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► **To cite this version:**

Takashi Kishimoto, Yuri Prokhorov, Mikhail Zaidenberg. UNIPOTENT GROUP ACTIONS ON DEL PEZZO CONES. 2012. hal-00766761v1

**HAL Id: hal-00766761**

**<https://hal.science/hal-00766761v1>**

Preprint submitted on 18 Dec 2012 (v1), last revised 31 Dec 2012 (v2)

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# UNIPOTENT GROUP ACTIONS ON DEL PEZZO CONES

TAKASHI KISHIMOTO, YURI PROKHOROV, AND MIKHAIL ZAIDENBERG

**ABSTRACT.** In our previous paper [KPZ11b] we showed that for any del Pezzo surface  $Y$  of degree  $d \geq 4$  and for any  $r \geq 1$ , the affine cone  $X = \text{cone}_{r(-K_Y)}(Y)$  admits an effective  $\mathbb{G}_a$ -action. In particular, the group  $\text{Aut}(X)$  is infinite dimensional. In this note we prove that for a del Pezzo surface  $Y$  of degree  $\leq 2$  the generalized cones  $X$  as above do not admit any non-trivial action of a unipotent algebraic group.

## 1. INTRODUCTION

We are working over an algebraically closed field  $\mathbb{k}$  of characteristic 0. Let  $Y$  be a smooth projective variety with a polarization  $H$ , where  $H$  is an ample Cartier divisor. A *generalized affine cone* over  $(Y, H)$  is the affine variety

$$\text{cone}_H(Y) = \text{Spec} \bigoplus_{m \geq 0} H^0(Y, mH).$$

If  $H$  is very ample then  $\text{cone}_H(Y)$  is the usual affine cone over  $Y$  embedded in a projective space by the linear system  $|H|$ . In this paper we let  $Y$  be a del Pezzo surface and  $H$  be a pluri-anticanonical divisor on  $Y$ ; we call then  $\text{cone}_H(Y)$  a *del Pezzo cone*.

It is known [KPZ11b, 3.1.13] that for any smooth rational surface equipped with a suitable polarization, the associate affine cone admits an effective  $\mathbb{G}_a$ -action. Furthermore, for any del Pezzo surface of degree  $\geq 4$  the corresponding del Pezzo cones  $\text{cone}_{-mK_Y}(Y)$  ( $m \geq 1$ ) admit such an action (*loc.cit*). The latter holds also for some smooth rational Fano threefolds with Picard number 1 [KPZ11b, KPZ11a]. However, for del Pezzo surfaces of small degrees the consideration turns out to be more complicated. It is unknown so far whether the affine cone over a smooth cubic surface in  $\mathbb{P}^3$  admits a  $\mathbb{G}_a$ -action (cf. [KPZ11b, §4]). In this paper we investigate the cases  $d = 1$  and  $d = 2$ . Our main result can be stated as follows.

**Theorem 1.1.** *Let  $Y$  be a del Pezzo surface of degree  $d = K_Y^2 \leq 2$ . Then there is no non-trivial action of a unipotent group on the generalized affine cone*

$$X = \text{cone}_{-rK_Y}(Y) = \text{Spec } A, \quad \text{where } A = \bigoplus_{m \geq 0} H^0(Y, -mrK_Y).$$

**Corollary 1.2.** *In the notation as before assume that  $d \leq 2$  and  $r \geq 4 - d$  so that  $X = \text{cone}_{-rK_Y}(Y)$  is a usual del Pezzo cone. Then any algebraic subgroup  $G \subset \text{Aut}(X)$  is isomorphic to a subgroup of  $\mathbb{G}_m \times \text{Aut}(Y)$ , where  $\text{Aut}(Y)$  is finite.*

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2010 *Mathematics Subject Classification.* Primary 14R20, 14J45; Secondary 14J50, 14R05.

*Key words and phrases.* affine cone, del Pezzo surface, additive group, group action.

The first author was supported by a Grant-in-Aid for Scientific Research of JSPS No. 24740003. The second author was partially supported by RFBR grants No. 11-01-00336-a, the grant of Leading Scientific Schools No. 4713.2010.1, Simons-IUM fellowship, and AG Laboratory SU-HSE, RF government grant ag. 11.G34.31.0023.

*Proof.* As follows from Theorem 1.1  $G$  is a reductive group. Thus by Lemma 2.3.1 and Proposition 2.2.6 in [KPZ11b] there are an injection and an isomorphism

$$G \hookrightarrow \mathrm{Lin}(X) \simeq \mathbb{G}_m \times \mathrm{Lin}(Y) \subset \mathbb{G}_m \times \mathrm{Aut}(Y),$$

where the group  $\mathrm{Aut}(Y)$  is finite, see [Dol].  $\square$

We suggest the following

**1.3. Conjecture.** If  $d \leq 2$  then the full automorphism group  $\mathrm{Aut}(X)$  is a finite extension of the multiplicative group  $\mathbb{G}_m$ .

