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Boundary-Layer Effect in Composite Beams with Interlayer Slip

Noël Challamel¹ and Ulf Arne Girhammar²

Abstract: An apparent analytical peculiarity or paradox in the bending behavior of elastic-composite beams with interlayer slip, sandwich beams, or other similar problems subjected to boundary moments exists. For a fully composite beam subjected to such end moments, the partial composite model will render a nonvanishing uniform value for the normal force in the individual subelement. This is from a formal mathematical point of view in apparent contradiction with the boundary conditions, in which the normal force in the individual subelement usually is assumed to vanish at the extremity of the beam. This mathematical paradox can be explained with the concept of boundary layer. The bending of the partially composite beam expressed in dimensionless form depends only on one structural parameter related to the stiffness of the connection between the two subelements. An asymptotic method is used to characterize the normal force and the bending moment in the individual subelement to this dimensionless connection parameter. The outer expansion that is valid away from the boundary and the inner expansion valid within the layer adjacent to the boundary (beam extremity) are analytically given. The inner and outer expansions are matched by using Prandtl's matching condition over a region located at the edge of the boundary layer. The thickness of the boundary layer is the inverse of the dimensionless connection parameter. Finite-element results confirm the analytical results and the sensitivity of the bending solution to the mesh density, especially in the edge zone with stress gradient. Finally, composite beams with interlayer slip can be treated in the same manner as nonlocal elastic beams. The fundamental differential equation appearing in the constitutive law associated with the partial-composite action in a nonlocal elasticity framework is discussed. Such an integral formulation of the constitutive equation encompassing the behavior of the whole of the beam allows the investigation of the mechanical problem with the boundary-element method.

Keywords: Composite beams; Sandwich beams; Partial interaction; Interlayer slip; Boundary layer; Asymptotic method; Nonlocal mechanics; Green operator.

Introduction

Composite structures of different materials have important applications in civil and mechanical engineering. The composite beams with interlayer slip studied in this paper represent a number of different classes of structures with the same type of basic behavior and governing equations, such as layered beams (i.e., those that are mechanically jointed), shear connections (i.e., those with lap joints mechanically or adhesively jointed), and sandwich constructions (i.e., those with weak shear cores). Also, there exist different types of buildings in which part or the whole of the structure is modeled as a partially composite or sandwich-type of structure.

Layered structural elements with interlayer slip are typically encountered in wood design in which wooden beams are composed of layers assembled by means of nailing, bolting, or gluing with a soft

shear modulus. Partially composite structures built up by subelements of different materials and connected by shear connectors to form an interacting unit, such as timber-concrete or steel-concrete elements, are widely used in building engineering. In the case of a flexible connection, the analysis procedure requires the consideration of the interlayer slip among the subelements, leading to the partial interaction concept. For a detailed background in the literature about the partial-composite theory, refer to Girhammar and Gopu (1993), Girhammar and Pan (2007), or Girhammar (2008). Details about the original developments of the fundamental differential equations are found in the work of Granholm (1949) and Newmark et al. (1951) (see also Stüssi 1947 and Pleshkov 1952). A central work summarizing these early theories is that of Goodman and Popov (1968). The dynamics of partially composite beams have been recently investigated by Girhammar et al. (2009) for the in-plane behavior and Challamel (2009) or Challamel et al. (2010a) for the out-of-plane behavior.

Sandwich beams are usually composed of three layers, two thin faces and a thick, weak, shear core (Norris et al. 1952; Plantema 1966; Allen 1969). The static in-plane behavior of sandwich beams was well established by Hoff (1956). Another application in building engineering was found for high-rise buildings, which can be modeled as equivalent sandwich beams (Potzta and Kollár 2003). The in-plane buckling problem of a three-layer sandwich beam was studied by Hoff and Mautner (1948) (see also Hoff 1956; Bauld 1967; Kollár 1986). A sixth-order differential equation is obtained for the deflection, and the similarity between this governing equation and that for composite beams with partial interaction was demonstrated by Heuer (2004). However, the equivalence

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between the slip modulus of shear connectors and the shear modulus of glue lines of finite thickness or the sandwich-type of cores was first demonstrated by McCutcheon (1977). The buckling of sandwich beams has been recently reconsidered by Bažant (2003) or Kardomateas (2010). Berdichevsky (2010) rigorously derived sandwich plate or beam models from an asymptotic theory that are also valid for hard-skin plates.

In this paper, we will discuss an interesting paradox present in the bending behavior of composite beams with interlayer slip (or sandwich beams or other kinds of similar problems) subjected to boundary moments. For a very stiff connection or very thin sandwich layer, the normal forces in each individual subelement tend toward a nonvanishing value in apparent contradiction with the boundary conditions, in which the normal force in the individual subelement usually is assumed to vanish at the extremity of the beam. In reality, a moment is defined as a force couple and is the formal mathematical representation of the resulting moment effect of forces of some origin that are applied on or transferred to the beam at the end of the beam, in this case.

We will discuss the behavior of the composite beam subjected to end moments when the origin of these end moments is not known; that is, from a formal mathematical perspective. In an ordinary beam, the analysis of the effect of these resulting end moments will render mathematically unique and consistent results. However, in a composite beam with interlayer slip, the distribution of the end moments among the individual subelements needs to be known for the analysis to be unique and consistent. In consistency with the way the theory for composite beams with interlayer slip is set up, no unique way to distribute a total resulting end moment applied at the boundary of the beam to the individual subelements exists. The problem is of a hyperstatic nature, and the distribution cannot be determined without knowing the conditions and the behavior of the whole composite beam, which in turn, will depend on the boundary loadings. So from a formal mathematical point of view, this problem cannot be solved in a unique and consistent way. In a real situation, the origin and the conditions for the forces that cause these end moments are known and the correct boundary loading conditions for the individual subelements can be stated. The partial-composite action theory is formulated according to the internal normal force for the individual subelement and the total moment for the whole cross section of the composite beam. To be consistent, the boundary loads need to be separated in a corresponding way. Not knowing the background for the resulting end moment, the natural choice for the value of the boundary condition for the normal force of the individual subelement is zero when the boundary is subjected to this total end moment. However, the fact that the boundary of the whole beam is subjected to a total end moment only means that the resulting normal force is zero at the boundary, not that the normal forces of the individual parts of the beam is zero. For example, in a two-component composite beam, the normal forces in the individual components are equal but of opposite sign.

Special cases exist in which both the real conditions and the formal analyses are such that the boundary conditions on the composite beam with interlayer slip are represented by a resulting end moment for the whole cross section and a zero normal force of one of the individual subelements. For example, this is the case when only one of the subelements is subjected to an axial load, which produces an eccentric end moment with respect to the centroid of the fully composite beam (Girhammar and Copu 1991). However, to be consistent with the partial-composite action theory, these boundary loadings need to be interpreted as an axial load applied at the centroid of the fully composite beam on the whole area of the cross section of the beam and as an end moment, also applied

on the whole cross section of the beam. An analysis with those formal boundary conditions will not render the actual stress conditions at the end of the beam. The real local load or stress distributions at the boundary of the partially composite beam need to be evaluated separately as the difference between the results of the formal analysis of the load distribution at the end and the real distribution of the applied loads at the boundary.

