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► **To cite this version:**

Radjesvarane Alexandre, Frederic Herau, Wei-Xi Li. Global hypoelliptic and symbolic estimates for the linearized Boltzmann operator without angular cutoff. 2013. hal-00766817v1

HAL Id: hal-00766817

<https://hal.science/hal-00766817v1>

Preprint submitted on 19 Dec 2012 (v1), last revised 22 Feb 2019 (v5)

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Global hypoelliptic and symbolic estimates for the linearized Boltzmann operator without angular cutoff

Radjesvarane Alexandre* Frédéric Hérau † Wei-Xi Li ‡

Abstract

In this article we provide global subelliptic estimates for the linearized inhomogeneous Boltzmann equation without angular cutoff, and show that some global gain in the spatial direction is available although the corresponding operator is not elliptic in this direction. The proof is based on a multiplier method and the so-called Wick quantization, together with a careful analysis of the symbolic properties of the Weyl symbol of the Boltzmann collision operator.

Keywords: global hypoellipticity, subellipticity, Boltzmann equation without cut-off, anisotropic diffusion, Wick quantization

2010 MSC: 35S05, 35H10, 35H20, 35B65, 82C40.

1 Introduction

We consider the non-cutoff inhomogeneous Boltzmann equation:

$$\partial_t F + v \cdot \partial_x F = Q(F, F), \quad (1)$$

with F standing for a probability density function, and a given Cauchy data at $t = 0$, while the position x and velocity v are in \mathbb{R}^3 , see [14, 35] and references therein for more details on Boltzmann equation. In (1), the collision kernel Q is defined by

$$Q(G, F)(t, x, v) = \int_{\mathbb{R}^3} \int_{S^2} B(v - v_*, \sigma) (F' G'_* - F G_*') dv_* d\sigma$$

where $F_{\#} = F(v_{\#})$ and $F'_{\#} = F(v'_{\#})$ for short. v and v_* are the velocities after collision, v' and v'_* before collision, with the following energy and momentum conservation rules

$$v' + v'_* = v + v_*, \quad |v'|^2 + |v'_*|^2 = |v|^2 + |v_*|^2, \quad (2)$$

and we will choose the so-called σ representation, for σ on the sphere S^2 ,

$$\begin{cases} v' = \frac{v+v_*}{2} + \frac{|v-v_*|}{2}\sigma \\ v'_* = \frac{v+v_*}{2} - \frac{|v-v_*|}{2}\sigma \end{cases}$$

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¶RA is supported by a starting grant from SJTU and by NSFC (11171211), WXL was supported in part by NSFC(11001207), WXL and FH were supported by ANR NOSEVOL (2011 BS01019 01)

We define the deviation angle by

$$\cos \theta = \left\langle \frac{v - v_*}{|v - v_*|}, \sigma \right\rangle.$$

In the case of inverse power laws, the collisional cross section B is approximatively given by

$$B(v - v_*, \sigma) = |v - v_*|^\gamma b(\cos \theta). \quad (3)$$

Without loss of generality, we assume $B(v - v_*, \sigma)$ is supported on the set $\langle v - v_*, \sigma \rangle > 0$ which corresponds to $\theta \in]0, \pi/2[$, since as usual B can be replaced by its symmetrized version

$$\overline{B}(v - v_*, \sigma) = B(v - v_*, \sigma) + B(v - v_*, -\sigma).$$

Moreover, we assume that b has the following expression and singular behavior when $\theta \in]0, \pi/2[$:

$$0 \leq \sin \theta b(\cos \theta) \approx C_0 \theta^{-1-2s}, \text{ as } \theta \rightarrow 0^+,$$

for some constant $C_0 > 0$, where the symbol \approx means bounded from below and above by some positive constants. In the preceding formulas, we will impose the following range of parameters, coming from the physical derivation,

$$s \in (0, 1), \quad \gamma \in (-3, \infty), \quad \gamma + 2s > -3.$$

Note that the last condition on $\gamma + 2s$ is weaker than in [7, 21] since we will deal only with the linearized part of Boltzmann collisional operator.

The behavior of this singular kernel is strongly related to the following non-integrability condition

$$\int_0^{\pi/2} \sin \theta b(\cos(\theta)) d\theta = \infty,$$

which implies some diffusion properties of the (linearized) Boltzmann kernel that we will explain in a moment.

In some expressions involving the integral kernels, it may therefore happen that some non-integrability happens and in this case the integrals in question have to be understood as principal values (following [11]). Anyway we shall do most of the computations as if B were integrable and use the principal value trick whenever needed.

In this work, we are interested in the linearized Boltzmann operator, around a normalized Maxwellian distribution, which is described as follows. Let this normalized maxwellian be

$$\mu(v) = (2\pi)^{-3/2} e^{-|v|^2/2}.$$

Setting $F = \mu + \sqrt{\mu}f$, the perturbation f satisfies the equation

$$\partial_t f + v \cdot \partial_x f - \mu^{-1/2} Q(\mu, \sqrt{\mu}f) - \mu^{-1/2} Q(\sqrt{\mu}f, \mu) = \mu^{-1/2} Q(\sqrt{\mu}f, \sqrt{\mu}f),$$

since $\partial_t F + v \cdot \partial_x F - Q(F, F) = 0$ and $Q(\mu, \mu) = 0$. Using the notation

$$\Gamma(g, f) = \mu^{-1/2} Q(\sqrt{\mu}g, \sqrt{\mu}f),$$

we may rewrite the above equation as

$$\partial_t f + \mathcal{P}f = \Gamma(f, f),$$

where the linearized Boltzmann operator \mathcal{P} takes the form

$$\mathcal{P} = v \cdot \partial_x - \mathcal{L} = v \cdot \partial_x - \mathcal{L}_1 - \mathcal{L}_2$$

with

$$\mathcal{L}_1 f = \Gamma(\sqrt{\mu}, f), \quad \mathcal{L}_2 f = \Gamma(f, \sqrt{\mu}).$$

The operator \mathcal{P} acts only in variables (x, v) , is non selfadjoint, and consists of a transport part which is skew-adjoint, and a diffusion part acting only in the v variable, see for example [7, 21] and the references therein for more comments.

Our main goal is to improve these previous results, and the main theorem of this paper deals with operator \mathcal{P} , viewed as an unbounded operator in $L^2(\mathbb{R}_x^3 \times \mathbb{R}_v^3)$ with norm denoted by $\|\cdot\|$.

Theorem 1.1. *For all $l \in \mathbb{R}$, there exists a constant C_l such that for all $f \in \mathcal{S}(\mathbb{R}^6)$, we have*

$$\begin{aligned} & \|\langle v \rangle^\gamma |D_v|^{2s} f\|^2 + \|\langle v \rangle^\gamma |v \wedge D_v|^{2s} f\|^2 + \|\langle v \rangle^{\gamma+2s} f\|^2 \\ & \quad + \|\langle v \rangle^{\gamma/(2s+1)} |D_x|^{2s/(2s+1)} f\|^2 + \|\langle v \rangle^{\gamma/(2s+1)} |v \wedge D_x|^{2s/(2s+1)} f\|^2 \\ & \leq C_l \left(\|\mathcal{P}f\|^2 + \|\langle v \rangle^l f\|^2 \right) \end{aligned}$$

where $D_v = \frac{1}{i} \partial_v$, $D_x = \frac{1}{i} \partial_x$.

Note that we can take the index l as negative as we wish on the r.h.s. Moreover, note carefully that we do not need to take into account the finite dimensional kernel associated with the linearized Boltzmann operator [7, 21].

The previous result can be extended to a time dependent version as follows, by considering the time dependent operator

$$\tilde{\mathcal{P}} = \partial_t + v \cdot \partial_x - \mathcal{L},$$

the functional spaces being now $L^2([0, T] \times \mathbb{R}_x^3 \times \mathbb{R}_v^3)$ and $\mathcal{S}([0, T] \times \mathbb{R}_x^3 \times \mathbb{R}_v^3)$ with norm denoted by $\|\cdot\|_{L_T^2}$. With this setting, one can show that

Theorem 1.2. *For all $l \in \mathbb{R}$, there exists a constant C_l such that for all $f \in \mathcal{S}([0, T] \times \mathbb{R}_{x,v}^6)$, we have*

$$\begin{aligned} & \|\langle v \rangle^{\frac{\gamma-2s}{1+2s}} |D_t|^{\frac{2s}{1+2s}} f\|_{L_T^2}^2 + \|\langle v \rangle^\gamma |D_v|^{2s} f\|_{L_T^2}^2 + \|\langle v \rangle^\gamma |v \wedge D_v|^{2s} f\|_{L_T^2}^2 + \|\langle v \rangle^{\gamma+2s} f\|_{L_T^2}^2 \\ & \quad + \|\langle v \rangle^{\gamma/(2s+1)} |D_x|^{2s/(2s+1)} f\|_{L_T^2}^2 + \|\langle v \rangle^{\gamma/(2s+1)} |v \wedge D_x|^{2s/(2s+1)} f\|_{L_T^2}^2 \\ & \leq C_l \left(\|\tilde{\mathcal{P}}f\|_{L_T^2}^2 + \|\langle v \rangle^l f\|_{L_T^2}^2 \right) \end{aligned}$$

The preceding theorems are consequences of the fundamental pseudo-differential properties of the linearized Boltzmann operator. Indeed, as we shall see in Section 3, the operator $\mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2$ can be splitted as

$$\mathcal{L}_1 = -a^w - \mathcal{K}_1, \quad \mathcal{L}_2 = -\mathcal{K}_2$$

where $a \geq 0$ is real, its Weyl quantization a^w being a pseudo-differential operator of order $2s$, and $\mathcal{K} = \mathcal{K}_1 + \mathcal{K}_2$ is controlled by a^w (see Proposition 1.3 below and the review about

Weyl Hörmander calculus in the appendix). Precise expressions of a and \mathcal{K}_i will be given in Section 3. The most significant part of \mathcal{L} is therefore of a pseudo differential type and by the next result, we have fundamental symbolic estimates for a , implying in particular that operator a^w is elliptic in its own calculus (although of infinite order). This very strong property allows to avoid the systematic use of Gårding type inequalities which are not available here.

In the following, η denotes the dual variable of v and $\Gamma \equiv dv^2 + d\eta^2$ is the flat metric in $\mathbb{R}_{v,\eta}^6$. Standard notions concerning symbolic estimates and the pseudo-differential calculus are explained at the beginning of section 4.

Proposition 1.3. *Define*

$$\tilde{a}(v, \eta) \equiv \langle v \rangle^\gamma (1 + |\eta|^2 + |\eta \wedge v|^2 + |v|^2)^s, \text{ for all } (v, \eta) \in \mathbb{R}_{v,\eta}^6.$$

Then we can write $\mathcal{L} = -a^w - \mathcal{K}$, where

i) the symbols a, \tilde{a} are temperate w.r.t. Γ , $a, \tilde{a} \in S(\tilde{a}, \Gamma)$, and there exists a positive constant C such that $C^{-1}\tilde{a}(v, \eta) \leq a(v, \eta) \leq C\tilde{a}(v, \eta)$;

ii) for all $\varepsilon > 0$ there exists C_ε such that

$$\|\mathcal{K}f\| + \|\mathcal{R}f\| \leq \varepsilon \|a^w f\| + C_\varepsilon \|\langle v \rangle^{\gamma+2s} f\|;$$

iii) for a sufficiently large constant K depending only on the dimension, $a_K \stackrel{\text{def}}{=} a + K \langle v \rangle^{\gamma+2s}$ belongs to $S(\tilde{a}, \Gamma)$, is invertible and its inverse $(a_K^w)^{-1}$ has the form

$$(a_K^w)^{-1} = H_1 (a_K^{-1})^w = (a_K^{-1})^w H_2,$$

with H_1, H_2 belonging to $\mathcal{B}(L^2)$, the space of bounded operators on L^2 .

Recall that in Hörmander's terminology, $a \in S(\tilde{a}, \Gamma)$ means that for all multi-indices α and β , there exists a constant $C_{\alpha,\beta}$ such that

$$|\partial_v^\alpha \partial_\eta^\beta a(v, \eta)| \leq C_{\alpha,\beta} \tilde{a}$$

and the same for \tilde{a} . The temperance then implies a correct definition for the associated operators. We postpone to section 3 and the appendix a review of these standard notions of pseudo-differential calculus.

The exponents of derivative terms and weight terms in Theorem 1.1 and Theorem 1.2 seem to be optimal, since the symbolic estimates provided by Proposition 1.3 implies that the operator \mathcal{P} should behave locally like a generalized Kolmogorov type operator

$$\partial_t + v \cdot \partial_x + |D_v|^{2s},$$

for which the exponent $2s/(2s+1)$ for the regularity in the time and space variables is indeed sharp by using a simple scaling argument (see also [28]). In the particular case when $s = 1$, the coercivity estimate shows that the Boltzmann collision operator, is a type of differential operator, just like the Landau equation which seems simpler to handle than fractional derivatives, and our exponents in regularity terms and weight terms match perfectly with thoses in [22].

The main ideas of our proofs of the above theorems rely on some formal computations in [1], on the method by multiplier used in [22, 30] and some microlocal techniques developed

by Lerner while using Wick quantization [26]. Let us note that functional estimates from a series of work of Alexandre et al. [9, 8, 7, 6] and Gressman et al. [21] are also helpful for a clear understanding of the structure of the collision operator, but a nice feature of our method is that we will be able to completely avoid the use of these previous estimates. Note that there are some other methods to study the regularity of the transport equation; for instance the average arguments used by Bouchut [13] and a version of the uncertainty principle used by Alexandre et al. [5] to prove the regularity in the time and space variables t, x . However, these results do not provide any optimal hypoelliptic estimate for the spatially inhomogeneous Boltzmann equation without angular cutoff.

Let us end this introduction by giving some bibliographical references about the hypoelliptic properties of the non cutoff Boltzmann equation and related kinetic models. Note that the angular cross-section b is not integrable on the sphere due to the singularity θ^{-2-2s} , which leads to the formal statement that the nonlinear collision operator should behave like a fractional Laplacian; that is,

$$Q(g, f) \approx -C_g(-\Delta_v)^s f + \text{lower order terms},$$

with $C_g > 0$ a constant depending only on the physical properties of g . Initiated by Desvillettes [17, 18], there have been extensive works around this result and regarding the smoothness of solutions for the homogeneous Boltzmann equation without angular cutoff, c.f. [4, 10, 15, 19, 20, 24, 31, 33]. For the inhomogeneous case the study becomes more complicated. We remark that there have been some related works concerned with the linear model of spatially inhomogeneous Boltzmann equation, which takes the following form

$$\partial_t + v \cdot \partial_x + e(t, x, v)(-\Delta_v)^s, \quad \inf_{t,x,v} e(t, x, v) > 0.$$

This model equation was firstly studied by Morimoto and Xu [32], where a global but non optimal hypoelliptic estimate was established. This study was then improved by Chen et al. [16], and also by Lerner et al. in [28] for an optimal local result. We also mention [3] where a simple proof of the subelliptic estimate for the above model operator is given. For general inhomogeneous Boltzmann equation we refer to [9, 8, 7, 6] for recent progress on its qualitative properties. Finally, let us also mention a recent global result by Lerner et al. [29] in the radially symmetric case and the maxwellian case (which corresponds to $\gamma = 0$ in our notations).

The paper is organized as follows. In Section 2, we provide precise estimates on the nice terms appearing in the splitting of the collision operator $\mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2$, involving compact parts and relatively bounded terms w.r.t. the operator of multiplication by $\langle v \rangle^{\gamma+2s}$. In Section 3 we deal with the main terms, which appear to be of pseudodifferential type, and give precise symbolic estimates in the sense of the Weyl-Hörmander calculus. Section 4 is devoted to the proof of the main theorems. An appendix is devoted to a short review of some tools used in this work (Wick quantization, cancellation Lemma and Carleman representation).

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2 Estimates on the linearized Boltzmann kernel

In this section we study the linearized kernel \mathcal{L} in order to get the full symbol as well as symbolic and other properties on the remaining non pseudodifferential parts. Recall that the linearized Boltzmann operator \mathcal{L} is defined by

$$\begin{aligned}
\mathcal{L}f &= \mu^{-1/2}Q(\mu, \mu^{1/2}f) + \mu^{-1/2}Q(\mu^{1/2}f, \mu) \\
&= \mu^{-1/2} \iint dv_* d\sigma B \left(\mu'_*(\mu')^{1/2}f' - \mu_*\mu^{1/2}f + \mu'(\mu'_*)^{1/2}f'_* - \mu(\mu_*)^{1/2}f_* \right) \\
&= \iint dv_* d\sigma B(\mu_*)^{1/2} \left((\mu'_*)^{1/2}f' - (\mu_*)^{1/2}f + (\mu')^{1/2}f'_* - (\mu)^{1/2}f_* \right) \\
&= \iint dv_* d\sigma B(\mu_*)^{1/2} \left((\mu'_*)^{1/2}f' - (\mu_*)^{1/2}f \right) + \iint dv_* d\sigma B(\mu_*)^{1/2} \left((\mu')^{1/2}f'_* - (\mu)^{1/2}f_* \right) \\
&= \mathcal{L}_1f + \mathcal{L}_2f.
\end{aligned}$$

We shall study more precisely each part of \mathcal{L} . Let us immediately point out that they have completely different behavior. The non local term \mathcal{L}_2 behaves essentially like a convolution term, with nice estimates, and is relatively compact w.r.t. the main part of \mathcal{L}_1 which will appear to be of pseudodifferential type.

To simplify the notation, by $A \lesssim B$ we mean that there exists a harmless constant $C > 0$, such that $A \leq CB$, and similarly for $A \gtrsim B$. While the notation $A \approx B$ means that both $A \lesssim B$ and $B \lesssim A$ hold.

2.1 Study of \mathcal{L}_2

Starting from the expression of \mathcal{L}_2 given by

$$\mathcal{L}_2f = \iint dv_* d\sigma B(\mu_*)^{1/2} \left((\mu')^{1/2}f'_* - (\mu)^{1/2}f_* \right),$$

we split it into four terms which make sense even for strong singularities of B , i.e. in particular for $s \geq 1/2$. This point will be clear from the proof of Lemma 2.1 below.

$$\begin{aligned}
\mathcal{L}_2 f &= \iint dv_* d\sigma B(\mu_*)^{1/2} \left((\mu')^{1/2} f'_* - (\mu)^{1/2} f_* \right) \\
&= \iint dv_* d\sigma B \left((\mu^{1/2} f)'_* (\mu')^{1/2} - (\mu^{1/2} f)_* \mu^{1/2} \right) + \iint dv_* d\sigma B(\mu')^{1/2} \left((\mu_*)^{1/2} - (\mu'_*)^{1/2} \right) f'_* \\
&= \iint dv_* d\sigma B(\mu^{1/2} f)'_* \left((\mu')^{1/2} - \mu^{1/2} \right) \\
&\quad + \mu^{1/2} \iint dv_* d\sigma B \left((\mu^{1/2} f)'_* - (\mu^{1/2} f)_* \right) \\
&\quad + \mu^{1/2} \iint dv_* d\sigma B \left((\mu_*)^{1/2} - (\mu'_*)^{1/2} \right) f'_* \\
&+ \iint dv_* d\sigma B \left((\mu')^{1/2} - (\mu)^{1/2} \right) \left((\mu_*)^{1/2} - (\mu'_*)^{1/2} \right) f'_* \\
&= \mathcal{L}_{2,r} f + \mathcal{L}_{2,ca} f + \mathcal{L}_{2,c} f + \mathcal{L}_{2,d} f.
\end{aligned}$$

$\mathcal{L}_{2,ca}$ involves essentially a convolution term and can be treated using the cancellation lemma (see [11] and the appendix herein), and the three other ones can be estimated by hands. Let us note that the analysis of \mathcal{L}_2 was already given by [7], Lemma 2.15, but we provide a somewhat direct and shorter proof.

Lemma 2.1. *For all $f \in \mathcal{S}(\mathbb{R}_v^3)$ and for all $\alpha, \beta \in \mathbb{R}$ there exists a constant $C_{\alpha,\beta}$ such that*

$$\| \langle v \rangle^\alpha \mathcal{L}_2 \langle v \rangle^\beta f \| \leq C_{\alpha,\beta} \| f \|.$$

Proof. We start with $\mathcal{L}_{2,ca} f$:

$$\mathcal{L}_{2,ca} f = \mu^{1/2} \iint dv_* d\sigma B \left((\mu^{1/2} f)'_* - (\mu^{1/2} f)_* \right).$$

Applying the Cancellation Lemma (see [11] or the appendix), we get, for some constant c depending only on b :

$$\mathcal{L}_{2,ca} f = c\mu^{1/2} \int dv_* |v - v_*|^\gamma (\mu^{1/2} f)_*.$$

This is an integral operator with the kernel $K(v, v_*) = c\mu^{1/2}(\mu_*)^{1/2}|v - v_*|^\gamma$ for which we can apply Schur's Lemma to get

$$\| \mathcal{L}_{2,ca} f \| \lesssim \| f \|.$$

Note that the assumption $\gamma > -3$ is needed at this point.

More generally, replacing $\mathcal{L}_{2,ca} f$ by $\langle v \rangle^\alpha \mathcal{L}_{2,ca} \langle v \rangle^\beta f$ leads to a kernel

$$K_{\alpha,\beta}(v, v_*) = c\mu^{1/2} \langle v \rangle^\alpha (\mu_*)^{1/2} \langle v^* \rangle^\beta |v - v_*|^\gamma$$

for which we can use the same argument to get

$$\| \langle v \rangle^\alpha \mathcal{L}_{2,ca} \langle v \rangle^\beta f \| \leq C_{\alpha,\beta} \| f \|.$$

Now we deal with $\mathcal{L}_{2,c}f$:

$$\mathcal{L}_{2,c}f = \mu^{1/2} \iint dv_* d\sigma B \left((\mu_*)^{1/2} - (\mu'_*)^{1/2} \right) f'_*.$$

We split this term into a singular and a non-singular parts. First consider the non singular part:

$$\mathcal{L}_{2,c,nonsing}f \stackrel{\text{def}}{=} \mu^{1/2} \iint dv_* d\sigma B \mathbb{1}_{|v'-v| \geq 1} \left((\mu_*)^{1/2} - (\mu'_*)^{1/2} \right) f'_*.$$

As noticed in [7], one has $\mu'_* \mu' = \mu_* \mu \leq (\mu'_* \mu)^{1/5}$ due to the kinetic and momentum relations in (2). Therefore

$$Af \equiv |\mathcal{L}_{2,c,nonsing}f| \lesssim \mu^{1/10} \iint dv_* d\sigma |B| \mathbb{1}_{|v'-v| \geq 1} \left| (\mu^{1/10} f)'_* \right|$$

which writes in Carleman representation

$$Af \lesssim \mu^{1/10} \int_{\mathbb{R}_h^3} dh \int_{E_{0,h}} d\alpha \mathbb{1}_{|h| \geq 1} \mathbb{1}_{|\alpha| \geq |h|} \frac{|\alpha + h|^{1+\gamma+2s}}{|h|^{3+2s}} |(\mu^{1/10} f)(\alpha + v)|.$$

Recall that $E_{0,h}$ denotes the hyperplane orthogonal to h and containing 0. By duality, we get, for all $g \in \mathcal{S}$,

$$\begin{aligned} |(Af, g)| &\lesssim \int_{\mathbb{R}_v^3} dv \int_{\mathbb{R}_h^3} dh \int_{E_{0,h}} d\alpha \mathbb{1}_{|h| \geq 1} \mathbb{1}_{|\alpha| \geq |h|} \frac{|\alpha + h|^{1+\gamma+2s}}{|h|^{3+2s}} |(\mu^{1/10} f)(\alpha + v)| \\ &\lesssim \int_{\mathbb{R}_v^3} dv \int_{\mathbb{R}_h^3} dh \int_{E_{0,h}} d\alpha \mathbb{1}_{|h| \geq 1} \mathbb{1}_{|\alpha| \geq |h|} \frac{|\alpha|^{1+\gamma+2s}}{|h|^{3+2s}} |(\mu^{1/10} f)(\alpha + v)| |(\mu^{1/10} g)(v)| \end{aligned}$$

which upon using (75) yields

$$\begin{aligned} |(Af, g)| &\lesssim \int_{\mathbb{R}_v^3} dv \int_{\mathbb{R}_\alpha^3} d\alpha \int_{E_{0,\alpha}} dh \mathbb{1}_{|h| \geq 1} \mathbb{1}_{|\alpha| \geq |h|} \frac{|\alpha|^{\gamma+2s}}{|h|^{2+2s}} |\mu^{1/10} f(\alpha + v)| |\mu^{1/10} g(v)| \\ &\lesssim \int_{\mathbb{R}_v^3} dv \int_{\mathbb{R}_\alpha^3} d\alpha |\alpha|^{(\gamma+2s)^+} |\mu^{1/10} f(\alpha + v)| |\mu^{1/10} g(v)|. \end{aligned}$$

Therefore

$$|(Af, g)| \lesssim \|\mu^{1/20} f\| \|\mu^{1/20} g\|$$

from which we directly get

$$\| \langle v \rangle^\alpha \mathcal{L}_{2,c,nonsing} \langle v \rangle^\beta f \| \leq C_{\alpha,\beta} \|f\| \quad (4)$$

for all real α and β .