Likewise in [KPZ11b, KPZ11a] we use a geometric criterion of existence of an effective  $\mathbb{G}_a$ -action on the affine cone  $\mathrm{cone}_H(Y)$  (see Theorem 2.1 below).

Sections 2, 3, and 4 contain necessary preliminaries. Theorem 1.1 is proven in section 5. The proof proceeds as follows. Assuming to the contrary that there exists a non-trivial unipotent group action on  $X = \mathrm{cone}_{r(-K_Y)}(Y)$ , there also exists an effective  $\mathbb{G}_a$ -action on  $X$ . By Theorem 2.1 there is an effective  $\mathbb{Q}$ -divisor  $D$  on  $Y$  such that  $D \sim_{\mathbb{Q}} -K_Y$  and  $U = Y \setminus D \cong Z \times \mathbb{A}^1$ , where  $Z$  is a smooth rational affine curve. Such a principal open subset  $U$  is called in [KPZ11b] a  $(-K_Y)$ -polar cylinder. One of the key points consists in an estimate for the singularities of the pair  $(Y, D)$ . More precisely, we consider the linear pencil  $\mathcal{L}$  on  $Y$  generated by the closures of the fibers of the projection  $U \cong Z \times \mathbb{A}^1 \rightarrow Z$ . Letting  $S$  be the last exceptional divisor appearing in the process of the minimal resolution of the base locus of  $\mathcal{L}$  we compute the discrepancy  $a(S; D)$ . Using this and some subtle geometrical properties of the pair  $(Y, D)$  we finally come to a contradiction.

## 2. CRITERION

Let  $Y$  be a projective variety and  $H$  be an ample divisor on  $Y$ . Recall [KPZ11b] that an  $H$ -polar cylinder in  $Y$  is an open subset  $U = Y \setminus \mathrm{supp}(D)$ , where  $D = \sum_i \delta_i \Delta_i$  with  $\delta_i > 0 \ \forall i$  is an effective  $\mathbb{Q}$ -divisor on  $Y$ , which is  $\mathbb{Q}$ -linearly equivalent to  $H$ , and  $U$  is isomorphic to  $Z \times \mathbb{A}^1$  for some affine variety  $Z$ . In [KPZ11b] and [KPZ] we provided the following useful criterion for existence of an effective  $\mathbb{G}_a$ -action on the affine cone.

**Theorem 2.1.** *Let  $Y$  be a projective algebraic variety with an ample polarization  $H$ , and let  $X = \mathrm{cone}_H(Y)$  be the corresponding generalized affine cone. Then  $X$  admits an effective  $\mathbb{G}_a$ -action if and only if  $Y$  contains an  $H$ -polar cylinder.*

We apply below this criterion to del Pezzo surfaces and their anticanonical divisors. Note that our cone  $\mathrm{Spec} \bigoplus_{i \geq 0} H^0(Y, -irK_Y)$  is a normal affine variety (see e.g. [Dol, Theorem 8.3.4]).

## 3. PRELIMINARIES ON WEAK DEL PEZZO SURFACES

A smooth projective surface  $Y$  is called a *del Pezzo surface* if the anticanonical divisor  $-K_Y$  is ample, and a *weak del Pezzo surface* if  $-K_Y$  is big and nef. The *degree* of such a surface is  $\deg Y = K_Y^2 \in \{1, \dots, 9\}$ .

**Lemma 3.1** (see e.g. [Dol, Proposition 8.1.23]). *Blowing up a point on a del Pezzo surface of degree  $d \geq 2$  yields a weak del Pezzo surface of degree  $d - 1$ .*

**Theorem 3.2** (see e.g. [Dol, Thm. 8.3.2]). *Let  $Y$  be a del Pezzo surface of degree  $d$ . Then the following hold.*

- (i) *If  $d \geq 3$  then  $|-K_Y|$  defines an embedding  $Y \hookrightarrow \mathbb{P}^d$ .*
- (ii) *If  $d = 2$  then  $|-K_Y|$  defines a double cover  $\Phi : Y \rightarrow \mathbb{P}^2$  branched along a smooth curve  $B \subset \mathbb{P}^2$  of degree 4.*
- (iii) *If  $d = 1$  then  $|-K_X|$  is a pencil with a single base point, say  $O$ . The linear system  $|-2K_Y|$  defines a double cover  $\Phi : Y \rightarrow Q' \subset \mathbb{P}^3$ , where  $Q'$  is a quadric cone with vertex at  $\Phi(O)$ . Furthermore  $\Phi$  is branched along a smooth curve  $B \subset Q'$  cut out on  $Q'$  by a cubic surface.*

The Galois involution  $\tau : Y \rightarrow Y$  associated to the double cover  $\Phi$  is a regular morphism. It is called *Geiser involution* in the case  $d = 2$  and *Bertini involution* in the case  $d = 1$ .

