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Very Strong Disorder for the Parabolic Anderson model in low dimensions

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Abstract

We study the free energy of the Parabolic Anderson Model, a time-continuous model of directed polymers in random environment. We prove that in dimension 1 and 2, the free energy is always negative, meaning that very strong disorder always holds. The result for discrete polymers in dimension two, as well as better bounds on the free energy on dimension 1, were first obtained by Hubert Lacoin in [10], and the goal of this paper is to adapt his proof to the Anderson Parabolic Model.

Keywords : Parabolic Anderson model; Brownian Directed Polymer; Free energy; Strong disorder; Coarse graining.

1 Model and known results

1.1 The Parabolic Anderson Model

The general model we consider in this paper is defined in terms of a random trajectory in a random environment :

- The random trajectory : let $((X_t)_{t\geq 0}, P^\kappa)$ be a nearest-neighbour random walk on the lattice $\mathbb{Z}^d$ with rate of jump κ starting from 0. Specifically, we let $(\mathcal{M}, \mathcal{F})$ be the path space of càdlàg trajectories from $\mathbb{R}_+$ to $\mathbb{Z}^d$, and the process (X_t, P^κ) has for infinitesimal generator $\kappa\triangle^d$, where $\triangle^d$ is the discrete Laplacian. Sometimes we will need to start the random walk from a point other than 0, let $((X_t), P^\kappa_x)$ be the walk starting from the point x.

- The random environment : let $((B^x_t(\omega))_{t\geq 0}, x \in \mathbb{Z}^d)$ be a family of independent Brownian motions defined on a probability space $(\Omega, \mathcal{G}, Q)$, one for each vertex of $\mathbb{Z}^d$. For $t > 0$, it is natural and convenient to introduce the sub-σ-field

$$\mathcal{G}_t = \sigma\{B^x_s, s \in [0,t], x \in \mathbb{Z}^d\}.$$

- The Gibbs measure : for a time horizon t we define the following Hamiltonian :

$$H_t = H_t(X) = \int_0^t dB_s^{X_s} = \sum_{x\in\mathbb{Z}^d} \int_0^t 1_{\{X_s=x\}} dB_s^x.$$

For any $t > 0$, we define a probability measure $\mu_t^{\kappa,\beta}$ on the path space $(\Omega, \mathcal{F})$

$$\mu_t^{\kappa,\beta}(dX) = \frac{\exp(\beta H_t(X) - t\beta^2/2)}{Z_t^{\kappa,\beta}} P^\kappa(dX),$$

where $\beta \geq 0$ is the inverse temperature and

$$Z_t^{\kappa,\beta} = P^\kappa\left[\exp(\beta H_t(X) - t\beta^2/2))\right].$$

Since the polymer measure is parametrized by the environment ω, it is random.

This model has been studied in [4] with the point of view of intermittency, and in [9] when interacting with a particle system. It has also been studied as an example of polymer model in [1], [2]. In [13] Moriarty and O'Connell studied a perfectly assymetrical version of the 1-dimensional model in which they could prove exact formulas.

In this paper, we investigate the asymptotic behavior of the partition function $Z_t^{\kappa,\beta}$ when $t \to \infty$.

Let us finish the definition of the model with some remarks on scales. The process (X_t, P^κ) has the same law than $(X_{\kappa t}, P^1)$, and if B_t is a Brownian motion, $B_{\kappa t}$ has the same law than $\sqrt{\kappa} B_t$. Therefore, $Z_t^{\kappa,\beta}$ has the same law than $Z_{\kappa t}^{1,\frac{\beta}{\sqrt{\kappa}}}$, and we only need to study the case of $\kappa = 1$, and the rest is obtain by rescaling.

We will denote by P, μ_t^β, Z_t^β,... the quantities P^κ, $\mu_t^{\kappa,\beta}$, $Z_t^{\kappa,\beta}$,... with $\kappa = 1$. In most case, when the value of β is not ambiguous, we will only write μ_t and Z_t.

