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# Asymptotic Analysis of a Non-Linear Non-Local Integro-Differential Equation Arising from Bosonic Quantum Field Dynamics

Sébastien Breteaux\*<sup>†</sup>

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## Abstract

We introduce a one parameter family of non-linear, non-local integro-differential equations and its limit equation. These equations originate from a derivation of the linear Boltzmann equation using the framework of bosonic quantum field theory. We show the existence and uniqueness of strong global solutions for these equations, and a result of uniform convergence on every compact interval of the solutions of the one parameter family towards the solution of the limit equation.

*Keywords:* Integro-differential equation, nonlinear equation, nonlocal equation, equation with memory.

*Mathematics Subject Classification (2010):* 34G20, 45J05, 47G20.

## 1 Introduction

Let  $d \geq 1$  be an integer,  $h$  a strictly positive parameter,  $f$  a function in the Schwartz space  $\mathcal{S}(\mathbb{R}^d; \mathbb{C})$  and  $\vec{\xi}_0$  a vector in  $\mathbb{R}^d \setminus \{0\}$ .

The spaces of bounded and continuous functions from  $X \subset \mathbb{R}$  to  $Y \subset \mathbb{R}^d$  are denoted by  $\mathcal{B}(X; Y)$  and  $\mathcal{C}(X; Y)$ . For a function  $\vec{u} \in \mathcal{B}(X; Y)$  we write  $\|\vec{u}\|_{\infty, X'} = \sup\{|\vec{u}(x)|, x \in X'\}$  for  $X' \subset X$ .

The applications  $\mathcal{F}^{(h)}$  and  $F^{(0)}$  from  $\mathcal{B}(\mathbb{R}^+; \mathbb{R}^d)$  to  $\mathcal{C}(\mathbb{R}^+; \mathbb{R}^d)$  are defined for

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$\vec{u} \in \mathcal{B}(\mathbb{R}^+; \mathbb{R}^d)$  by

$$\begin{aligned}\mathcal{F}^{(h)}(\vec{u})(t) &:= -2\Re \int_{\mathbb{R}^d} \int_0^{t/h} e^{-ir(\vec{\eta}^2 - 2\vec{\eta} \cdot \int_{t-hr}^t \vec{u}_\sigma \, d\sigma)} \vec{\eta} |f(\vec{\eta})|^2 \, dr \, d\vec{\eta} \\ F^{(0)}(\vec{u})(t) &:= -2\Re \int_0^\infty \int_{\mathbb{R}^d} e^{-ir(\vec{\eta}^2 - 2\vec{\eta} \cdot \vec{u}_t)} \vec{\eta} |f(\vec{\eta})|^2 \, d\vec{\eta} \, dr\end{aligned}$$

where  $\int_a^b \vec{u}(\sigma) \, d\sigma := \frac{1}{b-a} \int_a^b \vec{u}(\sigma) \, d\sigma$ .

**Theorem 1.** *For every  $h > 0$ , the non-linear integro-differential equation*

$$\begin{cases} \frac{d}{dt} \vec{\xi}_t^{(h)} = \mathcal{F}^{(h)}(\vec{\xi}_t^{(h)}) \\ \vec{\xi}_{t=0}^{(h)} = \vec{\xi}_0, \end{cases} \quad (1)$$

admits a unique solution  $\vec{\xi}^{(h)}$  in  $\mathcal{C}^1(\mathbb{R}^+; \mathbb{R}^d)$ .

The limit equation as  $h \rightarrow 0$

$$\begin{cases} \frac{d}{dt} \vec{\xi}_t^{(0)} = F^{(0)}(\vec{\xi}_t^{(0)}) \\ \vec{\xi}_{t=0}^{(0)} = \vec{\xi}_0. \end{cases} \quad (2)$$

admits a unique maximal solution  $\vec{\xi}^{(0)}$  in  $\mathcal{C}^1([0, T_{max}); \mathbb{R}^d)$  with  $T_{max} > 0$ .

- The norm of  $\vec{\xi}^{(0)}$  decreases with time.
- If  $\min\{|f(\vec{\eta})|, \vec{\eta} \in \bar{B}(0, 2|\vec{\xi}_0^{(0)}|)\}$  is strictly positive then  $T_{max} = +\infty$ , and  $\vec{\xi}_t^{(0)} \rightarrow 0$  as  $t \rightarrow +\infty$ .

For any  $T \in (0, T_{max})$ ,  $\vec{\xi}^{(h)}$  converges uniformly to  $\vec{\xi}^{(0)}$  on  $[0, T]$  as  $h \rightarrow 0$ .

*Remark 2.* The non-locality feature of Equation (1) disappears in the limit Equation (2).

