \right\}.$$

The connection between “réduite” and Snell envelope is a classical fact from Optimal Stopping Theory (see *e.g.* [12]), namely

$$\mathbb{E} U_0 = \sup \mathbb{E} \{ (F_\tau(X_{0:\tau}^x) | \mathcal{F}_k), \tau \text{ } \mathcal{F}\text{-stopping time, } \tau \geq k \}$$

(idem for  $V_0$  and  $Y^x$ ).

Following the lines of the proof of Step 1 of Proposition 2.1, one shows that the sequence of functions  $u_k : \mathbb{R}^{k+1} \rightarrow \mathbb{R}$  defined by

$$u_k(x_{0:k}) = \max \left( F_k(x_{0:k}), Qu_{k+1}(x_{0:k}, x_k + \cdot)(\sigma_k(x_k)) \right)$$

are convex owing to Lemma 2.1. On the other hand, relying on the Backward Dynamic Programming principle satisfied by the  $(\mathbb{P}, \mathcal{F})$ -Snell envelope

$$U_n = F_n(X_{0:n}), U_k = \max \left( F_k(X_{0:k}), \mathbb{E}(U_{k+1} | \mathcal{F}_k) \right), k = 0, \dots, n-1.$$

We prove by a backward induction that  $U_k = u_k(X_{0:k})$  *a.s.*,  $k = 0, \dots, n$ . One defines likewise the functions  $v_k : \mathbb{R}^{k+1} \rightarrow \mathbb{R}$  in connection with the  $(\mathbb{P}, \mathcal{F})$ -Snell envelope  $V$ .

Assume that  $0 \leq \sigma_k \leq \theta_k$ ,  $k = 0, \dots, n$  and  $\sigma_k$  are all convex. One shows by a backward induction that  $u_k \leq v_k$ ,  $k = 0, \dots, n$ . If  $k = n$  this is obvious. If it holds true with  $k+1 \leq n$ , then for every  $x_{0:k} \in \mathbb{R}^{k+1}$

$$\begin{aligned} u_k(x_{0:k}) &\leq \max \left( F_k(x_{0:k}), (Qu_{k+1}(x_{0:k}, x_k + \cdot))(\theta_k(x_k)) \right) \\ &\leq \max \left( F_k(x_{0:k}), (Qv_{k+1}(x_{0:k}, x_k + \cdot))(\theta_k(x_k)) \right) = v_k(x_{0:k}) \end{aligned}$$

where we used successively that  $u \mapsto (Qu_{k+1}(x_{0:k}, x_k + \cdot))(u)$  is non-decreasing on  $\mathbb{R}_+$  since  $u_{k+1}$  is convex and that  $u_{k+1} \leq v_{k+1}$ . Finally, the inequality for  $k = 0$  reads

$$u_0(x) = \mathbb{E} U_0 \leq \mathbb{E} V_0 = v_0(x)$$

which yields the announced result. Other cases follow from using similar arguments.  $\square$

## 3.2 American options

### 3.2.1 Brownian diffusions

This result is an generalization to path dependent “payoffs functionals of El Karoui-Jeanblanc-Shreve’s Theorem (see [3]) which mainly deal with convex functions of the marginal of the processes at time  $T$ .

**Proposition 3.2.** *Let  $(X_t^{(\sigma)})_{t \in [0, T]}$  and  $(X_t^{(\theta)})_{t \in [0, T]}$  be the martingale diffusions with  $\sigma$  and  $\theta$  as diffusion coefficients, continuous on  $[0, T] \times \mathbb{R}$ , Lipschitz in  $x$  uniformly in  $t \in [0, T]$ , driven by an  $(\mathcal{F}_t)_{t \geq 0}$ -Brownian motion  $W$  defined on a probability space  $(\Omega, \mathcal{A}, \mathbb{P})$ . Assume that there exists  $\kappa : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$  such that  $\kappa(t, \cdot)$  is convex for every  $t \in [0, T]$  and*

$$0 \leq \sigma(t, \cdot) \leq \kappa(t, \cdot) \leq \theta(t, \cdot), \quad t \in [0, T].$$

*Then for every functional  $F : [0, T] \times \mathcal{C}([0, T], \mathbb{R}) \rightarrow \mathbb{R}_+$  be a continuous functional (w.r.t. the  $\|\cdot\|_{\text{sup}}$ -norm), with polynomial growth uniformly in  $t$  in the sense*

$$\forall t \in [0, T], \forall \alpha \in \mathcal{C}([0, T], \mathbb{R}), \quad 0 \leq F(t, \alpha) \leq C(1 + \|\alpha\|_{\text{sup}}^r), \quad r \geq 0.$$

and such that, for every  $t \in [0, T]$ ,  $F(t, \cdot)$  is convex on  $\mathcal{C}([0, T], \mathbb{R})$ , one has

$$\sup \left\{ \mathbb{E}(F(\tau, X_{0:\tau}^{(\sigma), x})), \tau \in \mathcal{T}_{[0, T]}^{\mathcal{F}} \right\} \leq \sup \left\{ \mathbb{E}(F(\tau, X_{0:\tau}^{(\theta), x})), \tau \in \mathcal{T}_{[0, T]}^{\mathcal{F}} \right\}$$

where  $\mathcal{T}_{[0, T]}^{\mathcal{F}} = \{\tau : \Omega \rightarrow [0, T], \mathcal{F}\text{-stopping time}\}$ .