Thus, the paradox as observed from a formal mathematical point of view can be explained by using the concept of boundary layer, which is very instructive in the field of fluid mechanics (Schlichting and Gersten 2000). Boundary layer theory is also used in solid mechanics, especially at the interface modeling between two solids by using the plasticity theory (Fleck and Hutchinson 1993; Shu et al. 2001). Boundary layers are also typically observed for elastic problems constituted of dissimilar materials (e.g., a soft one associated with a stiff one), for three-dimensional media, or for very thin plate problems (Sanchez-Hubert and Sanchez-Palencia 1992). Everstine and Pipkin (1971) also studied the plane strain of ideal fiber-reinforced composites exhibiting boundary layer phenomenon for a large stiffness ratio between the extensional modulus and the shear modulus. The use of the boundary layer theory for the lateral-torsional buckling of composite elastic beams with small warping terms is also mentioned in Hodges and Peters (2001). This phenomenon may also be observed for two-dimensional elastic beam models in the vicinity of the applied boundary conditions (Gao et al. 2007). Boundary layers can also appear for very thin interfaces (Lebon and Rizzoni 2010).

The boundary layer typically arises in gradient elastic beams in which the small characteristic length term is multiplying the highest derivative of the problem (Challamel and Wang 2008). We will show in this paper that composite beams with interlayer slip belong to the class of nonlocal elasticity beams and that they also exhibit this specific boundary layer phenomenon, for example, when subjected to boundary moments. More specifically, the boundary layer appearing at the extremity of the composite beam for the normal force distribution in the individual subelement along the beam will be analyzed with the elastic connection parameter between the two connected subelements of the beam.

Basic Assumptions and Notations

The geometric parameters defining a typical composite beam with two subelements of different geometry and materials are shown in Fig. 1(a). The subscripts “1” and “2” refer to the top and bottom elements of the cross section, respectively. The x -axis of the coordinate system is located in the center of gravity (i.e., in the centroid) of the fully composite section. The displacements in the x -, y -, and z -direction are denoted u , v , and w , respectively.

A free-body diagram of an element in the composite beam is shown in Fig. 1(b), for which moments M_z , M_{z1} , and M_{z2} ; shear forces V_y , V_{y1} , and V_{y2} ; normal forces N_1 and N_2 ; and slip force per unit length $V_{s,x}$ are defined. The influence of shear forces will be neglected in this study. The effect of shear deformation can be formally considered for shear composite beams as considered by Qiao and Wang (2004), Wang and Qiao (2006), Hamey (2007), or Wang and Zhang (2009). The centroid of the fully composite section is given by $y_{cg,\infty} = E_2 A_2 h_0 / EA_0$ where $E_2 A_2$ is the axial stiffness of Subelement 2 and $EA_0 = E_1 A_1 + E_2 A_2$ is the total axial stiffness of the composite member.

The subelements are connected together by means of some kind of discrete shear connectors or a weak shear layer, which is assumed to produce uniformly distributed slip forces or interlayer shear stresses. The shear connector or shear layer force per unit

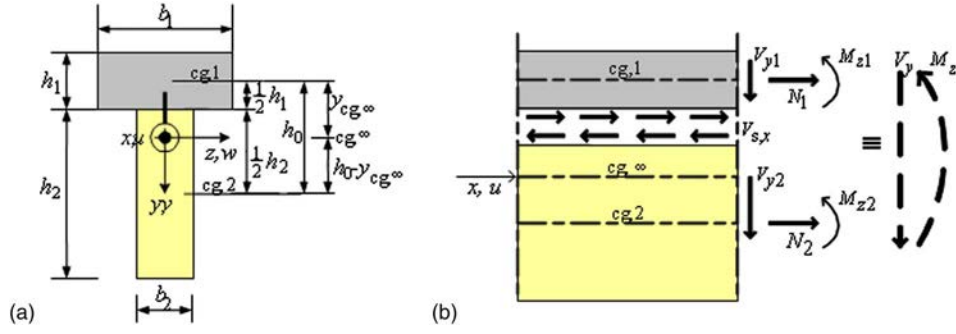


Fig. 1. (a) Geometric parameters of partial-composite beam where x , y , and z = coordinates; u , v , and w = displacements; $cg,1$, $cg,2$ and cg,∞ = centroid (center of gravity) of Subelements 1, 2 and of a fully composite section, respectively; $y_{cg,\infty} = E_2A_2h_0/EA_0$ = distance of the centroid of the fully composite section to the centroid of Subelement 1; (b) element in the partial-composite column where internal forces and moments are defined positive, as shown

length versus the slip or shear displacement behavior is linear elastic with a constant slip modulus, K_T [N/m²], with respect to bending in the transverse direction. Frictional effects and uplift at the shear interface are neglected. Full composite action (i.e., infinite slip modulus, $K_T \rightarrow \infty$) and noncomposite action (i.e., zero slip modulus, $K_T \rightarrow 0$) represent the upper and lower boundaries for the partial-composite action, respectively. Cohesive zone modeling can be potentially implemented for the delamination process (Wang 2007).

Consider the simply fork supported composite beam of length L subjected to end moments M_0 shown in Fig. 2. For each subelement, the Euler-Bernoulli beam theory, neglecting the effect of shear deformation, was assumed to be applicable. The curvature of the two subelements was assumed to be equal for the case of transverse bending. The differential equation expressed in the normal force $N_1(x)$ in Subelement 1 (with regard to equilibrium, the normal force in Subelement 2 is $N_2(x) = -N_1(x)$) is given by Girhammar and Gopu 1993 (see also the appendix).

$$N_1'' - \alpha_T^2 N_1 = \beta_T M_0 \quad \text{with} \quad N_1(0) = N_1(L) = 0 \quad (1)$$

The parameters of the differential equation are given by

$$\begin{aligned} \alpha_T^2 &= K_T \left(\frac{1}{E_1 A_1} + \frac{1}{E_2 A_2} + \frac{h_0^2}{EI_{z,0}} \right) & \beta_T &= K_T \frac{h_0}{EI_{z,0}} \\ EA_0 &= E_1 A_1 + E_2 A_2 & EA_p &= E_1 A_1 \cdot E_2 A_2 \\ EI_{z,0} &= E_1 I_{z1} + E_2 I_{z2} & EI_{z,\infty} &= EI_{z,0} + \frac{EA_p h_0^2}{EA_0} \end{aligned} \quad (2)$$

$$\frac{\beta_T h_0}{\alpha_T^2} = 1 - \frac{EI_{z,0}}{EI_{z,\infty}}$$

It is easy to obtain the exact solution of this differential equation

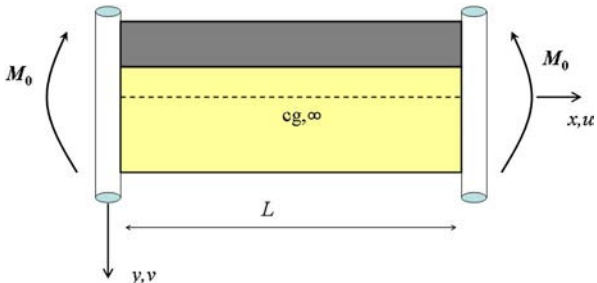


Fig. 2. Partially composite beam subjected to bending end moments

$$\begin{aligned} N_1(x) &= \left(1 - \frac{EI_{z,0}}{EI_{z,\infty}} \right) \frac{M_0}{h_0} \left[\cosh \alpha_T x - \tanh \frac{\alpha_T L}{2} \sinh \alpha_T x - 1 \right] \\ &= -N_2(x) \end{aligned} \quad (3)$$