**Origin of the equations** These equations come from a problem of derivation of the linear Boltzmann equation in dimension  $d$  from a model with a particle in a Gaussian random field (centered and invariant by translation), in the weak density limit, as presented in [4]. The parameter  $h$  has then the interpretation of the ratio of the microscopic typical length over the macroscopic one, and it is thus natural to be interested in the behaviour of the equations as  $h \rightarrow 0$ . Note that the same scaling is used to rescale the time. It is *not* the Planck constant (here taken equal to one by a suitable choice of units). It turns out that this stochastic problem can be translated in a deterministic one with more geometric features by using an isomorphism between the  $L^2(\Omega)$  space associated with the Gaussian random field and the symmetric Fock space  $\Gamma L^2 = \bigoplus_{n=0}^\infty (L^2)^{\otimes_s n}$  over

$L^2 := L^2(\mathbb{R}^d; \mathbb{C})$  (see [6] for information about this isomorphism). Up to some isomorphisms the Schrödinger equation can be rewritten as

$$ih \partial_t u = Q(z)^{Wick} u,$$

where

- The function  $u$  has values in  $\Gamma L^2$ .
- The polynomial  $Q$  in the variable  $z \in L^2$  is defined by

$$Q(z) = \xi^2 + \langle z, (\vec{\eta}^2 - 2\vec{\xi} \cdot \vec{\eta})z \rangle + \langle z, \vec{\eta}z \rangle^2 + 2\sqrt{\hbar} \Re \langle z, f \rangle,$$

for a fixed  $\vec{\xi}$  in  $\mathbb{R}^d \setminus \{0\}$ , with  $\vec{a} \cdot \vec{a} = \sum_{j=1}^d a_j^2$  for every vector  $\vec{a}$ ,  $\langle \cdot, \cdot \rangle$  is the scalar product in  $L^2$ ,  $\Re$  denotes the real part.  $\vec{\eta}$  denotes the multiplication operator which to a function  $v$  of the variable  $\vec{\eta} \in \mathbb{R}^d$  associates the function with  $d$  components  $\vec{\eta} \mapsto \vec{\eta}v(\eta)$ . We do here as if this operator was bounded on  $L^2$  since the small improvements needed to handle this case are irrelevant for this article.

- The Wick quantization is defined for a monomial  $b(z) = \langle z^{\otimes q}, \tilde{b}z^{\otimes p} \rangle$  with variable in  $z \in L^2$  and  $\tilde{b} \in \mathcal{L}((L^2)^{\otimes_s p}, (L^2)^{\otimes_s q})$  by the fomula

$$b^{Wick} \Big|_{L^2(\mathbb{R}^d)^{\otimes_s p+n}} = \frac{\sqrt{(n+p)!(n+q)!}}{n!} \tilde{b} \otimes_s Id_{L^2(\mathbb{R}^d)^{\otimes_s n}}.$$

In this framework the Hamilton equations associated with the polynomial  $Q(z)$  (see [1, 2, 3]) are  $ih \partial_t z_t^{(h)} = \partial_z Q(z_t^{(h)})$  with  $\partial_z Q(z) = (\vec{\eta}^2 - 2\vec{\xi} \cdot \vec{\eta})z + 2\langle z, \vec{\eta}z \rangle \cdot \vec{\eta}z + \sqrt{\hbar}f$  which can again be written as

$$ih \partial_t z_t^{(h)} = (\vec{\eta}^2 - 2\vec{\xi}_t \cdot \vec{\eta})z_t^{(h)} + \sqrt{\hbar}f$$

once we defined  $\vec{\xi}_t^{(h)} = \vec{\xi} - \langle z_t^{(h)}, \vec{\eta}z_t^{(h)} \rangle$ . We are interested in the solution of this equation with initial data  $z_{t=0}^h = 0$  which is

$$z_t^h = -\frac{i}{\hbar} \int_0^t e^{-\frac{i}{\hbar} \int_s^t (\vec{\eta}^2 - 2\vec{\xi}_s^{(h)} \cdot \vec{\eta}) d\sigma} \sqrt{\hbar}f ds.$$

This solution is fully determined by  $\vec{\xi}_t^{(h)}$  which satisfies Equation 1. The study of equation of  $\vec{\xi}_t^{(h)}$  is thus justified by the study of the hamiltonian equation associated with the polynom  $Q(z)$  which is itsel useful to give an approximate solution to the Schrödinger equation above in terms of coherent states (see also the Hepp method, introduced in [5] and presented in [1] whose viewpoint we use in this article).

**Intrinsic significance of the equations** Eventhough this approach did not until now allow us to improve our results on the derivation of the linear Boltzmann equation, these equations are interesting in themself since it is a case of non-linear, non-local equations and thus an already non trival problem for which a thorough study is possible.

**Organisation of the paper** We show in Section 2 the existence and uniqueness of a global solution to the parameter dependent Equation (1), along with some estimates about the solutions. Section 3 is devoted to the Equation (2). We first compute a more geometric form of the application  $F^{(0)}$  and then use it to prove the existence and uniqueness of the solution to Equation (2) through the usual theory of differential equations. Then we compare both solutions in Section 4 using a Grönwall type argument.

### Notation

1. For any time  $T > 0$  the applications  $\mathcal{F}^{(h)}$  and  $F^{(0)}$  induce applications from  $\mathcal{B}([0, T]; \mathbb{R}^d)$  to  $\mathcal{C}([0, T]; \mathbb{R}^d)$ .
2. We sometime write  $F^{(0)}(\vec{u})$  with  $\vec{u}$  a fixed vector in  $\mathbb{R}^d \setminus \{0\}$  for

$$-2\Re \int_0^\infty \int_{\mathbb{R}^d} e^{-ir(\vec{\eta}^2 - 2\vec{\eta} \cdot \vec{u})} \vec{\eta} |f(\vec{\eta})|^2 d\vec{\eta} dr.$$

3. We introduce the function  $g : \vec{\eta} \in \mathbb{R}^d \mapsto \vec{\eta} |f(\vec{\eta})|^2$  to simplify the notations.