**Proof.** STEP 1 (Euler schemes) We consider the Euler schemes  $\bar{X}^{(\sigma), n}$  and  $\bar{X}^{(\theta), n}$  of both diffusions (we drop the dependency on the (common) starting value  $x$ ). Both schemes are adapted to the filtration  $\mathcal{F}^{(n)} := (\mathcal{F}_{t_k^n})_{0 \leq k \leq n}$ .

It follows from the former Proposition 3.1, that the  $(\mathbb{P}, \mathcal{F}^{(n)})$ -Snell envelopes  $\bar{U}^{(n)}$  and  $\bar{V}^{(n)}$  of the  $\mathcal{F}^{(n)}$ -adapted ‘‘obstacle’’ processes  $F(t_k^n, [I_n(\bar{X}^{(\sigma), n})]_{t_k^n}^{t_k^n})$ ,  $k = 0, \dots, n$ , and  $F(t_k^n, [I_n(\bar{X}^{(\theta), n})]_{t_k^n}^{t_k^n})$ ,  $k = 0, \dots, n$ , satisfy

$$\mathbb{E} \bar{U}_0^n \leq \mathbb{E} \bar{V}_0^n.$$

STEP 2 (Convergence) Now set for convenience  $\bar{Y}_{t_k^n}^{(n)} = F(t_k^n, [I_n(\bar{X}^{(\sigma), n})]_{t_k^n}^{t_k^n})$ ,  $k = 0, \dots, n$ . We know that

$$\bar{U}_{t_k^n}^{(n)} = \mathbb{P}\text{-supess} \left\{ \mathbb{E}(\bar{Y}_\tau^{(n)} | \mathcal{F}_{t_k^n}^{(n)}), \tau \in \mathcal{T}_{t_k^n, T}^{(n)} \right\}, \quad k = 0, \dots, n,$$

where  $\mathcal{T}_{t_k^n, T}^{(n)} = \left\{ \tau : \Omega \rightarrow \{t_k^n, \dots, t_{\ell}^n, \dots, t_n^n\}, \mathcal{F}^{(n)}\text{-stopping time} \right\}$ ; we also know that the Snell envelope of the process  $Y_t = F(t, X^t)$ ,  $t \in [0, T]$  is defined by

$$U_t = \mathbb{P}\text{-supess} \left\{ \mathbb{E}(Y_\tau | \mathcal{F}_t), \tau \in \mathcal{T}_{t, T}^{\mathcal{F}} \right\}, \quad t \in [0, T],$$

where  $\mathcal{T}_{t, T}^{\mathcal{F}} = \left\{ \tau : \Omega \rightarrow [t, T], \mathcal{F}\text{-stopping time} \right\}$ . Note that this Snell envelope admits a version which is a right continuous non-negative super-martingale whose compensator is continuous (and non-decreasing) since  $t \mapsto F(t, X^t)$  is continuous.

For technical purpose we introduce an intermediate quantity defined by

$$\tilde{U}_{t_k^n} = \mathbb{P}\text{-supess} \left\{ \mathbb{E}(Y_\tau | \mathcal{F}_{t_k^n}^{(n)}), \tau \in \mathcal{T}_{t_k^n, T}^{(n)} \right\} \leq U_{t_k^n}, \quad k = 0, \dots, n,$$

Our aim is to prove that, after canonically extending  $\bar{U}^{(n)}$  as a stepwise constant process by setting  $\bar{U}_t^{(n)} = \bar{U}_{t_k^n}^{(n)}$ ,  $t \in [t_k^n, t_{k+1}^n)$ ,  $\bar{U}^{(n)}$  to  $U$  in  $L^p$ .

We start from the fact that

$$|U_t - \bar{U}_{\underline{t}_n}^{(n)}| \leq |U_t - U_{\underline{t}_n}| + U_{\underline{t}_n} - \tilde{U}_{\underline{t}_n}^{(n)} + |\tilde{U}_{\underline{t}_n}^{(n)} - \bar{U}_{\underline{t}_n}^{(n)}|.$$

First, the regularity of  $U$  implies in particular that it is left continuous in  $L^1$  i.e.  $\mathbb{E}|U_t - U_{t_k^n}| \rightarrow 0$  as  $n \rightarrow \infty$ .

As concerns the second term in the right hand side of the above equation, we proceed as follows

$$0 \leq U_{t_k^n} - \tilde{U}_{t_k^n}^{(n)} \leq \mathbb{P}\text{-supess} \left\{ \mathbb{E}(Y_\tau - Y_{\tau^{(n)}} | \mathcal{F}_{t_k^n}^{(n)}), \tau \in \mathcal{T}_{t, T} \right\}$$

where  $\tau^{(n)} = \sum_{k=1}^n \frac{kT}{n} \mathbf{1}_{\{\frac{(k-1)T}{n} < \tau \leq \frac{kT}{n}\}}$  so that

$$0 \leq U_{t_k^n} - \tilde{U}_{t_k^n}^{(n)} \leq \mathbb{E} \left( \sup_{t \geq t_k^n} |Y_t - Y_{\tau^{(n)}}| | \mathcal{F}_{t_k^n}^{(n)} \right) \leq \mathbb{E} \left( \sup_{t \in [0, T]} |Y_t - Y_{\tau^{(n)}}| | \mathcal{F}_{t_k^n}^{(n)} \right).$$