The normal force associated with the full composite beam can be introduced as

$$N_{1,\infty} = - \left(1 - \frac{EI_{z,0}}{EI_{z,\infty}} \right) \frac{M_0}{h_0}; \quad N_{2,\infty} = \left(1 - \frac{EI_{z,0}}{EI_{z,\infty}} \right) \frac{M_0}{h_0} \quad (4)$$

Therefore, the normal force can be normalized as

$$\frac{N_1(x)}{N_{1,\infty}} = \frac{N_2(x)}{N_{2,\infty}} = 1 - \cosh \alpha_T x + \tanh \frac{\alpha_T L}{2} \sinh \alpha_T x \quad (5)$$

The dimensionless parameters can be introduced as

$$\hat{\alpha} = \alpha_T L; \quad \hat{x} = \frac{x}{L}; \quad n = \frac{N_1}{N_{1,\infty}} = \frac{N_2}{N_{2,\infty}} \quad (6)$$

The dimensionless normal force is then written as

$$n(\hat{x}) = 1 - \cosh \hat{\alpha} \hat{x} + \tanh \frac{\hat{\alpha}}{2} \sinh \hat{\alpha} \hat{x} \quad (7)$$

It is easy to verify the boundary conditions

$$n(0) = n(1) = 0 \quad (8)$$

Furthermore, the normal force vanishes for the noncomposite beam

$$\hat{\alpha} \rightarrow 0 \Rightarrow n(\hat{x}) \rightarrow 0 \quad (9)$$

The normal force may also be expressed as

$$n(\hat{x}) = 1 - \frac{\cosh \left[\frac{\hat{\alpha}}{2} (2\hat{x} - 1) \right]}{\cosh \frac{\hat{\alpha}}{2}} \quad (10)$$

Eq. (10) clearly shows that the normal force is symmetrical about the median axis. This symmetrical portrait can be proven from the symmetrical properties of the differential equation coupled with symmetrical properties of the boundary conditions given by Eq. (1). Figs. 3 and 4 show the variation of the dimensionless normal force $n(\hat{x})$ parameterized by the dimensionless connection parameter $\hat{\alpha}$.

Analysis of the Boundary-Layer Effect with Respect to Normal Forces

The differential equation Eq. (1) can be expressed in the dimensionless form as

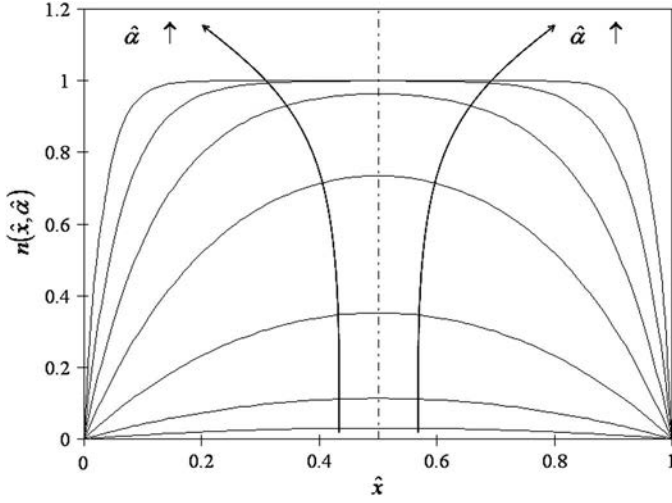


Fig. 3. Influence of the dimensionless connection parameter $\hat{\alpha}$ on the parameterized normal force $n(\hat{x}, \hat{\alpha})$ and $\hat{\alpha} \in \{0.5; 1; 2; 4; 8; 16; 32\}$

$$-\varepsilon^2 n'' + n = 1 \quad \text{with} \quad n(0) = n(1) = 0 \quad \text{and} \quad \varepsilon = \frac{1}{\hat{\alpha}} \quad (11)$$

where the prime now denotes the derivative with respect to the dimensionless variable \hat{x} . It appears that the fully composite beam ($\hat{\alpha} \rightarrow \infty$) is associated with a boundary layer phenomenon that we will study in more detail

$$\hat{\alpha} \rightarrow \infty \Rightarrow \varepsilon \rightarrow 0 \Rightarrow n(\hat{x}) \rightarrow 1 \quad \text{even if} \quad n(0) = n(1) = 0 \quad (12)$$

Boundary layers are regions in which a rapid change occurs in the value of a variable; in this case, the normal force in the individual subelement at the beam extremity for the fully composite beam. Classical physical examples of situations in which the boundary layer may occur are the fluid velocity near a solid wall. Ludwig Prandtl pioneered the subject of the boundary layer theory in his explanation of how a quantity as small as the viscosity of common fluids could nevertheless play a crucial role in determining their flow (Bush 1992; Nayfeh 2000). Mathematically, the occurrence of a boundary layer is associated with the presence of a small parameter ε multiplying the highest derivative [i.e., the second derivative in Eq. (11)] in the governing equation of the process.

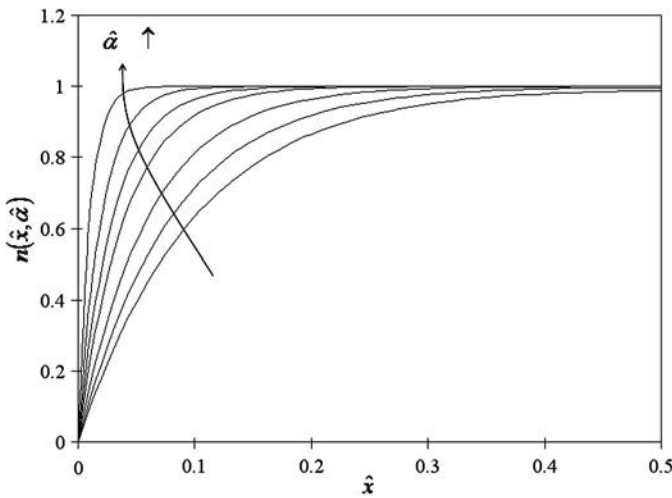


Fig. 4. Influence of the dimensionless connection parameter $\hat{\alpha}$ on the parameterized normal force $n(\hat{x}, \hat{\alpha})$; symmetrical portrait; and $\hat{\alpha} \in \{10; 12.5; 16.67; 25; 33.33; 50; 100\}$

A straightforward perturbation expansion by using an asymptotic sequence in the small parameter ε leads to differential equations of a lower order than the original governing equation. As a consequence, not all the boundary conditions can be satisfied by the perturbation expansion

$$n_{\text{out}}(\hat{x}, \varepsilon) = n_0(\hat{x}) + \varepsilon n_1(\hat{x}) + \varepsilon^2 n_2(\hat{x}) + \dots \quad (13)$$

Introducing Eq. (13) into Eq. (11) leads to the asymptotic derivation

$$n_0(\hat{x}) = 1, \quad n_1(\hat{x}) = 0, \quad -n_0'' + n_2 = 0 \Rightarrow n_2 = 0 \quad (14)$$