**Remark 3.3.** Recall the following facts (see e.g. [Dol]). For an irreducible curve  $C$  on  $Y$  we have  $C^2 \geq -1$  if  $Y$  is a del Pezzo surface and  $C^2 \geq -2$  if  $Y$  is a weak del Pezzo surface. In both cases  $C^2 = -1$  if and only if  $C$  is a  $(-1)$ -curve, if and only if  $-K_Y \cdot C = 1$ , and  $C^2 = -2$  if and only if  $C$  is a  $(-2)$ -curve, if and only if  $-K_Y \cdot C = 0$ . A weak del Pezzo surface is del Pezzo if and only if it has no  $(-2)$ -curve.

If  $d \geq 2$  then any curve  $C$  on  $Y$  such that  $-K_Y \cdot C = 1$  is an irreducible smooth rational curve by (i) and (ii). By the adjunction formula such  $C$  must be a  $(-1)$ -curve.

**Lemma 3.4.** *Let  $Y$  be a del Pezzo surface of degree  $d \leq 2$ . Then any member  $R \in |-K_Y|$  is reduced and  $p_a(R) = 1$ . Moreover,  $R$  is irreducible except in the case where*

- $d = 2$ ,  $R = R_1 + R_2$ ,  $R_i^2 = -1$ ,  $i = 1, 2$ ,  $R_1 \cdot R_2 = 2$ , and  $R_2 = \tau(R_1)$ .

*Furthermore,  $\text{Sing}(R) \subset \Phi^{-1}(B)$  and for any  $P \in \Phi^{-1}(B)$  there is a unique member  $R \in |-K_Y|$  singular at  $P$ .*

*Proof.* We have  $p_a(R) = 1$  by adjunction. Let  $R_1 \subsetneq R$  be a reduced irreducible component. Then  $(-K_Y) \cdot R_1 < (-K_Y) \cdot R = d$  and so  $d = 2$  and  $R_1$  is a  $(-1)$ -curve by Remark 3.3. Since  $R^2 = d = 2$ ,  $R \neq 2R_1$ . Therefore  $R = R_1 + R_2$ , where **the**  $R_i$  ( $i = 1, 2$ ) are  $(-1)$ -curves and  $R_1 \cdot R_2 = \frac{1}{2}(R^2 - R_1^2 - R_2^2) = 2$ . Finally, in both cases we have  $R = \Phi^{-1}(L)$ , where  $L$  is a line **in**  $\mathbb{P}^2$ . Thus  $R$  is singular at  $P$  if and only if  $\Phi(P) \in B$  and  $L$  is tangent to  $B$  at  $\Phi(P)$ .  $\square$

**Remark 3.5.** Let  $R_1$  and  $R_2$  be  $(-1)$ -curves on a del Pezzo surface  $Y$  of degree 2 such that  $R_1 \cdot R_2 \geq 2$ . Then  $R_2 = \tau(R_1)$ ,  $R_1 \cdot R_2 = 2$ , and  $R_1 + R_2 \in |-K_Y|$ . Indeed,  $R_1 + \tau(R_1) \sim -K_Y$ . Hence  $\tau(R_1) \cdot R_2 = -1$  and so  $\tau(R_1) = R_2$ .

#### 4. $(-K)$ -POLAR CYLINDERS ON DEL PEZZO SURFACES

We adjust here some lemmas in [KPZ11b, §4] to our setting.

**Notation 4.1.** Let  $Y$  be a del Pezzo surface of degree  $d$ . Suppose that  $Y$  admits a  $(-K_Y)$ -polar cylinder

$$(4.2) \quad U = Y \setminus \text{supp}(D) \cong Z \times \mathbb{A}^1, \quad \text{where} \quad D = \sum_{i=1}^n \delta_i \Delta_i \sim_{\mathbb{Q}} -K_Y \quad (\delta_i > 0)$$

and  $Z$  is a smooth rational affine curve. We let  $\mathcal{L}$  be the linear pencil on  $Y$  defined by the rational map  $\Psi : Y \dashrightarrow \mathbb{P}^1$  which extends the projection  $\text{pr}_1 : U \cong Z \times \mathbb{A}^1 \rightarrow Z$ .

Resolving, if necessary, the base locus of the pencil  $\mathcal{L}$  we obtain a diagram

$$(4.3) \quad \begin{array}{ccc} & W & \\ p \swarrow & & \searrow q \\ Y & \xrightarrow{\Psi} & \mathbb{P}^1 \end{array}$$

where we let  $p : W \rightarrow Y$  be the shortest succession of blowups such that the proper transform  $\mathcal{L}_W := p_*^{-1}\mathcal{L}$  is base point free. Let  $S$  be the last exceptional curve of the modification  $p$  unless  $p$  is the identity map, i.e.,  $\text{Bs } \mathcal{L} = \emptyset$ . Notice that  $S$  is a unique  $(-1)$ -curve in the exceptional locus  $p^{-1}(P)$  and a section of  $q$ . The restriction  $\Phi_{\mathcal{L}_W}|_U$  is an  $\mathbb{A}^1$ -fibration and its fibers are reduced, irreducible affine curves with one place at infinity, situated on  $S$ .