## 2 Parameter-Dependent Equation

**Proposition 3.** *In dimension  $d \geq 1$ , for any  $h > 0$  Equation (1) has a unique solution in  $\mathcal{C}^1(\mathbb{R}^+; \mathbb{R}^d)$ .*

*Proof.* Suppose we know that Equation (1) has a unique solution  $\vec{\xi}^{(h)}$  on an interval  $[0, T]$  and want to extend this solution to a larger interval  $[0, T + \delta]$ . We thus consider the application  $\Phi$  from  $\mathcal{C}^1([T, T + \delta]; \mathbb{R}^d)$  to itself defined by

$$\Phi(\vec{u})(t) = \vec{\xi}^{(h)}(T) + \int_T^t \mathcal{F}^{(h)}(\vec{u})(s) ds,$$

where the function  $\vec{u}$  is defined on  $[0, T + \delta]$  and coincide with  $\vec{\xi}^{(h)}$  on  $[0, T]$  and  $\vec{u} = \vec{v}$  on  $[T, T + \delta]$ . We want to prove that for  $\delta$  small enough  $\Phi$  is a contraction.

We then obtain

$$\begin{aligned} |\Phi(\vec{u})'(t) - \Phi(\vec{v})'(t)| &= |\mathcal{F}^{(h)}(\vec{u})(t) - \mathcal{F}^{(h)}(\vec{v})(t)| \\ &\leq 2 \int_{\mathbb{R}^d} \int_0^{t/h} \left| e^{ir2\vec{\eta} f_{t-hr}^t \vec{u}_\sigma} - e^{ir2\vec{\eta} f_{t-hr}^t \vec{v}_\sigma} \right| |g(\vec{\eta})| dr d\vec{\eta} \\ &\leq 4 \int_{\mathbb{R}^d} \int_0^{t/h} |r\vec{\eta} \cdot \int_{t-hr}^t (\vec{u}_\sigma - \vec{v}_\sigma) d\sigma| dr |g(\vec{\eta})| d\vec{\eta} \\ &\leq 4C_g \int_0^{t/h} r \frac{t-T}{hr} dr \|\vec{u} - \vec{v}\|_\infty \\ &\leq 4 \frac{t}{h^2} (t-T) C_g \|\vec{u} - \vec{v}\|_\infty \\ &\leq 4 \frac{T+\delta}{h^2} (t-T) C_g \|\vec{u} - \vec{v}\|_\infty \end{aligned}$$

with  $C_g = \int_{\mathbb{R}^d} |\vec{\eta}| |g(\vec{\eta})| d\vec{\eta}$ . Integrating  $\Phi(\vec{u})'(t) - \Phi(\vec{v})'(t)$  in time and taking the modulus we also get the estimate

$$\|\Phi(\vec{u}) - \Phi(\vec{v})\|_\infty \leq 2 \frac{T + \delta}{h^2} \delta^2 C_g \|\vec{u} - \vec{v}\|_\infty.$$

We can control the norm of  $\Phi(\vec{u}) - \Phi(\vec{v})$  by using  $\|\vec{u} - \vec{v}\|_\infty \leq \|\vec{u} - \vec{v}\|_{C^1}$

$$\|\Phi(\vec{u}) - \Phi(\vec{v})\|_{C^1} \leq (4\delta + 2\delta^2) \frac{T + \delta}{h^2} C_g \|\vec{u} - \vec{v}\|_{C^1},$$

for  $\delta(T + \delta) \leq \frac{h^2}{8C_g}$  we get a contraction, and we can apply a classical fixed point theorem.

Thus to prove a global existence result it is enough to observe that, for any positive constant  $C$ , the sequence  $(t_n)_n$  such that  $t_0 = 0$  and  $(t_{n+1} - t_n)^2 + t_n(t_{n+1} - t_n) = C$  with positive increments is such that

$$t_{n+1} - t_n = \frac{\sqrt{t_n^2 + 4C} - t_n}{2} = \frac{2C}{\sqrt{t_n^2 + 4C} + t_n} \geq \frac{C}{t_n + \sqrt{C}}.$$

using  $\sqrt{a+b} \leq \sqrt{a} + \sqrt{b}$  for  $a$  and  $b$  non-negative. A sum over  $n$  gives  $t_{N+1} \geq \sum_{n=0}^N \frac{C}{t_n + \sqrt{C}}$  and this shows that  $(t_n)$  is necessarily unbounded. We thus obtain the existence and the uniqueness of a solution on  $\mathbb{R}^+$ .  $\square$

**Proposition 4.** *In dimension  $d \geq 3$ , the family  $(\vec{\xi}^{(h)})_{h>0}$  is uniformly bounded, i.e.*

$$\|\vec{\xi}^{(h)'}\|_{\infty, [0, +\infty)} \leq 2C_g^1,$$

with  $C_g^1 = \|g\|_{L^1} + (\frac{d}{2} - 1)\pi^{d/2} \|\hat{g}\|_{L^1}$ . (The Fourier transform is defined by  $\hat{u}(x) = \frac{1}{\sqrt{2\pi}^d} \int_{\mathbb{R}^d} e^{-ix \cdot \eta} u(\vec{\eta}) d\vec{\eta}$ .)