This straightforward expansion is referred to as the outer expansion, which is valid away from the boundary. Therefore, the outer expansion is simply written as

$$n_{\text{out}}(\hat{x}, \varepsilon) = n_0(\hat{x}) = 1 \quad (15)$$

The counterpart, the inner expansion, is valid within the layer adjacent to the boundary (i.e., the beam extremity). The inner expansion associated with the boundary layer region is expressed as a stretched variable

$$s = \frac{\hat{x}}{\varepsilon} \quad (16)$$

This choice of a stretched variable is associated with the asymptotic property $\hat{x} = O(\varepsilon)$ in the boundary layer. The new differential equation is written as

$$-\frac{d^2 n}{ds^2} + n = 1 \quad (17)$$

whose solution is given by

$$n_{\text{in}}(s, \varepsilon) = A \cosh s + B \sinh s + 1 \quad (18)$$

An asymptotic solution is presented for the layer adjacent to the boundary of the left part of the beam. For symmetrical reasons, only half of the beam will be treated. The left boundary condition is written as

$$n_{\text{in}}(s = 0) = 0 \Rightarrow A = -1 \quad (19)$$

The inner and outer expansions are matched over a region located at the edge of the boundary layer. The technique is called the method of matched asymptotic expansion. Prandtl's matching condition is simply written as

$$\lim_{s \rightarrow \infty} n_{\text{in}}(s) = \lim_{\hat{x} \rightarrow 0} n_0(\hat{x}) = 1 \quad (20)$$

Prandtl's matching condition necessarily shows that

$$B = 1 \quad (21)$$

Therefore, the following result is obtained for the asymptotic expansion in both regions

$$\begin{aligned} \text{Outer region: } n_{\text{out}}(\hat{x}) &= 1 \quad \text{for } \hat{x} = O(1), \\ \text{Inner region: } n_{\text{in}}(\hat{x}) &= 1 + \sinh \frac{\hat{x}}{\varepsilon} - \cosh \frac{\hat{x}}{\varepsilon} \quad \text{for } \hat{x} = O(\varepsilon) \end{aligned} \quad (22)$$

Analysis of the Boundary-Layer Effect with Respect to Bending Moments

The equilibrium equations are simply written as [Fig. 1(b)]

$$M_0 = M_{z1} + M_{z2} - N_1 h_0, \quad N_1 + N_2 = 0 \quad (23)$$

The in-plane kinematics is derived from the curvature equality principle

$$-v'' = \chi = \frac{M_{z1}}{E_1 I_{z1}} = \frac{M_{z2}}{E_2 I_{z2}} \quad (24)$$

Introducing Eq. (24) into Eq. (23) leads to the bending moment identity

$$M_{z1} = \frac{E_1 I_{z1}}{E_1 I_{z1}} [M_0 + N_1(x)h_0] = \frac{E_1 I_{z1}}{E_2 I_{z2}} M_{z2} \quad (25)$$

where the normal force is given by Eq. (3).

It is convenient to introduce the asymptotic parameters

$$\begin{aligned} M_{z1,\infty} &= \frac{E_1 I_{z1}}{E_1 I_{z,\infty}} M_0 & M_{z2,\infty} &= \frac{E_2 I_{z2}}{E_1 I_{z,\infty}} M_0 \\ M_{z1,0} &= \frac{E_1 I_{z1}}{E_1 I_{z,0}} M_0 & M_{z2,0} &= \frac{E_2 I_{z2}}{E_1 I_{z,0}} M_0 \end{aligned} \quad (26)$$

where the different expressions for the bending stiffness are given by Eq. (2).

According to Eq. (24), the dimensionless bending moment can also be introduced as

$$m_z = \frac{M_{z1}}{M_{z1,\infty}} = \frac{M_{z2}}{M_{z2,\infty}} \quad (27)$$

According to Eqs. (4), (25), and (27), the dimensionless bending moment is related to the dimensionless normal force by

$$\begin{aligned} m_z &= \frac{EI_{z,\infty}}{EI_{z,0}} + \left(1 - \frac{EI_{z,\infty}}{EI_{z,0}}\right)n \\ \text{with } \frac{M_{z1,\infty}}{M_{z1,0}} &= \frac{M_{z2,\infty}}{M_{z2,0}} = \frac{EI_{z,0}}{EI_{z,\infty}} < 1 \end{aligned} \quad (28)$$

The case of the noncomposite beam is obtained from

$$\hat{\alpha} \rightarrow 0 \Rightarrow n(\hat{x}) \rightarrow 0 \Rightarrow m_z(\hat{x}) \rightarrow \frac{EI_{z,\infty}}{EI_{z,0}} > 1 \quad (29)$$

The fully composite beam is derived from the asymptotic behavior

$$\begin{aligned} \hat{\alpha} \rightarrow \infty &\Rightarrow n(\hat{x}) \rightarrow 1 \Rightarrow m_z(\hat{x}) \rightarrow 1 \\ \text{even if } m_z(0) &= m_z(1) = \frac{EI_{z,\infty}}{EI_{z,0}} \end{aligned} \quad (30)$$

according to Eqs. (12) and (28).

The dimensionless bending moment $m_z(\hat{x})$ is shown in Figs. 5 and 6.

Partially Composite Beam Considered as a Nonlocal Beam

In this section, we treat the partially composite beam as a nonlocal beam. The theory of nonlocal elasticity states that the stress at a reference point in the body depends not only on the strains at this reference point but also on the strains at all other points of the body. Eringen (1983) used a differential equation to express the stress as a spatial weight average of the strain by an integral operator. The same concept can be applied to nonlocal beam mechanics in which the bending moment at a given section depends not only on the curvature at this reference section but also on the curvature in the whole beam. For instance, Peddieson et al. (2003) investigated Eringen's model at the beam scale to obtain some specific scale effects inherent in small-scale structures.

The moment-curvature relationship of partially composite beams can be written as (Girhammar and Gopu 1993 and the appendix)

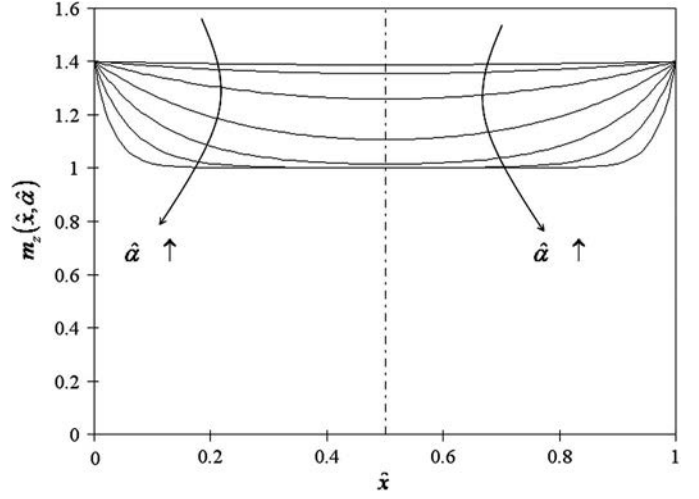


Fig. 5. Influence of the dimensionless connection parameter $\hat{\alpha}$ on the parameterized bending moment $m_z(\hat{x}, \hat{\alpha})$; $EI_{z,\infty}/EI_{z,0} = 1.4$; and $\hat{\alpha} \in \{0.5; 1; 2; 4; 8; 16; 32\}$

$$\frac{\alpha_T^2}{EI_{z,\infty}} M - \frac{1}{EI_{z,0}} M'' = \alpha_T^2 \chi - \chi'' \quad (31)$$

where χ = curvature of both subelements. In Eq. (31), the nonlocal constitutive relationship is recognized as considered by Challamel and Wang (2008) to model scale effects in elastic beams (see also Zhang et al. 2010)

$$\begin{aligned} M - l_c^2 M'' &= EI_{z,\infty} (\chi - a^2 \chi'') \quad \text{with } l_c = \frac{1}{\alpha_T} \sqrt{\frac{EI_{z,\infty}}{EI_{z,0}}} \quad \text{and} \\ a &= \frac{1}{\alpha_T} \end{aligned} \quad (32)$$