1.2 The partition function

Let us fix a $\beta \geq 0$ and begin by stating that the partition function Z_t is a martingale on $(\mathcal{M}, \mathcal{G}, Q)$. For any fixed path X, the process $\{H_t(X)\}_{t\geq 0}$ is himself a Brownian motion, and $\{\exp(\beta H_t - t\beta^2/2)\}_{t\geq 0}$ is its exponential martingale. Therefore, the partition function Z_t is a mean-one, continuous, positive martingale on $(\mathcal{M}, \mathcal{G}, Q)$ with respect to the filtration $(\mathcal{G}_t)_{t\geq 0}$. In particular, the following limit exists Q-a.s. :

$$Z_\infty = \lim_{t\to\infty} Z_t$$

Since $\exp(\beta H_t) > 0$, the event $\{Z_\infty = 0\}$ is measurable with respect to the tail σ-field

$$\bigcap_{t\geq 1} \sigma\{B_s^x, s > t, x \in \mathbb{Z}^d\}$$

and therefore by Kolmogorov's $0-1$ law,

$$Q\{Z_\infty = 0\} = 0 \text{ or } 1.$$

In the case $Z_\infty > 0$, we say that *weak disorder* holds, and in the case $Z_\infty = 0$ that *strong disorder* holds.

1.3 The phase transition

Theorem 1. *There exists a critical value $\beta_c = \beta_c(d) \in [0, \infty)$ such that :*

- *if $\beta < \beta_c$ then $Z_\infty > 0$ a.s.*
- *if $\beta > \beta_c$ then $Z_\infty = 0$ a.s.*

Moreover, if $d \geq 3$ then $\beta_c(d) > 0$.

A lot of work has been made to study these two different phases for random walk in random environment models, and it has strong links with the overlap of two independent trajectories in the same environment ω. If we call J_t the expected overlap of two independent replicas :

$$J_t = J_t(\beta) = \mu_t^{\beta \otimes 2} \left[\int_0^t 1_{\{X_s = \widehat{X}_s\}} ds \right],$$

then, roughly speaking :

$$\ln Z_t^\beta \approx Q[\ln Z_t^\beta] = -\int_0^\beta bQ[J_t(b)]db.$$

For precise results on the localization of the polymer, favorite end-point and favorite trajectory, I invite you to consult [2] and [5].

Empirically, in the strong disoder case, the polymer will be compelled to follow a most attractive path, while in the weak disorder phase, the probability will be more fairly distributed amongst all possible trajectories. This is why this phase transition is sometimes called the transition from delocalized phase to localized phase.

It is also interesting to know when *very strong disorder* holds, meaning that Z_t converges to 0 exponentially fast. It is a well-known fact in the PAM that the limit

$$\Psi(\kappa, \beta) = \lim_{t \to \infty} \frac{1}{t} \ln Z_t^{\kappa, \beta}$$

exists a.s. and in $\mathbf{L}^p$ for $p \in [1, \infty)$. It is called the *free energy* of the model. For more results on the free energy, consult [8], and read [7] for the large deviations point of view.

1.4 Main theorem

In this paper we prove that very strong disorder always holds in dimension 1 and 2.

Theorem 2. *In the Parabolic Anderson Model for dimension 1 and 2, for all $\beta > 0$, there exists $c > 0$ such that*

$$Z_t = O(e^{-ct}) \;\; Q\text{−a.s.}$$

Moreover, we have the following upper bounds for small β (c_1, c_2 are positive constants):

$$\Psi(1,\beta) \leq \begin{cases} -c_1\beta^4 & \textit{for } d=1 \\ -\exp\left(\frac{-c_2}{\beta^4}\right) & \textit{for } d=2 \end{cases}$$

In other terms, as $t \to \infty$, the partition function grows exponentially slower than its expectation.