*Proof.* It is enough to show that  $\mathcal{F}^{(h)}$  is bounded. Indeed, for  $d \geq 3$ , using Parseval equality we get

$$\begin{aligned} |\mathcal{F}^{(h)}(u)(t)| &\leq 2 \int_0^{t/h} \left| \int_{\mathbb{R}^d} e^{-ir(\vec{\eta}^2 - 2\vec{\eta} \cdot \int_{t-hr}^t \vec{u}_\sigma d\sigma)} g(\vec{\eta}) d\vec{\eta} \right| dr \\ &\leq 2 \|g\|_{L^1} + 2 \int_1^{t/h} \left| \int_{\mathbb{R}^d} e^{-ir\vec{\eta}^2} g\left(\vec{\eta} + \int_{t-hr}^t \vec{u}_\sigma d\sigma\right) d\vec{\eta} \right| dr \\ &\leq 2 \|g\|_{L^1} + 2 \int_1^{t/h} \left(\frac{\pi}{r}\right)^{d/2} \|\hat{g}\|_{L^1} dr \\ &\leq 2 \|g\|_{L^1} + (d-2)\pi^{d/2} \|\hat{g}\|_{L^1}. \end{aligned} \quad \square$$

### 3 Limit Equation

**Proposition 5.** For any  $\vec{u}$  in  $\mathbb{R}^d \setminus \{0\}$ , with  $\hat{u} = \frac{\vec{u}}{|\vec{u}|}$ ,

$$F^{(0)}(\vec{u}) = -\pi|\vec{u}|^{d-1} \int_{[0,2]} \int_{S^{d-2}} (\rho\hat{u} + \sqrt{2\rho - \rho^2}\vec{\omega}') \\ \times |f(|\vec{u}|(\rho\hat{u} + \sqrt{2\rho - \rho^2}\vec{\omega}'))|^2 d\mathcal{H}^{d-2}(\vec{\omega}') \sqrt{2\rho - \rho^2}^{d-2} d\rho,$$

where  $\vec{\omega}'$  is in the orthogonal of  $\vec{u}$ .

*Proof.* We introduce a parameter  $\varepsilon$  to permute the integrals

$$F^{(0)}(\vec{u}) = \lim_{\varepsilon \rightarrow 0^+} -2\Re \int_0^{+\infty} e^{-r\varepsilon} \int_{\mathbb{R}_\eta^d} e^{-ir(\vec{\eta}^2 - 2\vec{\eta} \cdot \vec{u})} g(\vec{\eta}) d\vec{\eta} dr \\ = \lim_{\varepsilon \rightarrow 0^+} -2\Re \int_0^{+\infty} \int_{\mathbb{R}_\eta^d} e^{-ir(\vec{\eta}^2 - 2\vec{\eta} \cdot \vec{u}) - r\varepsilon} g(\vec{\eta}) d\vec{\eta} dr \\ = \lim_{\varepsilon \rightarrow 0^+} -2\Re \int_{\mathbb{R}_\eta^d} \int_0^{+\infty} e^{-ir(\vec{\eta}^2 - 2\vec{\eta} \cdot \vec{u}) - r\varepsilon} dr g(\vec{\eta}) d\vec{\eta} \\ = \lim_{\varepsilon \rightarrow 0^+} -2 \int_{\mathbb{R}_\eta^d} \left( \Re \int_0^{+\infty} e^{-r[i(\vec{\eta}^2 - 2\vec{\eta} \cdot \vec{u}) + \varepsilon]} dr \right) g(\vec{\eta}) d\vec{\eta}.$$

The time integral can be explicitly computed

$$\int_0^{+\infty} e^{-r[i(\vec{\eta}^2 - 2\vec{\eta} \cdot \vec{u}) + \varepsilon]} dr = \frac{\varepsilon - i(\vec{\eta}^2 - 2\vec{\eta} \cdot \vec{u})}{(\vec{\eta}^2 - 2\vec{\eta} \cdot \vec{u})^2 + \varepsilon^2}$$

and taking the real part and the limit as  $\varepsilon \rightarrow 0^+$  we obtain

$$\lim_{\varepsilon \rightarrow 0^+} \Re \int_0^{+\infty} e^{-r[i(\vec{\eta}^2 - 2\vec{\eta} \cdot \vec{u}) + \varepsilon]} dr = \pi \delta(\vec{\eta}^2 - 2\vec{\eta} \cdot \vec{u}).$$