Doob's Inequality applied to the martingale in the right hand side of the above inequality then implies that for every  $p \in (1, +\infty)$ ,

$$\begin{aligned} \left\| \max_{0 \leq k \leq n} (U_{t_k^{n,n}} - \tilde{U}_{t_k^{(n)}}^{(n)}) \right\|_p &\leq \frac{p}{p-1} \left\| \sup_{t \in [0, T]} |Y_t - Y_{t_n}^r| \right\|_p \\ &\rightarrow 0 \text{ as } n \rightarrow \infty \end{aligned}$$

since  $Y^{\bar{t}_n}$  converges towards  $Y^t$  *a.s.* for the sup-norm which in turn implies that  $F(\bar{t}_n, Y^{\bar{t}_n})$  *a.s.* converges toward  $F(t, Y^t)$ . The  $L^p$ -convergence follows by uniform integrability since  $\|Y\|_{\text{sup}}$  has polynomial moments at any order.

$$\begin{aligned} |\tilde{U}_{t_k^{(n)}}^{(n)} - U_{t_k^{(n)}}^{(n)}| &\leq \text{supess} \left\{ \mathbb{E}(|\bar{Y}_{\tau}^{(n)} - Y_{\tau}| \mid \mathcal{F}_{t_k^{(n)}}), \tau \in \mathcal{T}_{t_k^{(n)}, T}^{(n)} \right\} \\ &\leq \mathbb{E} \left( \max_{0 \leq k \leq n} |\bar{Y}_{t_k^{(n)}}^{(n)} - Y_{t_k^{(n)}}^{(n)}| \mid \mathcal{F}_{t_k^{(n)}} \right). \end{aligned}$$

Now

$$|\bar{Y}_{t_k^{(n)}}^{(n)} - Y_{t_k^{(n)}}^{(n)}| = |F(t_k^n, (I_n(\bar{X}^{(\sigma, n)})^{t_k^n}) - F(t_k^n, (X^{(\sigma)})^{t_k^n})|.$$

Now, note that the functional defined from  $(\mathcal{C}([0, T], \mathbb{R}), \|\cdot\|_{\text{sup}})$  into itself by

$$\alpha \mapsto \left( t \mapsto F(t, I_n(\alpha)^t) \right)$$

is continuous owing to the continuity of  $F$  itself on  $[0, T] \times \mathcal{C}([0, T], \mathbb{R})$  since  $\alpha \mapsto I_n(\alpha)^t$  is 1-Lipschitz continuous with respect to the sup norm uniformly in  $t \in [0, T]$  and  $\|\alpha^t - \alpha^s\|_{\text{sup}} \leq w(\alpha, |t - s|)$ .  $\square$

APPLICATIONS TO COMPARISON THEOREMS FOR AMERICAN OPTIONS IN A LOCAL VOLATILITY MODELS. By specifying our diffusion dynamics as a local volatility model like (2.3) we can extend the comparison result (2.4) to path-dependent American options provided the ‘‘payoff’’ functionals  $F(t, \cdot)$  are convex with linear growth as specifies in the above theorem. This is again an extension of [3].

### 3.2.2 The case of jump martingale diffusions

In what follows the product space  $\mathcal{D}([0, T], \mathbb{R}) \times [0, T]$  is endowed with the product topology ‘‘ $Sk \otimes |\cdot|$ ’’. The canonical process on  $\mathcal{D}([0, T], \mathbb{R})$  is denoted  $X$  (*i.e.*  $X_t(\alpha) = \alpha(t)$ ) and  $\theta$  denotes the canonical r.v. on  $[0, T]$  (*i.e.*  $\theta(x) = x$ ).

Let  $(\mathcal{F}_t)_{t \in [0, T]}$  be a càd filtration and let  $Y$  be an  $(\mathcal{F}_t)_{t \in [0, T]}$ -adapted càdlàg process defined on a probability space  $(\Omega, \mathcal{A}, \mathbb{P})$ . We introduce the so-called  $(\mathcal{H})$ -assumption (also known as filtration enlargement assumption) which reads as follows:

$$(\mathcal{H}) \equiv \forall H \in \mathcal{F}_T^Y, \text{ bounded, } \mathbb{E}(H \mid \mathcal{F}_t) = \mathbb{E}(H \mid \mathcal{F}_t^Y) \text{ } \mathbb{P}\text{-a.s.}$$

This filtration enlargement assumption is equivalent to the the following more tractable condition: there exists  $D \subset [0, T]$  everywhere dense in  $[0, T]$ , with  $T \in D$ , such that

$$\forall n \geq 1, \forall t_1, \dots, t_n \in D, \forall h \in \mathcal{C}_0(\mathbb{R}^n, \mathbb{R}), \mathbb{E}(h(Y_{t_1}, \dots, Y_{t_n}) \mid \mathcal{F}_t) = \mathbb{E}(h(Y_{t_1}, \dots, Y_{t_n}) \mid \mathcal{F}_t^Y) \text{ } \mathbb{P}\text{-a.s.}$$

We still consider the jump diffusions of the form (2.5) *i.e.*

$$dX_t = \kappa(t, X_{t-})dZ_t$$

where  $\kappa : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ , is a continuous function, Lipschitz continuous in  $x$  uniformly in  $t \in [0, T]$ .