For the singular part $\mathcal{L}_{2,c,sing}$, Carleman's representation (see 76) gives

$$\begin{aligned} \mathcal{L}_{2,c,sing}f &= \mu^{1/2} \int_{\mathbb{R}_h^3} dh \int_{E_{0,h}} d\alpha \tilde{b}(\alpha, h) \mathbb{1}_{|\alpha| \geq |h|} \mathbb{1}_{|h| \leq 1} \\ &\quad \left(\mu^{1/2}(\alpha + v - h) - \mu^{1/2}(\alpha + v) \right) \frac{|\alpha + h|^{1+\gamma+2s}}{|h|^{3+2s}} f(\alpha + v). \end{aligned}$$

Changing $h \rightarrow -h$ and adding the resulting two formulas (so we see that formally we cancel higher singularities, recall also that $\tilde{b}(\alpha, h) = \tilde{b}(\pm\alpha, \pm h)$) yields

$$\begin{aligned} \mathcal{L}_{2,c,sing}f &= \frac{1}{2}\mu^{1/2} \int_h dh \int_{E_{0,h}} d\alpha \tilde{b} \mathbb{1}_{|\alpha| \geq |h|} \mathbb{1}_{|h| \leq 1} \times \\ &\quad \left(\mu^{1/2}(\alpha + v - h) + \mu^{1/2}(\alpha + v + h) - 2\mu^{1/2}(\alpha + v) \right) \frac{|\alpha + h|^{1+\gamma+2s}}{|h|^{3+2s}} f(\alpha + v). \end{aligned}$$

Factorizing by $\mu^{1/2}(\alpha + v)$ we get

$$\begin{aligned} \mathcal{L}_{2,c,sing}f &= \frac{1}{2}\mu^{1/2} \int_h dh \int_{E_{0,h}} d\alpha \tilde{b} \mathbb{1}_{|\alpha| \geq |h|} \mathbb{1}_{|h| \leq 1} \left(e^{-2(\alpha+v)\cdot h} + e^{-2(\alpha+v)\cdot h} - 2 \right) \\ &\quad \times \frac{|\alpha + h|^{1+\gamma+2s}}{|h|^{3+2s}} \mu^{1/2}(\alpha + v) f(\alpha + v). \end{aligned}$$

The term in parentheses is bounded by $|h|^2 \mu^{-1/4}(\alpha + v)$ thanks to the condition on the support for h , and since $|h| \leq |\alpha|$, one has

$$|\mathcal{L}_{2,c,sing}f| \lesssim \mu^{1/2} \int_h dh \int_{E_{0,h}} d\alpha \mathbb{1}_{|\alpha| \geq |h|} \mathbb{1}_{|h| \leq 1} \frac{|\alpha|^{1+\gamma+2s}}{|h|^{1+2s}} \mu^{1/4}(\alpha + v) |f(\alpha + v)|.$$

Using again (75) and the duality argument as in the non-singular case (now the singularity in h is integrable), we easily get

$$\| \langle v \rangle^\alpha \mathcal{L}_{2,c,sing} \langle v \rangle^\beta f \| \leq C_{\alpha,\beta} \|f\| \quad (5)$$

for all real α and β . At this point the assumption $\gamma + 2s > -3$ is needed.

We now analyze $\mathcal{L}_{2,r}f$. Recalling that

$$\mathcal{L}_{2,r}f = \iint dv_* d\sigma B(\mu^{1/2} f)'_* \left((\mu')^{1/2} - \mu^{1/2} \right)$$

we see immediately that, using the classical pre-post velocities change of variables that

$$(\mathcal{L}_{2,r}f, g) = (f, \mathcal{L}_{2,c}g)$$

and thus we are done for this term. It remains to study $\mathcal{L}_{2,d}f$ which is exactly

$$\mathcal{L}_{2,d}f = \iint dv_* d\sigma B \left((\mu')^{1/2} - (\mu)^{1/2} \right) \left((\mu_*)^{1/2} - (\mu'_*)^{1/2} \right) f'_*$$

Using the equality $a^2 - b^2 = (a - b)(a + b)$ for the gaussian functions in the above factors, we see again that we can put some power of a gaussian together with f , by using the argument of [7]: that means that for some $c > 0, d > 0$, one has

$$|\mathcal{L}_{2,d}f| \lesssim \mu^d \iint dv_* d\sigma B \left((\mu')^{1/4} - (\mu)^{1/4} \right) \left((\mu_*)^{1/4} - (\mu'_*)^{1/4} \right) (\mu^c)'_* |f'_*|$$

and then the remaining analysis is exactly similar to $\mathcal{L}_{2,c,sing}f$.

□

2.2 Splitting of \mathcal{L}_1

The operator \mathcal{L}_1 will also be cut into several pieces, which will require two different types of arguments. For some of the nice parts, tools similar to the ones in the previous section will be sufficient. The remaining pseudo-differential parts will be treated in the next Section.

Recall first that

$$\mathcal{L}_1 f = \iint dv_* d\sigma B(\mu_*)^{1/2} \left((\mu'_*)^{1/2} f' - (\mu_*)^{1/2} f \right).$$

Let $\delta > 0$ be a small parameter, say $\delta = 1/100$ fixed once and for all. We first split the above integral according to whether or not $|v' - v| \gtrsim \delta$. To this end, let φ be a positive radial function supported on the unit ball and say 1 in the $1/4$ ball. Consider $\varphi_\delta(v) = \varphi(|v|^2/\delta^2)$, which is therefore 0 for $|v| \geq \delta$ and 1 for $|v| \leq \delta/2$. By abuse of notations we shall also denote $\varphi_\delta(r) = \varphi_\delta(v)$ when $|v| = r$. Set $\bar{\varphi}_\delta(v) = 1 - \varphi_\delta(v)$, which is therefore 0 for small values and 1 for large values.

Then $\mathcal{L}_1 f$ can be decomposed as the sum of the following two terms

$$\bar{\mathcal{L}}_{1,\delta} f = \iint dv_* d\sigma B \bar{\varphi}_\delta(v' - v) (\mu_*)^{1/2} \left((\mu'_*)^{1/2} f' - (\mu_*)^{1/2} f \right)$$

and

$$\mathcal{L}_{1,\delta} f = \iint dv_* d\sigma B \varphi_\delta(v' - v) (\mu_*)^{1/2} \left((\mu'_*)^{1/2} f' - (\mu_*)^{1/2} f \right).$$

Note that $\bar{\mathcal{L}}_{1,\delta}$ is a cutoff type Boltzmann operator. We split it into two terms since there is no singularity any more

$$\begin{aligned} \bar{\mathcal{L}}_{1,\delta} f &= \iint dv_* d\sigma B \bar{\varphi}_\delta(v' - v) (\mu_*)^{1/2} (\mu'_*)^{1/2} f' \\ &\quad - \left(\iint dv_* d\sigma B \bar{\varphi}_\delta(v' - v) (\mu_*)^{1/2} (\mu_*)^{1/2} \right) f \\ &= \bar{\mathcal{L}}_{1,\delta,a} f + \bar{\mathcal{L}}_{1,\delta,b} f. \end{aligned}$$

As for $\mathcal{L}_{1,\delta}$, again we split it into four terms:

$$\begin{aligned} \mathcal{L}_{1,\delta} f &= \iint dv_* d\sigma B \varphi_\delta(v' - v) (\mu_*)^{1/2} \left((\mu'_*)^{1/2} f' - (\mu_*)^{1/2} f \right) \\ &= \iint dv_* d\sigma B \varphi_\delta(v' - v) (\mu'_*)^{1/2} (f' - f) \left((\mu_*)^{1/2} - (\mu'_*)^{1/2} \right) \\ &\quad + \left(\iint dv_* d\sigma B \varphi_\delta(v' - v) (\mu'_*)^{1/2} \left((\mu_*)^{1/2} - (\mu'_*)^{1/2} \right) \right) f \\ &\quad + \iint dv_* d\sigma B \varphi_\delta(v' - v) \mu'_* (f' - f) \\ &\quad + \left(\iint dv_* d\sigma B \varphi_\delta(v' - v) (\mu'_* - \mu_*) \right) f \\ &= \mathcal{L}_{1,1,\delta} f + \mathcal{L}_{1,4,\delta} f + \mathcal{L}_{1,2,\delta} f + \mathcal{L}_{1,3,\delta} f. \end{aligned}$$

Let us immediately notice that this splitting takes into account all values of s . However, for small singularities $0 < s < 1/2$, a simpler decomposition is available and avoids some of the issues dealt with below. We note that $\mathcal{L}_{1,4,\delta} f$ and $\mathcal{L}_{1,3,\delta} f$ are of multiplicative type,

and together with $\overline{\mathcal{L}}_{1,\delta,a}f$, they will be studied in the next subsection. They will appear later as relatively bounded terms with respect to $\mathcal{L}_{1,1,\delta} + \mathcal{L}_{1,2,\delta}f$. These last two parts will appear to be of pseudodifferential type, and we shall estimate them very precisely in section 3.

2.3 Relatively bounded terms in \mathcal{L}_1

Study of $\mathcal{L}_{1,3,\delta}$

Using some arguments from the proof of the cancellation lemma, see for example [11], we get the following

Lemma 2.2. *For all $f \in \mathcal{S}(\mathbb{R}_v^3)$, we have, for all $s < 1$*

$$\|\mathcal{L}_{1,3,\delta}f\|^2 \lesssim \delta^{2-2s} \|\langle v \rangle^\gamma f\|$$

and $\mathcal{L}_{1,3,\delta}$ commutes with the multiplication by $\langle v \rangle^\alpha$ for all $\alpha \in \mathbb{R}$.

Proof. The last assertion is trivial since $\mathcal{L}_{1,3,\delta}$ is a multiplication operator. For the inequality, recall first that

$$\mathcal{L}_{1,3,\delta}f(v) = \left(\iint dv_* d\sigma B\varphi_\delta(v' - v) (\mu'_* - \mu_*) \right) f$$

Going back to the proof of the cancellation Lemma, we get that

$$\left(\iint dv_* d\sigma B\varphi_\delta(v' - v) (\mu'_* - \mu_*) \right) = S *_{v_*} \mu(v)$$

where, letting $\psi_\delta(|z|) \stackrel{\text{def}}{=} \varphi_\delta(z)$ for all $z \in \mathbb{R}^3$, S has the following expression

$$\begin{aligned} S(z) &\equiv |z|^\gamma \int_0^{\pi/2} d\theta \sin \theta b(\cos \theta) \left(\psi_\delta\left(\frac{|z|}{\cos \frac{\theta}{2}} \sin \frac{\theta}{2}\right) \cos^{-3-\gamma} \frac{\theta}{2} - \psi_\delta(|z| \sin \frac{\theta}{2}) \right) \\ &= |z|^\gamma \int_0^{\pi/2} d\theta \sin \theta b(\cos \theta) \psi_\delta\left(\frac{|z|}{\cos \frac{\theta}{2}} \sin \frac{\theta}{2}\right) \left(\cos^{-3-\gamma} \frac{\theta}{2} - 1 \right) \\ &\quad + |z|^\gamma \int_0^{\pi/2} d\theta \sin \theta b(\cos \theta) \left(\psi_\delta\left(\frac{|z|}{\cos \frac{\theta}{2}} \sin \frac{\theta}{2}\right) - \psi_\delta(|z| \sin \frac{\theta}{2}) \right) \\ &= S_1(z) + S_2(z). \end{aligned}$$

The first part $S_1(z)$ is less than $|z|^\gamma$ since the integrand has small enough singularities in the θ variable, and we have

$$|S_1(z)| \lesssim |z|^\gamma. \tag{6}$$

The second part is zero if $|z| \leq \delta/2$, and we suppose therefore that $|z| \geq \delta/2$. Note also that for z bounded, say for $|z| \leq C$ where C is sufficiently large to be fixed later, $S_2(z)$ is also bounded. Since

$$\frac{|z|}{\cos \frac{\theta}{2}} \sin \frac{\theta}{2} \geq |z| \sin \frac{\theta}{2},$$

we get that if $|z| \sin \frac{\theta}{2} \geq \delta$, the integrand is 0, and similarly for small values of θ . In conclusion when $|z| \geq C$, the second integral can be estimated as follows :

$$S_2(z) = |z|^\gamma \int_{c'\delta|z|^{-1}}^{c\delta|z|^{-1}} d\theta \sin \theta b(\cos \theta) \left(\psi_\delta\left(\frac{|z|}{\cos \frac{\theta}{2}} \sin \frac{\theta}{2}\right) - \psi_\delta(|z| \sin \frac{\theta}{2}) \right),$$

where C is a posteriori chosen so that $C^{-1}c\delta \leq \pi/2$. Using Taylor formulae, we get

$$\begin{aligned} |S_2(z)| &\lesssim \delta^{-1}|z|^{\gamma+1} \int_{c'\delta|z|^{-1}}^{c\delta|z|^{-1}} d\theta \theta^2 b(\cos \theta) [\cos^{-1} \theta/2 - 1] \lesssim \delta^{-1}|z|^{\gamma+1} \int_{c'\delta|z|^{-1}}^{c\delta|z|^{-1}} d\theta \theta^4 b(\cos \theta) \\ &\lesssim \delta^{-1}|z|^{\gamma+1} \int_{c'\delta|z|^{-1}}^{c\delta|z|^{-1}} d\theta \theta^{2-2s} \sim \delta^{-1}|z|^{\gamma+1} \delta^{3-2s} |z|^{-3+2s} \\ &\lesssim \delta^{2-2s} |z|^{\gamma+2s-2}. \end{aligned}$$

From this estimate and (6), we get the result of the Lemma. \square

Study of $\bar{\mathcal{L}}_{1,\delta,a}$

We deal now with the non singular part $\bar{\mathcal{L}}_{1,\delta,a}$ of \mathcal{L}_1 for which we have the following result

Lemma 2.3. *For all $f \in \mathcal{S}(\mathbb{R}_v^3)$ and for all $\alpha, \beta \in \mathbb{R}$ such that $\alpha + \beta + \gamma \leq 0$, we have*

$$\| \langle v \rangle^\alpha \bar{\mathcal{L}}_{1,\delta,a} \langle v \rangle^\beta f \| \leq \delta^{-1-2s} C_{\alpha,\beta} \|f\|.$$

Proof. Recall that

$$\bar{\mathcal{L}}_{1,\delta,a} f = \iint dv_* d\sigma B \bar{\varphi}_\delta(v' - v) (\mu_*)^{1/2} (\mu'_*)^{1/2} f',$$

so that

$$\langle v \rangle^\alpha \bar{\mathcal{L}}_{1,\delta,a} \langle v \rangle^\beta f = \langle v \rangle^\alpha \iint dv_* d\sigma B \bar{\varphi}_\delta(v' - v) (\mu_*)^{1/2} (\mu'_*)^{1/2} (\langle v \rangle^\beta f).$$

An application of Carlemann's representation (see the appendix for instance) shows that

$$\begin{aligned} | \langle v \rangle^\alpha \bar{\mathcal{L}}_{1,\delta,a} \langle v \rangle^\beta f | &\lesssim \langle v \rangle^\alpha \int_h dh \int_{E_{0,h}} d\alpha \mathbb{1}_{|\alpha| \geq |h|} \mathbb{1}_{|h| \geq \delta/2} \mu^{1/2} (\alpha + v) \mu^{1/2} (\alpha + v - h) \\ &\quad \frac{|h + \alpha|^{1+\gamma+2s}}{|h|^{3+2s}} \langle v - h \rangle^\beta |f(v - h)| \\ &\lesssim \langle v \rangle^\alpha \int_h dh \int_{E_{0,h}} d\alpha \mathbb{1}_{|h| \geq \delta/2} \mu^{1/2} (\alpha + v) \mu^{1/2} (\alpha + v - h) \\ &\quad \frac{|\alpha|^{1+\gamma+2s}}{|h|^{3+2s}} \langle v - h \rangle^\beta |f(v - h)|, \end{aligned} \tag{7}$$

where we used that $|\alpha| \geq |h|$ for the second inequality, and $E_{0,h}$ denotes the vector plane containing 0 and orthogonal to h . Now we can write

$$e^{-|\alpha+v|^2} = e^{-|\alpha+S(h)v|^2} e^{|S(h)v|^2 - |v|^2}$$

and similarly

$$e^{-|\alpha+v-h|^2} = e^{-|\alpha+S(h)v|^2} e^{|S(h)(v-h)|^2 - |v-h|^2},$$

so that

$$\begin{aligned} \mu^{1/2}(\alpha+v)\mu^{1/2}(\alpha+v-h) &= e^{-|\alpha+S(h)v|^2} \left(e^{2(|S(h)v|^2 - |v|^2) + |v|^2 - |v-h|^2} \right)^{1/2} \\ &= e^{-|\alpha+S(h)v|^2} \left(e^{2(|S(h)v|^2 - |v|^2) + 2v \cdot h - |h|^2} \right)^{1/2}. \end{aligned}$$

Putting this in (7) gives

$$\begin{aligned} |\langle v \rangle^\alpha \bar{\mathcal{L}}_{1,\delta,a} \langle v \rangle^\beta f| &\lesssim \langle v \rangle^\alpha \int_{\mathbb{R}_h^3} dh \int_{E_{0,h}} d\alpha \mathbb{1}_{|\alpha| \geq |h|} \mathbb{1}_{|h| \geq \delta/2} \frac{|\alpha|^{1+\gamma+2s}}{|h|^{3+2s}} \langle v-h \rangle^\beta |f(v-h)| \\ &\quad e^{-|\alpha+S(h)v|^2} \left(e^{2(|S(h)v|^2 - |v|^2) + 2v \cdot h - |h|^2} \right)^{1/2}. \end{aligned}$$

Integrating with respect to α we have

$$\begin{aligned} |\langle v \rangle^\alpha \bar{\mathcal{L}}_{1,\delta,a} \langle v \rangle^\beta f| &\lesssim \langle v \rangle^\alpha \int_{\mathbb{R}_h^3} dh \mathbb{1}_{|h| \geq \delta/2} \langle S(h)v \rangle^{1+\gamma+2s} \frac{1}{|h|^{3+2s}} \langle v-h \rangle^\beta |f(v-h)| \\ &\quad \left(e^{2(|S(h)v|^2 - |v|^2) + 2v \cdot h - |h|^2} \right)^{1/2} \\ &\lesssim \int_{\mathbb{R}_z^3} dz \mathbb{1}_{|v-z| \geq \delta/2} \langle v \rangle^\alpha \langle S(v-z)v \rangle^{1+\gamma+2s} \frac{1}{|v-z|^{3+2s}} \langle z \rangle^\beta |f(z)| \\ &\quad \left(e^{2(|S(v-z)v|^2 - |v|^2) + 2v \cdot (v-z) - |v-z|^2} \right)^{1/2} \\ &\stackrel{\text{def}}{=} \int_{\mathbb{R}_z^3} K_{\alpha,\beta}(v,z) |f(z)| dz \end{aligned}$$

with

$$\begin{aligned} K_{\alpha,\beta}(v,z) &= \mathbb{1}_{|v-z| \geq \delta/2} \langle v \rangle^\alpha \langle z \rangle^\beta \langle S(v-z)v \rangle^{1+\gamma+2s} \frac{1}{|v-z|^{3+2s}} \\ &\quad \left(e^{2(|S(v-z)v|^2 - |v|^2) + 2v \cdot (v-z) - |v-z|^2} \right)^{1/2}. \end{aligned}$$

We want to apply Schur's Lemma. To this end, let's first integrate w.r.t. to z , to get

$$\begin{aligned} \int_{\mathbb{R}_z^3} dz K_{\alpha,\beta}(v,z) &= \int_{\mathbb{R}_z^3} dz \mathbb{1}_{|v-z| \geq \delta/2} \langle v \rangle^\alpha \langle z \rangle^\beta \langle S(v-z)v \rangle^{1+\gamma+2s} \frac{1}{|v-z|^{3+2s}} \\ &\quad \left(e^{2(|S(v-z)v|^2 - |v|^2) + 2v \cdot (v-z) - |v-z|^2} \right)^{1/2} \\ &= \int_{\mathbb{R}_h^3} dh \mathbb{1}_{|h| \geq \delta/2} \langle v \rangle^\alpha \langle v-h \rangle^\beta \langle S(h)v \rangle^{1+\gamma+2s} \frac{1}{|h|^{3+2s}} \\ &\quad \left(e^{2(|S(h)v|^2 - |v|^2) + 2v \cdot h - |h|^2} \right)^{1/2}, \end{aligned}$$

so that

$$\begin{aligned}
& \int_{\mathbb{R}_z^3} dz K_{\alpha,\beta}(v, z) \\
&= \int_{\mathbb{R}_h^3} dh \mathbb{1}_{|h| \geq \delta/2} \langle v \rangle^\alpha \left(1 + |v|^2 - |v \cdot \frac{h}{|h|}|^2 \right)^{(1+\gamma+2s)/2} \frac{1}{|h|^{3+2s}} \langle v-h \rangle^\beta \\
&\quad \left(e^{-2|v \cdot \frac{h}{|h|}|^2 + 2v \cdot h - |h|^2} \right)^{1/2} \\
&= \int_{\mathbb{R}_h^3} dh \mathbb{1}_{|h| \geq \delta/2} \langle v \rangle^\alpha \left(1 + |v|^2 - \frac{|v|^2}{|h|^2} \left| \frac{v}{|v|} \cdot h \right|^2 \right)^{(1+\gamma+2s)/2} \frac{1}{|h|^{3+2s}} \langle v-h \rangle^\beta \\
&\quad \left(1 + |v|^2 - 2|v| \frac{v}{|v|} \cdot h + |h|^2 \right)^{\beta/2} \left(e^{-2 \frac{|v|^2}{|h|^2} \left| \frac{v}{|v|} \cdot h \right|^2 + 2|v| \frac{v}{|v|} \cdot h - |h|^2} \right)^{1/2}.
\end{aligned}$$

Shifting to polar coordinates, with an axis along direction $v/|v|$, we obtain

$$\begin{aligned}
\int_{\mathbb{R}_z^3} dz K_{\alpha,\beta}(v, z) &\lesssim \int_0^\pi \int_\delta^\infty dr d\varphi \langle v \rangle^\alpha \sin \varphi (1 + |v|^2 - |v|^2 \cos^2 \varphi)^{(1+\gamma+2s)/2} \frac{1}{r^{1+2s}} \\
&\quad (1 + |v|^2 - 2|v|r \cos \varphi + r^2)^{\beta/2} \left(e^{-2|v|^2 \cos^2 \varphi + 2|v|r \cos \varphi - r^2} \right)^{1/2}.
\end{aligned}$$