Thus

$$F^{(0)}(\vec{u}) = -2\pi \int_{\mathbb{R}_\eta^d} \delta(\vec{\eta}^2 - 2\vec{\eta} \cdot \vec{u}) g(\vec{\eta}) d\vec{\eta} \\ = -2\pi \int_{\vec{u} + |\vec{u}|S^{d-1}} g(\vec{\eta}) \frac{d\mathcal{H}^{d-1}(\vec{\eta})}{|\nabla(\vec{\eta}^2 - 2\vec{\eta} \cdot \vec{u})|} \\ = -\pi|\vec{u}|^{-1} \int_{\vec{u} + |\vec{u}|S^{d-1}} g(\vec{\eta}) d\mathcal{H}^{d-1}(\vec{\eta})$$

with  $d\mathcal{H}^{d-1}$  the  $(d-1)$ -dimensional Hausdorff measure. Then, using  $\hat{u} = \vec{u}/|\vec{u}|$ ,

$$F^{(0)}(\vec{u}) = -\pi|\vec{u}|^{-1} \int_{\vec{u} + |\vec{u}|S^{d-1}} \vec{\eta} |f(\vec{\eta})|^2 d\mathcal{H}^{d-1}(\vec{\eta}) \\ = -\pi|\vec{u}|^{-1} \int_{\hat{u} + S^{d-1}} |\vec{u}|\vec{\eta}' |f(|\vec{u}|\vec{\eta}')|^2 |\vec{u}|^{d-1} d\mathcal{H}^{d-1}(\vec{\eta}') \\ = -\pi|\vec{u}|^{d-1} \int_{\hat{u} + S^{d-1}} \vec{\eta} |f(|\vec{u}|\vec{\eta})|^2 d\mathcal{H}^{d-1}(\vec{\eta}) \quad (3)$$

we can now parametrize the sphere in a fashion which shows the particular importance of the direction of  $\vec{u}$ . We define the change of variable

$$\begin{aligned} [0, 2] \times S^{d-2} &\rightarrow \hat{u} + S^{d-1} \\ (\rho, \vec{\omega}') &\mapsto \rho \hat{u} + \sqrt{2\rho - \rho^2} \vec{\omega}' \end{aligned}$$

whose inverse transformation is given by

$$\rho = \vec{\eta} \cdot \hat{u} \quad \text{and} \quad \vec{\omega}' = \frac{\vec{\eta} - (\vec{\eta} \cdot \hat{u}) \hat{u}}{|\vec{\eta} - (\vec{\eta} \cdot \hat{u}) \hat{u}|}.$$

The jacobian under this change of variable is  $\sqrt{2\rho - \rho^2}^{d-2}$  as  $|\vec{\eta} - (\vec{\eta} \cdot \hat{u}) \hat{u}| = \sqrt{2\rho - \rho^2}$  and thus we get the result.  $\square$

**Proposition 6.** *In dimension  $d \geq 3$  the limit differential equation (2) has a unique maximal solution  $\vec{\xi}^{(0)}$ .*

*The norm of this solution decreases with the time.*

*If  $\min\{|f(\vec{\eta})|, \vec{\eta} \in \bar{B}(0, 2|\xi_0^{(0)}|)\}$  is strictly positive then this solution is defined on  $\mathbb{R}^+$ , and  $\xi_t^{(0)}$  converges to 0 as  $t \rightarrow +\infty$ .*

*Proof.* We first show that on any compact subset of  $\mathbb{R}^d \setminus \{0\}$  the map  $F^{(0)}$  is Lipschitz so that the theorem of Cauchy-Lipschitz applies. Indeed on  $C(r, R) \times \bar{B}(0, 2)$ , with  $C(r, R) = \{\vec{x} \in \mathbb{R}^d, r \leq |\vec{x}| \leq R\}$  and  $0 < r < R$ , the application

$$(\vec{u}, \vec{\eta}) \mapsto \vec{\eta} |f(|\vec{u}| \vec{\eta})|^2 |\vec{u}|^{d-1}$$

is smooth and thus Lipschitz with a constant  $L$ , and

$$\begin{aligned} |F^{(0)}(\vec{u}) - F^{(0)}(\vec{v})| &= \pi \left| |\vec{u}|^{d-1} \int_{\hat{u} + S^{d-1}} \vec{\eta} |f(|\vec{u}| \vec{\eta})|^2 d\mathcal{H}^{d-1}(\vec{\eta}) \right. \\ &\quad \left. - |\vec{v}|^{d-1} \int_{\hat{v} + S^{d-1}} \vec{\eta} |f(|\vec{v}| \vec{\eta})|^2 d\mathcal{H}^{d-1}(\vec{\eta}) \right| \\ &\leq \pi L (|\hat{u} - \hat{v}| + |\vec{u} - \vec{v}|) \\ &\leq L' |\vec{u} - \vec{v}| \end{aligned}$$

where we used an equivalent formulas for  $F^{(0)}$  derived in Equation (3) in the proof of Proposition 5. Thus there exists a unique solution  $\vec{\xi}^{(0)}$  defined on a maximal interval  $[0, T_{\max})$ .



We then prove that the norm of this solution is decreasing, indeed

$$\begin{aligned}
\frac{d}{dt} \frac{1}{2} |\vec{\xi}_t^{(0)}|^2 &= \vec{\xi}_t^{(0)} \cdot \vec{\xi}_t^{(0)'} \\
&= \vec{\xi}_t^{(0)} \cdot F^{(0)}(\vec{\xi}_t^{(0)}) \\
&= -\pi |\vec{\xi}_t^{(0)}|^{d-1} \int_{[0,2]} \int_{S^{d-2}} \rho (\hat{\vec{\xi}}_t^{(0)} + \sqrt{2\rho - \rho^2 \vec{\omega}'}) \cdot \vec{\xi}_t^{(0)} \\
&\quad |f(|\vec{\xi}_t^{(0)}|(\rho \vec{u}' + \sqrt{2\rho - \rho^2 \vec{\omega}'}))|^2 d\mathcal{H}^{d-2}(\vec{\omega}') \sqrt{2\rho - \rho^2}^{d-2} d\rho \\
&= -\pi |\vec{\xi}_t^{(0)}|^d \int_{[0,2]} \int_{S^{d-2}} \rho |f(|\vec{\xi}_t^{(0)}|(\rho \vec{u}' + \sqrt{2\rho - \rho^2 \vec{\omega}'}))|^2 \\
&\quad d\mathcal{H}^{d-2}(\vec{\omega}') \sqrt{2\rho - \rho^2}^{d-2} d\rho \\
&\leq 0.
\end{aligned}$$