The aim of this section is to extend the result obtained for convex order of Brownian diffusions to jump diffusions. We will rely on an abstract convergence result for “réduites” established in [10] (Theorem 3.7 and the remark that follows) that we recall below. To this end, we need to recall two classical definitions on stochastic processes.

**Definition 3.1.** (a) ( $D$ )-class processes: A càdlàg process  $(Y_t)_{t \in [0, T]}$  is of class ( $D$ ) if

$$\{Y_\tau, \tau \in \mathcal{T}_{[0, T]}\} \text{ is uniformly integrable.} \quad (3.10)$$

(b) Aldous criterion: A sequence of  $\mathcal{F}^n$ -adapted càdlàg processes  $(Y_t^n)_{t \in [0, T]}$  defined on a stochastic space  $(\Omega^n, \mathcal{F}^n, \mathbb{P}^n)$ ,  $n \geq 1$ , satisfies Aldous criterion with respect to  $\mathcal{F}$  if

$$\forall \eta > 0, \lim_{\delta \rightarrow 0} \limsup_n \sup_{\tau \leq \tau' \leq (\tau + \delta) \wedge T} \mathbb{P}^n(|Y_\tau^n - Y_{\tau'}^n| \geq \eta) = 0 \quad (3.11)$$

where  $\tau$  and  $\tau'$  run over  $\mathcal{T}_{[0, T]}^{\mathcal{F}^{Y^n}}$ . Then (see e.g. [8], chapter VI), the sequence  $(Y^n)_{n \geq 1}$  is tight for the Skorokhod topology.

**Theorem 3.1.** (a) Let  $(X^n)_{n \geq 1}$  be a sequence of quasi-left càdlàg processes defined on a probability spaces  $(\Omega^n, \mathcal{F}^n, \mathbb{P}^n)$  of ( $D$ )-class and satisfying the Aldous criterion (3.11). Let  $(\tau_n^*)_{n \geq 1}$  be a sequence of  $(\mathcal{F}^{X^n}, \mathbb{P}^n)$ -optimal stopping times. If furthermore,  $(X^n)_{n \geq 1}$  satisfies

$$X^n \xrightarrow{\mathcal{L}} \mathbb{P}, \mathbb{P} \text{ probability measure on } (\mathcal{ID}([0, T], \mathbb{R}), \mathcal{D}_T) \text{ such that } \sup_{t \in [0, T]} |X_t| \in L^1(\mathcal{ID}([0, T], \mathbb{R}), \mathcal{D}_T, \mathbb{P})$$

If every limiting value  $\mathbb{Q}$  of  $\mathcal{L}(X^n, \tau_n^*)$  on  $\mathcal{ID}([0, T], \mathbb{R}) \times [0, T]$  satisfies the ( $\mathcal{H}$ ) property, then the “réduites” converge i.e.

$$\lim_n \mathbb{E} U_0^n = \mathbb{E} U_0.$$

If the optimal stopping problem related to  $(X, \mathbb{Q}, \mathcal{D}^\theta)$  has a unique solution in distribution, say  $\mu_{\tau^*}^*$ , not depending on  $\mathbb{Q}$ , then  $\tau_n^* \xrightarrow{[0, T]} \mu_{\tau^*}^*$ .

(b) The same result holds with a sequence of “companion processes”  $Y^n$  taking values in a Polish space  $E$  i.e. we consider that the filtration of interest at finite range is now  $(\mathcal{F}_t^{(X^n, Y^n)})_{t \in [0, T]}$  such that  $X^n$  is quasi-left continuous with respect to this enlarged filtration. We will only need that the couple  $(X^n, Y^n)$  converges for the product topology i.e. on  $(\mathcal{ID}([0, T], \mathbb{R}), Sk_{\mathbb{R}}) \times (\mathcal{ID}([0, T], E), Sk_E)$  since this product topology spans the same Borel  $\sigma$ -field as the true Skorokhod topology on  $\mathcal{ID}([0, T], \mathbb{R} \times E)$ .

We aim at establishing the following result

**Theorem 3.2.** Let  $F : [0, T] \times \mathcal{ID}([0, T], \mathbb{R}) \rightarrow \mathbb{R}_+$  be a functional satisfying the following local Lipschitz assumption (w.r.t. to the sup norm) combined with a Skorokhod continuity assumption, namely

$$\begin{cases} (i) & F : [0, T] \times \mathcal{ID}([0, T], \mathbb{R}) \rightarrow \mathbb{R}_+ \text{ is Sk-continuous,} \\ (ii) & |F(t, \beta) - F(s, \alpha)| \leq C \left( |t - s|^{\rho'} + \|\alpha - \beta\|_{\sup}^\rho (1 + \|\alpha\|_{\sup}^{r-\rho} + \|\beta\|_{\sup}^{r-\rho}) \right), \rho, \rho' \in (0, 1], r \in (0, p), \end{cases} \quad (3.12)$$

Let  $Z = (Z_t)_{t \in [0, T]}$  be a Lévy process with Lévy measure  $\nu$  satisfying  $\nu(|z|^p) < +\infty$  for an exponent  $p \geq 2$ . Moreover assume  $\mathbb{E} Z_1 = 0$  so that the process  $Z$  is an  $L^2$ -martingale null at 0.