Note here that if $|v| \leq 1$, then we directly get that $\int_{\mathbb{R}_z^3} dz K_{\alpha,\beta}(v, z) \lesssim 1$. Assuming now that $|v| \geq 1$ and setting $t = \cos \varphi$, we get

$$\begin{aligned}
& \int_{\mathbb{R}_z^3} dz K_{\alpha,\beta}(v, z) \\
&\lesssim \int_{-1}^1 \int_\delta^\infty dr dt \langle v \rangle^\alpha (1 + |v|^2 - |v|^2 t^2)^{(1+\gamma+2s)/2} e^{(-2|v|^2 t^2 + 2|v|rt - r^2)/2} \frac{1}{r^{1+2s}} \\
&\quad (1 + |v|^2 - 2|v|rt + r^2)^{\beta/2} \\
&\sim \langle v \rangle^\alpha |v|^{-1} \int_{-|v|}^{|v|} \int_\delta^\infty dr dt (1 + |v|^2 - t^2)^{(1+\gamma+2s)/2} e^{-(r-t)^2/2} \frac{1}{r^{1+2s}} (1 + |v|^2 - 2rt + r^2)^{\beta/2} \\
&\sim \langle v \rangle^\alpha |v|^{-1} \int_{-|v|}^{|v|} \int_\delta^\infty dr dt (1 + |v|^2 - t^2)^{(1+\gamma+2s)/2} e^{-(r-t)^2/2} \frac{1}{r^{1+2s}} (1 + |v|^2 - t^2 + (r-t)^2)^{\beta/2}.
\end{aligned}$$

In the inner term, note that $|v|^2 - t^2 \geq 0$. We now use Peetre's inequality

$$\langle u \rangle^\beta \langle u+w \rangle^{-|\beta|} \lesssim \langle w \rangle^\beta \lesssim \langle u \rangle^\beta \langle u+w \rangle^{|\beta|},$$

to get here

$$(1 + |v|^2 - t^2 + (r-t)^2)^{\beta/2} \lesssim (1 + |v|^2 - t^2)^{\beta/2} \langle r-t \rangle^{|\beta|}.$$

In addition, since $\delta = 1/100$, then $r \geq \delta$ implies that $r \geq C\delta \langle r \rangle$, for some C . Thus

$$\begin{aligned}
& \int_{\mathbb{R}_z^3} dz K_{\alpha,\beta}(v, z) \\
& \lesssim \delta^{-1-2s} \langle v \rangle^\alpha |v|^{-1} \int_{-|v|}^{|v|} \int_{-\infty}^{\infty} dr dt (1 + |v|^2 - t^2)^{(1+\gamma+2s+\beta)/2} \left(\langle r - t \rangle^{|\beta|} e^{-(r-t)^2/2} \right) \frac{1}{\langle r \rangle^{1+2s}} \\
& \lesssim \delta^{-1-2s} \langle v \rangle^\alpha |v|^{-1} \int_{-|v|}^{|v|} dt (1 + |v|^2 - t^2)^{(1+\gamma+2s+\beta)/2} \frac{1}{\langle t \rangle^{1+2s}} \\
& \lesssim \delta^{-1-2s} \langle v \rangle^{\alpha-1} \int_0^{|v|} dt (1 + |v|^2 - t^2)^{(1+\gamma+2s+\beta)/2} \frac{1}{\langle t \rangle^{1+2s}}.
\end{aligned} \tag{8}$$

Now for evaluating this quantity, we split the integral into two parts. First note that

$$\begin{aligned}
\int_0^{|v|/2} dt (1 + |v|^2 - t^2)^{(1+\gamma+2s+\beta)/2} \frac{1}{\langle t \rangle^{1+2s}} & \lesssim \langle v \rangle^{1+\gamma+2s+\beta} \int_0^{|v|/2} dt \frac{1}{\langle t \rangle^{1+2s}} \\
& \lesssim \langle v \rangle^{1+\gamma+2s+\beta} \langle v \rangle^{-2s} = \langle v \rangle^{1+\gamma+\beta}.
\end{aligned} \tag{9}$$

For the remaining part, we write

$$\begin{aligned}
& \int_{|v|/2}^{|v|} dt (1 + |v|^2 - t^2)^{(1+\gamma+2s+\beta)/2} \frac{1}{\langle t \rangle^{1+2s}} \\
& \lesssim \langle v \rangle^{-1-2s} \int_{|v|/2}^{|v|} dt (1 + |v|^2 - t^2)^{(1+\gamma+2s+\beta)/2} \\
& \lesssim \langle v \rangle^{-1-2s} \int_{|v|/2}^{|v|} dt (1 + (|v| - t)(|v| + t))^{(1+\gamma+2s+\beta)/2} \\
& \lesssim \langle v \rangle^{-1-2s} \int_{|v|/2}^{|v|} dt (1 + |v|(|v| - t))^{(1+\gamma+2s+\beta)/2}
\end{aligned}$$

Posing $s = |v|(|v| - t)$, $ds = -|v|dt$, we get

$$\begin{aligned}
& \lesssim \langle v \rangle^{-1-2s} |v|^{-1} \int_0^{|v|^2/2} ds (1 + s)^{(1+\gamma+2s+\beta)/2} \\
& \lesssim \langle v \rangle^{-1-2s} |v|^{-1} \langle v \rangle^{(1+\gamma+2s+\beta)+2} \\
& \lesssim \langle v \rangle^{1+\gamma+\beta}.
\end{aligned} \tag{10}$$

Putting estimates (9) and (10) in (8) we get

$$\begin{aligned}
\int_{\mathbb{R}_z^3} dz K_{\alpha,\beta}(v, z) & \lesssim \delta^{-1-2s} \langle v \rangle^{\alpha-1} \langle v \rangle^{1+\gamma+\beta} \\
& \lesssim \delta^{-1-2s} \langle v \rangle^{\alpha+\gamma+\beta} \\
& \lesssim \delta^{-1-2s} \quad \text{if } \alpha + \beta + \gamma \leq 0.
\end{aligned}$$

In conclusion, we have obtained that if $\alpha + \beta + \gamma \leq 0$, then

$$\int_{\mathbb{R}_z^3} dz K_{\alpha,\beta}(v, z) \lesssim \delta^{-1-2s}. \tag{11}$$

Now we look for the integration w.r.t. variable v of $K_{\alpha,\beta}$. We have

$$\int_{\mathbb{R}_z^3} dv K_{\alpha,\beta}(v, z) = \int_{\mathbb{R}_z^3} dz \mathbb{1}_{|v-z| \geq \delta/2} \langle v \rangle^\alpha \langle z \rangle^\beta \langle S(v-z)v \rangle^{1+\gamma+2s} \\ \left(e^{|S(v-z)v|^2 - |v|^2 + |S(v-z)(z)|^2 - |z|^2} \right)^{1/2} \frac{1}{|v-z|^{3+2s}},$$

since by direct computation

$$2(|S(v-z)v|^2 - |v|^2) + 2v \cdot (v-z) - |v-z|^2 \\ = |S(v-z)v|^2 - |v|^2 + |S(v-z)(z)|^2 - |z|^2.$$

Taking $h = v - z$, $dh = dv$, we get

$$\int_{\mathbb{R}_z^3} dv K_{\alpha,\beta}(v, z) = \int_{\mathbb{R}_h^3} dh \mathbb{1}_{|h| \geq \delta/2} \langle z+h \rangle^\alpha \langle z \rangle^\beta \langle S(h)(z+h) \rangle^{1+\gamma+2s} \\ \left(e^{|S(h)(z+h)|^2 - |z+h|^2 + |S(h)(z)|^2 - |z|^2} \right)^{1/2} \frac{1}{|h|^{3+2s}} \\ = \int_{\mathbb{R}_h^3} dh \mathbb{1}_{|h| \geq \delta/2} \langle z+h \rangle^\alpha \langle z \rangle^\beta \langle S(h)z \rangle^{1+\gamma+2s} \\ \left(e^{|S(h)z|^2 - |z+h|^2 + |S(h)z|^2 - |z|^2} \right)^{1/2} \frac{1}{|h|^{3+2s}},$$

so that expanding again the brackets, we get

$$\int_{\mathbb{R}_z^3} dv K_{\alpha,\beta}(v, z) \\ = \int_{\mathbb{R}_h^3} dh \mathbb{1}_{|h| \geq \delta/2} (1 + |z|^2 + 2z \cdot h + |h|^2)^{\alpha/2} \langle z \rangle^\beta \left(1 + |z|^2 - |z \cdot \frac{h}{|h|}|^2 \right)^{(1+\gamma+2s)/2} \\ \left(e^{-|z \cdot \frac{h}{|h|}|^2 - 2z \cdot h - |h|^2} e^{-|z \cdot \frac{h}{|h|}|^2} \right)^{1/2} \frac{1}{|h|^{3+2s}}.$$

We shift to spherical coordinates (along axis w.r.t z) ($h = r\omega$) to get

$$\int_{\mathbb{R}_z^3} dv K_{\alpha,\beta}(v, z) \\ = \int_0^\pi \int_\delta^\infty d\varphi \sin \varphi dr \langle z \rangle^\beta (1 + |z|^2 + 2|z|r \cos \varphi + r^2)^{\alpha/2} (1 + |z|^2 - |z|^2 \cos^2 \varphi)^{(1+\gamma+2s)/2} \\ \left(e^{-|z|^2 \cos^2 \varphi - 2|z|r \cos \varphi - r^2} e^{-|z|^2 \cos^2 \varphi} \right)^{1/2} \frac{1}{r^{1+2s}}.$$

Set $t = \cos \varphi$ to get

$$\int_{\mathbb{R}_z^3} dv K_{\alpha,\beta}(v, z) \\ = \int_{-1}^1 dt \int_\delta^\infty dr \langle z \rangle^\beta (1 + |z|^2 + 2|z|rt + r^2)^{\alpha/2} (1 + |z|^2 - |z|^2 t^2)^{(1+\gamma+2s)/2} \\ \left(e^{-|z|^2 t^2 - 2|z|rt - r^2} e^{-|z|^2 t^2} \right)^{1/2} \frac{1}{r^{1+2s}}.$$

We note again that if $|z| \leq 1$, this integral is bounded uniformly. We therefore assume in the following that $|z| \geq 1$ and change variable $t' = |z|t$ to deduce that

$$\begin{aligned} & \int_{\mathbb{R}_z^3} dv K_{\alpha,\beta}(v, z) \\ &= |z|^{-1} \int_{-|z|}^{|z|} dt \int_{\delta}^{\infty} dr \langle z \rangle^{\beta} (1 + |z|^2 + 2rt + r^2)^{\alpha/2} (1 + |z|^2 - t^2)^{(1+\gamma+2s)/2} \\ & \quad e^{-(t+r)^2/2} e^{-t^2/2} \frac{1}{r^{1+2s}} \\ & \lesssim |z|^{-1} \int_{-|z|}^{|z|} dt \int_{\delta}^{\infty} dr \langle z \rangle^{\beta} (1 + |z|^2 - t^2)^{(1+\gamma+2s+\alpha)/2} \langle r+t \rangle^{|\alpha|} e^{-(t+r)^2/2} e^{-t^2/2} \frac{1}{r^{1+2s}}, \end{aligned}$$

where the last inequality is a consequence of Peetre's inequalities. With exactly the same argument as before for the integration w.r.t. r , for small δ , we obtain

$$\int_{\mathbb{R}_z^3} dv K_{\alpha,\beta}(v, z) \lesssim \delta^{-1-2s} \langle z \rangle^{\alpha+\beta+\gamma}$$

and thus

$$\int_{\mathbb{R}_z^3} dv K_{\alpha,\beta}(v, z) \lesssim \delta^{-1-2s} \quad (12)$$

when $\alpha + \beta + \gamma \leq 0$. From (11) and (12), Schur's Lemma applies and this concludes the proof of Lemma 2.3. \square

Study of $\mathcal{L}_{1,4,\delta}$

Recalling that

$$\mathcal{L}_{1,4,\delta} f = f \iint dv_* d\sigma B \varphi_{\delta}(|v' - v|^2) (\mu'_*)^{1/2} \left((\mu_*)^{1/2} - (\mu'_*)^{1/2} \right)$$

one has the following Lemma

Lemma 2.4. *For all $f \in \mathcal{S}(\mathbb{R}_v^3)$, we have, for all $s < 1$*

$$\|\mathcal{L}_{1,4,\delta} f\|^2 \lesssim \delta^{2-2s} \| \langle v \rangle^{\gamma+2s} f \|^2,$$

and $\mathcal{L}_{1,4,\delta}$ commutes with the multiplication by $\langle v \rangle^{\alpha}$ for all $\alpha \in \mathbb{R}$.

Proof. The last assertion is again trivial since $\mathcal{L}_{1,3,\delta}$ is a multiplication operator. Using the formula $2a(b-a) = b^2 - a^2 - (b-a)^2$, we get

$$\begin{aligned} \mathcal{L}_{1,4,\delta} f &= \frac{1}{2} f \iint dv_* d\sigma B \varphi_{\delta}(v' - v) \left((\mu_*) - (\mu'_*) \right) \\ & \quad - \frac{1}{2} f \iint dv_* d\sigma B \varphi_{\delta}(v' - v) \left((\mu_*)^{1/2} - (\mu'_*)^{1/2} \right)^2 \\ &= A(v)f + D(v)f. \end{aligned}$$

We first deal with the first term. Using Carleman representation, we have

$$A(v) = \frac{1}{2} \int_{\mathbb{R}_h^3} dh \int_{E_{0,h}} d\alpha \tilde{\mathbb{I}}_{|\alpha| \geq |h|} \varphi_\delta(h) (\mu(\alpha + v - h) - \mu(\alpha + v)) \frac{|\alpha + h|^{1+\gamma+2s}}{|h|^{3+2s}}.$$

Again we have to kill high singularities and give a sense to this expression. For this purpose we change h to $-h$ and add the resulting two formulas; this gives

$$A(v) = \frac{1}{4} \int_h dh \int_{E_{0,h}} d\alpha \tilde{\mathbb{I}}_{|\alpha| \geq |h|} \varphi_\delta(h) \times (\mu(\alpha + v - h) + \mu(\alpha + v + h) - 2\mu(\alpha + v)) \frac{|\alpha + h|^{1+\gamma+2s}}{|h|^{3+2s}},$$

and thus

$$\begin{aligned} |A(v)| &\lesssim \int_h dh \int_{E_{0,h}} d\alpha \varphi_\delta(h) \times \left| \left(e^{2(\alpha+v) \cdot h - h^2} + e^{-2(\alpha+v) \cdot h - h^2} - 2 \right) \right| \\ &\quad \times |\alpha|^{1+\gamma+2s} \mu(\alpha + v) \frac{1}{|h|^{3+2s}} \\ &\lesssim \int_h dh \int_{E_{0,h}} d\alpha \varphi_\delta(h) \times \left(|\alpha|^{1+\gamma+2s} \mu^{1/2}(\alpha + v) \right) \frac{1}{|h|^{1+2s}}, \end{aligned}$$

where we used that the big parenthesis is bounded by a constant times $|h|^2 \mu^{-1/2}(\alpha + v)$ thanks to the condition on the support of h . Using (75) we get

$$\begin{aligned} |A(v)| &\lesssim \int_{\mathbb{R}_\alpha^3} d\alpha \int_{E_{0,\alpha}} dh \varphi_\delta(h) |\alpha|^{\gamma+2s} \mu^{1/2}(\alpha + v) \frac{1}{|h|^{2s}} \\ &\lesssim \delta^{2-2s} \int_{\mathbb{R}_\alpha^3} d\alpha |\alpha|^{\gamma+2s} \mu^{1/2}(\alpha + v) \lesssim \delta^{2-2s} \langle v \rangle^{\gamma+2s}. \end{aligned}$$

Again the condition $\gamma + 2s > -3$ is needed in the last step.

For the second term $D(v)$ we essentially follow the same procedure, except that we don't need to use a symmetrizing argument to kill singularities. We write

$$\begin{aligned} |D(v)| &= \frac{1}{2} \int_{\mathbb{R}_h^3} dh \int_{E_{0,h}} d\alpha \mathbb{I}_{|\alpha| \geq |h|} \varphi_\delta(h) \left(\mu^{1/2}(\alpha + v - h) - \mu^{1/2}(\alpha + v) \right)^2 \frac{|\alpha + h|^{1+\gamma+2s}}{|h|^{3+2s}} \\ &\lesssim \int_{\mathbb{R}_h^3} dh \int_{E_{0,h}} d\alpha \mathbb{I}_{|\alpha| \geq |h|} \varphi_\delta(h) \left(e^{(\alpha+v) \cdot h - h^2/2} - 1 \right)^2 \mu(\alpha + v) \frac{|\alpha + h|^{1+\gamma+2s}}{|h|^{3+2s}} \\ &\lesssim \int_{\mathbb{R}_h^3} dh \int_{E_{0,h}} d\alpha \mathbb{I}_{|\alpha| \geq |h|} \varphi_\delta(h) \mu^{1/2}(\alpha + v) \frac{|\alpha|^{1+\gamma+2s}}{|h|^{1+2s}} \\ &\lesssim \delta^{2-2s} \langle v \rangle^{\gamma+2s}, \end{aligned}$$

following the same arguments as before. From the estimates on $A(v)$ and $D(v)$, the proof is complete. \square

3 Pseudodifferential parts

In this section we deal with the remaining parts of \mathcal{L}_1 , namely a multiplicative operator $\overline{\mathcal{L}}_{1,\delta,b}$, the main term $\mathcal{L}_{1,2,\delta}$ which will appear to be of pseudo-differential type, and the term $\mathcal{L}_{1,1,\delta}$ are also of pseudo-differential type but with lower order, our goal being to prove Proposition 1.3.

In the following, we keep the notation for φ_δ , the positive compactly supported function equal to 1 in a δ -neighborhood of 0 as introduced previously in the definitions of the operators, and let $E_{0,\omega}$ denote the vector plane containing 0 and orthogonal to ω . We study each operator separately. Proposition 1.3 will be obtained as a direct consequence of Proposition 3.4 and Proposition 3.1 below.

3.1 Study of the principal term $\mathcal{L}_{1,2,\delta}$

Recall that

$$\mathcal{L}_{1,2,\delta}f = \iint dv_* d\sigma B \varphi_\delta(v' - v) \mu'_*(f' - f)$$

where

$$B(v, \sigma) = |v - v_*|^\gamma b \left(\left\langle \frac{v - v_*}{|v - v_*|}, \sigma \right\rangle \right).$$

This will appear to be a genuine pseudo differential operator of order $2s$ for which we can control the weights. Namely one has

Proposition 3.1. *We can write*

$$\mathcal{L}_{1,2,\delta}f = -a_p(v, D_v)f,$$

where a_p is a real symbol in (v, η) satisfying:

i) there exists $C > 0$ such that for all $0 < \kappa < 1$,

$$\begin{aligned} C^{-1} \left(-\kappa \langle v \rangle^{\gamma+2s} + \kappa \langle v \rangle^\gamma (|\eta|^{2s} + |\eta \wedge v|^{2s}) \right) \\ \leq a_p(v, \eta) \leq C \langle v \rangle^\gamma (1 + |\eta|^{2s} + |\eta \wedge v|^{2s}); \end{aligned} \quad (13)$$

ii) $a_p \in S(\langle v \rangle^\gamma (1 + |v|^{2s} + |\eta|^{2s} + |\eta \wedge v|^{2s}), \Gamma)$.

Proof. From the expression of $\mathcal{L}_{1,2,\delta}$, using Carleman's transformation as in previous arguments and as in [1] (see also the Appendix), we get

$$\mathcal{L}_{1,2,\delta}f = \int_{\mathbb{R}_h^3} dh \int_{E_{0,h}} \mathbb{1}_{|\alpha| \geq |h|} \varphi_\delta(h) d\alpha \mu(\alpha + v) |\alpha + h|^{1+\gamma+2s} (f(v-h) - f(v)) \frac{1}{|h|^{3+2s}}.$$

This integral is typically undefined for large values of s , and we have to use its symmetrized version in order to give a meaning in the principal value sense: for this, we change h to $-h$ and add the two expressions so that

$$\begin{aligned} \mathcal{L}_{1,2,\delta}f &= \frac{1}{2} \int_h dh \int_{E_{0,h}} d\alpha \tilde{\mathbb{1}}_{|\alpha| \geq |h|} \varphi_\delta(h) \mu(\alpha + v) |\alpha + h|^{1+\gamma+2s} \\ &\quad (f(v-h) + f(v+h) - 2f(v)) \frac{1}{|h|^{3+2s}} \\ &\equiv -a_p(v, D_v)f(v) \equiv - \int_{\mathbb{R}_\eta^3} a_p(v, \eta) \hat{f}(\eta) e^{i\eta \cdot v} d\eta \end{aligned}$$

with

$$\begin{aligned}
a_p(v, \eta) &\equiv -\frac{1}{2} \int_{\mathbb{R}_h^3} dh \int_{E_{0,h}} d\alpha \tilde{b} \mathbb{1}_{|\alpha| \geq |h|} \varphi_\delta(h) \mu(\alpha + v) |\alpha + h|^{1+\gamma+2s} \left(e^{-i\eta \cdot h} + e^{i\eta \cdot h} - 2 \right) \frac{1}{|h|^{3+2s}} \\
&= \int_{\mathbb{R}_h^3} dh \int_{E_{0,h}} d\alpha \tilde{b} \mathbb{1}_{|\alpha| \geq |h|} \varphi_\delta(h) \mu(\alpha + v) |\alpha + h|^{1+\gamma+2s} (1 - \cos(\eta \cdot h)) \frac{1}{|h|^{3+2s}}.
\end{aligned} \tag{14}$$

The non-negativity of $a_p(v, \eta)$ is clear and we work now with this symbol. First recall that on the support of the integrand, we have $|h| \leq \delta \leq 1$ and that $\alpha \perp h$, so that

$$0 \leq a_p(v, \eta) \lesssim \int_{\mathbb{R}^3} dh \int_{E_{0,h}} \mathbb{1}_{|\alpha| \geq |h|} \mathbb{1}_{|h| \leq \delta} d\alpha \mu(\alpha + v) \langle \alpha \rangle^{1+\gamma+2s} (1 - \cos(\eta \cdot h)) \frac{1}{|h|^{3+2s}}.$$

Now we can shift to spherical coordinates $h = r\omega$, and (forgetting the truncation in α) we get

$$a_p(v, \eta) \lesssim \int_0^\delta \int_{S_\omega^2} dr d\omega \int_{E_{0,\omega}} d\alpha \mu(\alpha + v) \langle \alpha \rangle^{1+\gamma+2s} (1 - \cos(r\eta \cdot \omega)) \frac{1}{r^{1+2s}}.$$

It is possible to integrate directly w.r.t. r , and use the fact that

$$\int_0^\delta (1 - \cos(r\eta \cdot \omega)) \frac{1}{r^{1+2s}} dr \leq C_s |\eta \cdot \omega|^{2s}.$$

In fact, note that

$$\int_0^\delta (1 - \cos(r\eta \cdot \omega)) \frac{1}{r^{1+2s}} dr = |\omega \cdot \eta|^{2s} \int_0^{\delta|\omega \cdot \eta|} (1 - \cos(r)) \frac{1}{r^{1+2s}} dr$$

Next, we choose a small constant c such that $1 - \cos r \gtrsim r^2$ if $r \leq c$.