**Lemma 4.4.** *One of the following holds.*

- (i)  $\text{Bs } \mathcal{L}$  consists of a single point, say  $P$ ;
- (ii)  $\text{Bs } \mathcal{L} = \emptyset$  and  $5 \leq d \leq 8$ .

*Proof.* Since the general members of  $\mathcal{L}$  are disjoint in  $U$  and each one meets the cylinder  $U$  along an  $\mathbb{A}^1$ -curve,  $\text{Bs } \mathcal{L}$  consists of at most one point, which we denote by  $P$ . Suppose that  $\text{Bs } \mathcal{L} = \emptyset$ . Then the pencil  $\mathcal{L}$  yields a conic bundle  $\Psi : Y \rightarrow \mathbb{P}^1$  with a section, which is a component of  $D$ , say  $\Delta_0$ . In particular  $d \leq 8$ . For a general fiber  $L$  of  $\Psi$  we have

$$L^2 = 0, \quad -K_Y \cdot L = 2 = D \cdot L = \delta_0.$$

Note that  $\Psi$  has exactly  $8 - d$  degenerate fibers  $L_1, \dots, L_{8-d}$ . Each of these fibers is reduced and consists of two  $(-1)$ -curves meeting transversally at a point. Let  $C_i$  be the component of  $L_i$  that meets  $\Delta_0$ . We claim that each  $C_i$  is a component of  $D$ . Indeed, otherwise

$$1 = -K_Y \cdot C_i = D \cdot C_i \geq \delta_0 \Delta_0 \cdot C_i = \delta_0 = 2,$$

a contradiction. Therefore we may assume that  $C_i = \Delta_i$  and so

$$1 = D \cdot C_i \geq \delta_0 \Delta_0 \cdot C_i + \delta_i C_i^2 = 2 - \delta_i.$$

Hence  $\delta_i \geq 1$ ,  $i = 1, \dots, 8 - d$ . We obtain

$$d = -K_Y \cdot D \geq \sum \delta_i \geq \delta_0 + \sum_{i=1}^{8-d} \delta_i \geq 2 + 8 - d = 10 - d.$$

Thus  $d \geq 5$  as stated. □

**Remark 4.5.** If  $\text{Bs } \mathcal{L} = \{P\}$  ( $\text{Bs } \mathcal{L} = \emptyset$ , respectively) then all components of  $D$  (all components of  $D$  except for  $\Delta_0$ , respectively) are contained in the fibers of  $\Psi$ . Indeed, otherwise not all the fibers of  $\Psi|_U$  were  $\mathbb{A}^1$ -curves, contrary to the definition of a cylinder.

**Lemma 4.6.** *The number of irreducible components of the reduced curve  $\text{supp}(D)$ , say  $n$ , is greater than or equal to  $10 - d$ .*

*Proof.* Consider the exact sequence

$$\bigoplus_{i=1}^n \mathbb{Z}[\Delta_i] \longrightarrow \text{Pic}(Y) \longrightarrow \text{Pic}(U) \longrightarrow 0.$$

Since  $\text{Pic}(Z) = 0$  and  $U \cong Z \times \mathbb{A}^1$  we have  $\text{Pic}(U) = 0$ . Hence  $n \geq \rho(Y) = 10 - d$ , as stated.  $\square$

**Lemma 4.7.** *Assume that  $\text{Bs } \mathcal{L} = \{P\}$ . Let  $L$  be a member of  $\mathcal{L}$  and  $C$  be an irreducible component of  $L$ . Then the following hold.*

- (i)  $\text{supp}(L)$  is simply connected and  $\text{supp}(L) \setminus \{P\}$  is an SNC divisor;
- (ii)  $C$  is rational and smooth outside  $P$ ;
- (iii) if  $P \in C$  then  $C \setminus \{P\} \simeq \mathbb{A}^1$ .

*Proof.* All the assertions follow from the fact that  $q$  in (4.3) is a rational curve fibration and the exceptional locus of  $p$  coincides with  $p^{-1}(P)$ .  $\square$

In the next lemma we study the singularities of the pair  $(Y, D)$ . We refer to [Kol97] or to [KM98, Chapter 2] for the standard terminology on singularities of pairs.