If there exists  $\kappa_i : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ ,  $i = 1, 2$ , continuous and Lipschitz continuous in  $x \in \mathbb{R}$ , uniformly in  $t \in [0, T]$ , and a continuous function  $\kappa_c : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ , convex and Lipschitz in  $x \in \mathbb{R}$ , uniformly in  $t \in [0, T]$ , such that

$$\kappa_1 \leq \kappa_c \leq \kappa_2.$$

Let  $X^{(\kappa_i, x)}$  be the martingale jump diffusion driven by  $Z$  starting at the same  $x \in \mathbb{R}$ . Let  $U^{(\kappa_i)}$  their Snell envelope.

$$U_0^{(\kappa_1)} \leq U_0^{(\kappa_2)}$$

**Remark.** One proves likewise that

$$\mathbb{E}(U_t^{(\kappa_1)}) \leq \mathbb{E}(U_t^{(\kappa_2)}), \quad t \in [0, T].$$

**Lemma 3.1.** Let  $G : [0, T] \times \mathcal{D}([0, T], \mathbb{R}) \rightarrow \mathbb{R}_+$  be a Skorokhod continuous functional such that  $|G(\alpha)| \leq C(1 + \|\alpha\|_{\text{sup}}^r)$ ,  $r \in (0, p)$ . Then the ‘‘obstacle process’’  $(G(t, X^t))_{t \in [0, T]}$  is regular for optimal stopping (i.e. QLC).

**Proof.** First we note that if  $\alpha_n \xrightarrow{Sk} \alpha$  and  $t_n \rightarrow t \in \text{Cont}(\alpha)$  then  $(\alpha, \alpha_{t_n}^{Sk}) \xrightarrow{Sk} (\alpha, \alpha^t)$  (see Appendix?).

As  $X$  is QLC, if a sequence of  $\mathcal{F}^Z$ -stopping times satisfies  $\tau_n \uparrow \tau$ , then  $X_\tau = X_{\tau_n}$   $\mathbb{P}$ -a.s. i.e.  $\tau(\omega) \in \text{Cont}(X(\omega))$   $\mathbb{P}(d\omega)$ -a.s.. The continuity of  $G$  implies that  $G(\tau_n, X^{\tau_n}) \xrightarrow{Sk} G(\tau, X^\tau)$ . One concludes by a uniform integrability argument that  $\mathbb{E}G(\tau_n, X^{\tau_n}) \rightarrow \mathbb{E}G(\tau, X^\tau)$  since  $(G(\tau_n, X^{\tau_n}))^{\frac{p}{r}}$  is  $L^1$ -bounded owing to the fact that  $\|X\|_{\text{sup}} \in L^1$ .  $\square$

**Proof.** STEP 1 (Aldous criterion) We denote by  $\bar{X}^n = (\bar{X}_t^n)_{t \in [0, T]}$  the stepwise constant Euler scheme with step  $\frac{T}{n}$  (i.e.  $\bar{X}_t^n = \bar{X}_{t_n}^n$ ) defined by

$$\bar{X}_{t_k}^n = \bar{X}_{t_{k-1}}^n + \kappa(t_{k-1}^n, \bar{X}_{t_{k-1}}^n)(Z_{t_k} - Z_{t_{k-1}}^n), \quad k = 1, \dots, n, \quad \bar{X}_0^n = X_0.$$

Let  $\sigma_n, \tau_n \in \mathcal{T}_{[0, T]}^{\mathcal{F}^n}$ , such that  $\sigma_n \leq \tau_n \leq (\sigma_n + \delta) \wedge T$ . In fact following Lemma 2.3, we may assume w.l.g. that  $\sigma_n$  and  $\tau_n$  take values in  $\{t_k^n, k = 0, \dots, n\}$ . Then

$$\mathbb{E}|F(\tau_n, (\bar{X}^n)^{\tau_n}) - F(\sigma_n, (\bar{X}^n)^{\sigma_n})| \leq C\delta\rho' + C\mathbb{E}(\|(\bar{X}^n)^{\tau_n} - (\bar{X}^n)^{\sigma_n}\|_{\text{sup}}(1 + 2\|\bar{X}^n\|_{\text{sup}})).$$

Hölder Inequality applied with the conjugate exponents  $a = \frac{r}{\rho}$  and  $b = \frac{r}{r-\rho}$ .

$$\mathbb{E}(\|(\bar{X}^n)^{\tau_n} - (\bar{X}^n)^{\sigma_n}\|_{\text{sup}}(1 + 2\|\bar{X}^n\|_{\text{sup}})) \leq \left\| \sup_{\sigma_n \leq s \leq (\sigma_n + \delta) \wedge T} |\bar{X}_s^n - \bar{X}_{\sigma_n}^n| \right\|_{L^r}^\rho \left(1 + 2\left\| \sup_{t \in [0, T]} |\bar{X}_t^n| \right\|_{L^r}^{r-\rho}\right).$$