If $|\omega \cdot \eta| \geq c$, then we get

$$\int_0^\delta (1 - \cos(r\eta \cdot \omega)) \frac{1}{r^{1+2s}} dr \gtrsim |\eta \cdot \omega|^{2s} \int_0^{c\delta} (1 - \cos(r)) \frac{1}{r^{1+2s}} dr \gtrsim \delta^{2-2s} |\omega \cdot \eta|^{2s},$$

while if $|\omega \cdot \eta| \leq c$, then we get

$$\int_0^\delta (1 - \cos(r\eta \cdot \omega)) \frac{1}{r^{1+2s}} dr \gtrsim |\omega \cdot \eta|^2 \int_0^\delta r^2 \frac{1}{r^{1+2s}} dr \gtrsim \delta^{2-2s} |\omega \cdot \eta|^2.$$

On the whole, we get

$$\int_0^\delta (1 - \cos(r\eta \cdot \omega)) \frac{1}{r^{1+2s}} dr \gtrsim \delta^{2-2s} \min\{|\omega \cdot \eta|^2, |\omega \cdot \eta|^{2s}\}. \tag{15}$$

In fact the same type of arguments show that we get a similar upper bound, and eventually

$$\delta^{2-2s} \min\{|\omega \cdot \eta|^2, |\omega \cdot \eta|^{2s}\} \lesssim \int_0^\delta (1 - \cos(r\eta \cdot \omega)) \frac{1}{r^{1+2s}} dr \lesssim \delta^{2-2s} |\omega \cdot \eta|^{2s}. \tag{16}$$

Next, we deal with the upper bound on a_p . A crude estimate is enough and we get

$$a_p(v, \eta) \lesssim \int_{S_\omega^2} d\omega \int_{E_{0,\omega}} d\alpha \mu(\alpha + v) |\omega \cdot \eta|^{2s} \langle \alpha \rangle^{1+\gamma+2s}. \tag{17}$$

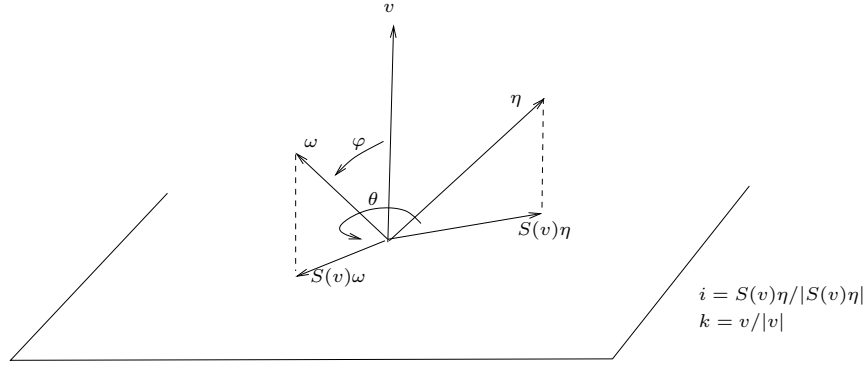


Figure 1: Polar coordinates

Splitting $v = S(\omega)v + (\omega \cdot v)\omega$, we have

$$|\alpha + v|^2 = |\alpha + S(\omega)v + (\omega \cdot v)\omega|^2 = |\alpha + S(\omega)v|^2 + |(\omega \cdot v)|^2 \quad (18)$$

since α and ω are orthogonal. We can therefore write

$$\mu(\alpha + v) = e^{-|\alpha + S(\omega)v|^2} e^{-|(\omega \cdot v)|^2}$$

to get

$$a_p(v, \eta) \lesssim \int_{S_\omega^2} d\omega \int_{E_{0,\omega}} d\alpha e^{-|\alpha + S(\omega)v|^2} e^{-|(\omega \cdot v)|^2} |\omega \cdot \eta|^{2s} \langle \alpha \rangle^{1+\gamma+2s}. \quad (19)$$

Next, note that

$$\beta(v, \omega) = \int_{E_{0,\omega}} d\alpha e^{-|\alpha + S(\omega)v|^2} \langle \alpha \rangle^{1+\gamma+2s} \sim \langle S(\omega)v \rangle^{1+\gamma+2s}$$

and thus

$$a_p(v, \eta) \lesssim \int_{S_\omega^2} d\omega e^{-|(\omega \cdot v)|^2} \langle S(\omega)v \rangle^{1+\gamma+2s} |\omega \cdot \eta|^{2s}. \quad (20)$$

We introduce polar coordinates in a coordinate system where $\mathbf{i} = S(v)\eta / |S(v) \cdot \eta|$, $\mathbf{k} = v / |v|$.

In this system, we note that $(\omega \cdot \mathbf{k}) = \cos(\varphi)$. Besides we have $\eta = (\eta \cdot \mathbf{k})\mathbf{k} + S(v)\eta$ so that

$$\begin{aligned} \omega \cdot \eta &= (\eta \cdot \mathbf{k})(\mathbf{k} \cdot \omega) + (S(v)\eta) \cdot \omega \\ &= (\eta \cdot \mathbf{k})(\mathbf{k} \cdot \omega) + (S(v)\eta) \cdot (S(v)\omega) \\ &= (\eta \cdot \mathbf{k})(\mathbf{k} \cdot \omega) + (\mathbf{i} \cdot (S(v)\omega)) |S(v)\eta| \\ &= \eta \cdot \mathbf{k} \cos(\varphi) + |S(v)\eta| \sin(\varphi) \cos(\theta). \end{aligned}$$

and in a similar way

$$|S(\omega)v|^2 = |v|^2 - |(v \cdot \omega)|^2 = |v|^2(1 - \cos^2(\varphi)) = |v|^2 \sin^2(\varphi).$$

We therefore get

$$\begin{aligned} a_p(v, \eta) \lesssim \int_0^\pi d\varphi \int_0^{2\pi} d\theta \sin(\varphi) e^{-|v|^2 \cos^2(\varphi)} (1 + |v|^2 \sin^2(\varphi))^{(1+\gamma+2s)/2} \\ |\eta \cdot \mathbf{k} \cos(\varphi) + |S(v)\eta| \sin(\varphi) \cos(\theta)|^{2s}. \end{aligned}$$

Setting $\cos \varphi = t$ in the preceding formula, we get

$$a_p(v, \eta) \lesssim \int_0^{2\pi} d\theta \int_0^1 dt e^{-|v|^2 t^2} (1 + |v|^2(1 - t^2))^{(1+\gamma+2s)/2} |\eta \cdot \mathbf{k}t + |S(v)\eta| \sqrt{1 - t^2} \cos(\theta)|^{2s}. \quad (21)$$

If we bound roughly $1 - t^2$ and $\cos(\varphi)$ by 1 we get

$$a_p(v, \eta) \lesssim \int_0^{2\pi} d\theta \int_0^1 dt e^{-|v|^2 t^2} (1 + |v|^2)^{(1+\gamma+2s)/2} (|\eta \cdot \mathbf{k}t|^{2s} + |S(v)\eta|^{2s}).$$

If we set $y = |v|t$, we get

$$\begin{aligned} a_p(v, \eta) &\lesssim \frac{1}{|v|} \langle v \rangle^{1+\gamma+2s} \int_0^{2\pi} d\theta \int_0^{|v|} dy e^{-y^2} \left(|\eta \cdot \mathbf{k}|^{2s} \frac{y^{2s}}{|v|^{2s}} + |S(v)\eta|^{2s} \right) \\ &\lesssim \frac{1}{|v|} \langle v \rangle^{1+\gamma+2s} \left(|\eta \cdot \mathbf{k}|^{2s} \frac{1}{|v|^{2s}} + |S(v)\eta|^{2s} \right) \\ &\lesssim \frac{\langle v \rangle^{1+\gamma+2s}}{|v|^{1+2s}} |\eta|^{2s} + \frac{\langle v \rangle^{1+\gamma+2s}}{|v|} |S(v)\eta|^{2s}. \end{aligned} \quad (22)$$

For $|v| \geq 1$, we therefore get

$$a_p(v, \eta) \lesssim \langle v \rangle^\gamma |\eta|^{2s} + \langle v \rangle^{\gamma+2s} |S(v)\eta|^{2s},$$

and thus

$$a_p(v, \eta) \lesssim \langle v \rangle^\gamma (|\eta|^{2s} + |v \wedge \eta|^{2s}),$$

since $|v \wedge \eta| = |v||S(v)\eta|$. For $|v| \leq 1$, a rough estimate gives directly $|a(v, \eta)| \leq \langle \eta \rangle^{2s}$ so that the preceding estimate is also true. The proof of the upper bound is complete.

Now we deal with the lower bound. In order to prove this result we shall use the formula (14) that we recall

$$a_p(v, \eta) = \int_{\mathbb{R}_h^3} dh \int_{E_{0,h}} d\alpha \tilde{b} \varphi_\delta(h) \mathbb{1}_{|\alpha| \geq |h|} \mu(\alpha + v) |\alpha + h|^{1+\gamma+2s} (1 - \cos(\eta \cdot h)) \frac{1}{|h|^{3+2s}}.$$

As we want a lower bound we can restrict the integration range to $\{|\alpha| \geq 10\}$ since the integrand is non negative. We use also the facts that \tilde{b} is bounded from below by a positive constant and that $|\alpha + h| \sim |\alpha|$ since $\alpha \perp h$ and $|h| \leq |\alpha|$ in the preceding integral. This gives

$$a_p(v, \eta) \gtrsim \int_{\mathbb{R}_h^3} dh \int_{E_{0,h}} \varphi_\delta(h) \mathbb{1}_{|\alpha| \geq 10} d\alpha \mu(\alpha + v) \langle \alpha \rangle^{1+\gamma+2s} (1 - \cos(\eta \cdot h)) \frac{1}{|h|^{3+2s}}.$$

We can use some of the previous computations, and from (15)-(16) we get as in (19), with here $C_\delta \sim \delta^{2-2s}$,

$$a_p(v, \eta) \gtrsim C_\delta \int_{S_\omega^2} d\omega \int_{E_{0,\omega}} d\alpha \mathbb{1}_{|\alpha| \geq 10} e^{-|\alpha+S(\omega)v|^2} e^{-|(\omega \cdot v)|^2} \min\{|\omega \cdot \eta|^2, |\omega \cdot \eta|^{2s}\} \langle \alpha \rangle^{1+\gamma+2s}.$$

Consider

$$\beta_{10}(v, \omega) = \int_{E_{0,\omega}} d\alpha 1_{|\alpha| \geq 10} e^{-|\alpha + S(\omega)v|^2} \langle \alpha \rangle^{1+\gamma+2s} \sim \langle S(\omega)v \rangle^{1+\gamma+2s}.$$

Then

$$a_p(v, \eta) \gtrsim C_\delta \int_{S_\omega^2} d\omega e^{-|(\omega \cdot v)|^2} \langle S(\omega)v \rangle^{1+\gamma+2s} \min\{|\omega \cdot \eta|^2, |\omega \cdot \eta|^{2s}\}. \quad (23)$$

We now consider an arbitrary real $0 < \kappa < 1$. Using the fact that

$$\min\{|\omega \cdot \eta|^2, |\omega \cdot \eta|^{2s}\} \geq |\omega \cdot \eta|^{2s} - 1,$$

and that the right member in (23) is non-negative, we get that

$$\begin{aligned} a_p(v, \eta) &\gtrsim \kappa C_\delta \int_{S_\omega^2} d\omega e^{-|(\omega \cdot v)|^2} \langle S(\omega)v \rangle^{1+\gamma+2s} \min\{|\omega \cdot \eta|^2, |\omega \cdot \eta|^{2s}\} \\ &\gtrsim \kappa C_\delta \int_{S_\omega^2} d\omega e^{-|(\omega \cdot v)|^2} \langle S(\omega)v \rangle^{1+\gamma+2s} (|\omega \cdot \eta|^{2s} - 1) \\ &\gtrsim \kappa C_\delta \int_{S_\omega^2} d\omega e^{-|(\omega \cdot v)|^2} \langle S(\omega)v \rangle^{1+\gamma+2s} |\omega \cdot \eta|^{2s} \\ &\quad - \kappa C_\delta \int_{S_\omega^2} d\omega e^{-|(\omega \cdot v)|^2} \langle S(\omega)v \rangle^{1+\gamma+2s} \\ &\equiv \kappa C_\delta a_{pp} - \kappa C_\delta a_{pr}. \end{aligned} \quad (24)$$

We split the study of the two terms a_{pp} and a_{pr} . For a_{pp} , we can use the computations thereafter and similarly to (20). This gives

$$a_{pp}(v, \eta) = \int_0^{2\pi} d\theta \int_0^1 dt e^{-|v|^2 t^2} (1 + |v|^2(1 - t^2))^{(1+\gamma+2s)/2} \left| \eta \cdot \mathbf{k}t + |S(v)\eta| \sqrt{1 - t^2} \cos(\theta) \right|^{2s}.$$

Now an easy remark is that the symbol a has the following parity properties:

$$a_{pp}(\pm v, \pm \eta) = a_{pp}(v, \eta).$$

We can therefore suppose that $\eta \cdot \mathbf{k} \geq 0$ in all the computations. Moreover we can restrict the above integration to the following subsets

$$t \in [0, \sqrt{3}/2], \quad \theta \in [0, \pi/3], \quad (25)$$

which implies that all terms inside the absolute value

$$|\eta \cdot \mathbf{k}t + |S(v)\eta| \sqrt{1 - t^2} \cos(\theta)|$$

are non negative. We therefore get, when (25) is fulfilled, that

$$(1 + |v|^2(1 - t^2))^{(1+\gamma+2s)/2} \geq \left(1 + \frac{|v|^2}{4}\right)^{(1+\gamma+2s)/2} \geq c_{s,\gamma} \langle v \rangle^{1+\gamma+2s}$$

and

$$\begin{aligned} |\eta \cdot \mathbf{k}t + |S(v)\eta|\sqrt{1-t^2}\cos(\theta)|^{2s} &\geq 4^{-2s}|\eta \cdot \mathbf{k}t + |S(v)\eta|^{2s} \\ &\geq c_s (|\eta \cdot \mathbf{k}|^{2s} + |S(v)\eta|^{2s}). \end{aligned}$$

Therefore we have

$$a_{pp}(v, \eta) \gtrsim \int_0^{\pi/3} d\theta \int_0^{\sqrt{3}/2} dt e^{-|v|^2 t^2} \langle v \rangle^{1+\gamma+2s} (|\eta \cdot \mathbf{k}|^{2s} + |S(v)\eta|^{2s}).$$

As in the case of the upper bound, we set $y = |v|t$, and get for $|v| \geq 1$ that

$$\begin{aligned} a_{pp}(v, \eta) &\gtrsim \frac{1}{|v|} \langle v \rangle^{1+\gamma+2s} \int_0^{\pi/3} d\theta \int_0^{\sqrt{3}|v|/2} dy e^{-y^2} \left(|\eta \cdot \mathbf{k}|^{2s} \frac{y^{2s}}{|v|^{2s}} + |S(v)\eta|^{2s} \right) \\ &\gtrsim \frac{1}{|v|} \langle v \rangle^{1+\gamma+2s} \int_0^{\pi/3} d\theta \int_0^{\sqrt{3}/2} dy e^{-y^2} \left(|\eta \cdot \mathbf{k}|^{2s} \frac{y^{2s}}{|v|^{2s}} + |S(v)\eta|^{2s} \right) \\ &\gtrsim \frac{1}{|v|} \langle v \rangle^{1+\gamma+2s} \left(|\eta \cdot \mathbf{k}|^{2s} \frac{1}{|v|^{2s}} + |S(v)\eta|^{2s} \right) \\ &\gtrsim \left(\langle v \rangle^\gamma |\eta|^{2s} + \langle v \rangle^{\gamma+2s} |S(v)\eta|^{2s} \right), \end{aligned}$$

where in the last inequality we use that $\eta \cdot \mathbf{k} \geq 0$ and the fact that if $\eta \cdot \mathbf{k} \leq |\eta|/2$ then

$$|S(v)\eta| \geq \sqrt{3}|\eta|/2.$$

Since $|v \wedge \eta| = |v||S(v)\eta|$ we get for $|v| \geq 1$ the desired result

$$a_{pp}(v, \eta) \gtrsim \langle v \rangle^\gamma (|\eta|^{2s} + |v \wedge \eta|^{2s}). \quad (26)$$

For $|v| \leq 1$, a direct verification, without the change of variables $|v|t \rightarrow y$, gives

$$\begin{aligned} a_{pp}(v, \eta) &\gtrsim \int_0^{\pi/3} d\theta \int_0^{\sqrt{3}/2} dt e^{-t^2} (|\eta \cdot \mathbf{k}t|^{2s} + |S(v)\eta|^{2s}) \gtrsim |\eta \cdot \mathbf{k}|^{2s} + |S(v)\eta|^{2s} \\ &\gtrsim |\eta|^{2s} + |v \wedge \eta|^{2s}. \end{aligned}$$

So the preceding estimate (26) is also true for $|v| \leq 1$.

For the remainder term in (24), we can use similar computations as the ones done for the upper bound for a_p , and we easily get

$$a_{pr} \lesssim \langle v \rangle^{\gamma+2s}.$$

Putting this estimate and (26) together into (24) completes the proof of the lower bound in (13).

Now we deal with estimates on the derivatives in η and v of a_p . Recall that

$$a_p(v, \eta) = \int_{\mathbb{R}_h^3} dh \int_{E_{0,h}} d\alpha \tilde{b} \mathbb{1}_{|\alpha| \geq |h|} \varphi_\delta(h) \mu(\alpha + v) |\alpha + h|^{1+\gamma+2s} (1 - \cos(\eta \cdot h)) \frac{1}{|h|^{3+2s}}$$

which is clearly smooth with respect to v and η . Let us consider for $\nu_1, \nu_2 \in \mathbb{N}^3$ the derivative

$$\begin{aligned} \partial_v^{\nu_1} \partial_\eta^{\nu_2} a_p(v, \eta) &= \int_h dh \int_{E_{0,h}} d\alpha \tilde{b} \mathbb{1}_{|\alpha| \geq |h| \varphi_\delta(h)} (\partial_v^{\nu_1} \mu(\alpha + v)) |\alpha + h|^{1+\gamma+2s} \\ &\quad (\partial_\eta^{\nu_2} (1 - \cos(\eta \cdot h))) \frac{1}{|h|^{3+2s}}. \end{aligned}$$

Setting again $h = r\omega$, and (forgetting the truncation in α) we get

$$|\partial_v^{\nu_1} \partial_\eta^{\nu_2} a_p(v, \eta)| \lesssim \int_0^\delta \int_{S_\omega^2} dr d\omega \int_{E_{0,\omega}} d\alpha |\partial_v^{\nu_1} \mu(\alpha + v)| \langle \alpha \rangle^{1+\gamma+2s} |\partial_\eta^{\nu_2} (1 - \cos(r\eta \cdot \omega))| \frac{1}{r^{1+2s}}. \quad (27)$$

Since $r \in [0, \delta]$ we claim that we have the following rough estimate

$$\mathbf{Lemma 3.2.} \quad \forall \nu_2 \in \mathbb{N}^3, \int_0^\delta dr |\partial_\eta^{\nu_2} (1 - \cos(r\omega \cdot \eta))| \frac{1}{r^{1+2s}} \leq C_{\delta,s} \langle \eta \cdot \omega \rangle^{2s}.$$

Proof of the Lemma. This is clear for $\nu_2 = 0$ from the previous upper bound computation. For $|\nu_2| = 1$ we have to estimate

$$I(\nu_2) = \int_0^\delta dr |(\partial_\eta^{\nu_2} (1 - \cos(r\omega \cdot \eta)))| \frac{1}{r^{1+2s}} \leq \int_0^\delta dr |\sin(r\omega \cdot \eta)| \frac{1}{r^{2s}}.$$

Firstly, when $s < 1/2$, we directly get

$$I(\nu_2) \leq \int_0^\delta dr \frac{1}{r^{2s}} \leq C_{s\delta} \leq C_{s\delta} \langle \eta \cdot \omega \rangle^{2s}.$$

When $s \geq 1/2$ we have

$$I(\nu_2) \leq |\eta \cdot \omega|^{2s-1} \int_0^\infty \frac{\sin(t)}{t^{2s}} dt \leq C_s |\eta \cdot \omega|^{2s-1} \leq C_s \langle \eta \cdot \omega \rangle^{2s}.$$

Eventually considering the case when $|\nu_2| \geq 2$ then

$$I(\nu_2) = \int_0^\delta dr |(\partial_\eta^{\nu_2} (1 - \cos(r\omega \cdot \eta)))| \frac{1}{r^{1+2s}} \leq \int_0^\delta dr |\sin(r\omega \cdot \eta)| \frac{1}{r^{2s-1}} \leq C_{s\delta} \leq C_{s\delta} \langle \eta \cdot \omega \rangle^{2s}.$$

The proof of the lemma is complete. \square

End of the proof of Proposition 3.1 Now we go back to (27). We have also to estimate the term $(\partial_v^{\nu_1} \mu(\alpha + v))$ in this integral. For this purpose, we directly use the fact that for all ν_1 ,

$$|\partial_v^{\nu_1} \mu(\alpha + v)| \leq C_{\nu_1} \mu^{1/2}(\alpha + v). \quad (28)$$

Thanks to Lemma 3.2 and the preceding estimate we get from (27) that

$$|\partial_v^{\nu_1} \partial_\eta^{\nu_2} a_p(v, \eta)| \lesssim \int_\omega d\omega \int_{E_{0,\omega}} d\alpha \mu^{1/2}(\alpha + v) \langle \alpha \rangle^{1+\gamma+2s} \langle \eta \cdot \omega \rangle^{2s}.$$

For the final estimates, we can repeat exactly the proof of the case $\nu_1 = \nu_2 = 0$, to get the desired result. The proof of Proposition (3.1) is complete. \square

For further use we shall also need the following estimate.

Proposition 3.3. *The symbol a_p also satisfies the following estimate:*

$$\begin{aligned} \partial_\eta a_p &\in S\left(\langle v \rangle^{\gamma+1} (1 + |v|^{2s-1} + |\eta|^{2s-1} + |\eta \wedge v|^{2s-1}), \Gamma\right) \text{ if } s \geq 1/2, \\ \text{and } \partial_\eta a_p &\in S\left(\langle v \rangle^{\gamma+2s}, \Gamma\right) \text{ if } s < 1/2. \end{aligned}$$

Proof. We can partially use the preceding arguments. We begin again with (27) and we can write for $|\nu_2| \geq 1$,

$$\begin{aligned} |\partial_v^{\nu_1} \partial_\eta^{\nu_2} a_p(v, \eta)| &\leq C \int_0^\delta \int_{S_\omega^2} dr d\omega \int_{E_{0,\omega}} d\alpha (\partial_v^{\nu_1} \mu(\alpha + v)) | \\ &\qquad \qquad \qquad \langle \alpha \rangle^{1+\gamma+2s} (\partial_\eta^{\nu_2} (1 - \cos(r\eta \cdot \omega))) \frac{1}{r^{1+2s}}. \end{aligned}$$

An inspection of the proof of Lemma 3.2 also shows that we have in fact for $|\nu_2| \geq 1$

$$\int_0^\delta dr |(\partial_\eta^{\nu_2} (1 - \cos(r\omega \cdot \eta)))| \frac{1}{r^{1+2s}} \leq C_{\delta,s} (1 + \langle \eta \cdot \omega \rangle^{2s-1}),$$

and we therefore get using also (28) that

$$|\partial_v^{\nu_1} \partial_\eta^{\nu_2} a_p(v, \eta)| \lesssim \int_{S_\omega^2} d\omega \int_{E_{0,\omega}} d\alpha \mu^{1/2}(\alpha + v) \langle \alpha \rangle^{1+\gamma+2s} (1 + \langle \eta \cdot \omega \rangle^{2s-1}).$$

For the remaining parts of the proof we can proceed exactly as in the case of the upper bound without derivative.

Suppose first that $s \geq 1/2$. Then we can write

$$|\partial_v^{\nu_1} \partial_\eta^{\nu_2} a_p(v, \eta)| \lesssim \int_{S_\omega^2} d\omega \int_{E_{0,\omega}} d\alpha \mu^{1/2}(\alpha + v) \langle \alpha \rangle^{1+\gamma+2s} \langle \eta \cdot \omega \rangle^{2s-1}.$$

We then follow the computations from (17) with $|\omega \cdot \eta|^{2s}$ there replaced here by $\langle \omega \cdot \eta \rangle^{2s-1}$. With the same computations, we then get after (22) that when $|\nu_2| \geq 1$,

$$|\partial_v^{\nu_1} \partial_\eta^{\nu_2} a_p(v, \eta)| \lesssim \langle v \rangle^\gamma (1 + |\eta|^{2s-1} + |v \wedge \eta|^{2s-1}),$$

which is the result.

Suppose now that $s < 1/2$. Then we have

$$|\partial_v^{\nu_1} \partial_\eta^{\nu_2} a_p(v, \eta)| \lesssim \int_{S_\omega^2} d\omega \int_{E_{0,\omega}} d\alpha \mu^{1/2}(\alpha + v) \langle \alpha \rangle^{1+\gamma+2s},$$

and the preceding computations from (17) are valid with $|\omega \cdot \eta|^{2s}$ there replaced here by 1. We therefore get that when $|\nu_2| \geq 1$,

$$|\partial_v^{\nu_1} \partial_\eta^{\nu_2} a_p(v, \eta)| \lesssim \langle v \rangle^{\gamma+2s}$$

which is the result. The proof of Proposition 3.1 is complete. \square

3.2 Study of the multiplicative term $\overline{\mathcal{L}}_{1,\delta,b}$

Recall that the multiplicative part $\overline{\mathcal{L}}_{1,\delta,b}$ has the following form

$$\overline{\mathcal{L}}_{1,\delta,b}f = - \left(\iint dv_* d\sigma B \overline{\varphi}_\delta(v' - v) \mu_* \right) f.$$

A nice feature of the multiplicative function defining $\overline{\mathcal{L}}_{1,\delta,b}$ is its good symbolic properties.