Thus  $\vec{\xi}^{(0)}$  is necessarily bounded and cannot blow up. The maximal interval is thus necessarily  $[0, +\infty)$ .

If  $f$  does not vanish we have even more information, we know that

$$-C_{M,f} |\vec{\xi}_t^{(0)}|^d \leq \frac{d}{dt} |\vec{\xi}_t^{(0)}|^2 \leq -C_{m,f} |\vec{\xi}_t^{(0)}|^d$$

where  $C_{M,f} = C_d \max_{\bar{B}(0,2|\vec{\xi}^{(0)}|)} |f|$  and  $C_{m,f} = C_d \min_{\bar{B}(0,2|\vec{\xi}^{(0)}|)} |f|$  with  $C_d = 2\pi \int_{[0,2]} \rho \sqrt{2\rho - \rho^2}^{d-2} d\rho \mathcal{H}^{d-2}(S^{d-2})$ . By an integration we conclude that  $|\vec{\xi}_t^{(0)}|^2$  is in the interval bounded by the quantities

$$\left( |\vec{\xi}_0|^{1-d/2} + \left(\frac{d}{2} - 1\right) C t \right)^{-\frac{2}{d-2}}$$

for  $C = C_{M,f}$  and  $C_{m,f}$ . Thus  $\vec{\xi}_t^{(0)}$  doesn't exit from any compact set of  $\mathbb{R}^d \setminus \{0\}$  in finite time, and this *a priori* estimate implies  $T_{\max} = +\infty$ .  $\square$

*Remark 7.* We have the control, for  $0 < hr < t$ ,

$$\left| \int_{t-hr}^t \vec{\xi}^{(h)}(\sigma) d\sigma - \vec{\xi}^{(h)}(t) \right| \leq \frac{1}{2} \|\vec{\xi}^{(h)'}\|_{\infty, [t-hr, t]} hr \leq C_g^1 hr.$$

*Proof.* Indeed

$$\int_{t-hr}^t \vec{\xi}^{(h)}(\sigma) d\sigma - \vec{\xi}^{(h)}(t) = \int_{t-hr}^t \int_t^\sigma \vec{\xi}^{(h)'}(v) dv d\sigma,$$

and by taking the norm we get

$$\left\| \int_{t-hr}^t \vec{\xi}^{(h)}(\sigma) d\sigma - \vec{\xi}^{(h)}(t) \right\| \leq \|\vec{\xi}^{(h)'}\|_{\infty, [t-hr, t]} \int_{t-hr}^t (t - \sigma) d\sigma$$

which gives the result.  $\square$

## 4 Control of the Difference Between the Two Equations

We set the application  $F^{(h)}$  from  $\mathcal{B}(\mathbb{R}^+; \mathbb{R}^d)$  to  $\mathcal{C}(\mathbb{R}^+; \mathbb{R}^d)$ , defined for  $\vec{u} \in \mathcal{B}(\mathbb{R}^+; \mathbb{R}^d)$  by

$$F^{(h)}(\vec{u})(t) = -2\Re \int_{\mathbb{R}^d} \int_0^{t/h} e^{-ir(\vec{\eta}^2 - 2\vec{\eta} \cdot \vec{u}_t)} \vec{\eta} |f(\vec{\eta})|^2 \, d\vec{\eta} \, dr.$$

**Lemma 8.** *In dimension  $d \geq 3$ ,*

$$\|\mathcal{F}^{(h)}(\vec{\xi}^{(h)}) - F^{(h)}(\vec{\xi}^{(h)})\|_\infty \leq C_g^2 h^{\nu(d)-\delta}$$

with  $0 < \delta < \nu(d) = \frac{d-2}{4}$  and  $C_g^2 = 4\|g\|_{L^1} + 2\pi^{d/2}(C_g^1 \|\hat{g}'\|_{L^1} + \|\hat{g}\|_{L^1})$ .