• If  $r \in (0, 1]$

$$\begin{aligned} \left\| \sup_{\sigma_n \leq t_k^n \leq (\sigma_n + \delta) \wedge T} |\bar{X}_{t_k}^n - \bar{X}_{\sigma_n}^n| \right\|_{L^r}^r &\leq \mathbb{E} \sum_k \mathbf{1}_{\{\sigma_n < t_k^n \leq (\sigma_n + \delta) \wedge T\}} |\kappa(t_{k-1}^n, \bar{X}_{t_{k-1}}^n)|^r |Z_{t_k} - Z_{t_{k-1}}^n|^r \\ &= \mathbb{E}|Z_{t_k}^n - Z_{t_{k-1}}^n|^r \mathbb{E} \sum_k \mathbf{1}_{\{\sigma_n < t_k^n \leq (\sigma_n + \delta) \wedge T\}} |\kappa(t_{k-1}^n, \bar{X}_{t_{k-1}}^n)|^r \\ &= \mathbb{E}|Z_{\frac{T}{n}}|^r \mathbb{E} \left( \max_{0 \leq k \leq n-1} |\kappa(t_{k-1}^n, \bar{X}_{t_k}^n)|^r \times \text{card}\{k : \sigma_n < t_k^n \leq (\sigma_n + \delta) \wedge T\} \right) \\ &= \mathbb{E}|Z_{\frac{T}{n}}|^r \mathbb{E} \left( \max_{0 \leq k \leq n-1} |\kappa(t_{k-1}^n, \bar{X}_{t_k}^n)|^r \right) \frac{\delta n}{T} \\ &\leq \delta \left( \frac{n}{T} \mathbb{E}|Z_{\frac{T}{n}}|^r \right) \mathbb{E} \left( \max_{0 \leq k \leq n-1} |\kappa(t_{k-1}^n, \bar{X}_{t_k}^n)|^r \right) \\ &\leq C_{Z, T} \delta \left\| \max_{0 \leq k \leq n-1} |\kappa(t_{k-1}^n, \bar{X}_{t_k}^n)| \right\|_{L^r}^r \end{aligned}$$

where we used that  $t \mapsto \frac{1}{t} \mathbb{E}|Z_t|^r$  is bounded on  $[0, T]$ .

Finally, owing to the linear growth of  $\kappa$  and the fact that  $\sup_{0 \leq k \leq n} |\bar{X}_{t_k^n}^n| \in L^r(\mathbb{P}) \subset L^p(\mathbb{P})$ , we get

$$\left\| \sup_{\sigma_n \leq t_k^n \leq (\sigma_n + \delta) \wedge T} |\bar{X}_{t_k^n}^n - \bar{X}_{\sigma_n}^n| \right\|_{L^r} \leq C_{\rho, r, \kappa, Z, T} \delta^{\frac{1}{r}}$$

which implies in turn that

$$\limsup_n \sup_{\sigma_n < \tau_n \leq (\sigma_n + \delta) \wedge T} \mathbb{E} |F(\tau_n, \bar{X}^{n, \tau_n}) - F(\sigma_n, \bar{X}^{n, \sigma_n})| \leq C_{\rho, r, \kappa, Z, T} \delta^{\frac{1}{r}}.$$

The conclusion follows.

• If  $r \in [1, 2]$ : it follows from the B.D.G. Inequality applied to the local martingale  $(\bar{X}_{\sigma_n + \frac{iT}{n}} - \bar{X}_{\sigma_n}^n)_{i \geq 0}$  that

$$\begin{aligned} \left\| \sup_{\sigma_n < t_k^n \leq (\sigma_n + \delta) \wedge T} |\bar{X}_{t_k^n}^n - \bar{X}_{\sigma_n}^n| \right\|_{L^r}^r &\leq C_r \left\| \sum_{\sigma_n < t_k^n \leq (\sigma_n + \delta) \wedge T} \kappa(t_{k-1}^n, \bar{X}_{t_{k-1}^n}^n)^2 (Z_{t_k^n}^n - Z_{t_{k-1}^n}^n)^2 \right\|_{L^{\frac{r}{2}}}^{\frac{r}{2}} \\ &\leq C_r \mathbb{E} \left( \sum_k \mathbf{1}_{\{\sigma_n < t_k^n \leq (\sigma_n + \delta) \wedge T\}} |\kappa(t_{k-1}^n, \bar{X}_{t_{k-1}^n}^n)|^r |Z_{t_k^n}^n - Z_{t_{k-1}^n}^n|^r \right) \end{aligned}$$

and one concludes like in the former case.