Challamel and Wang (2008) showed that this model for ordinary beams can be considered as a nonlocal integral model on the basis of the combination of Eringen's nonlocal constitutive model with a local model. This can be shown from a strain-based approach on the basis of the nonlocal curvature $\bar{\chi}$ defined from

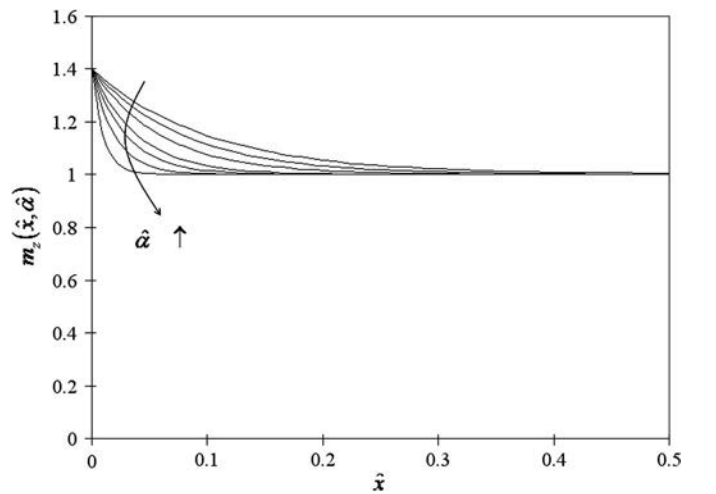


Fig. 6. Influence of the dimensionless connection parameter $\hat{\alpha}$ on the parameterized bending moment $m_z(\hat{x}, \hat{\alpha})$; $EI_{z,\infty}/EI_{z,0} = 1.4$; symmetrical portrait; and $\hat{\alpha} \in \{10; 12.5; 16.67; 25; 33.33; 50; 100\}$

$$\bar{\chi} - l_c^2 \bar{\chi}'' = \chi \Rightarrow \bar{\chi}(x) = \int_0^L G(x,t) \chi(t) dt \quad (33)$$

where the weighting function $G(x,t) = \text{Green's function}$ of the differential system associated with the associated curvature boundary conditions of the problem. The nonlocal curvature $\bar{\chi}(x)$ can be understood as a spatial average of the local curvature along the beam.

In a partially composite beam, the interlayer slip, and hence, the force or stress distribution in the beam, cannot be determined without knowing the complete behavior and all conditions of the whole beam. This indicates that one cannot solve a partially composite beam problem only knowing the local constitutive equation; one needs to know the global equation. Thus, in such composite beams, a kind of nonlocal beam behavior exists. The distribution of the stresses is of a hyperstatic type.

By analogy with Eq. (33), Eq. (32) can also be reformulated as

$$\begin{aligned} M &= EI_{z,\infty} \left\{ \left[1 - \left(\frac{a}{l_c} \right)^2 \right] \bar{\chi} + \left(\frac{a}{l_c} \right)^2 \chi \right\} \\ &= EI_{z,\infty} \left[\left(1 - \frac{EI_{z,0}}{EI_{z,\infty}} \right) \bar{\chi} + \frac{EI_{z,0}}{EI_{z,\infty}} \chi \right] \\ &= EI_{z,0} \left[\left(\frac{EI_{z,\infty}}{EI_{z,0}} - 1 \right) \bar{\chi} + \chi \right] \end{aligned} \quad (34)$$

A similar approach can be presented by using a stress-based approach on the basis of the nonlocal bending moment \tilde{M} defined from

$$\tilde{M} - a^2 \tilde{M}'' = M \Rightarrow \tilde{M}(x) = \int_0^L H(x,t) M(t) dt \quad (35)$$

where the weighting function $H(x,t) = \text{Green's function}$ of the differential system associated with the associated bending moment boundary conditions of the problem. The nonlocal constitutive law can then be written by using a combination of the local and nonlocal bending moment variables

$$\begin{aligned} \chi &= \frac{1}{EI_{z,\infty}} \left\{ \left[1 - \left(\frac{l_c}{a} \right)^2 \right] \tilde{M} + \left(\frac{l_c}{a} \right)^2 M \right\} \\ &= \frac{1}{EI_{z,\infty}} \left[\left(1 - \frac{EI_{z,\infty}}{EI_{z,0}} \right) \tilde{M} + \frac{EI_{z,\infty}}{EI_{z,0}} M \right] \\ &= \frac{1}{EI_{z,0}} \left[\left(\frac{EI_{z,0}}{EI_{z,\infty}} - 1 \right) \tilde{M} + M \right] \end{aligned} \quad (36)$$

In the absence of distributed forces, the nonlocal law is simplified as

$$M''(x) = 0 \Rightarrow \chi = \frac{1}{EI_{z,\infty}} \tilde{M} \quad (37)$$

In the present case studied in the paper, the bending moment is uniform. Eq. (32) then leads to the simplified nonlocal constitutive law

$$M(x) = M_0 \Rightarrow M = EI_{z,\infty} (\chi - a^2 \chi'') \quad (38)$$

Such a constitutive relationship is typically encountered in the case of gradient elastic beams (Challamel and Wang 2008). The boundary layer typically arises for gradient elastic beams in which the small length term is multiplied into the highest derivative of the problem. However, for a partially composite beam, the boundary conditions are different from the ones of usual gradient elasticity models. Under the conditions for the bending case studied in this paper, the boundary conditions are assumed to be

$$N_1(0) = N_2(0) = N_1(L) = N_2(L) = 0 \quad (39)$$

Introducing Eq. (39) into the equilibrium equation Eq. (23) leads to the following boundary conditions for the moments at the beam extremity:

$$M_0 = M_{z1} + M_{z2} \quad (40)$$

According to Eq. (24), ($\chi = -v''$), the boundary conditions can then be expressed for the curvature as

$$\chi(0) = \chi(L) = \frac{M_0}{EI_{z,0}} \quad (41)$$

Finally, the static problem of a partially composite beam under uniform bending moment is reduced to the following second-order differential equation of the curvature:

$$\chi - \frac{1}{\alpha_7^2} \chi'' = \frac{M_0}{EI_{z,\infty}} \quad (42)$$

with the pertaining boundary conditions according to Eq. (41).

The dimensionless curvature $\hat{\chi}$ can be introduced as

$$\hat{\chi} = \frac{\chi}{\frac{M_0}{EI_{z,\infty}}} \quad (43)$$

Solving Eq. (42) and noting the expression for the normal force according to Eq. (7), the dimensionless curvature is found to be equivalent to Eq. (28)

$$\hat{\chi} = \frac{EI_{z,\infty}}{EI_{z,0}} + \left(1 - \frac{EI_{z,\infty}}{EI_{z,0}} \right) n = m_z \quad (44)$$

Hence, Figs. 5 and 6 also include the curvature evolution along the beam and the specific boundary-layer effect obtained for large values of the connection parameter. A partially composite beam model as discussed in this paper can be considered to correspond to a nonlocal beam model. The existence of a boundary layer in gradient elasticity beams has already been demonstrated in Challamel and Wang (2008) in the case of an ordinary cantilever beam.