Remark 1 : In higher dimensions, as a corollary of theorem 1, the free energy $\Psi(1,\beta)$ is constant equal to 0 on the (non trivial) interval $[0, \beta_c(d))$ (at least).

Remark 2 : A lot of work has also be done to study the asymptotics of $\psi(1,\beta)$ for β big. It is well known in the Parabolic Anderson model (see [1] [3] [5] [8]) that when $\beta \to \infty$

$$\Psi(1,\beta) = \frac{\beta^2}{2} - \frac{\alpha^2\beta^2}{4\log(\beta^2)} + o\left(\frac{\beta^2}{\log(\beta^2)}\right),$$

where α is a constant depending on the dimension.

Remark 3 : In [13], O'Connell and Moriarty studied a perfectly assymetrical version of the 1-dimensional model : the random walk X_t always jumps in the same direction. They were able, thanks to reversibility properties of this new model, to prove an exact formula for the free energy. The asymptotics of this free energy when $\beta \to 0$ are $-\frac{\beta^4}{24} + O(\beta^6)$, which is the order found in our theorem. Moreno in [12] also showed that an assymetric Parabolic Anderson Model with very high drift had the same behavior than O'Connell and Moriarty's perfectly assymetric model.

The goal of this paper is to adapt the proof of the same results for directed polymers in random environment for discrete times, which can be found in [10] to the Parabolic Anderson Model.

The same proof can be adapted for other directed polymer models : a Brownian directed polymer in a Poisson environment as seen in [6], the Linear Stochastic Evolutions model introduced by Yoshida in [14, 15]. Similar techniques are also used for non-directed models, Lacoin shows in [11] that the number of self avoiding walks of length n in the infinite cluster of the supercritical percolation in $\mathbb{Z}^2$ is exponentially smaller than its expectation.

2 Proof in dimension 1

Before starting, we sketch the proof and how it is decomposed in different steps. These steps are the same as in the section 3 of [10]:

1. We reduce the problem to the exponential decay of $Q[Z_t^\theta]$ for some $\theta \in (0,1)$.
2. We use a decomposition of Z_t by splitting it into different contributions that corresponds to trajectories that stay in a large corridor.
3. To estimate the fractional moment terms appearing in the decomposition, we will change the measure Q around the path.

4. We use some basic properties of a random walk in $\mathbb{Z}$ to compute the expectation with the new measure.

2.1 The fractional moment method

To show the exponential decay of Z_t, we will use the trick known as the fractional moment. We want to show that

$$\lim_{t\to\infty} \frac{1}{t} Q[\log Z_t] \leq -c < 0$$

but it is not easy to handle the expectation of a log. So we pick $\theta \in (0,1)$, and by Jensen inequality :

$$Q[\log Z_t] = \frac{1}{\theta} Q[\log Z_t^\theta] \leq \frac{1}{\theta} \log Q[Z_t^\theta] \tag{1}$$

We are left with showing that $Q[Z_t^\theta] = \mathcal{O}(e^{-c't})$, which is easier. In the rest of the proof, we will take $\theta = \frac{1}{2}$.

2.2 Decomposition of Z_t

We fix n to be a very large number. The meaning of very large will be seen at the end of the proof, . The number n will be used in the sequel as a scaling factor. We also take n such that $\sqrt{n}$ is an integer. In the following, when we define n, the notation $n = f(\beta)$ will mean that n is the smallest squared integer bigger than $f(\beta)$. If $f(\beta) \geq 5$, then $f(\beta) \leq n < 2f(\beta)$.

For $z \in \mathbb{Z}$, we define $I_z = [z\sqrt{n}, (z+1)\sqrt{n})$ so that the I_z are disjoint and cover $\mathbb{Z}$. For $T = mn$, we sort the paths $(X_s)_{s\in[0,T]}$ of the random walk by the position of the points $(X_{in})_{i=1}^m$. More precisely, we look at the m-uplet $(z_1, \ldots, z_m)$ such that $X_{in} \in I_{z_i}$ for $i = 1, \ldots, m$.