**Lemma 4.8** (Key Lemma). *Assume that  $\text{Bs } \mathcal{L} = \{P\}$ . Then the pair  $(Y, D)$  is not log canonical at  $P$ . More precisely, in notation as in 4.1 the discrepancy  $a(S; D)$  of  $S$  with respect to  $K_Y + D$  is equal to  $-2$ .*

*Proof.* We write

$$(4.9) \quad K_W + D_W \sim_{\mathbb{Q}} p^*(K_Y + D) + a(S; D)S + \sum a(E; D)E,$$

where the summation on the right hand side ranges over the components of the exceptional divisor of  $p$  except for  $S$ , and  $D_W$  is the proper transform of  $D$  on  $W$ . Letting  $l$  be a general fiber of  $q$ , by (4.9) we obtain

$$-2 = (K_W + D_W) \cdot l = a(S; D).$$

Indeed,  $K_Y + D \sim_{\mathbb{Q}} 0$  and  $l$  does not meet the curve  $\text{supp}(D_W + p^*(P) - S)$ . This proves the assertion.  $\square$

**Corollary 4.10.** *If  $\text{Bs } \mathcal{L} = \{P\}$  then  $\text{mult}_P(D) > 1$ .*

*Proof.* Indeed, otherwise the pair  $(Y, D)$  would be canonical by [Kol97, Ex. 3.14.1], and in particular, log canonical at  $P$ , which contradicts Lemma 4.8.  $\square$

**Corollary 4.11.** *If  $\text{Bs } \mathcal{L} = \{P\}$  then every  $(-1)$ -curve  $C$  on  $Y$  passing through  $P$  is contained in  $\text{supp}(D)$ .*

*Proof.* Assume to the contrary that  $C$  is not a component of  $D$ . Then

$$\text{mult}_P D \leq C \cdot D = -K_Y \cdot C = 1,$$

which contradicts Corollary 4.10.  $\square$

**Convention 4.12.** From now on we assume that  $d \leq 3$ . By Lemma 4.4 we have  $\text{Bs } \mathcal{L} = \{P\}$ .

**Lemma 4.13.** *We have  $[D] = 0$  i.e.  $\delta_i < 1$  for all  $i = 1, \dots, n$ .*

*Proof.* For the case  $d = 3$  see [KPZ11b, Lemma 4.1.5]. Consider the case  $d = 1$ . By Lemma 4.6  $n \geq 9$ . For any  $i = 1, \dots, n$  we have

$$1 = -K_Y \cdot D = \sum_{j=1}^n \delta_j (-K_Y) \cdot \Delta_j > \delta_i (-K_Y) \cdot \Delta_i.$$

Since the anticanonical divisor  $-K_Y$  is ample, it follows that  $\delta_i < 1$ , as required.

Let further  $d = 2$ . Assuming that  $\delta_1 \geq 1$  we obtain:

$$(4.14) \quad 2 = -K_Y \cdot D = \sum_{i=1}^n \delta_i (-K_Y) \cdot \Delta_i > \delta_1 (-K_Y) \cdot \Delta_1 \geq -K_Y \cdot \Delta_1,$$

where  $n \geq 8$  by Lemma 4.6. It follows that  $-K_Y \cdot \Delta_1 = 1$ , i.e.  $\Delta_1$  is a  $(-1)$ -curve. Then  $C := \tau(\Delta_1)$  is also a  $(-1)$ -curve, where  $\tau$  is the Geiser involution, and  $\Delta_1 + C \sim -K_Y$ . If  $C \subset \text{supp}(D)$ , e.g.  $C = \Delta_2$ , then by (4.14) we obtain that  $\delta_2 < 1$ . Now  $\Delta_1 + \Delta_2 \sim_{\mathbb{Q}} D$  yields a relation with positive coefficients

$$(1 - \delta_2)\Delta_2 \sim_{\mathbb{Q}} (\delta_1 - 1)\Delta_1 + \sum_{i=3}^n \delta_i \Delta_i.$$

This implies that  $C^2 = \Delta_2^2 \geq 0$ , a contradiction.

Hence  $C \not\subset \text{supp}(D)$ . Thus  $C \sim_{\mathbb{Q}} D - \Delta_1$ , where the right hand side is effective. This leads to a contradiction as before.  $\square$

**Lemma 4.15.** <sup>1</sup> *For a member  $L$  of  $\mathcal{L}$ , any irreducible component of  $L$  passes through the base point  $P$  of  $\mathcal{L}$ .*

*Proof.* Assume to the contrary that there exists a component  $C$  of  $L$  such that  $P \notin C$ . Then clearly  $C^2 < 0$  (see the proof of Lemma 4.4). Since also  $-K_Y \cdot C > 0$ ,  $C$  is a  $(-1)$ -curve. Let  $C'$  be a component of  $L$  meeting  $C$ . If  $P \notin C'$ , then  $C$  and  $C'$  are both  $(-1)$ -curves and so  $L = C + C'$ . Thus  $\mathcal{L} = |C + C'|$  is base point free, which contradicts Lemma 4.4. Hence  $C'$  passes through  $P$ . Since  $P$  is a unique base point of  $\mathcal{L}$ ,  $C$  does not meet any member  $L' \in \mathcal{L}$  different from  $L$ . By Lemma 4.7  $L$  is simply connected, so  $C'$  is the only component of  $L$  meeting  $C$ . Note that  $\text{supp}(D)$  is connected because  $D$  is ample. Hence  $C'$  must be contained in  $\text{supp}(D)$ . In fact, supposing to the contrary that  $C'$  is not contained in  $\text{supp}(D)$ , the curve  $C$  must be contained in  $\text{supp}(D)$ . Indeed, the affine surface  $U = Y \setminus \text{supp}(D)$  does not contain any complete curve. Since  $\text{supp}(D)$  is connected there is an irreducible component of  $\text{supp}(D)$  intersecting  $C$  and passing through  $P$ . This contradicts Lemma 4.7. Thus we may suppose that  $C' = \Delta_1$ .