The stress-approach would, in the case of a uniform moment, lead to

$$M(x) = M_0 \Rightarrow \chi = \frac{1}{EI_{z,\infty}} \tilde{M}_0 = \frac{M_0}{EI_{z,\infty}} \int_0^L H(x,t) dt \neq \frac{M_0}{EI_{z,\infty}} \quad (45)$$

This result is attributable to the fact that the boundary conditions of the nonlocal bending moments may differ from the ones of the uniform bending values M_0 . The Green operator of the stress-based approach is deduced from the boundary conditions

$$\frac{\tilde{M}(0)}{EI_{z,\infty}} = \frac{\tilde{M}(L)}{EI_{z,\infty}} = \frac{M_0}{EI_{z,0}} \quad (46)$$

The stress-based approach is interesting from an engineering point of view and allows the calculation of the curvature field directly from the bending stress variation by using the integral operator.

Calculations for the integral operator are derived with the following differential equation:

$$\tilde{M} - a^2 \tilde{M}'' = M \quad \text{with} \quad \tilde{M}(0) = \tilde{M}(L) = \frac{EI_{z,\infty}}{EI_{z,0}} M_0 \quad (47)$$

It can be useful to introduce the change of variable to impose homogeneous boundary conditions

$$M^* = \tilde{M} - \frac{EI_{z,\infty}}{EI_{z,0}} M_0 \quad (48)$$

Therefore, the differential problem is converted into the equivalent one

$$M^* - \alpha^2 M^{*''} = M - \frac{EI_{z,\infty}}{EI_{z,0}} M_0 \quad \text{with } M^*(0) = M^*(L) = 0 \quad (49)$$

leading to the integral formulation

$$M^* = \int_0^L H^*(x,t) \left[M(t) - \frac{EI_{z,\infty}}{EI_{z,0}} M_0 \right] dt \quad (50)$$

where the Green's function, associated with the differential operator with the specific associated homogeneous boundary conditions according to Eq. (49), is given by (Polyanin and Manzhirov 2007)

$$H^*(x,t) = \frac{1}{a} \frac{\sinh(\frac{L-t}{a})}{\sinh(\frac{L}{a})} \sinh \frac{x}{a} \quad \text{if } x < t \quad \text{and}$$

$$H^*(x,t) = \frac{1}{a} \frac{\sinh(\frac{L-x}{a})}{\sinh(\frac{L}{a})} \sinh \frac{t}{a} \quad \text{if } x > t \quad (51)$$

This integral operator can be understood as an interacting weight function that tends toward a Dirac distribution for the full composite action (i.e., $\varepsilon = 1/\hat{\alpha} \rightarrow 0$)—see Fig. 7 or Fig. 8. This indicates that the two interacting beams behave like a single beam with nonlocal interacting effects strongly dependent on the connection parameter.

The curvature is finally computed from the Green's operator considering Eqs. (37) and (48)

$$\chi = \frac{1}{EI_{z,\infty}} \int_0^L H^*(x,t) \left[M(t) - \frac{EI_{z,\infty}}{EI_{z,0}} M_0 \right] dt + \frac{M_0}{EI_{z,0}} \quad (52)$$

Such a formulation is typically compatible with the boundary-element method. In the particular case of a uniform bending moment, Eq. (52) is reduced to

$$\chi = - \left(\frac{EI_{z,\infty}}{EI_{z,0}} - 1 \right) \frac{M_0}{EI_{z,\infty}} \int_0^L H^*(x,t) dt + \frac{M_0}{EI_{z,0}}$$

$$= - \left[\left(1 - \frac{EI_{z,0}}{EI_{z,\infty}} \right) \int_0^L H^*(x,t) dt - 1 \right] \frac{M_0}{EI_{z,0}} \quad (53)$$

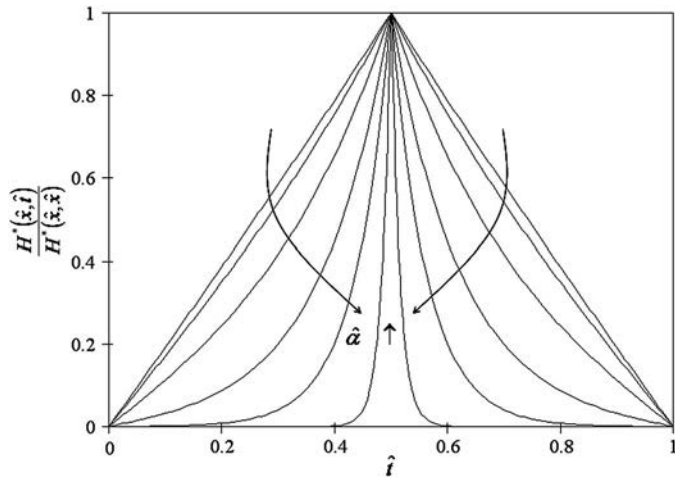


Fig. 7. Influence of the dimensionless connection parameter $\hat{\alpha}$ on the dimensionless weight function $H^*(\hat{x}, \hat{t})/H^*(\hat{x}, \hat{x})$; $\hat{x} = 1/2$; and $\hat{\alpha} \in \{1; 2; 4; 8; 16; 64\}$

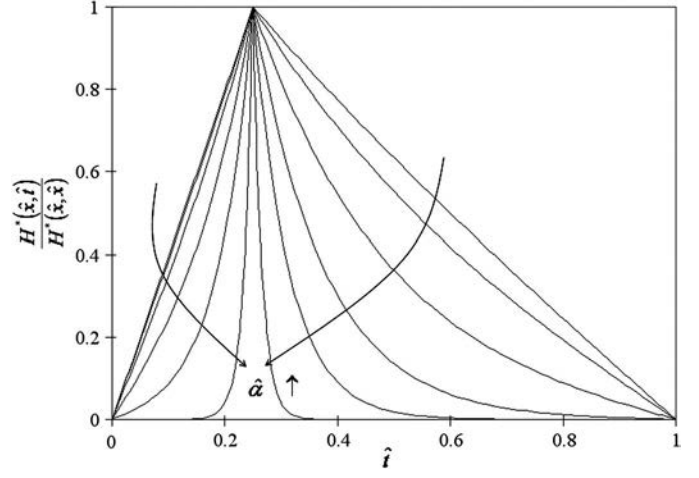


Fig. 8. Influence of the dimensionless connection parameter $\hat{\alpha}$ on the dimensionless weight function $H^*(\hat{x}, \hat{t})/H^*(\hat{x}, \hat{x})$; $\hat{x} = 1/4$; and $\hat{\alpha} \in \{1; 2; 4; 8; 16; 64\}$

It is easily checked that

$$\int_0^L H^*(x,t) dt = 1 - \cosh \frac{x}{a} + \frac{-1 + \cosh \frac{L}{a}}{\sinh \frac{L}{a}} \sinh \frac{x}{a} \quad (54)$$

Introducing Eq. (54) into Eq. (53) leads to the curvature equation Eq. (44) with the dimensionless normal forces given by Eq. (7). Such an integral representation is strictly equivalent to the direct approach first investigated in the paper.