Proposition 3.4. *We can write*

$$\overline{\mathcal{L}}_{1,\delta,b}uf = -a_m(v)f,$$

where a_m is a function in v satisfying the following symbolic estimates:

- i) there exists $C > 0$ such that $C^{-1} \langle v \rangle^{\gamma+2s} \leq a_m(v, \eta) \leq C \langle v \rangle^{\gamma+2s}$;
- ii) $a_m \in S(\langle v \rangle^{\gamma+2s}, \Gamma)$.

Proof. Let us again use Carleman's representation. We get

$$a_m(v) = \int_{\mathbb{R}_h^3} dh \int_{E_{0,h}} d\alpha \tilde{\mathbb{1}}_{|\alpha| \geq |h|} \overline{\varphi}_\delta(h) \mu(v + \alpha - h) |\alpha + h|^{1+\gamma+2s} \frac{1}{|h|^{3+2s}}. \quad (29)$$

In this integral $h \perp \alpha$ and $|\alpha| \geq |h|$ so that there exists C_s such that

$$C_s^{-1} |\alpha|^{1+\gamma+2s} \leq |\alpha + h|^{1+\gamma+2s} \leq C_s |\alpha|^{1+\gamma+2s}. \quad (30)$$

Using this and shifting to spherical coordinates gives

$$a_m(v) \lesssim \iint d\omega dr \int_{E_{0,\omega}} d\alpha \mathbb{1}_{|\alpha| \geq r} \overline{\varphi}_\delta(r^2) \mu(v + \alpha - r\omega) |\alpha|^{1+\gamma+2s} \frac{1}{r^{1+2s}}.$$

Now we know that

$$|v + \alpha - r\omega|^2 = |\alpha + S(\omega)v|^2 + |(\omega \cdot v) - r|^2$$

exactly as in (18) so that

$$e^{-|v+\alpha-r\omega|^2} = e^{-|\alpha+S(\omega)v|^2} e^{-|(\omega \cdot v) - r|^2}.$$

Now we use that

$$\int_{E_{0,\omega}} d\alpha |\alpha|^{1+\gamma+2s} \mu(\alpha + S(\omega)v) \sim \langle S(\omega)v \rangle^{1+\gamma+2s},$$

and we get (forgetting the truncation function in α)

$$a_m(v) \lesssim \iint d\omega dr \overline{\varphi}_\delta(r^2) \langle S(\omega)v \rangle^{1+\gamma+2s} e^{-|(\omega \cdot v) - r|^2} \frac{1}{r^{1+2s}}.$$

We can now integrate w.r.t. r and compute by virtue of Peetre's inequality (forgetting now the dependance on δ for the constants)

$$\begin{aligned} \int dr \overline{\varphi}_\delta(r^2) e^{-|(\omega \cdot v) - r|^2} \frac{1}{r^{1+2s}} &\lesssim \int dr \overline{\varphi}_\delta(r^2) e^{-|(\omega \cdot v) - r|^2} \langle r - \omega \cdot v \rangle^{1+2s} \langle \omega \cdot v \rangle^{-(1+2s)} \\ &\lesssim \langle \omega \cdot v \rangle^{-(1+2s)}, \end{aligned}$$

and thus

$$a_m(v) \lesssim \int_{S_\omega^2} d\omega \langle \omega \cdot v \rangle^{-(1+2s)} \langle S(\omega)v \rangle^{1+\gamma+2s}.$$

We therefore have a similar integral as in (20) and using exactly the same change of polar coordinates and computations as therein with $e^{-|\omega \cdot v|^2}$ replaced by $\langle \omega \cdot v \rangle^{-(1+2s)}$ (see Figure 1), we get

$$a_m(v) \lesssim \int_0^\pi d\varphi \int_0^{2\pi} d\theta \langle |v| \cos \varphi \rangle^{-(1+2s)} \sin \varphi (1 + |v|^2 \sin^2 \varphi)^{(1+\gamma+2s)/2} \lesssim \langle v \rangle^{\gamma+2s}.$$

For the lower bound we can do essentially the same computations : because of the non-negative sign of a_m we can restrict the computations to subdomains in (α, h) namely

$$\{|\alpha| \geq 10\} \quad \text{and} \quad \{|h| \leq 10\},$$

and following (29) and using (30) we get

$$\begin{aligned} a_m(v) &\gtrsim \iint d\omega dr \int_{E_{0,\omega}} d\alpha \mathbb{1}_{|\alpha| \geq 10} \mathbb{1}_{1 \leq r \leq 10} \mu(v + \alpha - r\omega) |\alpha|^{1+\gamma+2s} \frac{1}{r^{1+2s}} \\ &\gtrsim \iint d\omega dr \int_{E_{0,\omega}} d\alpha \mathbb{1}_{|\alpha| \geq 10} \mathbb{1}_{1 \leq r \leq 10} \mu(\alpha + S(\omega)v) e^{-|\omega \cdot v|^2} |\alpha|^{1+\gamma+2s} \frac{1}{r^{1+2s}} \end{aligned}$$

since $\bar{\varphi}_\delta = 1$ in the set $\{1 \leq r \leq 10\}$ (recall $\delta = 1/100$), and

$$|v + \alpha - r\omega|^2 = |S(\omega)v + \alpha|^2 + |\omega \cdot v - r|^2 \leq |S(\omega)v + \alpha|^2 + |\omega \cdot v|^2 + 100$$

for $r \leq 10$. Then as before we can use the fact that

$$\int d\alpha \mathbb{1}_{|\alpha| \geq 10} |\alpha|^{1+\gamma+2s} \mu(\alpha + S(\omega)v) \sim \langle S(\omega)v \rangle^{1+\gamma+2s}$$

and

$$\int dr \mathbb{1}_{1 \leq r \leq 10} \frac{1}{r^{1+2s}} \sim C$$

and we get for a new constant C that

$$a_m(v) \geq C^{-1} \int d\omega \langle S(\omega)v \rangle^{1+\gamma+2s} e^{-|\omega \cdot v|^2},$$

and again we can follow the computations as in (23) and after to get

$$a_m(v) \geq C^{-1} \langle v \rangle^{\gamma+2s}.$$

The proof is thus complete. □

3.3 Proof of Proposition 1.3 i)

In this subsection we prove part i) of Proposition 1.3 concerning the so-called symbol a . We first give its definition, then prove the Proposition, and we shall end this section by giving additional properties of a needed in the sequel.

Definition 3.5. *We define a to be the following real symbol:*

$$a = a_p + a_m,$$

where a_p is defined in Proposition 3.1 and a_m is defined in Proposition 3.4.

We now give the proof of Proposition 1.3 i). From Proposition 3.1 and Proposition 3.4 we know respectively that

$$C^{-1} \langle v \rangle^{\gamma+2s} \leq a_m(v, \eta) \leq C \langle v \rangle^{\gamma+2s}$$

and for all $0 < \kappa \leq 1$,

$$C^{-1} \left(-\kappa \langle v \rangle^{\gamma+2s} + \kappa \langle v \rangle^\gamma (1 + |\eta|^{2s} + |\eta \wedge v|^{2s}) \right) \leq a_p(v, \eta) \leq C \langle v \rangle^\gamma (1 + |\eta|^{2s} + |\eta \wedge v|^{2s}),$$

where in both cases C denotes a constant independent of κ (but depending on δ, s). Choosing κ sufficiently small and fixed from now on, and adding the two inequalities gives

$$C^{-1} \left(\langle v \rangle^{\gamma+2s} + \langle v \rangle^\gamma (|\eta|^{2s} + |\eta \wedge v|^{2s}) \right) \leq a(v, \eta) \leq C \langle v \rangle^{\gamma+2s} + C \langle v \rangle^\gamma (1 + |\eta|^{2s} + |\eta \wedge v|^{2s}).$$

so that

$$C^{-1} \langle v \rangle^\gamma \left(1 + |v|^2 + |\eta|^2 + |\eta \wedge v|^2 \right)^s \leq a(v, \eta) \leq C \langle v \rangle^\gamma \left(1 + |v|^2 + |\eta|^2 + |\eta \wedge v|^2 \right)^s$$

for a new constant C . This proves the lower and upper bounds for a . Using the definition of \tilde{a}

$$\tilde{a}(v, \eta) = \langle v \rangle^\gamma \left(1 + |v|^2 + |\eta|^2 + |\eta \wedge v|^2 \right)^s \quad (31)$$

we get

$$C^{-1} \tilde{a} \leq a \leq C \tilde{a}. \quad (32)$$

From the same Proposition 3.1 and Proposition 3.4, we also directly get by addition that

$$a \in S(\tilde{a}, \Gamma).$$

A direct computation also gives that

$$\tilde{a} \in S(\tilde{a}, \Gamma),$$

and it only remains to check the temperance of a and \tilde{a} . From (32) it is sufficient to verify that there exists N such that for all $Y = (y, \eta)$, $Y' = (y', \eta')$ there exists $C_N > 0$ such that

$$\tilde{a}(Y') \leq C_N \tilde{a}(Y) (1 + \Gamma(Y - Y'))^N.$$

This is a direct consequence of Peetre's inequality since we have powers of polynomial type quantities. The proof is complete. \square

For further use we also give here two propositions concerning a and \tilde{a} , which will be of great interest in the next section.

Proposition 3.6. *We have*

i) for any $|\alpha| \geq 0$ and any $|\beta| \geq 1$, there exists a constant $C_{\alpha,\beta} > 0$ such that

$$\left| \partial_v^\alpha \partial_\eta^\beta a(v, \eta) \right| \leq C_{\alpha,\beta} \begin{cases} \langle v \rangle^{1+\gamma} \left(1 + |v|^2 + |\eta|^2 + |\eta \wedge v|^2 \right)^{s-\frac{1}{2}}, & \frac{1}{2} \leq s \leq 1; \\ \langle v \rangle^{2s+\gamma}, & 0 < s < \frac{1}{2} \end{cases}$$

and

$$\left| \partial_\eta^\beta \tilde{a} \right| \leq C_{\alpha,\beta} \langle v \rangle^{\gamma+1} \left(1 + |v|^2 + |\eta|^2 + |\eta \wedge v|^2 \right)^{s-1/2};$$

ii) the following estimate is true for any $0 < \varepsilon \leq 1$, with semi-norms independent of ε :

$$\partial_\eta \tilde{a}, \partial_\eta a \in S \left(a^{1/2} \langle v \rangle^{\gamma/2+s}, \Gamma \right) \subset S \left(\varepsilon a + \varepsilon^{-1} \langle v \rangle^{2s+\gamma}, \Gamma \right); \quad (33)$$

iii) there exists a constant such that

$$|\xi \cdot \partial_\eta \tilde{a}| \lesssim \langle v \rangle^\gamma \left(1 + |v|^2 + |\eta|^2 + |\eta \wedge v|^2 \right)^{s-\frac{1}{2}} \left(|\xi|^2 + |v \wedge \xi|^2 \right)^{1/2}. \quad (34)$$

Proof. The point i) holds because of Proposition 3.3 and direct verification for \tilde{a} , while the first conclusion in (33) follows from the estimates in i) and the second one is obvious. Point iii) in Proposition 3.6 is a direct consequence of the estimates on \tilde{a} . The proof is complete. \square

3.4 Study of the subprincipal term $\mathcal{L}_{1,1,\delta}$

Proposition 3.7. *We can write*

$$\mathcal{L}_{1,1,\delta} f = -a_s(v, D_v) f,$$

where a_s is a (complex valued) classical symbol in (v, η) satisfying

i) we have

$$a_s(v, \eta) \in \begin{cases} S \left(\langle v \rangle^{\gamma+1} \left(\langle v \rangle^{2s-1} + \langle \eta \rangle^{2s-1} + \langle v \wedge \eta \rangle^{2s-1} \right), \Gamma \right) & \text{if } s \geq 1/2; \\ S \left(\langle v \rangle^{\gamma+2s}, \Gamma \right) & \text{if } s < 1/2; \end{cases} \quad (35)$$

ii) for all $0 < s < 1$ and any $0 < \varepsilon \leq 1$, we have, with semi-norms independent of ε ,

$$a_s(v, \eta) \in S \left(a^{1/2} \langle v \rangle^{\gamma/2+s}, \Gamma \right) \subset S \left(\varepsilon a + \varepsilon^{-1} \langle v \rangle^{\gamma+2s}, \Gamma \right). \quad (36)$$

Proof. We recall that

$$\mathcal{L}_{1,1,\delta} f = \iint dv_* d\sigma B \varphi_\delta (|v' - v|^2) (\mu'_*)^{1/2} [f' - f] \left((\mu_*)^{1/2} - (\mu'_*)^{1/2} \right).$$

We shift to Carleman's representation and get

$$\begin{aligned}\mathcal{L}_{1,1,\delta}f &= \int_{\mathbb{R}_h^3} dh \int_{E_{0,h}} d\alpha \tilde{b} 1_{|\alpha| \geq |h|} |\alpha + h|^{1+\gamma+2s} \varphi_\delta(|h|^2) \mu^{\frac{1}{2}}(\alpha + v) [f(v-h) - f(v)] \\ &\quad \left(\mu^{\frac{1}{2}}(\alpha + v - h) - \mu^{\frac{1}{2}}(\alpha + v) \right) \frac{1}{|h|^{3+2s}} \\ &= - \int_{\mathbb{R}_\eta^3} \hat{f}(\eta) e^{iv \cdot \eta} a_s(v, \eta) d\eta\end{aligned}$$

with

$$\begin{aligned}a_s(v, \eta) &= - \int_{\mathbb{R}_h^3} dh \int_{E_{0,h}} d\alpha \tilde{b} 1_{|\alpha| \geq |h|} |\alpha + h|^{1+\gamma+2s} \varphi_\delta(|h|^2) \mu^{\frac{1}{2}}(\alpha + v) [e^{-ih \cdot \eta} - 1] \\ &\quad \left(\mu^{\frac{1}{2}}(\alpha + v - h) - \mu^{\frac{1}{2}}(\alpha + v) \right) \frac{1}{|h|^{3+2s}}.\end{aligned}$$

For the study of this symbol, we shall essentially follow the same computations as in the $\mathcal{L}_{1,2,\delta}$ case. We first note that we have the following bound for all $h \neq 0$

$$\left| \left(\mu^{\frac{1}{2}}(\alpha + v - h) - \mu^{\frac{1}{2}}(\alpha + v) \right) \frac{1}{|h|} \right| \leq C.$$

So that using also that $|\alpha + h| \leq 2|\alpha|$ we get

$$|a_s(v, \eta)| \lesssim \int_{\mathbb{R}_h^3} dh \int_{E_{0,h}} \langle \alpha \rangle^{1+\gamma+2s} \mu^{\frac{1}{2}}(\alpha + v) \varphi_\delta(|h|^2) \frac{|e^{-ih \cdot \eta} - 1|}{|h|^{2+2s}}.$$

Now we shift to spherical coordinates taking $h = r\omega$ and we get

$$|a_s(v, \eta)| \lesssim \iint_{\omega} d\omega dr \int_{E_{0,\omega}} \langle \alpha \rangle^{1+\gamma+2s} \mu^{\frac{1}{2}}(\alpha + v) \varphi_\delta(r^2) \frac{|e^{-ir\omega \cdot \eta} - 1|}{r^{2s}}.$$

We can directly integrate w.r.t. r and this gives

$$\int_0^\infty \varphi_\delta(r^2) \frac{|e^{-ir\omega \cdot \eta} - 1|}{r^{2s}} \lesssim \begin{cases} C_s |\eta \cdot \omega|^{2s-1}, & \text{if } s \geq 1/2; \\ C_s, & \text{if } 0 < s < 1/2, \end{cases}$$

which yields

$$|a_s(v, \eta)| \lesssim \begin{cases} \int_{S_\omega^2} d\omega dr \int_{E_{0,\omega}} \langle \alpha \rangle^{1+\gamma+2s} \mu^{\frac{1}{2}}(\alpha + v) |\eta \cdot \omega|^{2s-1} & \text{if } s \geq 1/2; \\ \int_{S_\omega^2} d\omega dr \int_{E_{0,\omega}} \langle \alpha \rangle^{1+\gamma+2s} \mu^{\frac{1}{2}}(\alpha + v) & \text{if } 0 < s < 1/2. \end{cases} \quad (37)$$

Suppose now that $s \geq 1/2$.

In this case we can do exactly the same computations as in the $\mathcal{L}_{1,2,\delta}$ case, with the factors $\mu(\alpha + v)$ and $|\eta \cdot \omega|^{2s}$ in formula (17) replaced respectively by $\mu^{1/2}(\alpha + v)$ and $|\eta \cdot \omega|^{2s-1}$ here. We directly get, following the computations after (17), that

$$|a_s(v, \eta)| \lesssim \langle v \rangle^{\gamma+1} (|\eta|^{2s-1} + |v \wedge \eta|^{2s-1})$$

for $|v| > 1$. For $|v| \leq 1$, a rough estimate gives directly $|a(v, \eta)| \leq \langle \eta \rangle^{2s}$ so that the preceding estimate is also true in this case.

Suppose now that $s \leq 1/2$.

We can follow the same type of computations as after (17), and we get easily

$$|a_s(v, \eta)| \lesssim \langle v \rangle^{\gamma+2s}. \quad (38)$$

Again the proof of the estimates for higher order derivatives of a_s is similar to the one of order 0, and we skip this part of the proof. This completes part i) of proposition 3.7. The second part ii) is just an immediate consequence of the estimates in i). The proof of Proposition 3.7 is complete. \square

4 Proof of the main results

This section is devoted to the proof of the main Theorems 1.1 and 1.2. We shall use extensively properties of the classical, Weyl and Wick quantization, for which we postpone a brief review in the Appendix. In Section 5.2, we give some coercivity estimates, and recover a result of coercivity of [7] implying the so-called triple norm. In 4.1 we make the reduction to the hypoelliptic problems for a simplified operator, by virtue of Proposition 1.3 whose proof is also presented in this subsection. In Subsection 4.2 we give a direct proof of the well-known coercivity. The proof of the main results is then achieved in the last subsection, Subsection 4.3.

4.1 Proof of Proposition 1.3 ii) and iii) and related results

In the previous sections, we splitted operator \mathcal{L} into several pieces in the following way, with $a = a_p + a_m$ defined in Proposition 3.1 and Proposition 3.4, and a_s defined in Proposition 3.7,

$$\begin{aligned} \mathcal{L} &= -a(v, D_v) + \mathcal{L}_2 + \overline{\mathcal{L}}_{1,\delta,a} + \mathcal{L}_{1,3,\delta} + \mathcal{L}_{1,4,\delta} + \overline{\mathcal{L}}_{1,\delta,a} - a_s(v, D_v) \\ &= -a^w - \underbrace{(-\mathcal{L}_2 - \overline{\mathcal{L}}_{1,\delta,a} - \mathcal{L}_{1,3,\delta} - \mathcal{L}_{1,4,\delta} + a_s(v, D_v) + (a(v, D_v) - a^w))}_{\mathcal{K}}, \end{aligned}$$

so that we can write

$$P = v \cdot \partial_x + a^w + \mathcal{K}.$$

Notice that the diffusion term $a^w + \mathcal{K}$ above is only an operator with respect to the velocity variable v . So we only work on the resulting operator after performing partial Fourier transform in the x variables, considering the dual variables ξ of x as parameter. More precisely we will study the operator

$$\hat{P}_K = i(v \cdot \xi) + a_K^w,$$

where

$$a_K = a + K \langle v \rangle^{2s+\gamma}.$$

with K a fixed number, constructed in Lemma 4.2 and Lemma 4.8 below, depending only on the integer N in (77). Accordingly we also introduce the weight function

$$\tilde{a}_K = \tilde{a} + K \langle v \rangle^{2s+\gamma},$$

where \tilde{a} is the weight function given in Proposition 1.3. We note that $a_K \in S(\tilde{a}_K, \Gamma)$ uniformly in K , and more generally $(a_K)^r \in S(\tilde{a}_K^r, \Gamma)$ uniformly w.r.t. K for all $r \in [0, 1]$. The advantage of working on a_K^w instead of a^w is that we can construct the inverse of the former; see Lemma 4.2 below. This is of big importance in the following, in the treatment of hypoelliptic estimates.

Notations. In what follows let K be fixed, satisfying the assumptions in Lemma 4.2 and Lemma 4.8 below, and let $\ell \in \mathbb{R}$ be an arbitrary number, fixed and as small as we want. To simplify the notation, by $A \lesssim B$ we mean there exists a positive constant C , which may depend on K and ℓ but is *independent of the parameters* ξ , such that $A \leq CB$, and similarly for $A \gtrsim B$. While the notation $A \approx B$ means both $A \lesssim B$ and $B \lesssim A$ hold. Given a symbol q and a weight function M , by $q \in S(M, \Gamma)$ we always mean, in the following discussions, q lies in $S(M, \Gamma)$ *uniformly w.r.t. K and ξ* .

Now we state the main result of this subsection, which shows it is sufficient to study the operator \hat{P}_K instead of the original one.

Proposition 4.1. *The conclusion in Theorem 1.1 follows if the estimate*

$$\|\tilde{a}(v, \xi)^{\frac{1}{1+2s}} f\| + \|a_K^w f\| \lesssim \|\hat{P}_K f\|_{L^2} + \|f\|_{L^2_\xi} \quad (39)$$

holds uniformly with respect to ξ .

We proceed to prove the above proposition through several lemmas. Firstly we begin with constructing the inverses of operators.

Lemma 4.2. *There exists a K_0 large sufficiently, which depends only on the dimension, such that for all $K \geq K_0$ we have*

(i) a_K^w is invertible and its inverse $(a_K^w)^{-1}$ has the form

$$(a_K^w)^{-1} = H_1 (a_K^{-1})^w = (a_K^{-1})^w H_2,$$

with H_1, H_2 belonging to $\mathcal{B}(L^2)$, the space of bounded operators on L^2 , and $\|H_j\|_{\mathcal{B}(L^2)}$ independent of K for $j = 1, 2$;

(ii) $(a_K^{1/2})^w$ is invertible and its inverse $[(a_K^{1/2})^w]^{-1}$ has the form

$$[(a_K^{1/2})^w]^{-1} = G_1 (a_K^{-1/2})^w = (a_K^{-1/2})^w G_2$$

with $G_1, G_2 \in \mathcal{B}(L^2)$ and $\|G_j\|_{\mathcal{B}(L^2)}$ independent of K for $j = 1, 2$;

(iii) $(\tilde{a}_K^{1/2} a_K^{1/2})^w$ is invertible and its inverse $[(\tilde{a}_K^{1/2} a_K^{1/2})^w]^{-1}$ has the form

$$[(\tilde{a}_K^{1/2} a_K^{1/2})^w]^{-1} = Q_1 (\tilde{a}_K^{-1/2} a_K^{-1/2})^w = (\tilde{a}_K^{-1/2} a_K^{-1/2})^w Q_2$$

with $Q_1, Q_2 \in \mathcal{B}(L^2)$ and $\|Q_j\|_{\mathcal{B}(L^2)}$ independent of K for $j = 1, 2$.