If  $C \subset \text{supp}(D)$ , say,  $C = \Delta_2$ , then

$$1 = -K_Y \cdot C = \left( \sum_{i=1}^n \delta_i \Delta_i \right) \cdot \Delta_2 = \delta_1 - \delta_2.$$

Hence  $\delta_1 = \delta_2 + 1 > 1$ , which contradicts Lemma 4.13.

Therefore  $C \not\subset \text{supp}(D)$  and so

$$1 = -K_Y \cdot C = \left( \sum_{i=1}^n \delta_i \Delta_i \right) \cdot C = \delta_1,$$

which again gives a contradiction by Lemma 4.13.  $\square$

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<sup>1</sup>Cf. [KPZ11b, Lemma 4.1.6].

## 5. PROOF OF THEOREM 1.1

According to our geometric criterion 2.1, Theorem 1.1 is a consequence of the following proposition.

**Proposition 5.1.** *Let  $Y$  be a del Pezzo surface of degree  $d \leq 2$ . Then  $Y$  does not admit any  $(-K_Y)$ -polar cylinder.*

**Convention 5.2.** We let  $Y$  be a del Pezzo surface of degree  $d \leq 2$ . We assume to the contrary that  $Y$  possesses a  $(-K_Y)$ -polar cylinder  $U$  as in (4.2). By Lemma 4.4 we have  $\text{Bs } \mathcal{L} = \{P\}$ .

**Lemma 5.3.** *For any  $R \in |-K_Y|$  we have  $\text{supp}(R) \not\subseteq \text{supp}(D)$ .*

*Proof.* Suppose to the contrary that  $\text{supp}(R) \subset \text{supp}(D)$ . Let  $\lambda \in \mathbb{Q}_{>0}$  be maximal such that  $D - \lambda R$  is effective. We can write

$$D = \lambda R + D_{\text{res}},$$

where  $D_{\text{res}}$  is an effective  $\mathbb{Q}$ -divisor such that  $\text{supp}(R) \not\subseteq \text{supp}(D_{\text{res}})$ . For  $t \in \mathbb{Q}_{\geq 0}$  we consider the following linear combination

$$D_t := D - tR + \frac{t}{1-\lambda} D_{\text{res}} \sim_{\mathbb{Q}} -K_Y.$$

We have  $D_0 = D$  and  $D_\lambda = \frac{1}{1-\lambda} D_{\text{res}}$ . For  $t < \lambda$ , the  $\mathbb{Q}$ -divisor  $D_t$  is effective with  $\text{supp}(D_t) = \text{supp}(D)$ . By Lemma 4.8 applied to  $D_t$  instead of  $D$ , for any  $t < \lambda$  the pair  $(Y, D_t)$  is not log canonical at  $P$ , with discrepancy  $a(S; D_t) = -2$ . Since the function  $t \mapsto a(S; D_t)$  is continuous, passing to the limit we obtain  $a(S; D_\lambda) = -2$ . Hence the pair  $(Y, D_\lambda)$  is not log canonical at  $P$  either and so  $\text{mult}_P(D_\lambda) > 1$ .

Assume that  $R$  is irreducible. Since  $R \subset \text{supp}(D)$ ,  $R$  is a component of a member of  $\mathcal{L}$ . Hence the curve  $R$  is smooth outside  $P$  and rational (see Lemma 4.7(ii)). Since  $p_a(R) = 1$ ,  $R$  is singular at  $P$  and  $\text{mult}_P(R) = 2$ . Since  $R$  is different from the components of  $D_\lambda$  and  $\text{mult}_P(D_\lambda) > 1$  we obtain

$$(5.4) \quad 2 \geq K_Y^2 = D_\lambda \cdot R \geq \text{mult}_P(D_\lambda) \text{mult}_P(R) > 2,$$

a contradiction.