Finally, we would like to discuss the structure of the differential equation appearing in the constitutive law of the nonlocal beam (or partially composite beam, or sandwich beam) from the uniaxial stress-strain relationship deduced from Eq. (32)

$$\sigma - l_c^2 \sigma'' = E(e - a^2 e'') \quad (55)$$

where σ and e = uniaxial stress and the uniaxial strain, respectively. This model comprises the nonlocal integral model of Eringen (1983) when the parameter a is vanishing (i.e., when $a = 0$) and the gradient elasticity model when the parameter l_c is vanishing (i.e., when $l_c = 0$). The behavior of such nonlocal constitutive law was recently discussed by Challamel et al. (2009b) with respect to elastic dynamics wave properties. Aifantis (2003) postulated such a model, generalized to three-dimensional media, to preclude the strain and the stress singularities in dislocation and crack problems. Challamel et al. (2008) also used a similar model at the beam scale for plastic softening problems in the nonlocal softening constitutive law (see also Challamel et al. 2009a for the nonlocal damage problem or Challamel et al. 2010b for the nonlocal hardening constitutive law). As recently shown by di Paola et al. (2009), models of elastic foundation can also involve nonlocality. In fact, the model of Reissner is derived from the following differential equation (Reissner 1958; Kerr 1964; Karnovsky and Lebed 2001):

$$p - l_c^2 p'' = k_0(y - a^2 y'') \quad (56)$$

where p and y = foundation reaction and deflection, respectively. The Reissner model typically belongs to the class of nonlocal models studied in this paper. The model of Pasternak (1954) is recognized when the parameter l_c is vanishing (i.e., when $l_c = 0$), which is analogous to a gradient elasticity model. Recently, Challamel et al. (2010c) studied the buckling of elastic columns on a Reissner nonlocal foundation. Therefore, it seems that the fundamental constitutive differential equation of Eq. (32), has been

widely studied, even if the treatment in a nonlocal framework is certainly more recent.

Finite-Element Results

Even if the exact solution of this theoretical problem is easily obtained analytically, the computation of the curvature by using the finite-element method can be instructive to characterize the sensitivity of the boundary layer to the mesh density, especially for stiff connection. The exact finite-element model of the in-plane partially composite problem was studied by Faella et al. (2002), including the axial displacements and the deflection. In this section, we focus on the curvature distribution and present a simple variational formulation leading to the finite-element resolution of the bending problem. The differential equation Eq. (42) dealing with the curvature with the boundary conditions obtained for this problem can be reformulated by the energy functional

$$U[\chi] = \int_0^L \frac{1}{2} EI_{z,\infty} (\chi^2 + a^2 \chi'^2) - M_0 \chi dx - \lambda_1 \left[\chi(0) - \frac{M_0}{EI_{z,0}} \right] - \lambda_2 \left[\chi(L) - \frac{M_0}{EI_{z,0}} \right] \quad (57)$$

where λ_1 and λ_2 = two additional Lagrange multipliers associated with the boundary conditions of Eq. (41). The stationarity of $U[\chi]$ with respect to the curvature function is written as

$$\delta U[\chi] = \int_0^L EI_{z,\infty} (\chi \delta \chi + a^2 \chi' \delta \chi') - M_0 \delta \chi dx - \lambda_1 \delta \chi(0) - \lambda_2 \delta \chi(L) = 0 \quad (58)$$

The gradient elasticity solution of Eq. (42) is found again by integrating Eq. (58) by part (see also Challamel and Wang 2008 for a similar discussion). A two-degrees-of-freedom element is presented for the resolution of the discretized bending problem. The linear interpolation for the bending curvature is expressed by

$$\chi(x) = \mathbf{P}^T(x) \mathbf{X} \quad \text{with } \mathbf{P}(x) = \begin{pmatrix} 1 - x/l \\ x/l \end{pmatrix} \quad \text{and } \mathbf{X} = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \quad (59)$$

where l = length of the finite element. This interpolation field leads to the usual third-order polynomial interpolation for the deflection. Introducing the approximated solution Eq. (59) into the first variation of the energy functional leads to the expression of the local matrix and local nodal forces

$$\begin{aligned} \mathbf{K}_i &= EI_{z,\infty} \int_0^l \mathbf{P}^T \mathbf{P} dx + a^2 EI_{z,\infty} \int_0^l \mathbf{P}'^T \mathbf{P}' dx \\ &= EI_{z,\infty} l \begin{pmatrix} 1/3 & 1/6 \\ 1/6 & 1/3 \end{pmatrix} + EI_{z,\infty} \frac{a^2}{l} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \quad \text{and} \\ \mathbf{F}_i &= M_0 \int_0^l \mathbf{P} dx = \frac{M_0 l}{2} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \end{aligned} \quad (60)$$

The curvature approximation along the beam, denoted by \mathbf{X} , is then computed from the global linear system $\mathbf{K} \mathbf{X} = \mathbf{F}$ where the curvatures at the beam extremity are imposed. \mathbf{K} is the global stiffness matrix and \mathbf{F} is the global force vector. Fig. 9 shows the sensitivity of the numerical solution to the number of finite elements. It is expected that the boundary layer is well described with a sufficient number of finite elements within this small zone. Such accuracy is computationally not efficient if the size of the finite elements

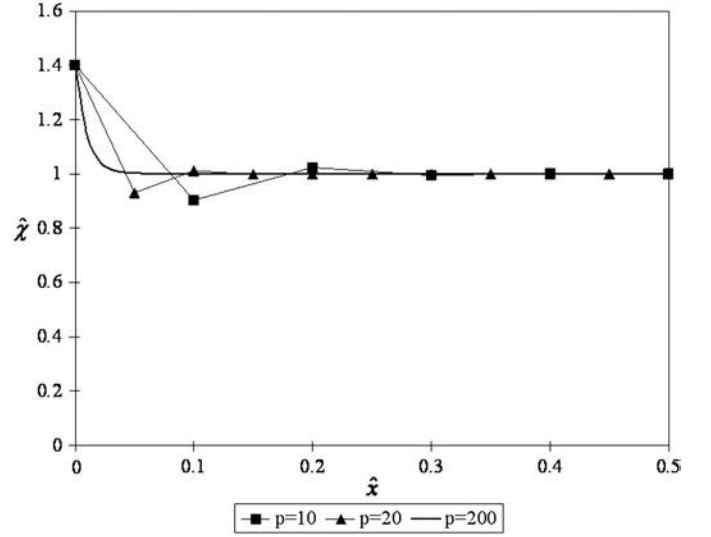


Fig. 9. Influence of the number p of finite elements on the numerical solution of the curvature; $EI_{z,\infty}/EI_{z,0} = 1.4$; and $\hat{\alpha} = 100$

is fixed as in the case considered in this study. In the simulation, the dimensionless connection parameter ε was chosen equal to 0.01, which is also equal to a/L . This is clearly a stiff connection associated with the boundary layer phenomenon. At least 100 finite elements are needed for a good description of the boundary-layer zone in this case. For such a stiff connection (i.e., $\hat{\alpha} = \alpha_T L = 100$), a number p of 200 finite elements has been chosen for the reference solution (Note: the difference between the theoretical solution and the numerical one never exceeds 0.2% for such a discretization). Generally speaking, the rate of convergence of the numerical method is lower for a stiffer connection, as explained previously, with the boundary zone with a meaningful stress gradient. An optimization of the numerical process could be envisaged as an extension of this study, with finite elements of variable sizes along the beam or with enriched finite elements in the high gradient zone.