Proof. We firstly prove the conclusion in (i). Using (78) and (79), we may write

$$a_K^w (a_K^{-1})^w = \text{Id} - R_K^w, \quad (40)$$

where

$$R_K = - \int_0^1 (\partial_\eta a_K) \#_\theta (\partial_v(a_K^{-1})) d\theta + \int_0^1 (\partial_v a_K) \#_\theta (\partial_\eta(a_K^{-1})) d\theta$$

with $g \#_\theta h$ defined by

$$g \#_\theta h(Y) = \iint e^{-2i\sigma(Y-Y_1, Y-Y_2)/\theta} \frac{1}{2i} g(Y_1) h(Y_2) dY_1 dY_2 / (\pi\theta)^6. \quad (41)$$

In what follows let N be the integer which is given in (77) and depends only on the dimension. By [12, Proposition 1.1] we can find a constant C_N and a positive integer ℓ_N , both depending only on N but independent of K and θ , such that

$$\| (\partial_\eta a_K) \#_\theta (\partial_v(a_K^{-1})) \|_{N;S(1,\Gamma)} \leq C_N \| \partial_\eta a_K \|_{\ell_N;S(\bar{a}_K,\Gamma)} \| (\partial_v(a_K^{-1})) \|_{\ell_N;S(\bar{a}_K^{-1},\Gamma)}.$$

Moreover, using (33) for $\varepsilon = K^{-1/4}$ yields

$$\| \partial_\eta a_K \|_{\ell_N;S(\bar{a}_K,\Gamma)} \leq \tilde{C}_N K^{-\frac{1}{4}}$$

and

$$\| \partial_v(a_K^{-1}) \|_{\ell_N;S(\bar{a}_K^{-1},\Gamma)} \leq \tilde{C}_N$$

with \tilde{C}_N a constant depending only on N but independent of K . As a result,

$$\| (\partial_\eta a_K) \#_\theta (\partial_v(a_K^{-1})) \|_{N;S(1,\Gamma)} \leq C_N \tilde{C}_N^2 K^{-\frac{1}{4}}.$$

Similarly,

$$\| (\partial_v a_K) \#_\theta (\partial_\eta(a_K^{-1})) \|_{N;S(1,\Gamma)} \leq C_N \tilde{C}_N^2 K^{-\frac{1}{4}}.$$

Then

$$\| R_K \|_{N;S(1,\Gamma)} \leq 2C_N \tilde{C}_N^2 K^{-\frac{1}{4}},$$

and thus by (77)

$$\| R_K^w \|_{\mathcal{B}(L^2)} \leq 2CC_N \tilde{C}_N^2 K^{-\frac{1}{4}}$$

with C a constant depending only on the dimension. This implies $\text{Id} - R_K^w$ is invertible in the space $\mathcal{B}(L^2)$ of bounded operators on L^2 if we choose K in such a way that $K \geq (4CC_N \tilde{C}_N^2)^4$. Moreover

$$(\text{Id} - R_K^w)^{-1} = \sum_{j=0}^{\infty} (R_K^w)^j \in \mathcal{B}(L^2).$$

Taking into account (40), we conclude

$$a_K^w \left((a_K^{-1})^w (\text{Id} - R_K^w)^{-1} \right) = \text{Id}.$$

Similarly we can find a $\tilde{R}_K \in S(1, \Gamma)$ such that

$$\left(\left(\text{Id} - \tilde{R}_K^w \right)^{-1} (a_K^{-1})^w \right) a_K^w = \text{Id}.$$

These imply a_K^w is invertible and its inverse $(a_K^w)^{-1}$ has the form

$$(a_K^w)^{-1} = (a_K^{-1})^w (\text{Id} - \tilde{R}_K^w)^{-1} = \left(\text{Id} - \tilde{R}_K^w \right)^{-1} (a_K^{-1})^w.$$

We have proven the conclusion in (i) in Lemma 4.2. The remaining ones in (ii) and (iii) can be deduced quite similarly. So we omit it here for brevity. The proof of Lemma 4.2 is thus complete. \square

In what follows we always let K be fixed satisfying the condition in the above lemma 4.2.

Corollary 4.3. *Let ε be an arbitrarily small number and let $g \in S\left(\varepsilon a + \varepsilon^{-1} \langle v \rangle^{2s+\gamma}, \Gamma\right)$ uniformly with respect to ε . Then*

$$\|g(v, D_v)f\|_{L^2} + \|g^w f\|_{L^2} \lesssim \varepsilon \|a_K^w f\| + \varepsilon^{-1} \|\langle v \rangle^{2s+\gamma} f\|_{L^2}.$$

Proof. This is just a consequence of the conclusion (i) in Lemma 4.2. In fact by Lemma 4.2 we see the operator $a_{K+\varepsilon^{-1-2s}} = a_K + \varepsilon^{-2} \langle v \rangle^{2s+\gamma}$ is invertible and its inverse satisfies that

$$\forall h \in S(a_{K+\varepsilon^{-2}}, \Gamma), \quad h^w \left(a_K + \varepsilon^{-2} \langle v \rangle^{2s+\gamma} \right)^{-1} \in \mathcal{B}(L^2).$$

Thus the assumption on g allows us write

$$g^w = \underbrace{(\varepsilon^{-1}g)^w \left(a_K + \varepsilon^{-2} \langle v \rangle^{2s+\gamma} \right)^{-1}}_{\in \mathcal{B}(L^2)} \varepsilon \left(a_K + \varepsilon^{-2} \langle v \rangle^{2s+\gamma} \right),$$

which yields the desired estimate for g^w . The estimate for $g(v, D_v)$ is similar, since $g(v, D_v) = (J^{-1/2}g)^w$ with $J^{-1/2}g$ belonging to the same symbol class as g . We have obtained the desired estimate in Corollary 4.3. The proof is complete. \square

We will apply the preceding lemma to precise pseudodifferential operators:

Lemma 4.4. *The symbols of $a_s(v, D_v)$ and $a^w - a(v, D_v)$ lie in $S\left(\varepsilon a + \varepsilon^{-1} \langle v \rangle^{2s+\gamma}, \Gamma\right)$ for all $\varepsilon > 0$ with seminorms independent of ε .*

Proof. For the first operator $a_s(v, D_v)$, this is point ii) of Proposition 3.7. For the second one $a^w - a(v, D_v)$, we enter more deeply into the theory of Weyl and classical quantization. In order to get the result, we use the expansion of $J^{1/2}a$, which reads (c.f. [25, Lemma 4.1.5] and the appendix)

$$a^w - a(v, D_v) = \left(J^{1/2}a \right) (v, D_v) - a(v, D_v) = R(v, D_v)$$

with

$$R(v, \eta) = \frac{1}{2} \int \left(J^{\theta/2} (D_\eta \cdot \partial_v a) \right) (v, \eta) d\theta.$$

Proposition 3.6 implies $D_\eta \cdot \partial_v a \in S(M_\varepsilon, \Gamma)$ uniformly with respect to ε , where

$$M_\varepsilon = \varepsilon \tilde{a} + \varepsilon^{-1} \langle v \rangle^{2s+\gamma}.$$

Then proceeding as in the proof of [25, Lemma 4.1.2], we conclude $J^{\theta/2}(D_\eta \cdot \partial_v a)$ belongs to the same symbol class $S(M_\varepsilon, \Gamma)$ as $D_\eta \cdot \partial_v a$, due to the fact that

$$M_\varepsilon(v+z, \eta+\zeta) \leq CM_\varepsilon(v, \eta)H(\langle z \rangle, \langle \zeta \rangle)$$

with $H(\langle z \rangle, \langle \zeta \rangle)$ being some polynomial of $\langle z \rangle, \langle \zeta \rangle$ and C a constant independent of ε . Observe $\tilde{a} \lesssim a_K$ due to Proposition 3.1. Then we have proven that the classical symbol of the difference $a(v, D_v) - a^w$ lies in $S(\varepsilon a + \varepsilon^{-1} \langle v \rangle^{2s+\gamma}, \Gamma)$. The Weyl symbol therefore also belongs to this class by direct transformation. The proof is complete.

Proposition 4.5. *For any ε there exists a constant C_ε such that*

$$\forall f \in \mathcal{S}(\mathbb{R}_v^n), \quad \|\langle v \rangle^{2s+\gamma} f\|_{L^2} \leq \varepsilon \|a_K^w f\|_{L^2} + C_\varepsilon \left(\|(iv \cdot \xi - \mathcal{L})f\|_{L^2} + \|f\|_{L_t^2} \right). \quad (42)$$

Proof. The starting point is the weight estimate

$$\forall f \in \mathcal{S}(\mathbb{R}^6), \quad \|\langle v \rangle^{s+\frac{\gamma}{2}} f\|_{L^2}^2 \lesssim \operatorname{Re}((iv \cdot \xi - \mathcal{L})f, f)_{L^2} + \|\langle v \rangle^{\ell-s-\gamma/2} f\|_{L^2}^2, \quad (43)$$

which is just an immediate consequence of the coercivity estimate established in [7]. Thanks to the explicit symbolic estimates for a^w , we are able to give here a direct proof of such a coercivity estimate. Indeed, let

$$\mathcal{L} = \mathcal{L}_2 + \mathcal{L}_{1,\delta} + \overline{\mathcal{L}}_{1,\delta,a} + \overline{\mathcal{L}}_{1,\delta,b}$$

be the splitting given in Section 2.2. Straightforward computation shows that

$$(-\mathcal{L}_{1,\delta} f, f)_{L^2} = \frac{1}{2} \iint dv_* d\sigma B \varphi_\delta(v' - v) \left((\mu'_*)^{1/2} f' - (\mu_*)^{1/2} f \right)^2 \geq 0,$$

and

$$(-\overline{\mathcal{L}}_{1,\delta,b} f, f)_{L^2} \gtrsim \|\langle v \rangle^{\gamma/2+s} f\|_{L^2}^2$$

because of Proposition 3.4, and

$$|(\mathcal{L}_{2,\delta} f, f)_{L^2}| + |(\overline{\mathcal{L}}_{1,\delta,a} f, f)_{L^2}| \lesssim \|\langle v \rangle^{\gamma/2} f\|_{L^2}^2$$

due to Lemma 2.1 and Lemma 2.3. These estimates, together with the fact that $s > 0$, yield

$$\|\langle v \rangle^{\gamma/2+s} f\|_{L^2}^2 \lesssim (-\mathcal{L} f, f)_{L^2} + \varepsilon \|\langle v \rangle^{\gamma/2+s} f\|_{L^2}^2 + C_{\varepsilon, \tilde{\ell}} \|\langle v \rangle^{\tilde{\ell}} f\|_{L^2}^2$$

for any $\varepsilon > 0$ and any $\tilde{\ell} \in \mathbb{R}$. This gives the estimate (43).

Now applying the estimate (43) to the function $\langle v \rangle^{s+\frac{\gamma}{2}} f$ yields

$$\begin{aligned} & \|\langle v \rangle^{2s+\gamma} f\|_{L^2}^2 \lesssim \operatorname{Re} \left((iv \cdot \xi - \mathcal{L}) \langle v \rangle^{s+\frac{\gamma}{2}} f, \langle v \rangle^{s+\frac{\gamma}{2}} f \right)_{L^2} + \|f\|_{L_t^2}^2 \\ & \lesssim \left| \left((iv \cdot \xi - \mathcal{L}) f, \langle v \rangle^{2s+\gamma} f \right)_{L^2} \right| + \left| \left(\langle v \rangle^{s+\frac{\gamma}{2}} [\mathcal{L}, \langle v \rangle^{s+\frac{\gamma}{2}}] f, f \right)_{L^2} \right| + \|f\|_{L_t^2}^2. \end{aligned}$$

This yields

$$\|\langle v \rangle^{2s+\gamma} f\|_{L^2}^2 \lesssim \|(iv \cdot \xi - \mathcal{L})f\|_{L^2}^2 + \|f\|_{L_\xi^2}^2 + \left| \left(\langle v \rangle^{s+\frac{\gamma}{2}} [\mathcal{L}, \langle v \rangle^{s+\frac{\gamma}{2}}] f, f \right)_{L^2} \right|. \quad (44)$$

We have to treat the last term in the above estimate, which is bounded from above by

$$\left| \left(\langle v \rangle^{s+\frac{\gamma}{2}} [a^w, \langle v \rangle^{s+\frac{\gamma}{2}}] f, f \right)_{L^2} \right| + \left| \left(\langle v \rangle^{s+\frac{\gamma}{2}} [\mathcal{K}, \langle v \rangle^{s+\frac{\gamma}{2}}] f, f \right)_{L^2} \right|. \quad (45)$$

We apply (33) and [25, Theorem 2.3.19] to conclude that for any $\varepsilon \in]0, 1[$ the symbol of the operator

$$\langle v \rangle^{-(2s+\gamma-1)} \langle v \rangle^{s+\frac{\gamma}{2}} [a^w, \langle v \rangle^{s+\frac{\gamma}{2}}]$$

belongs to

$$S\left(\varepsilon a_K + \varepsilon^{-1} \langle v \rangle^{2s+\gamma}, \Gamma\right)$$

uniformly with respect to ε . Then Corollary 4.3 gives, with $\tilde{\varepsilon}$ arbitrarily small,

$$\begin{aligned} \left| \left(\langle v \rangle^{s+\frac{\gamma}{2}} [a^w, \langle v \rangle^{s+\frac{\gamma}{2}}] f, f \right)_{L^2} \right| &\lesssim \left(\varepsilon \|a_K^w f\|_{L^2} + \varepsilon^{-1} \|\langle v \rangle^{2s+\gamma} f\|_{L^2} \right) \|\langle v \rangle^{2s+\gamma-1} f\|_{L^2} \\ &\lesssim \varepsilon \|a_K^w f\|_{L^2}^2 + \tilde{\varepsilon} \|\langle v \rangle^{2s+\gamma} f\|_{L^2}^2 + C_{\varepsilon, \tilde{\varepsilon}} \|f\|_{L_\xi^2}^2, \end{aligned}$$

where in the last inequality we used the interpolation inequality:

$$\|\langle v \rangle^{2s+\gamma-1} f\|_{L^2} \leq \tilde{\varepsilon} \|\langle v \rangle^{2s+\gamma} f\|_{L^2} + C_{\tilde{\varepsilon}} \|f\|_{L_\xi^2}.$$

Now we have to deal with operator

$$\langle v \rangle^{s+\frac{\gamma}{2}} [\mathcal{K}, \langle v \rangle^{s+\frac{\gamma}{2}}]$$

in (45). For this we split \mathcal{K} into three parts :

$$\mathcal{K} = \underbrace{-\mathcal{L}_2 - \overline{\mathcal{L}}_{1,\delta,a}}_{\mathcal{K}_{small}} - \underbrace{\mathcal{L}_{1,3,\delta} - \mathcal{L}_{1,4,\delta}}_{\mathcal{K}_{mult}} + \underbrace{a_s(v, D_v) + (a(v, D_v) - a^w)}_{\mathcal{K}_{pseudo}}. \quad (46)$$

For the second part \mathcal{K}_{mult} , the estimate is easy since, as recalled in lemma 2.2 and 2.4, operators $\mathcal{L}_{1,3,\delta}$ and $\mathcal{L}_{1,4,\delta}$ commute with the multiplication with a function of v . We therefore have

$$\left| \left(\langle v \rangle^{s+\frac{\gamma}{2}} [\mathcal{K}_{mult}, \langle v \rangle^{s+\frac{\gamma}{2}}] f, f \right)_{L^2} \right| = 0.$$

For the first part \mathcal{K}_{small} of \mathcal{K} in (46), we develop the commutators and use Cauchy Schwartz inequality to get

$$\begin{aligned} &\left| \left(\langle v \rangle^{s+\frac{\gamma}{2}} [\mathcal{K}_{small}, \langle v \rangle^{s+\frac{\gamma}{2}}] f, f \right)_{L^2} \right| \\ &\lesssim C_\varepsilon \|\langle v \rangle^{-s-\frac{\gamma}{2}} \mathcal{L}_2 \langle v \rangle^{s-\frac{\gamma}{2}} \langle v \rangle^\gamma f\|^2 + C_\varepsilon \|\mathcal{L}_2 \langle v \rangle^{-\gamma} \langle v \rangle^\gamma f\|^2 \\ &\quad + C_\varepsilon \|\langle v \rangle^{-s-\frac{\gamma}{2}} \overline{\mathcal{L}}_{1,\delta,a} \langle v \rangle^{s-\frac{\gamma}{2}} \langle v \rangle^\gamma f\|^2 + C_\varepsilon \|\overline{\mathcal{L}}_{1,\delta,a} \langle v \rangle^{-\gamma} \langle v \rangle^\gamma f\|^2 \\ &\quad + \varepsilon \|\langle v \rangle^{2s+\gamma} f\|^2. \end{aligned}$$

We use then Lemmas 2.3 and 2.1 with either $\alpha = -s - \gamma/2$, $\beta = s - \gamma/2$ or $\alpha = 0$, $\beta = -\gamma$ (for which we have in both cases $\alpha + \beta + \gamma \leq 0$) and we get

$$\begin{aligned} \left| \left(\langle v \rangle^{s+\frac{\gamma}{2}} [\mathcal{K}_{small}, \langle v \rangle^{s+\frac{\gamma}{2}}] f, f \right)_{L^2} \right| &\lesssim \tilde{C}_\varepsilon \|\langle v \rangle^\gamma f\|^2 + \varepsilon \|\langle v \rangle^{2s+\gamma} f\|^2 \\ &\lesssim \tilde{C}_\varepsilon \|f\|_{L^2_\ell}^2 + 2\varepsilon \|\langle v \rangle^{2s+\gamma} f\|^2 \end{aligned}$$

since $s > 0$.

Next we deal with the last part \mathcal{K}_{pseudo} of \mathcal{K} in (46). From Lemma 4.4, we already know that \mathcal{K}_{pseudo} belongs to

$$S \left(\varepsilon a + \varepsilon^{-1} \langle v \rangle^{2s+\gamma}, \Gamma \right)$$

with uniform semi-norms with respect to ε . We follow the same strategy that after (45) for commutators involving a^w . Using that $\partial_v \langle v \rangle^\mu = \mathcal{O}(\langle v \rangle^{\mu-1})$ for all $\mu \in \mathbb{R}$, and applying [25, Theorem 2.3.19] (see also appendix), we get that for any $\varepsilon \in]0, 1[$ the symbol of the operator

$$\langle v \rangle^{-(2s+\gamma-1)} \langle v \rangle^{s+\frac{\gamma}{2}} [\mathcal{K}_{pseudo}, \langle v \rangle^{s+\frac{\gamma}{2}}]$$

belongs to

$$S \left(\varepsilon a_K + \varepsilon^{-1} \langle v \rangle^{2s+\gamma}, \Gamma \right)$$

uniformly with respect to ε . Then Corollary 4.3 gives, with $\tilde{\varepsilon}$ arbitrarily small,

$$\begin{aligned} \left| \left(\langle v \rangle^{s+\frac{\gamma}{2}} [\mathcal{K}_{pseudo}, \langle v \rangle^{s+\frac{\gamma}{2}}] f, f \right)_{L^2} \right| &\lesssim \left(\varepsilon \|a_K^w f\|_{L^2} + \varepsilon^{-1} \|\langle v \rangle^{2s+\gamma} f\|_{L^2} \right) \|\langle v \rangle^{2s+\gamma-1} f\|_{L^2} \\ &\lesssim \varepsilon \|a_K^w f\|_{L^2}^2 + \tilde{\varepsilon} \|\langle v \rangle^{2s+\gamma} f\|_{L^2}^2 + C_{\varepsilon, \tilde{\varepsilon}} \|f\|_{L^2_\ell}^2. \end{aligned}$$

Combining these estimates we obtain

$$\left| \left(\langle v \rangle^{s+\frac{\gamma}{2}} [\mathcal{K}, \langle v \rangle^{s+\frac{\gamma}{2}}] f, f \right)_{L^2} \right| \lesssim \varepsilon \|a_K^w f\|_{L^2}^2 + \tilde{\varepsilon} \|\langle v \rangle^{2s+\gamma} f\|_{L^2}^2 + C_{\varepsilon, \tilde{\varepsilon}} \|f\|_{L^2_\ell}^2.$$

Now taking into account (44), the desired estimate (42) follows if we choose $\tilde{\varepsilon}$ small enough. The proof is thus complete. \square

In order to prove the main result, Proposition 4.1, we will need the conclusion in Proposition 1.3. So let us firstly present the latter's proof.

Proof of Proposition 1.3 ii) and iii). We have shown Proposition 1.3 iii) in Lemma 4.2. For the conclusion ii), let us rewrite the linearized Boltzmann operator \mathcal{L} as

$$\mathcal{L} = -a^w + \underbrace{\mathcal{L}_2 + \overline{\mathcal{L}}_{1,\delta,a} + \mathcal{L}_{1,3,\delta} + \mathcal{L}_{1,4,\delta} - a_s(v, D_v) - (a(v, D_v) - a^w)}_{-\mathcal{K}}.$$

As a direct consequence of Lemma 2.1, Lemma 2.3, Lemma 2.2, Lemma 2.4 we have

$$\| (\mathcal{L}_2 + \overline{\mathcal{L}}_{1,\delta,a} + \mathcal{L}_{1,3,\delta} + \mathcal{L}_{1,4,\delta}) f \|_{L^2} \lesssim \|\langle v \rangle^{2s+\gamma} f\|_{L^2}.$$

Moreover from Lemma 4.4 we know that for any $\varepsilon > 0$,

$$\mathcal{K}_{pseudo} \equiv -a_s(v, D_v) - (a^w - a(v, D_v)) \in Op_{weyl} \left(\varepsilon a + \varepsilon^{-1} \langle v \rangle^{2s+\gamma}, \Gamma \right)$$

uniformly with respect to ε , and thus

$$\|\mathcal{K}_{pseudo}f\|_{L^2} \lesssim \varepsilon \|a_K^w f\| + \varepsilon^{-1} \|\langle v \rangle^{2s+\gamma} f\|_{L^2}$$

due to Corollary 4.3.

The proof of point ii) of Proposition 1.3 is complete. \square

The rest of this subsection is occupied by the

Proof of Proposition 4.1. Now supposing (39) holds, we have

$$\|\tilde{a}(v, \xi)^{\frac{1}{1+2s}} f\| + \|a_K^w f\| \lesssim \|(iv \cdot \xi - \mathcal{L})f\|_{L^2} + \|f\|_{L^2_\xi} + \|(iv \cdot \xi - \mathcal{L} - \hat{P}_K)f\|_{L^2}.$$

On the other hand, note that

$$iv \cdot \xi - \mathcal{L} - \hat{P}_K = a^w + \mathcal{K} - (a + K \langle v \rangle^{2s+\gamma})^w = \mathcal{K} - K \langle v \rangle^{2s+\gamma},$$

and thus Proposition 1.3 yields, with ε arbitrarily small,

$$\begin{aligned} \|(iv \cdot \xi - \mathcal{L} - \hat{P}_K)f\|_{L^2} &\lesssim \varepsilon \|a_K^w f\|_{L^2} + C_\varepsilon \|\langle v \rangle^{2s+\gamma} f\|_{L^2} \\ &\lesssim \varepsilon \|a_K^w f\| + C_\varepsilon \left(\|(iv \cdot \xi - \mathcal{L})f\|_{L^2} + \|f\|_{L^2_\xi} \right), \end{aligned}$$

the last inequality following from (42). Combining these inequalities and letting the above ε be small sufficiently, we get

$$\|\tilde{a}(v, \xi)^{\frac{1}{1+2s}} f\| + \|a_K^w f\| \lesssim \|(iv \cdot \xi - \mathcal{L})f\|_{L^2} + \|f\|_{L^2_\xi}.$$

Taking into account the relation that

$$\langle v \rangle^{\gamma/(2s+1)} \langle \xi \rangle^{2s/(2s+1)} + \langle v \rangle^{\gamma/(2s+1)} \langle v \wedge \xi \rangle^{2s/(2s+1)} \approx \tilde{a}(v, \xi)^{2s/(2s+1)}$$

and that

$$\|\langle v \rangle^\gamma \langle D_v \rangle^{2s} f\|_{L^2} + \|\langle v \rangle^\gamma \langle v \wedge D_v \rangle^{2s} f\|_{L^2} + \|\langle v \rangle^{2s+\gamma} f\|_{L^2} \lesssim \|a_K^w\|_{L^2}$$

due to the conclusion (i) in Lemma 4.2, we obtain the desired estimate in Theorem 1.1. The proof of Proposition 4.1 is complete. \square

4.2 Coercivity and boundedness estimates

In this section we prove the following result that can be understood as an exact estimate for the so called triple norm introduced in [7] and recalled in Remark 4.7 below. It involves the pseudodifferential part a^w , for which we have elliptic properties stated in Proposition 1.3.