Let further  $R$  be reducible. By Lemma 3.4 we have  $d = 2$  and  $R = R_1 + R_2$ , where, say,  $R_i = \Delta_i$ ,  $i = 1, 2$ , are  $(-1)$ -curves passing through  $P$  (see Lemma 4.15). We may assume that  $\delta_1 \leq \delta_2$  and so  $\lambda = \delta_1$ . Since  $\Delta_1$  is not a component of  $D_\lambda$  we obtain

$$1 = -K_Y \cdot R_1 = D_\lambda \cdot \Delta_1 \geq \text{mult}_P(D_\lambda) > 1,$$

a contradiction. This finishes the proof. □

*Proof of Proposition 5.1 in the case  $d = 1$ .* Since  $\dim |-K_Y| = 1$  there is  $C \in |-K_Y|$  passing through  $P$ . Furthermore, by Lemma 3.4  $C$  is irreducible. By Lemma 5.3  $C$  is not contained in  $\text{supp}(D)$ . Likewise in (5.4) we get a contradiction. Indeed, by Corollary 4.10 we have

$$1 = C^2 = D \cdot C \geq \text{mult}_P D \cdot \text{mult}_P C > 1.$$

□

**Convention 5.5.** We assume in the remaining part that  $d = 2$ .

**Lemma 5.6.** *A member  $R \in |-K_Y|$  cannot be singular at  $P$ .*



*Proof.* Assume that  $P \in \text{Sing}(R)$ . By Lemma 3.4 we have two possibilities for  $R$ . Suppose first that  $R$  is irreducible. By Lemma 5.3  $R \not\subset \text{supp}(D)$  and we get a contradiction likewise in (5.4). In the second case  $R = R_1 + R_2$ , where  $R_1$  and  $R_2$  are  $(-1)$ -curves passing through  $P$ . Hence  $R_1, R_2 \subset \text{supp}(D)$  by Corollary 4.11. The latter contradicts Lemma 5.3.  $\square$

**Notation 5.7.** We let  $f : Y' \rightarrow Y$  be the blowup of  $P$  and  $E' \subset Y'$  be the exceptional divisor. By Lemma 3.1  $Y'$  is a weak del Pezzo surface of degree 1.

**5.8.** Applying Proposition 5.1 with  $d = 1$ , we can conclude that  $Y'$  is not del Pezzo because it contains a  $-K_{Y'}$ -polar cylinder. Indeed, let  $D'$  be the crepant pull-back of  $D$  on  $Y'$ , that is,

$$K_{Y'} + D' = f^*(K_Y + D) \quad \text{and} \quad f_*D' = D.$$

Then

$$(5.9) \quad D' = \sum_{i=1}^6 \delta_i \Delta'_i + \delta_0 E', \quad \text{where} \quad \delta_0 = \text{mult}_P(D) - 1 > 0$$

(see Lemma 4.10) and  $\Delta'_i$  is the proper transform of  $\Delta_i$  on  $Y'$ . Thus  $D'$  is an effective  $\mathbb{Q}$ -divisor on  $Y'$  such that  $D' \sim_{\mathbb{Q}} -K_{Y'}$  and  $Y' \setminus \text{supp } D' \simeq U \simeq Z \times \mathbb{A}^1$  is a  $-K_{Y'}$ -polar cylinder.

**Lemma 5.10.** *We have  $\text{mult}_P(D) < 2$  and  $\lfloor D' \rfloor = 0$ .*

*Proof.* Suppose first that all components of  $D$  are  $(-1)$ -curves. Then  $\Delta_i \cdot \Delta_j = 1$  for  $i \neq j$  by Remark 3.5 and Lemma 5.3. Hence  $f$  is a log resolution of the pair  $(Y, D)$ . Therefore  $1 - \sum \delta_i = a(Y, E') < -1$  by Lemma 4.8, so  $\sum \delta_i > 2$ . On the other hand  $2 = -K_Y \cdot D = \sum \delta_i$ , a contradiction. This shows that there exists a component  $\Delta_i$  of  $D$  which is not a  $(-1)$ -curve. By the dimension count there exists an effective divisor  $R \in |-K_Y|$  passing through  $P$  and a general point  $Q \in \Delta_i$ . On the other hand, there is no  $(-1)$ -curve in  $Y$  passing through  $Q$ . So by Lemma 3.4 we may assume that  $R$  is reduced and irreducible. By Lemma 5.3  $R$  is different from the components of  $D$ . Assuming that  $\text{mult}_P(D) \geq 2$  we obtain

$$2 = R \cdot D \geq \text{mult}_P(D) + \delta_i > 2,$$

a contradiction. This proves the first assertion. Now the second follows since  $\delta_0 > 0$  in (5.9).  $\square$

**Corollary 5.11.** *The pair  $(Y', D')$  is Kawamata log terminal in codimension one and is not log canonical at some point  $P' \in E'$ .*

*Proof.* This follows from Lemma 5.10 taking into account that  $D'$  is the crepant pull-back of  $D$ , see [Kol97, L. 3.10].  $\square$

Since  $\dim |-K_{Y'}| = 1$  there exists an element  $C' \in |-K_{Y'}|$  passing through the point  $P'$  as in Corollary 5.11.