Summary and Conclusions

The bending of an elastic-composite beam with interlayer slip has been theoretically and numerically investigated. These beams represent a number of different classes of structures with the same type of basic behavior and governing equations, such as layered beams (i.e., mechanically jointed), shear connections (i.e., lap joints mechanically or adhesively jointed) and sandwich constructions (i.e., with weak shear cores). The specific case of the elastic beam subjected to end moments was considered. It was shown that a boundary layer prevails at the beam extremity for a very stiff connection. This boundary-layer phenomenon was analytically characterized from an asymptotic point of view. The outer expansion, which was valid away from the boundary, and the inner expansion, which was valid within the layer adjacent to the boundary (i.e., beam extremity), were analytically given. The inner and outer expansions were matched over a region located at the edge of the boundary layer by using Prandtl's matching condition. We showed that the thickness of the boundary layer was of order ε , the inverse of the dimensionless connection parameter. The thickness of the boundary layer was decreasing with an increasing value of the connection parameter $\hat{\alpha}$. This specific phenomenon has to be numerically considered for the case of a stiff connection with a discretization process in concordance with the thickness of the

boundary layer. (see the section “Finite-Element Results”). The boundary layer phenomenon was extremely sensitive to the boundary conditions imposed at the boundary of the elastic beam, a phenomenon well accepted for two-dimensional beam theory (Gao et al. 2007).

Finally, it was shown that partially composite beams can be treated as nonlocal beams in the sense that the bending moment at a specific point depends on the overall curvature along the beam. This integral presentation can be useful to investigate the problem with the boundary-element method. Some other engineering applications were given for the nonlocal character of the constitutive law exhibited in this study.

Appendix. Derivation of the Partially Composite Model by Using the Variational Method

The fundamental differential equations of the partially composite beam with interlayer slip or sandwich beam subjected to a uniform bending moment can be obtained from the total energy functional

$$U[u_1, u_2, v] = \int_0^L \left[\frac{1}{2} EI_{z,0} (v'')^2 + \frac{1}{2} E_1 A_1 (u_1')^2 + \frac{1}{2} E_2 A_2 (u_2')^2 + \frac{1}{2} b e G \gamma^2 \right] dx + M_0 [v'(L) - v'(0)] - \int_0^L q v dx \quad (61)$$

where b = width of the beam; e = thickness of the interlayer or core; and G = shear modulus of the interlayer or core. Eq. (61) implicitly assumes that the curvature of the two subelements is equal. It is postulated that the flexible connection will only be associated with interface shear interaction without any normal interaction among the subelements. This last assumption can be relaxed, as shown by Adekola (1968) (see more recently Gara et al. 2006; Ranzi et al. 2006; Ranzi and Bradford 2007; Kroflik et al. 2010a). A considerable number of experimental verifications in the literature exist with respect to the applicability of the classical composite action theory, neglecting the effect of the vertical separation between each subelement and assuming equal curvatures for the two subelements. In addition, for example, in Kroflik et al. (2010b), nailed shear connections in composite timber structures were tested with respect to both slip and uplift, and no significant effect of including the transverse relationship was found. If transverse partial interaction can be neglected, the difference between the curvatures of the two subcomponents can be neglected. It will not affect the results obtained in this paper.

The shear strain γ in the soft interlayer or core is given by

$$\gamma = \frac{u_2 - u_1}{e} + \frac{h_0}{e} v' \quad (62)$$

The following equations also include the case of a distributed load q acting along the partially composite beam or the sandwich beam. The stationarity of the energy functional leads to the principle of virtual work

$$\delta U = \int_0^L (EI_{z,0} v'' \delta v'' + N_1 \delta u_1' + N_2 \delta u_2' + b e \tau \delta \gamma) dx + M_0 [\delta v'(L) - \delta v'(0)] - \int_0^L q \delta v dx = 0 \quad (63)$$

where the constitutive equations are

$$N_1 = E_1 A_1 u_1' \quad N_2 = E_2 A_2 u_2' \quad \tau = G \gamma \quad (64)$$

where γ is given by Eq. (62). The correspondence between the partially composite beam-columns with slip modulus K_T and the present sandwich beam with parameters b , G , and e , is immediate from the following identity:

$$\frac{bG}{e} = K_T \quad (65)$$

The equivalence between the slip modulus of shear connectors and the shear modulus of glue lines of finite thickness or the sandwich-type of cores was first demonstrated by McCutcheon (1977). The equivalence was also discussed for the in-plane dynamics problem by Girhammar et al. (2009), or for the out-of-plane behavior by Challamel (2009) (see also Challamel et al. 2010a). It is not surprising that boundary layers are present for a very stiff connection that is with the sandwich analogy for very thin layers.

An integration by part leads to the following system of differential equations:

$$\begin{cases} EI_{z,0} v^{(4)} - b h_0 (\tau)' - q = 0 \\ (N_1)' + b \tau = 0 \\ (N_2)' - b \tau = 0 \end{cases} \quad (66)$$

with the following boundary conditions:

$$[(-EI_{z,0} v''' + b h_0 \tau) \delta v]_0^L = 0 \quad (67a)$$

$$[(EI_{z,0} v'' + M_0) \delta v]_0^L = 0 \quad (67b)$$

$$[N_1 \delta u_1]_0^L = 0 \quad (67c)$$

$$[N_2 \delta u_2]_0^L = 0 \quad (67d)$$

In the case studied for this paper, these boundary conditions are simply reduced to

$$\begin{aligned} v(0) = v(L) = 0 \quad EI_{z,0} v''(0) = EI_{z,0} v''(L) = -M_0 \\ N_1(0) = N_2(0) = 0 \quad N_1(L) = N_2(L) = 0 \end{aligned} \quad (68)$$

The second and the third differential equations of Eq. (66) with the last boundary conditions lead to the simplification

$$\begin{aligned} (N_1 + N_2)' = 0 \Rightarrow N_1 + N_2 = N_1(L) + N_2(L) \\ = N_1(0) + N_2(0) = 0 \end{aligned} \quad (69)$$

After some manipulations, the differential equation for the deflection is obtained

$$v^{(6)} - \alpha_T^2 v^{(4)} + \alpha_T^2 \frac{q}{EI_{z,\infty}} - \frac{q''}{EI_{z,0}} = 0 \quad (70)$$

where the different parameters are given by Eq. (2). This differential equation can be also expressed for the bending moment and the curvature as

$$\begin{aligned} \chi^{(4)} - \alpha_T^2 \chi'' + \alpha_T^2 \frac{M''}{EI_{z,\infty}} - \frac{M^{(4)}}{EI_{z,0}} = 0 \\ \text{with } \chi = -v'' \quad \text{and } M'' = -q \end{aligned} \quad (71)$$

Integrating twice, this differential equation leads to the nonlocal bending curvature relationship

$$\frac{\alpha_T^2}{EI_{z,\infty}} M - \frac{1}{EI_{z,0}} M'' = \alpha_T^2 \chi - \chi'' \quad (72)$$

Eq. (31) is clearly recognized in this constitutive relationship.

By differentiating the second equation of Eq. (66) and by using the Eqs. (24), (25), and (64), the differential equation for the normal force N_1 can be expressed as

$$N_1'' - \alpha_7^2 N_1 = \beta_7 M_0 \quad (73)$$

where the different parameters are given by Eq. (2). This equation is exactly Eq. (1) of the problem.

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