Lemma 4.6. *We have for a sufficiently large constant C and for all $l \in \mathbb{R}$ with $l \leq \gamma/2 + s$,*

$$\begin{aligned} &\|\langle v \rangle^{\gamma/2} \langle D_v \rangle^s f\|^2 + \|\langle v \rangle^{\gamma/2} \langle v \wedge D_v \rangle^s f\|^2 + \|\langle v \rangle^{\gamma/2+s} f\|^2 \\ &\quad \sim (a^w f, f) + C \|\langle v \rangle^{\gamma/2+s} f\|^2 \sim -(\mathcal{L}f, f) + \|\langle v \rangle^l f\|^2, \end{aligned}$$

where in the last equivalence the constant depends on l .

Proof. The second equivalence is a consequence of Proposition 1.3 and estimate (43), while the first one holds because of the coercivity estimate (47) below, since

$$\| (a_K^{1/2})^w f \|_{L^2}^2 \sim \| \langle v \rangle^{\gamma/2} \langle D_v \rangle^s f \|^2 + \| \langle v \rangle^{\gamma/2} \langle v \wedge D_v \rangle^s f \|^2 + \| \langle v \rangle^{\gamma/2+s} f \|^2$$

due to the conclusion (ii) in Lemma 4.2. \square

Remark 4.7 In [7], the authors introduced the following non-isotropic norm

$$\|f\|^2 \stackrel{\text{def}}{=} \iint \Phi(|v - v_*|) b(\cos \theta) \mu_* (f - f')^2 + \iiint \Phi(|v - v_*|) b(\cos \theta) f_*^2 \left(\sqrt{\mu'} - \sqrt{\mu} \right)^2,$$

where the integration is over $\mathbb{R}_v^3 \times \mathbb{R}_{v_*}^3 \times \mathbb{S}_\sigma^2$. For such a norm Theorem 1.1 of [7]) says, with $l \in \mathbb{R}$ arbitrary (and equivalence norm depending on l),

$$\| \langle v \rangle^{\gamma/2} \langle D_v \rangle^s f \|^2 + \| \langle v \rangle^{\gamma/2+s} f \|^2 \lesssim \|f\|^2 \lesssim -(\mathcal{L}f, f) + C_2 \| \langle v \rangle^l f \|^2,$$

provided the Boltzmann cross-section B satisfies (3) with $0 < s < 1$ and $\gamma > -3$. In Proposition 4.6 above, we were able to exhibit the complete form of this triple norm $\|f\|$.

Now we focus on the more difficult subelliptic estimate stated in 1.1. we begin with an other coercivity estimate for the Weyl quantization a_K^w .

Lemma 4.8. *Let \hat{P}_K be the operator defined at the beginning of Subsection 4.1. Then there exists a positive number $k_0 > 0$ such that for all $K \geq k_0$ and any $f \in \mathcal{S}(\mathbb{R}^3)$, we have*

$$\| (a_K^{1/2})^w f \|^2 \sim (a_K^w f, f)_{L^2} = \text{Re} \left(\hat{P}_K f, f \right)_{L^2} \quad (47)$$

and

$$\| (\tilde{a}_K^{1/2} a_K^{1/2})^w f \|^2 \sim ((\tilde{a}_K a_K)^w f, f)_{L^2}. \quad (48)$$

Proof. The argument is similar as in the proof of Lemma 4.2. Using (78) and (79), we may write

$$(a_K^{1/2})^w (a_K^{1/2})^w = a_K^w - R^w, \quad (49)$$

where

$$R = - \int_0^1 \left(\partial_\eta (a_K^{1/2}) \right) \#_\theta \left(\partial_v (a_K^{1/2}) \right) d\theta + \int_0^1 \left(\partial_v (a_K^{1/2}) \right) \#_\theta \left(\partial_\eta (a_K^{1/2}) \right) d\theta$$

with $g \#_\theta h$ defined in (41). Using (33) for $\varepsilon = K^{-1/4}$, we conclude

$$\partial_\eta (a_K^{1/2}) \in S(K^{-1/4} a_K^{1/2}, \Gamma)$$

uniformly with respect to K . On the other hand, it is clear that $\partial_v (a_K^{1/2}) \in S(a_K^{1/2}, \Gamma)$. As a result, [12, Proposition 1.1] yields

$$\left(\partial_\eta (a_K^{1/2}) \right) \#_\theta \left(\partial_v (a_K^{1/2}) \right), \left(\partial_v (a_K^{1/2}) \right) \#_\theta \left(\partial_\eta (a_K^{1/2}) \right) \in S(K^{-1/4} a_K, \Gamma)$$

uniformly w.r.t. K . Thus $R \in S(K^{-1/4}a_K, \Gamma)$ uniformly w.r.t. K . Then the conclusion (ii) in Lemma 4.2 allows us to rewrite R^w as

$$R^w = K^{-1/4}(a_K^{1/2})^w \underbrace{K^{1/2}[(a_K^{1/2})^w]^{-1}R^w[(a_K^{1/2})^w]^{-1}(a_K^{1/2})^w}_{\in \mathcal{B}(L^2) \text{ uniformly w.r.t. } K},$$

which gives

$$|(R^w f, f)_{L^2}| \leq C_0 K^{-1/4} \|(a_K^{1/2})^w f\|^2$$

with C_0 some constant independent of K . Taking into account the relation (49) we obtain

$$(a_K^w f, f)_{L^2} \leq \left((a_K^{1/2})^w (a_K^{1/2})^w f, f \right)_{L^2} + C_0 K^{-1/4} \|(a_K^{1/2})^w f\|^2 \leq (C_0 + 1) \|(a_K^{1/2})^w f\|^2$$

and

$$\left((a_K^{1/2})^w (a_K^{1/2})^w f, f \right)_{L^2} \leq (a_K^w f, f)_{L^2} + C_0 K^{-1/4} \|(a_K^{1/2})^w f\|^2.$$

The desired estimate (47) follows if we take K sufficiently large such that $K \geq k_0 \stackrel{\text{def}}{=} 16C_0^4$. Since the second estimate (48) can be deduced similarly by virtue of (iii) in Lemma 4.2, we omit it here. The proof is thus complete. \square

Corollary 4.9. *The following estimate*

$$\forall f \in \mathcal{S}(\mathbb{R}_v^3), \quad \|\langle v \rangle^{2s+\gamma} f\|_{L^2} + \|(a_K^{1/2})^w \langle v \rangle^{s+\gamma/2} f\|_{L^2} \lesssim \|\hat{P}_K f\|_{L^2} + \|f\|_{L_v^2} \quad (50)$$

holds uniformly with respect to ξ .

Proof. We use the above coercivity estimate (47) to the function $\langle v \rangle^{s+\gamma/2} f$, and argue as in the proof of Lemma 4.5; this gives the conclusions. \square

Corollary 4.10.

$$\left((\langle v \rangle^{2s+\gamma})^{\text{Wick}} f, f \right)_{L^2} \lesssim \left((\tilde{a}(v, \eta))^{\text{Wick}} f, f \right)_{L^2} \lesssim \left| \left(\hat{P}_K f, f \right)_{L^2} \right|. \quad (51)$$

Proof. The first inequality is due to the positivity of Wick quantization. The second one is just an immediate consequence of (47) and Lemma 4.2, since we may write

$$(\tilde{a}(v, \eta))^{\text{Wick}} = (a_K^{1/2})^w \underbrace{[(a_K^{1/2})^w]^{-1} (\tilde{a}(v, \eta))^{\text{Wick}} [(a_K^{1/2})^w]^{-1}}_{\in \mathcal{B}(L^2)} (a_K^{1/2})^w.$$

The proof is complete. \square

4.3 Hypoelliptic estimates

We prove in this last subsection the main results, Theorem 1.1 and Theorem 1.2. As explained in Proposition 4.1, we only work on \hat{P}_K instead of P . So in this subsection ξ and τ are considered as parameters. Recall \tilde{a} is defined in (31), whose explicit form, as to be seen below, will be convenient for use. The main result we will show here can be stated as follows.

Proposition 4.11. *Under the conditions in Theorem 1, we have, for any $\ell \in \mathbb{R}$,*

$$\|\tilde{a}(v, \xi)^{\frac{1}{1+2s}} f\| + \|a_K^w f\| \lesssim \|\hat{P}_K f\|_{L^2} + \|\langle v \rangle^\ell f\|_{L^2}.$$

We will prove the above proposition in several steps, following the multiplier strategy announced in [22]. To this end we let, throughout this section, $\chi \in C_0^\infty(\mathbb{R}; [0, 1])$ such that $\chi = 1$ in $[-1, 1]$ and $\text{supp } \chi \subset [-2, 2]$, and let g be a symbol given by

$$g(v, \eta) = g_\xi(v, \eta) = \frac{a_3(v, \eta)}{\tilde{a}(v, \xi)^{\frac{2s}{1+2s}}} \psi(v, \eta), \quad (52)$$

where

$$\psi(v, \eta) = \chi \left(\frac{\tilde{a}(v, \eta)}{\tilde{a}(v, \xi)^{\frac{1}{1+2s}}} \right) \quad (53)$$

and

$$a_3(v, \eta) = \langle v \rangle^\gamma \left(1 + |v|^2 + |\xi|^2 + |v \wedge \xi|^2 \right)^{s-1} \left(\xi \cdot \eta + (v \wedge \xi) \cdot (v \wedge \eta) \right). \quad (54)$$

The main property linking a_3 and \tilde{a} is that

$$\{a_3(v, \eta), v \cdot \xi\} = \tilde{a}(v, \xi) - \langle v \rangle^{\gamma+2} \left(1 + |v|^2 + |\xi|^2 + |v \wedge \xi|^2 \right)^{s-1}. \quad (55)$$

where $\{\cdot, \cdot\}$ is the Poisson bracket defined in (84). Thanks to the explicit symbolic estimates for \tilde{a} , g and ψ also have good behavior as symbols; that is,

$$g, \psi \in S(1, |dv|^2 + |d\eta|^2)$$

uniformly with respect to ξ . Moreover direct computation shows that

$$|\xi \cdot \partial_\eta \psi| \lesssim \tilde{a}(v, \eta). \quad (56)$$

Lemma 4.12. *Under the conditions in Theorem 1, we have*

$$\forall f \in \mathcal{S}(\mathbb{R}^3), \quad \|\tilde{a}(v, \xi)^{\frac{1}{1+2s}} f\|^2 \lesssim \|\hat{P}_K f\|_{L^2}^2 + \|f\|_{L_\ell^2}.$$

Proof. The proof is divided into three steps.

Step 1) Let g^{Wick} be the Wick quantization of the symbol g given in (52). We claim

$$\left| \left(a_K^w f, g^{\text{Wick}} f \right)_{L^2} \right| \lesssim \left| \left(\hat{P}_K f, f \right)_{L^2} \right|. \quad (57)$$

To confirm this, let's write, denoting by H the inverse of $(a_K^{1/2})^w$,

$$\left(a_K^w f, g^{\text{Wick}} f \right)_{L^2} = \left(H a_K^w H (a_K^{1/2})^w f, (a_K^{1/2})^w g^{\text{Wick}} H (a_K^{1/2})^w f \right)_{L^2}.$$

Note that $Ha_K^w H$ and $(a_K^{1/2})^w g^{\text{Wick}} H$ are bounded operators on L^2 due to Lemma 4.2. Then one has

$$\left| (a_K^w f, g^{\text{Wick}} f)_{L^2} \right| \lesssim \left\| (a_K^{1/2})^w f \right\|_{L^2}^2 \lesssim \left| (\hat{P}_K f, f)_{L^2} \right|,$$

the last inequality following from (47).

Step 2) We now prove

$$\left\| \tilde{a}(v, \xi)^{\frac{1}{2+4s}} f \right\|_{L^2} \lesssim \left\| \tilde{a}(v, \xi)^{-\frac{1}{2+4s}} \hat{P}_K f \right\|_{L^2}. \quad (58)$$

Note that $g \in S(1, \Gamma)$ and $\tilde{a}(v, \xi)^r \in S(\tilde{a}(v, \xi)^r, \Gamma)$ for any $r \in \mathbb{R}$. Then the above estimate will follow if we can show that

$$\left\| \tilde{a}(v, \xi)^{\frac{1}{2+4s}} f \right\|_{L^2}^2 \lesssim \left| (\hat{P}_K f, f)_{L^2} \right| + \left| (\hat{P}_K f, g^{\text{Wick}} f)_{L^2} \right|. \quad (59)$$

To prove the above inequality we make use of the relation

$$\text{Re} \left(i(v \cdot \xi) f, g^{\text{Wick}} f \right)_{L^2} = \text{Re} \left(\hat{P}_K f, g^{\text{Wick}} f \right)_{L^2} - \text{Re} \left(a_K^w f, g^{\text{Wick}} f \right)_{L^2}$$

and (57), to conclude

$$\text{Re} \left(i(v \cdot \xi) f, g^{\text{Wick}} f \right)_{L^2} \lesssim \left| (\hat{P}_K f, f)_{L^2} \right| + \left| (\hat{P}_K f, g^{\text{Wick}} f)_{L^2} \right|. \quad (60)$$

Next we will give a lower bound of the term on the left hand side. Observe that by (82),

$$v \cdot \xi = (v \cdot \xi)^{\text{Wick}}.$$

Then we have, by (83),

$$\text{Re} \left(i(v \cdot \xi) f, g^{\text{Wick}} f \right)_{L^2} = \frac{1}{4\pi} \left(\{g, v \cdot \xi\}^{\text{Wick}} f, f \right)_{L^2}. \quad (61)$$

Using (55) we compute

$$\begin{aligned} & \{g, v \cdot \xi\} \\ &= \tilde{a}(v, \xi)^{\frac{1}{1+2s}} \psi - \frac{\langle v \rangle^{\gamma+2} \left(1 + |v|^2 + |\xi|^2 + |v \wedge \xi|^2 \right)^{s-1}}{\tilde{a}(v, \xi)^{\frac{2s}{1+2s}}} \psi + \frac{a_3(v, \eta)}{\tilde{a}(v, \xi)^{\frac{2s}{1+2s}}} \xi \cdot \partial_\eta \psi \\ &= \tilde{a}(v, \xi)^{\frac{1}{1+2s}} - \tilde{a}(v, \xi)^{\frac{1}{1+2s}} (1 - \psi) - \frac{\langle v \rangle^{\gamma+2} \left(1 + |v|^2 + |\xi|^2 + |v \wedge \xi|^2 \right)^{s-1}}{\tilde{a}(v, \xi)^{\frac{2s}{1+2s}}} \psi \\ & \quad + \frac{a_3(v, \eta)}{\tilde{a}(v, \xi)^{\frac{2s}{1+2s}}} \xi \cdot \partial_\eta \psi. \end{aligned}$$

This along with (60) and (61) yields

$$\left((\tilde{a}(v, \xi)^{\frac{1}{1+2s}})^{\text{Wick}} f, f \right)_{L^2} \lesssim \sum_{j=1}^3 T_j + \left| (\hat{P}_K f, f)_{L^2} \right| + \left| (\hat{P}_K f, g^{\text{Wick}} f)_{L^2} \right|, \quad (62)$$

with

$$\begin{aligned}
T_1 &= \left(\left(\tilde{a}(v, \xi)^{\frac{1}{1+2s}} (1 - \psi) \right)^{\text{Wick}} f, f \right)_{L^2}, \\
T_2 &= \left(\left(\langle v \rangle^{\gamma+2} \left(1 + |v|^2 + |\xi|^2 + |v \wedge \xi|^2 \right)^{s-1} \tilde{a}(v, \xi)^{-\frac{2s}{1+2s}} \psi \right)^{\text{Wick}} f, f \right)_{L^2}, \\
T_3 &= \left(\left(-\frac{a_3(v, \eta)}{\tilde{a}(v, \xi)^{\frac{2s}{1+2s}}} \xi \cdot \partial_\eta \psi \right)^{\text{Wick}} f, f \right)_{L^2}.
\end{aligned}$$

Note that $\tilde{a}(v, \xi)^{\frac{1}{1+2s}} \leq \tilde{a}(v, \eta)$ on the support of $1 - \psi$, and thus

$$\tilde{a}(v, \xi)^{\frac{1}{1+2s}} (1 - \psi) \leq \tilde{a}(v, \eta).$$

Then the positivity of Wick quantization gives

$$T_1 \lesssim \left((\tilde{a}(v, \eta))^{\text{Wick}} f, f \right)_{L^2} \lesssim \left| \left(\hat{P}_K f, f \right)_{L^2} \right|, \quad (63)$$

where the last inequality follows from (51). Similarly, observing

$$\langle v \rangle^{\gamma+2} \left(1 + |v|^2 + |\xi|^2 + |v \wedge \xi|^2 \right)^{s-1} \tilde{a}(v, \xi)^{-\frac{2s}{1+2s}} \psi \leq \langle v \rangle^{2s+\gamma},$$

we have

$$T_2 \lesssim \left(\left(\langle v \rangle^{2s+\gamma} \right)^{\text{Wick}} f, f \right)_{L^2} \lesssim \left| \left(\hat{P}_K f, f \right)_{L^2} \right|. \quad (64)$$

As for T_3 , it follows from (56) that

$$-\frac{a_3(v, \eta)}{\tilde{a}(v, \xi)^{\frac{2s}{1+2s}}} \xi \cdot \partial_\eta \psi \lesssim \tilde{a}(v, \eta).$$

Thus

$$T_3 \lesssim \left((\tilde{a}(v, \eta))^{\text{Wick}} f, f \right)_{L^2} \lesssim \left| \left(\hat{P}_K f, f \right)_{L^2} \right|.$$

This, together with (62), (63) and (64), gives

$$\left((\tilde{a}(v, \xi)^{\frac{1}{1+2s}})^{\text{Wick}} f, f \right)_{L^2} \lesssim \left| \left(\hat{P}_K f, f \right)_{L^2} \right| + \left| \left(\hat{P}_K f, g^{\text{Wick}} f \right)_{L^2} \right|.$$

As a result the desired estimate (59) follows, since by (82),

$$\left(\tilde{a}(v, \xi)^{\frac{1}{1+2s}} \right)^{\text{Wick}} = \int \tilde{a}(v - \tilde{v}, \xi)^{\frac{1}{1+2s}} e^{-2\pi \tilde{v}^2} 2^3 d\tilde{v},$$

which is bounded from below by $\tilde{a}(v, \xi)^{1/(1+2s)}$ by direct verification (see for instance the arguments used in the proof of [22, Lemma 3.14]).

Step 3) Now applying the inequality (59) to the function $\tilde{a}(v, \xi)^{\frac{1}{2+4s}} f$, we get

$$\begin{aligned}
\left\| \tilde{a}(v, \xi)^{\frac{1}{1+2s}} f \right\|_{L^2} &\lesssim \left\| \tilde{a}(v, \xi)^{-\frac{1}{2+4s}} \hat{P}_K \tilde{a}(v, \xi)^{\frac{1}{2+4s}} f \right\|_{L^2} \\
&\lesssim \left\| \hat{P}_K f \right\|_{L^2} + \left\| \tilde{a}(v, \xi)^{-\frac{1}{2+4s}} [a_K^w, \tilde{a}(v, \xi)^{\frac{1}{2+4s}}] f \right\|_{L^2}.
\end{aligned}$$

In view of (79), the symbol of $\tilde{a}(v, \xi)^{-1/(2+4s)} [a_K^w, \tilde{a}(v, \xi)^{1/(2+4s)}]$ has the form

$$\tilde{a}(v, \xi)^{-\frac{1}{2+4s}} \int_0^1 (\partial_\eta a_K) \#_\theta \left(\partial_v (\tilde{a}^{1/(2+4s)}) \right) d\theta,$$

which, arguing as in the proof of Lemma 4.2, belongs to

$$S \left(a^{1/2} \langle v \rangle^{s+\gamma/2}, \Gamma \right).$$

As a result, we can use (ii) in Lemma 4.2 to write

$$\begin{aligned} & \tilde{a}(v, \xi)^{-\frac{1}{2+4s}} [a_K^w, \tilde{a}(v, \xi)^{\frac{1}{2+4s}}] \\ = & \underbrace{\tilde{a}(v, \xi)^{-\frac{1}{2+4s}} [a_K^w, \tilde{a}(v, \xi)^{\frac{1}{2+4s}}] \langle v \rangle^{-(s+\gamma/2)} \left((a_K^{1/2})^w \right)^{-1} (a_K^{1/2})^w \langle v \rangle^{s+\gamma/2}}_{\in \mathcal{B}(L^2)}. \end{aligned}$$

This gives

$$\begin{aligned} \|\tilde{a}(v, \xi)^{-\frac{1}{2+4s}} [a_K^w, \tilde{a}(v, \xi)^{\frac{1}{2+4s}}] f\|_{L^2} & \lesssim \| (a_K^{1/2})^w \langle v \rangle^{s+\gamma/2} f \|_{L^2} \\ & \lesssim \|\hat{P}_K f\|_{L^2} + \|f\|_{L^2_\ell}, \end{aligned}$$

where the last inequality follows from (50). Combining these inequalities, we get the desired estimate

$$\|\tilde{a}(v, \xi)^{\frac{1}{1+2s}} f\|_{L^2} \lesssim \|\hat{P}_K f\|_{L^2} + \|f\|_{L^2_\ell}.$$

The proof of Lemma 4.12 is thus complete. \square

Lemma 4.13. *Under the conditions in Theorem 1, we have, for any $\ell \in \mathbb{R}$,*

$$\|a_K^w f\|_{L^2} \lesssim \|\hat{P}_K f\|_{L^2} + \|\langle v \rangle^\ell f\|_{L^2}.$$

Proof. The proof is divided into four steps. In what follows let $\varepsilon > 0$ be an arbitrarily small number, which is to be determined later, and denote by C_ε the different suitable constants depending only on ε .

Step 1) We define ρ_ε by

$$\rho_\varepsilon(v, \eta) = \chi \left(\frac{\tilde{a}(v, \xi)^{\frac{1}{1+2s}}}{\varepsilon \tilde{a}(v, \eta)} \right),$$

where $\chi \in C_0^\infty(\mathbb{R}; [0, 1])$ such that $\chi = 1$ in $[-1, 1]$ and $\text{supp } \chi \subset [-2, 2]$. Let $\lambda_{1,\varepsilon}$ and $\lambda_{2,\varepsilon}$ be two symbols defined by

$$\lambda_{1,\varepsilon}(v, \eta) = \rho_\varepsilon(v, \eta) \tilde{a}(v, \eta) \tag{65}$$

and

$$\lambda_{2,\varepsilon}(v, \eta) = (1 - \rho_\varepsilon(v, \eta)) \tilde{a}(v, \eta). \tag{66}$$

Then $\rho_\varepsilon(v, \eta) \in S(1, \Gamma)$,

$$\lambda_{1,\varepsilon}, \lambda_{2,\varepsilon} \in S(\tilde{a}(v, \eta), \Gamma) \quad \text{and} \quad \lambda_{2,\varepsilon} \in S\left(\varepsilon^{-1}\tilde{a}(v, \xi)^{\frac{1}{1+2s}}, \Gamma\right), \quad (67)$$

uniformly with respect to ξ and ε , due to the conclusion (i) in Proposition 1.3 and the fact that $\tilde{a}(v, \eta) \leq \varepsilon^{-1}\tilde{a}(v, \xi)^{\frac{1}{1+2s}}$ on the support of $\lambda_{2,\varepsilon}$.