*Proof.* Let  $t$  be in  $\mathbb{R}^+$  and  $\Lambda > 0$ , by a change of variable

$$\begin{aligned} & |\mathcal{F}^{(h)}(\vec{\xi}^{(h)})(t) - F^{(h)}(\vec{\xi}^{(h)})(t)| \\ & \leq 4 \int_0^\Lambda \int_{\mathbb{R}^d} |g(\vec{\eta})| \, d\vec{\eta} \, dr \\ & \quad + 2 \int_\Lambda^{t/h} \left| \int_{\mathbb{R}^d} e^{-ir(\vec{\eta}^2 - 2\vec{\eta} \cdot \vec{\xi}_t^{(h)})} (e^{2ir\vec{\eta} \cdot (\int_{t-hr}^t \vec{\xi}_\sigma^{(h)} \, d\sigma - \vec{\xi}_t^{(h)})} - 1) g(\vec{\eta}) \, d\vec{\eta} \right| \, dr \\ & \leq 4\Lambda \|g\|_{L^1} \\ & \quad + 2 \int_\Lambda^{t/h} \left| \int_{\mathbb{R}^d} e^{-ir\vec{\eta}' \cdot 2} (e^{2ir(\vec{\eta}' + \vec{\xi}_t^{(h)}) \cdot (\int_{t-hr}^t \vec{\xi}_\sigma^{(h)} \, d\sigma - \vec{\xi}_t^{(h)})} - 1) g(\vec{\eta}' + \vec{\xi}_t^{(h)}) \, d\vec{\eta}' \right| \, dr \end{aligned}$$

as  $|e^{ir|\vec{\xi}_t^{(h)}|^2}| = 1$ . We then control the internal integral using Parseval's equality. Indeed we have the Fourier transforms

$$\mathcal{F}(e^{-ir\vec{\eta}' \cdot 2})(y) = \left( \frac{\pi}{-ir} \right)^{d/2} e^{i\frac{y^2}{4r}},$$

$$\mathcal{F}(g(\vec{\eta}' + \vec{\xi}_t^{(h)}))(y) = e^{i\vec{\xi}_t^{(h)} \cdot \vec{y}} \hat{g}(\vec{y})$$

and

$$\begin{aligned} & \mathcal{F}(e^{2ir(\vec{\eta}' + \vec{\xi}_t^{(h)}) \cdot (\int_{t-hr}^t \vec{\xi}_\sigma^{(h)} \, d\sigma - \vec{\xi}_t^{(h)})} g(\vec{\eta}' + \vec{\xi}_t^{(h)}))(\vec{y}) \\ & = e^{i\vec{\xi}_t^{(h)} \cdot \vec{y}} \hat{g}(\vec{y} - 2r(\int_{t-hr}^t \vec{\xi}_\sigma^{(h)} \, d\sigma - \vec{\xi}_t^{(h)})). \end{aligned}$$

The difference between the two last Fourier transforms can be estimated using Remark 7, and thus

$$\|\hat{g}(\vec{y} - 2r(\int_{t-hr}^t \vec{\xi}_\sigma^{(h)} \, d\sigma - \vec{\xi}_t^{(h)})) - \hat{g}(\vec{y})\|_{L^1} \leq 2 \min \left\{ C_g^1 \|\hat{g}'\|_{L^1} r^2 h, \|\hat{g}\|_{L^1} \right\}.$$

The internal integral is then controled as

$$\begin{aligned} & \left| \int_{\mathbb{R}^d} e^{-ir\vec{\eta}' \cdot 2} \left( e^{2ir(\vec{\eta}' + \vec{\xi}_t^{(h)}) \cdot (f_{t-h,r}^t \vec{\xi}_\sigma^{(h)} d\sigma - \vec{\xi}_t^{(h)})} - 1 \right) g(\vec{\eta}' + \vec{\xi}_t^{(h)}) d\vec{\eta}' \right| \\ & \leq 2\pi^{d/2} (C_g^1 \|\hat{g}'\|_{L^1} + \|\hat{g}\|_{L^1}) \min(r^2 h, 1) r^{-d/2}. \end{aligned}$$

By interpolation,  $\min(r^2 h, 1) r^{-d/2} \leq h^\theta r^{2\theta - \frac{d}{2}}$ , and we then obtain

$$\|\mathcal{F}^{(h)}(\vec{\xi}^{(h)}) - F^{(h)}(\vec{\xi}^{(h)})\|_\infty \leq 4\Lambda \|g\|_{L^1} + 2\pi^{d/2} (C_g^1 \|\hat{g}'\|_{L^1} + \|\hat{g}\|_{L^1}) h^\theta \Lambda^{2\theta - \frac{d}{2} + 1}.$$

An optimization of the parameter  $\theta$  and of  $\mu$  in  $\Lambda = h^\mu$  shows that we can take  $\mu$  as high as  $\frac{d-2}{4} - \delta$  for any  $\delta > 0$ . (We then get  $\theta = \frac{\mu d}{2(1+2\mu)}$ .) This concludes the proof.  $\square$

**Lemma 9.** *In dimension  $d \geq 3$ , for  $h, t > 0$ ,*

$$\|F^{(h)}(\vec{\xi}^{(h)}) - F^{(0)}(\vec{\xi}^{(h)})\|_{\infty, [T, +\infty)} \leq C_g^3 \left(\frac{h}{T}\right)^{\frac{d}{2}-1},$$

with  $C_g^3 = \left(\frac{d}{2} - 1\right) \pi^{d/2} \|\hat{g}\|_{L^1}$ .