**Lemma 5.12.** *The point  $P \in Y$  is a smooth point of the image  $C = f_*C'$ .*

*Proof.* This follows by Lemma 5.6 since  $C \in |-K_Y|$  passes through  $P$ .  $\square$

**Corollary 5.13.**  *$E'$  is not a component of  $C'$ .*

*Proof.* We can write  $f^*C = C' + kE'$  for some  $k \in \mathbb{Z}$ . Then  $k = -kE'^2 = C' \cdot E' = 1$ . By Lemma 5.12 the coefficient of  $E'$  in  $f^*C$  is equal to 1 as well. Now the assertion follows.  $\square$

**Lemma 5.14.**  *$C$  is reducible.*

*Proof.* Indeed, otherwise  $C'$  is irreducible by Corollary 5.13. Since  $\text{mult}_{P'} D' > 1$  by Corollary 5.11 and  $D' \cdot C' = K_Y^2 = 1$ ,  $C'$  is a component of  $D'$ . Hence  $C$  is a component of  $D$ . This contradicts Lemma 5.3.  $\square$

**Lemma 5.15.** *We have  $C' = C'_1 + C'_2$ , where  $C_1$  is a  $(-1)$ -curve,  $C'_2$  is a  $(-2)$ -curve, and  $C'_1 \cdot C'_2 = 2$ . Furthermore,  $P' \in C'_2 \setminus C'_1$ , and  $C_2 = f(C'_2)$  is a  $(-1)$ -curve.*

*Proof.* Since  $C$  is reducible and  $C \in |-K_Y|$ , by Lemma 3.4  $C = C_1 + C_2$ , where  $C_1, C_2$  are  $(-1)$ -curves with  $C_1 \cdot C_2 = 2$ . By Lemma 5.12  $P \notin C_1 \cap C_2$ , where  $C_2$  is a component of  $D$  by Corollary 4.11, while by Lemma 5.3  $C_1$  is not. So we may assume that  $P \in C_2 \setminus C_1$ . Now the lemma follows from Corollary 5.11.  $\square$

**5.16.** Letting in the sequel  $C_2 = \Delta_1$  we can write  $D = \delta_1 C_2 + D_{\text{res}}$ , where  $\delta_1 > 0$ ,  $D_{\text{res}}$  is an effective  $\mathbb{Q}$ -divisor, and  $C_2$  is not a component of  $D_{\text{res}}$ . Similarly

$$D' = \delta_1 C'_2 + D'_{\text{res}} + \delta_0 E',$$

where  $D'_{\text{res}}$  is the proper transform of  $D_{\text{res}}$  and  $\delta_0 = \text{mult}_P(D) - 1$  (cf. (5.9)).

**Lemma 5.17.** *We have  $2\delta_1 \leq 1$ .*

*Proof.* This follows from

$$0 \leq D_{\text{res}} \cdot C_1 = (D - \delta_1 C_2) \cdot C_1 = 1 - 2\delta_1.$$

$\square$

**Lemma 5.18.** *In the notation as before  $\delta_0 + D'_{\text{res}} \cdot C'_2 > 1$ .*

*Proof.* Let us show first that  $\{P'\} = C'_2 \cap E' = C'_2 \cap \text{supp}(D'_{\text{res}})$ . Indeed,  $P' \in E'$  by construction,  $P' \in C'_2$  by Lemma 5.15, and  $P' \in \text{supp}(D'_{\text{res}})$  because otherwise  $P'$  would be a node of  $D'$  (indeed,  $E'$  meets  $C'_2$  transversally at  $P'$ ) and so the pair  $(Y', D')$  would be log canonical at  $P'$  contrary to Corollary 5.11. On the other hand, the curves  $C'_2$  and  $D'_{\text{res}}$  have only one point in common by Lemma 4.7(i).

Since  $\delta_1 < 1$  the pair  $(Y', C'_2 + D'_{\text{res}} + \delta_0 E')$  is not log canonical at  $P'$ . Now applying [KM98, Corollary 5.57] we obtain

$$1 < (D'_{\text{res}} + \delta_0 E') \cdot C'_2 = \delta_0 + D'_{\text{res}} \cdot C'_2,$$

as stated.  $\square$

*Proof of Proposition 5.1 in the case  $d = 2$ .* We use the notation as above. Since  $C'_2$  is a  $(-2)$ -curve, by virtue of Lemmas 5.17 and 5.18 we obtain

$$1 - \delta_0 < D'_{\text{res}} \cdot C'_2 = (D' - \delta_1 C'_2 - \delta_0 E') \cdot C'_2 = 2\delta_1 - \delta_0 \leq 1 - \delta_0,$$

a contradiction. Now the proof of Proposition 5.1 is completed.  $\square$

**Acknowledgements.** This work was done during a stay of the second and third authors at the Max Planck Institute für Mathematik at Bonn and a stay of the first and the second authors at the Institute Fourier, Grenoble. The authors thank these institutions for their hospitality and support.

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