Step 2) Let $\lambda_{1,\varepsilon}(v, \eta)$ be given in (65). In this step we show

$$\left| \left([v \cdot \xi, \lambda_{1,\varepsilon}^w] f, f \right)_{L^2} \right| \leq \varepsilon \|a_K^w f\|_{L^2}^2. \quad (68)$$

In fact, the symbol of the above commutator $[v \cdot \xi, \lambda_{1,\varepsilon}^w]$ is

$$-\frac{1}{2i\pi} \xi \cdot \partial_\eta \lambda_{1,\varepsilon}(v, \eta),$$

which belongs to $S(\varepsilon^{(1+2s)/2s} \tilde{a}(v, \eta)^2, \Gamma)$ uniformly with respect to ξ and ε , due to (34) and the fact that

$$|\xi| + |v \wedge \xi| \lesssim \tilde{a}(v, \xi)^{\frac{1}{2s}} \langle v \rangle^{-\frac{\gamma}{2s}} \leq \varepsilon^{\frac{1+2s}{2s}} \tilde{a}(v, \eta)^{\frac{1+2s}{2s}} \langle v \rangle^{-\frac{\gamma}{2s}}$$

on the support of $\lambda_{1,\varepsilon}$. Thus writing

$$[v \cdot \xi, \lambda_{1,\varepsilon}^w] = \varepsilon a_K^w \underbrace{(a_K^w)^{-1} [v \cdot \xi, \lambda_{1,\varepsilon}^w] (a_K^w)^{-1}}_{\in \mathcal{B}(L^2)} a_K^w,$$

we obtain

$$\left| \left([v \cdot \xi, \lambda_{1,\varepsilon}^w] f, f \right)_{L^2} \right| \lesssim \varepsilon \|a_K^w f\|_{L^2}^2.$$

This gives the desired upper bound. We have proven (68).

Step 3) Let $\lambda_{2,\varepsilon}(v, \eta)$ be given in (66). We claim

$$\left| \left([v \cdot \xi, \lambda_{2,\varepsilon}^w] f, f \right)_{L^2} \right| \lesssim \varepsilon \|(v \cdot \xi) f\|_{L^2}^2 + C_\varepsilon \left(\|\hat{P}_K f\|_{L^2}^2 + \|f\|_{L_t^2}^2 \right). \quad (69)$$

To confirm this, we write $[v \cdot \xi, \lambda_{2,\varepsilon}^w] = v \cdot \xi \lambda_{2,\varepsilon}^w - \lambda_{2,\varepsilon}^w v \cdot \xi$; this gives

$$\left| \left([v \cdot \xi, \lambda_{2,\varepsilon}^w] f, f \right)_{L^2} \right| \leq 2 \|(v \cdot \xi) f\|_{L^2} \|\lambda_{2,\varepsilon}^w f\|_{L^2}.$$

Moreover it follows from (67) that

$$\|\lambda_{2,\varepsilon}^w f\|_{L^2} \lesssim \varepsilon^{-1} \|\tilde{a}(v, \xi)^{1/(1+2s)} f\|_{L^2} \lesssim \varepsilon^{-1} \left(\|\tilde{\mathcal{P}} f\|_{L^2} + \|f\|_{L_t^2} \right),$$

the last inequality using Lemma 4.12. Combining these inequalities, we obtain the desired estimate (69).

Step 4) Now we are ready to prove

$$\|a_K^w f\|_{L^2}^2 \lesssim \|\hat{P}_K f\|_{L^2}^2 + \|f\|_{L_t^2}^2, \quad (70)$$

which will follow if we can show that

$$|\operatorname{Re}(i(v \cdot \xi) f, \tilde{a}_K^w f)_{L^2}| \lesssim \varepsilon \|a_K^w f\|_{L^2}^2 + C_\varepsilon \left(\|\hat{P}_K f\|_{L^2}^2 + \|f\|_{L_t^2}^2 \right) \quad (71)$$

and

$$\|a_K^w f\|^2 \leq \operatorname{Re} (a_K^w f, \tilde{a}_K^w f)_{L^2} + \varepsilon \|a_K^w f\|^2 + C_\varepsilon \left(\|\hat{P}_K f\|^2 + \|f\|^2 \right), \quad (72)$$

due to the relation

$$\operatorname{Re} \left(\hat{P}_K f, \tilde{a}_K^w f \right)_{L^2} = \operatorname{Re} (i(v \cdot \xi) f, \tilde{a}_K^w f)_{L^2} + \operatorname{Re} (a_K^w f, \tilde{a}_K^w f)_{L^2}.$$

To prove (71), we compute

$$\begin{aligned} |\operatorname{Re} (i(v \cdot \xi) f, \tilde{a}_K^w f)_{L^2}| &= \left| \frac{i}{2} \left([v \cdot \xi, \tilde{a}_K^w] f, f \right)_{L^2} \right| = \left| \frac{i}{2} \left([v \cdot \xi, \tilde{a}^w] f, f \right)_{L^2} \right| \\ &\lesssim \left| \left([v \cdot \xi, \lambda_{1,\varepsilon}^w] f, f \right)_{L^2} \right| + \left| \left([v \cdot \xi, \lambda_{2,\varepsilon}^w] f, f \right)_{L^2} \right| \end{aligned}$$

with $\lambda_{1,\varepsilon}, \lambda_{2,\varepsilon}$ defined in (65) and (66). Combining the above inequalities and the conclusion in the previous two steps, we have

$$|\operatorname{Re} (i(v \cdot \xi) f, \tilde{a}_K^w f)_{L^2}| \lesssim \varepsilon \|a_K^w f\|_{L^2}^2 + \varepsilon \|(v \cdot \xi) f\|_{L^2}^2 + C_\varepsilon \left(\|\hat{P}_K f\|_{L^2}^2 + \|f\|_{L_t^2}^2 \right).$$

This along with the relation

$$\|(v \cdot \xi) f\|_{L^2}^2 \lesssim \|\hat{P}_K f\|_{L^2}^2 + \|a_K^w f\|_{L^2}^2$$

implies the desired estimate (71).

We now prove (72). In view of (79) we may write

$$(\tilde{a}_K \sharp a_K)^w = (\tilde{a}_K a_K)^w + r^w, \quad (73)$$

where

$$r(Y) = \int_0^1 \iint e^{-2i\sigma(Y-Y_1, Y-Y_2)/\theta} \frac{1}{2i} \sigma(\partial_{Y_1}, \partial_{Y_2}) \tilde{a}(Y_1) a_K(Y_2) dY_1 dY_2 d\theta / (\pi\theta)^6.$$

Note that (33) also holds with a replaced by \tilde{a}_K or a_K . Then by view of [12, Proposition 1.1], we can verify that

$$r \in S \left(a_K^{3/2} \langle v \rangle^{s+\gamma/2}, \Gamma \right),$$

and thus we may use Lemma 4.2 to rewrite r^w as

$$r^w = \varepsilon^{1/2} a_K^w \underbrace{(a_K^w)^{-1} r^w \langle v \rangle^{-(s+\gamma/2)}}_{\in \mathcal{B}(L^2)} \left[(a_K^{1/2})^w \right]^{-1} \varepsilon^{-1/2} (a_K^{1/2})^w \langle v \rangle^{s+\gamma/2}.$$

This gives

$$\begin{aligned} |(r^w f, f)_{L^2}| &\lesssim \varepsilon \|a_K^w f\|_{L^2}^2 + \varepsilon^{-1} \left\| (a_K^{1/2})^w \langle v \rangle^{s+\gamma/2} f \right\|_{L^2}^2 \\ &\lesssim \varepsilon \|a_K^w f\|_{L^2}^2 + \varepsilon^{-1} \left(\|\hat{P}_K f\|_{L^2}^2 + \|f\|_{L_t^2}^2 \right), \end{aligned}$$

the last inequality following from (50). Taking into account (73), one has

$$\operatorname{Re} ((\tilde{a}_K a_K)^w f, f)_{L^2} \lesssim \operatorname{Re} (a_K^w f, \tilde{a}_K f)_{L^2} + \varepsilon \|a_K^w f\|_{L^2}^2 + \varepsilon^{-2} \left(\|\hat{P}_K f\|_{L^2}^2 + \|f\|_{L_t^2}^2 \right),$$

which along with (48) yields

$$\| (\tilde{a}_K^{1/2} a_K^{1/2})^w f \|_{L^2}^2 \lesssim \operatorname{Re} (a_K^w f, \tilde{a}_K f)_{L^2} + \varepsilon \| a_K^w f \|_{L^2}^2 + \varepsilon^{-2} \left(\| \hat{P}_K f \|_{L^2}^2 + \| f \|_{L_\ell^2}^2 \right).$$

Moreover note that

$$\| a_K^w f \|_{L^2}^2 \lesssim \| (\tilde{a}_K^{1/2} a_K^{1/2})^w f \|_{L^2}^2$$

due to the conclusion (iii) in Lemma 4.2. Then the desired estimate (72) follows from the above inequalities, completing the proof of Lemma 4.13. \square

Combining the conclusions in Lemma 4.12 and Lemma 4.13, we obtain Proposition 4.11. Thus Theorem 1.1 follows due to Proposition 4.1. Now it remains to do the

Proof of Theorem 1.2. Let τ be the dual variable of t and let \hat{P}_τ be the operator defines as follows

$$\hat{P}_\tau = i\tau + iv \cdot \xi - \mathcal{L} = i(\tau + v \cdot \xi) + a^w + \mathcal{K}.$$

Just proceeding as in the proof of Lemma 4.12 and Lemma 4.13, we have the maximal hypoelliptic estimate

$$\| \langle v \rangle^{2s+\gamma} f \|_{L^2} + \| (\tau + v \cdot \xi) f \|_{L^2} + \| a^w f \|_{L^2} + \| \langle v \rangle^{\frac{\gamma}{1+2s}} |\xi|^{\frac{2s}{1+2s}} f \|_{L^2} \lesssim \| \hat{P}_\tau f \|_{L^2} + \| f \|_{L_\ell^2}. \quad (74)$$

Now it remains to prove

$$\| \langle v \rangle^{\frac{\gamma-2s}{1+2s}} \langle \tau \rangle^{\frac{2s}{1+2s}} f \|_{L^2} \lesssim \| \hat{P}_\tau f \|_{L^2} + \| f \|_{L_\ell^2}.$$

To do so, we compute

$$\begin{aligned} \langle v \rangle^{\frac{\gamma-2s}{1+2s}} |\tau|^{\frac{2s}{1+2s}} &\lesssim \langle v \rangle^{\frac{\gamma-2s}{1+2s}} |\tau + v \cdot \xi|^{\frac{2s}{1+2s}} + \langle v \rangle^{\frac{\gamma-2s}{1+2s}} |v \cdot \xi|^{\frac{2s}{1+2s}} \\ &\lesssim \langle v \rangle^{\frac{\gamma-2s}{1+2s}} |\tau + v \cdot \xi|^{\frac{2s}{1+2s}} + \langle v \rangle^{\frac{\gamma}{1+2s}} |\xi|^{\frac{2s}{1+2s}} \\ &\lesssim \langle v \rangle^{\gamma-2s} + |\tau + v \cdot \xi| + \langle v \rangle^{\frac{\gamma}{1+2s}} |\xi|^{\frac{2s}{1+2s}}, \end{aligned}$$

where the last inequality follows from the Young's inequality:

$$\langle v \rangle^{\frac{\gamma-2s}{1+2s}} |\tau + v \cdot \xi|^{\frac{2s}{1+2s}} \leq \frac{\left(\langle v \rangle^{\frac{\gamma-2s}{1+2s}} \right)^{1+2s}}{1+2s} + \frac{2s}{1+2s} \left(|\tau + v \cdot \xi|^{\frac{2s}{1+2s}} \right)^{(1+2s)/(2s)}.$$

As a result we have,

$$\begin{aligned} \| \langle v \rangle^{\frac{\gamma-2s}{1+2s}} |\tau|^{\frac{2s}{1+2s}} f \|_{L^2} &\lesssim \| (\tau + v \cdot \xi) f \|_{L^2} + \| \langle v \rangle^{\gamma-2s} f \|_{L^2} + \| \langle v \rangle^{\frac{\gamma}{1+2s}} |\xi|^{\frac{2s}{1+2s}} f \|_{L^2} \\ &\lesssim \| \hat{P}_\tau f \|_{L^2} + \| f \|_{L_\ell^2}, \end{aligned}$$

where the last inequality follows from (74). The proof of Theorem 1.2 is complete. \square

5 Appendix

In this section we briefly review some tools used through the proofs. The first section is devoted to the links between some integral concerning the Boltzmann kernel. In the second one recall some basic facts about the Weyl-Hörmander quantization, and in the last one we recall some ideas and results about the Wick quantization.

5.1 Integral representations

Throughout parts 2 and 3 of this paper, we change any time needed of integral representation for the singular Boltzmann kernel.

A basic formula

The first tool we use is the following Fubini-type formula, derived by rather explicit computation:

Consider a measurable function $0 \leq F(\alpha, h)$ of variables h and $\alpha \in \mathbb{R}^3$. For any $h \in \mathbb{R}^3$, we denote by $E_{0,h}$ the (hyper-)vector plane orthogonal to h . Then

$$\int_{\mathbb{R}_h^3} dh \int_{E_{0,h}} d\alpha f(\alpha, h) = \int_{\mathbb{R}_\alpha^3} d\alpha \int_{E_{0,\alpha}} dh \frac{|h|}{|\alpha|} f(\alpha, h). \quad (75)$$

Carleman representation

The second formula is the so-called ω -representation. It says that we have the following (almost) equalities when all sides are well-defined :

$$\begin{aligned} & \iint dv_* d\sigma b(\cos\theta) |v - v_*|^\gamma F(v, v^*, v', v'_*) \\ &= 4 \int_{\mathbb{R}_h^3} dh \int_{E_{0,h}} d\alpha \frac{1}{|\alpha + h| |h|} b(\cos\theta) \Phi(|h|) F(v, v + \alpha - h, v - h, v + \alpha) \\ &\sim \int_{\mathbb{R}_h^3} dh \int_{E_{0,h}} d\alpha \mathbb{1}_{|\alpha| \geq |h|} \frac{1}{|\alpha + h| |h|} b\left(\frac{|\alpha|^2 - |h|^2}{|\alpha + h|^2}\right) |\alpha + h|^\gamma F(v, v + \alpha - h, v - h, v + \alpha). \end{aligned}$$

These formulae are consequences of the following properties (see picture 2):

1. We make the change of variables $(v_*, \sigma) \mapsto (\alpha, h)$ with $v' = v - h$, $v_* = v + \alpha - h$, $v'_* = v + \alpha$.
2. Since we restricted by symmetrization to the case $\sigma \cdot (v - v_*) \geq 0$ (which is equivalent to $\cos\theta \geq 0$), this implies $|\alpha| \geq |h|$. Note also that $h \perp \alpha$ and therefore $|\alpha + h|^2 = |\alpha - h|^2 = |\alpha|^2 + |h|^2$.
3. By immediate trigonometric properties we have $\cos\theta = \frac{|\alpha|^2 - |h|^2}{|\alpha + h|^2}$ and $\sin\theta = \frac{2|\alpha||h|}{|\alpha + h|^2}$.

From the singular behavior of the singular kernel we deduce

$$0 \leq b(\cos\theta) \sim K\theta^{-2-2s} \sim \tilde{K}(\sin\theta)^{-2-2s} \sim \tilde{K} \frac{|\alpha + h|^{4+4s}}{|\alpha|^{2+2s}|h|^{2+2s}} \equiv \frac{|\alpha + h|^{2+2s}}{|h|^{2+2s}},$$

since $|\alpha|^2 \leq |\alpha + h|^2 \leq 2|\alpha|^2$, and where \approx means that the ratio is bounded from below and above by universal constants. At the end we get

$$\begin{aligned} & \iint dv_* d\sigma b(\cos\theta) \Phi(|v - v_*|) F(v, v^*, v', v'_*) \\ &= \int_{\mathbb{R}_h^3} dh \int_{E_{0,h}} d\alpha \tilde{b}(\alpha, h) \mathbb{1}_{|\alpha| \geq |h|} \frac{|\alpha + h|^{\gamma+1+2s}}{|h|^{3+2s}} F(v, v + \alpha - h, v - h, v + \alpha). \quad (76) \end{aligned}$$

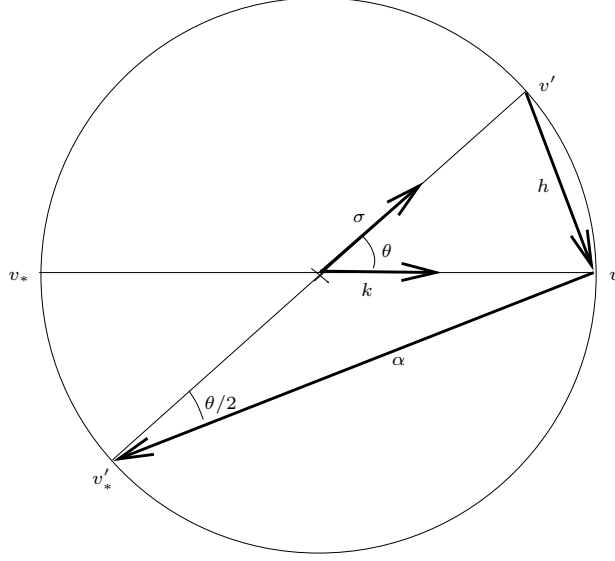


Figure 2: σ and Carleman representations

where $\tilde{b}(\alpha, h)$ is bounded from below and above by positive constants, and $\tilde{b}(\alpha, h) = \tilde{b}(\pm\alpha, \pm h)$. Figure 2 shows the preceding relations between all vectors and angles.

The cancellation lemma

We give here an other formula, in a slightly different version than the original one presented in [11]. We consider a function $G(|v - v_*|, |v - v'|)$. Then for smooth f , we have

$$\left(\iint dv_* d\sigma G(|v - v_*|, |v - v'|) b(\cos \theta) (f'_* - f_*) \right) = S *_{v_*} f(v),$$

where for all $z \in \mathbb{R}^3$, S has the following expression

$$S(z) = 2\pi \int_0^{\pi/2} d\theta \sin \theta b(\cos \theta) \left(G\left(\frac{|z|}{\cos \frac{\theta}{2}}, \frac{|z|}{\cos \frac{\theta}{2}} \sin \frac{\theta}{2}\right) \cos^{-3} \frac{\theta}{2} - G(|z|, |z| \sin \frac{\theta}{2}) \right)$$

This applies in particular to functions of type

$$G(|v - v_*|, |v - v'|, \cos \theta) = b(\cos \theta) |v - v_*|^\gamma \varphi(v - v').$$

5.2 Weyl-Hörmander calculus

We recall here some notations and basic facts of symbolic calculus, and refer to [23, Chapter 18] for detailed discussion on the pseudo-differential calculus.

From now on we pose $\Gamma = |dv|^2 + |d\eta|^2$, and let M be an admissible weight function. Considering symbols $q(\xi, v, \eta)$ as a function of (v, η) with parameters ξ , we say that $q \in S(M, \Gamma)$ uniformly with respect to ξ , if

$$\forall \alpha, \beta \in \mathbb{Z}_+^3, \quad \forall v, \eta \in \mathbb{R}^3, \quad \left| \partial_v^\alpha \partial_\eta^\beta q(\xi, v, \eta) \right| \leq C_{\alpha, \beta} M,$$

with $C_{\alpha, \beta}$ a constant depending only on α and β , but independent of ξ . For simplicity of notations, in the sequel discussion we omit the parameters in symbols, and by $q \in S(M, \Gamma)$

we always mean that q satisfies the above inequality uniformly with respect to ξ . The space $S(M, \Gamma)$ endowed with the seminorms

$$\|q\|_{k; S(m, \Gamma)} = \max_{0 \leq |\alpha| + |\beta| \leq k} \sup_{(v, \eta) \in \mathbb{R}^6} \left| M(v, \eta)^{-1} \partial_v^\alpha \partial_\eta^\beta q(v, \eta) \right|,$$

becomes a Frchet space. Denote by $\text{Op}(S(M, \Gamma))$ the class of pseudo-differential operators q^w with $q \in S(M, \Gamma)$. Here q^w stands for the Weyl quantization of symbol q , defined by

$$q^w u(v) = \frac{1}{(2\pi)^6} \int_{\mathbb{R}^6} e^{i(v-z) \cdot \eta} q\left(\frac{v+z}{2}, \eta\right) u(z) dz d\eta.$$

An elementary property to be used frequently is the L^2 continuity theorem in the class $S(1, g)$, which says that there exists a constant C and a positive integer N , depending only the dimension, such that

$$\forall u \in L^2, \quad \|q^w u\|_{L^2} \leq C \|q\|_{N; S(1, \Gamma)} \|u\|_{L^2}. \quad (77)$$

We just recall here the composition formula of Weyl quantization. Given $p_i \in S(M_i, \Gamma)$ we have

$$p_1^w p_2^w = (p_1 \# p_2)^w \quad (78)$$

with $p_1 \# p_2 \in S(M_1 M_2, \Gamma)$ admitting the expansion

$$p_1 \# p_2 = p_1 p_2 + \int_0^1 \iint e^{-2i\sigma(Y-Y_1, Y-Y_2)/\theta} \frac{1}{2i} \sigma(\partial_{Y_1}, \partial_{Y_2}) p_1(Y_1) p_2(Y_2) dY_1 dY_2 d\theta / (\pi\theta)^6, \quad (79)$$

where σ is a symplectic form in \mathbb{R}^6 given by

$$\sigma\left((z, \zeta), (\tilde{z}, \tilde{\zeta})\right) = \zeta \cdot \tilde{z} - \tilde{\zeta} \cdot z.$$

Given $q \in S(M, \Gamma)$, we have the formula of changing quantization (see Proposition 1.1.10 and Lemma 4.1.2 of [25]):

$$q^w = (J^{1/2} q)(v, D_v), \quad (80)$$

where $J^{1/2} : \mathcal{S}' \rightarrow \mathcal{S}'$ is defined by

$$(J^{1/2} q)(v, \eta) = 2^n \iint e^{-4i\pi y \cdot \zeta} q(v+z, \eta+\zeta) dz d\zeta. \quad (81)$$

5.3 Wick quantization

Finally let's recall some basic properties of the Wick quantization, and refer the reader to the works of Lerner [27, 26, 25] for thorough and extensive presentations of this quantization and some of its applications. Let $Y = (v, \eta)$ be a point in \mathbb{R}^6 . The wave-packets transform of a function $u \in \mathcal{S}(\mathbb{R}_v^3)$ is defined by

$$Wu(Y) = (u, \varphi_Y)_{L^2(\mathbb{R}_v^3)} = 2^{3/4} \int_{\mathbb{R}^3} u(z) e^{-\pi|z-v|^2} e^{2i\pi(z-v/2) \cdot \eta} dz,$$

with

$$\varphi_Y(z) = 2^{3/4} e^{-\pi|z-v|^2} e^{2i\pi(z-v/2) \cdot \eta}, \quad z \in \mathbb{R}^3.$$

Then W is an isometric mapping from $L^2(\mathbb{R}_v^3)$ to $L^2(\mathbb{R}^6)$ with adjoint W^* . We define the Wick quantization of any L^∞ symbol q as

$$p^{\text{Wick}} = W^* p W.$$

The main property of the Wick quantization is the positivity, i.e.,

$$q(v, \eta) \geq 0 \quad \text{for all } (v, \eta) \in \mathbb{R}^6 \text{ implies } q^{\text{Wick}} \geq 0.$$

According to Proposition 2.4.3 in [25], the Wick and Weyl quantizations of a symbol q are linked by the following identities

$$q^{\text{Wick}} = \left(q * 2^3 e^{-2\pi|\cdot|^2} \right)^w = q^w + r^w \quad (82)$$

with

$$r(Y) = \int_0^1 \int_{\mathbb{R}^6} (1 - \theta) q''(Y + \theta Z) Z^2 e^{-2\pi|Z|^2} 2^3 dZ d\theta.$$

We also recall the following composition formula obtained in the proof of Proposition 3.4 in [27]

$$q_1^{\text{Wick}} q_2^{\text{Wick}} = \left[q_1 q_2 - \frac{1}{4\pi} q_1' \cdot q_2' + \frac{1}{4i\pi} \{q_1, q_2\} \right]^{\text{Wick}} + T, \quad (83)$$

with T a bounded operator in $L^2(\mathbb{R}^{2n})$, when $q_1 \in L^\infty(\mathbb{R}^{2n})$ and q_2 is a smooth symbol whose derivatives of order ≥ 2 are bounded on \mathbb{R}^6 . The notation $\{q_1, q_2\}$ denotes the Poisson bracket defined by

$$\{q_1, q_2\} = \frac{\partial q_1}{\partial \eta} \cdot \frac{\partial q_2}{\partial v} - \frac{\partial q_1}{\partial v} \cdot \frac{\partial q_2}{\partial \eta}. \quad (84)$$

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