• If  $r \in [2, 4]$ : One writes

$$\begin{aligned} \sum_{\sigma_n < t_k^n \leq (\sigma_n + \delta) \wedge T} \kappa(t_{k-1}^n, \bar{X}_{t_{k-1}^n}^n)^2 (Z_{t_k^n}^n - Z_{t_{k-1}^n}^n)^2 &= \sum_{\sigma_n < t_k^n \leq (\sigma_n + \delta) \wedge T} \kappa(t_{k-1}^n, \bar{X}_{t_{k-1}^n}^n)^2 ((Z_{t_k^n}^n - Z_{t_{k-1}^n}^n)^2 - \mathbb{E}|Z_{\frac{T}{n}}|^2) \\ &\quad + \mathbb{E}|Z_{\frac{T}{n}}|^2 \sum_{\sigma_n < t_k^n \leq (\sigma_n + \delta) \wedge T} \kappa(t_{k-1}^n, \bar{X}_{t_{k-1}^n}^n)^2 (Z_{t_k^n}^n - Z_{t_{k-1}^n}^n)^2 \end{aligned}$$

The second term of the sum in the right hand side can be treated as above (it corresponds to  $r = 2$ ). As concerns the first one, the *i.i.d. sequence*  $((Z_{t_k^n}^n - Z_{t_{k-1}^n}^n)^2 - \mathbb{E}|Z_{\frac{T}{n}}|^2)_{1 \leq k \leq n}$  is centered and lies in  $L^{\frac{r}{2}}$  with  $\frac{r}{2} \in [0, 2]$  which can be solved by the former two cases. By carrying on the process by induction one can always lower  $r$  to  $r/2$  thanks to B.D.G. inequality.

STEP 3. It follows from Step 1 of Proposition 2.3 (adapted to a vector valued framework with  $(\kappa, \mathbf{1})$  as a drift) that

$$\left( \bar{X}^n, I_n(Z) \right) \xrightarrow{\mathbb{P}(S^k)} (X, Z) \quad \text{as } n \rightarrow \infty.$$

It is clear that in the discrete time framework, the existence of the optimal stopping time  $\tau_n$  taking values in  $\{t_k^n, k = 0, \dots, n\}$  is obvious.

STEP 2: Let  $(\mathcal{D}([0, T], \mathbb{R}^2) \times [0, T], Sk^{\otimes 2} \otimes | \cdot |)$  be the canonical space of the distribution of the sequence  $(\bar{X}^n, I_n(Z), \tau_n^*)_{n \geq 1}$ . For every  $(\alpha, u) \in \mathcal{D}([0, T], \mathbb{R}^2) \times [0, T]$ , the canonical process is defined by  $\Xi_t(\alpha, u) = \alpha(t) = (\alpha^1(t), \alpha^2(t)) \in \mathbb{R}^2$  and the canonical random times is given by  $\theta(\alpha, u) = u$ . Furthermore we will denote  $\Xi = (\Xi^1, \Xi^2)$  the components of  $\Xi$ .

Let

$$\mathcal{D}_t^\theta = \cap_{s > t} \sigma(\Xi_s, \{\theta \leq u\}, 0 \leq u \leq s) \text{ if } t \in [0, T) \text{ and } \mathcal{D}_t^T = \sigma(\Xi_s, \{\theta \leq s\}, 0 \leq s \leq T)$$

the canonical right-continuous filtration. This canonical space is equipped with a structure of metric space given by  $Sk^{\otimes 2} \otimes | \cdot |$  (product of the product scalar Skorokhod topology on  $\mathcal{D}([0, T], \mathbb{R}^2)$  and the standard topology on  $[0, T]$ ).

In order to conclude to the convergence of the “réduites”, we need, following Theorem 3.1 established in [10], to show that any limiting distribution  $\mathbb{Q} = \lim_n \mathbb{P}_{(\bar{X}^n, I_n(Z), \tau_n^*)}$  on the canonical space  $(\mathcal{D}([0, T], \mathbb{R}^2) \times [0, T], Sk^{\otimes 2} \otimes |\cdot|)$  satisfies the classical  $H$ -assumption

$$\mathbb{E}_{\mathbb{Q}}(H | \mathcal{D}_t^\theta) = \mathbb{E}_{\mathbb{Q}}(H | \mathcal{D}_t) \text{ a.s.}$$

for every random variable  $H$  defined on the canonical space.

Let  $\text{Cont}(\theta)$  be the complementary of  $\mathbb{Q}$ -atoms of  $\theta$ . Let  $\Phi : \mathcal{D}([0, T], \mathbb{R}^2) \rightarrow \mathbb{R}$  and  $\Psi : \mathcal{D}([0, T], \mathbb{R}) \rightarrow \mathbb{R}$  two bounded functionals,  $Sk^{\otimes 2}$  and  $Sk$ -continuous respectively and let  $u \in \text{Cont}(\theta)$ ,  $u \leq s \leq T$ .

$$\begin{aligned} \mathbb{E}_{\mathbb{Q}}(\Phi(\Xi)\Psi(\Xi^{2,s})\mathbf{1}_{\{\theta \leq u\}}) &= \lim_n \mathbb{E} \left( \Phi(\bar{X}^n, I_n(Z)) \underbrace{\Psi(I_n(Z)^s)\mathbf{1}_{\{\tau_n^* \leq u\}}}_{\in \mathcal{F}_s^n} \right) \\ &= \lim_n \mathbb{E} \left( \mathbb{E}[\Phi(\bar{X}^n, I_n(Z)) | \mathcal{F}_s^Z] \Psi(I_n(Z)^s)\mathbf{1}_{\{\tau_n^* \leq u\}} \right). \end{aligned}$$

Up to an extraction we may assume that  $\mathbb{E}[\Phi(\bar{X}^n, I_n(Z)) | \mathcal{F}_s^Z]$  a.s. converges to  $\mathbb{E}[\Phi(X, Z) | \mathcal{F}_s^Z]$  since  $\Phi(\bar{X}^n, I_n(Z))$  converges in probability toward  $\Phi(X, Z)$ . Up to a new extraction, we may assume that  $\Psi(I_n(Z)^s)$  a.s. converges toward  $\Psi(Z^s)$  for the Skorokhod topology since  $\mathbb{P}(\Delta Z_s \neq 0) = 0$  (stopping operator at time  $s$   $\alpha \mapsto \alpha^s$  is  $Sk$ -continuous at functions  $\alpha$  continuous at  $s$ ).