*Proof.* Indeed

$$|F^{(h)}(\vec{\xi}^{(h)})(t) - F^{(0)}(\vec{\xi}^{(h)})(t)| \leq \pi^{d/2} \|\hat{g}\|_{L^1} \int_{t/h}^{+\infty} \frac{dr}{r^{d/2}}$$

which gives the result.  $\square$

As a corollary we get:

**Lemma 10.** *In dimension  $d \geq 3$ , for  $h, T > 0$  and  $0 < \delta < \frac{d-2}{4}$*

$$\|F^{(0)}(\vec{\xi}^{(h)}) - \vec{\xi}^{(h)'}\|_{\infty, [\sqrt{h}, +\infty)} \leq C_g^2 h^{\frac{d-2}{4} - \delta} + C_g^3 h^{\frac{d-2}{4}} \leq \delta(h),$$

$$\begin{aligned} \Gamma_t^{(h)} &= \int_0^t |F^{(0)}(\vec{\xi}_s^{(h)}) - \vec{\xi}_s^{(h)'}| ds \\ &\leq \sqrt{h} + t\delta(h) = \delta_1(h). \end{aligned}$$

with  $\delta(h) \rightarrow 0$  as  $h \rightarrow 0$ .

We now compare  $\vec{\xi}^{(0)}$  and  $\vec{\xi}^{(h)}$ .

**Proposition 11.** *In dimension  $d \geq 3$ , let  $T \in (0, T_{max})$  then  $(\vec{\xi}_t^{(h)})_{t \in [0, T]}$  converges uniformly to  $(\vec{\xi}_t^{(0)})_{t \in [0, T]}$  as  $h \rightarrow 0$ .*

*Proof.* Let  $r, R > 0$  and  $\varepsilon > 0$  be such that  $r < |\vec{\xi}_t^{(0)}| - \varepsilon$  and  $|\vec{\xi}_t^{(0)}| + \varepsilon < R$  for all  $t \in [0, T]$ . Thus the open tubular neighbourhood

$$A_\varepsilon = \bigcup_{t \in [0, T]} B(\vec{\xi}_t, \varepsilon)$$

of the trajectory  $(\vec{\xi}_t^{(0)})_{t \in [0, T]}$  is included in the compact ring

$$C(r, R) = \{\vec{x} \in \mathbb{R}^d, r \leq |\vec{x}| \leq R\}.$$

On this ring the application  $F^{(0)}$  is Lipschitz with the Lipschitz constant  $L$ . We show that for  $h$  small enough  $\vec{\xi}_t^{(h)}$  remains in  $A_\varepsilon$ . We first observe that by continuity, for small  $t$ ,  $\vec{\xi}_t^{(h)}$  remains in  $A_\varepsilon$ . And that if, on an interval  $[0, t_0)$ ,  $\vec{\xi}_t^{(h)}$  remains in  $C(r, R)$  then we can compute a Gronwall type estimate. Let  $G(t) = \int_0^t |\vec{\xi}_s^{(0)} - \vec{\xi}_s^{(h)}| ds$ , then

$$\begin{aligned} G'(t) &= |\vec{\xi}_t^{(0)} - \vec{\xi}_t^{(h)}| \\ &= \left| \int_0^t (F^{(0)}(\vec{\xi}_s^{(0)}) - F^{(0)}(\vec{\xi}_s^{(h)}) + F^{(0)}(\vec{\xi}_s^{(h)}) - \vec{\xi}_s^{(h)'}) ds \right| \\ &\leq LG(t) + \Gamma_t^{(h)}. \end{aligned}$$

As  $G(0) = 0$  we get

$$G(t) \leq \int_0^t e^{L(t-s)} \Gamma_s^{(h)} ds \leq \int_0^t e^{Lt} \delta_1(h) ds \leq te^{Lt} \delta_1(h)$$

and

$$|\vec{\xi}_t^{(0)} - \vec{\xi}_t^{(h)}| = G'(t) \leq LG(t) + \Gamma_t^{(h)} \leq (Lte^{Lt} + 1)\delta_1(h).$$

for  $t \in [0, t_0)$ . Let  $h_0$  such that, for  $h$  smaller than  $h_0$ ,  $(LTe^{LT} + 1)\delta_1(h) < \frac{\varepsilon}{2}$ . Suppose there exists a time  $t$  in  $[0, T]$  such that  $\vec{\xi}_t^{(h)} \notin A_\varepsilon$  we define  $t_0$  the inferior bound of such times. As  $A_\varepsilon$  is open and  $\vec{\xi}_t^{(0)}$  and  $\vec{\xi}_t^{(h)}$  are continuous then  $\vec{\xi}_{t_0}^{(h)} \notin A_\varepsilon$ . But as  $|\vec{\xi}_t^{(0)} - \vec{\xi}_t^{(h)}| \leq (LTe^{LT} + 1)\delta_1(h) \leq \frac{\varepsilon}{2}$  on  $[0, t_0)$ , by continuity  $|\vec{\xi}_{t_0}^{(0)} - \vec{\xi}_{t_0}^{(h)}| \leq \frac{\varepsilon}{2}$  and so  $\vec{\xi}_{t_0}^{(h)} \in A_\varepsilon$  we thus get a contradiction and, for any  $t \in [0, T]$  and  $h \leq h_0$ ,  $\vec{\xi}_t^{(h)}$  remains in  $A_\varepsilon$ . And thus we get the uniform convergence of  $(\vec{\xi}_t^{(h)})_{t \in [0, T]}$  to  $(\vec{\xi}_t)_{t \in [0, T]}$  as  $h \rightarrow 0$ .  $\square$

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