Consequently, going back on the canonical space

$$(\mathbb{E}[\Phi(\bar{X}^n, I_n(Z)) | \mathcal{F}_s^Z], \Psi(I_n(Z)^s), \mathbf{1}_{\{\tau_n^* \leq u\}}) \xrightarrow{\mathcal{L}} \mathcal{L}_{\mathbb{Q}} \left( \mathbb{E}[\Phi(\Xi) | \mathcal{F}_s^{\Xi^2}], \Psi(\Xi^{2,s}), \mathbf{1}_{\{\theta \leq u\}} \right).$$

which ensures that

$$\mathbb{E}_{\mathbb{Q}}(\Phi(\Xi)\Psi(\Xi^{2,s})\mathbf{1}_{\{\theta \leq u\}}) = \mathbb{E}_{\mathbb{Q}}(\mathbb{E}_{\mathbb{Q}}[\Phi(\Xi) | \mathcal{D}_{s-}] \Psi(\Xi^{2,s})\mathbf{1}_{\{\theta \leq u\}}).$$

One concludes by standard functional monotone approximation arguments that the equality holds true for any bounded measurable functional  $\Phi, \Psi$  and any  $u \in [0, T]$ . Then by considering a sequence  $s_n \downarrow s$ ,  $s_n > s$ , we derive that

$$\mathbb{E}_{\mathbb{Q}}(\Phi(\Xi) | \mathcal{D}_s^\theta) = \mathbb{E}_{\mathbb{Q}}(\Phi(\Xi) | \mathcal{D}_s).$$

This shows that the  $H$ -assumption is fulfilled so that by Theorem 3.1,  $U_0^n$  converges toward  $U_0$ .  $\square$

## References

- [1] P. BILLINGSLEY (1999). *Convergence of probability measures*. Second edition. Wiley Series in Probability and Statistics: Probability and Statistics. A Wiley-Interscience Publication. John Wiley & Sons, Inc., New York, 277p.
- [2] P. CARR, C.-O. EWALD, Y. XIAO (2008). On the qualitative effect of volatility and duration on prices of asian options, *Finance Research Letters*, **5**, 162-171.
- [3] N. EL KAROU, M. JEANBLANC, S. SHREVE (1998). Robustness of the Black-Scholes formula, *Mathematical Finance*, **8**(2):93-126.
- [4] N. EL KAROU (1981). Les aspects probabilistes du contrôle stochastique. (French) [The probabilistic aspects of stochastic control] Ninth Saint Flour Probability Summer School 1979 (Saint Flour, 1979), pp. 73-238, *Lecture Notes in Math.*, **876**, Springer, Berlin-New York.



- [5] B. HAJEK (1985). Mean stochastic comparison of diffusions, *Probability and Related Fields*. **68**(3):315-329, DOI: 10.1007/BF00532643.
- [6] F. HIRSCH, C. PROFETA, B. ROYNETTE, M. YOR (2011). *Peacocks and Associated Martingales, With Explicit Constructions*. Bocconi & Springer, 430p.
- [7] J. JACOD (2004). The Euler scheme for Lévy driven stochastic differential equations: limit theorems, *Ann. Probab.*, **32**(3):1830-1872.
- [8] J. JACOD, A.N. SHIRYAEV (2003). *Limit theorems for stochastic processes*. Second edition. Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], **288**, Springer-Verlag, Berlin, 661p.
- [9] I. KARATZAS, S.E. SHREVE (1991). *Brownian motion and stochastic calculus*, Second edition. Graduate Texts in Mathematics, **113**, Springer-Verlag, New York, 470p.
- [10] D. LAMBERTON, G. PAGÈS (1990). Sur la convergence des réduites, *Annales de l'IHP*, (série B), **26**(2):331-35.
- [11] C. MARTINI (1999). Propagation of convexity by Markovian and Martingalian semi-groups, *Potential Analysis*, **10**(2):133-199.
- [12] J. NEVEU (1972). *Martingales à temps discret*, Masson, Paris, 218p.
- [13] G. PAGÈS, Functional co-monotony of processes with an application to peacocks, to appear in *Séminaire de Probabilités XLIV*, Springer.
- [14] D. REVUZ, M. YOR (1999). *Continuous martingales and Brownian motion*, third edition. Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], 293. Springer-Verlag, Berlin, 560p.
- [15] L. RUSCHENDORF (2007). Comparison of semi-martingales and Lévy processes, *Annals of Probability*, **35**:228-254.
- [16] L. RUSCHENDORF (2008). Comparison results fo path-dependent options, *Statistics and Decisions*, **26**:53-72.
- [17] K.I. SATO (1999). *Lévy distributions and Infinitely Divisible Distributions*, Cambridge studies in advanced mathematics, Cambridge University Press, UK, 486p.