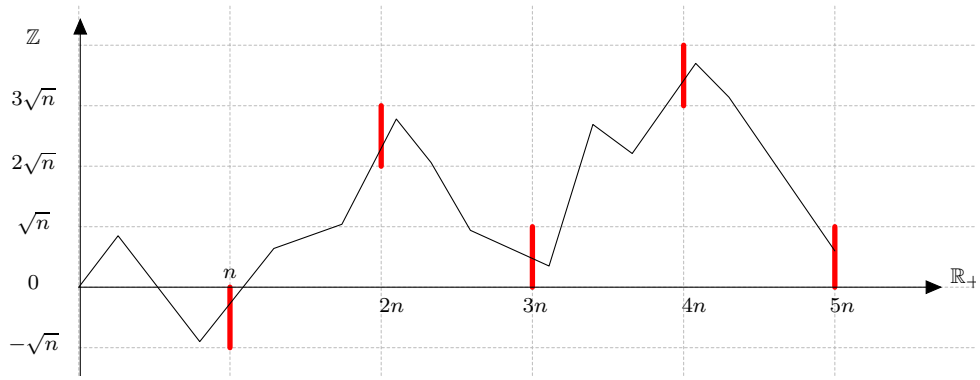
Now we can decompose Z_t :

$$Z_t = \sum_{(z_1,\ldots,z_m)\in\mathbb{Z}^m} \overline{Z}_{(z_1,\ldots,z_m)} \tag{2}$$

where

$$\overline{Z}_{(z_1,\ldots,z_m)} = P\left[\exp(\beta H_T(X) - T\beta^2/2) 1_{\{X_{in}\in I_{z_i},\ i=1\ldots m\}}\right] .$$

In the sequel, for shorter notations, we will write $\mathcal{Z} = (z_1, \ldots, z_m)$ and use the convention that $z_0 = 0$.



a trajectory for $\overline{Z}_{-1,2,0,3,0}$

Now we use our θ and the inequality $(\sum a_i)^\theta \leq \sum a_i^\theta$ (which holds for any countable collection of positive real numbers) to get :

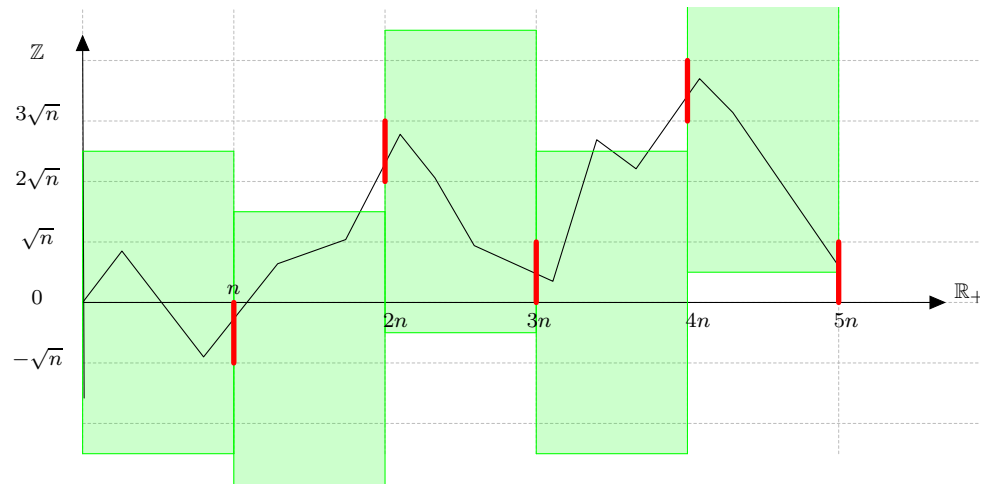
$$Q[Z_t^\theta] \leq \sum_{\mathcal{Z}\in\mathbb{Z}^m} Q[\overline{Z}_{\mathcal{Z}}^\theta]. \tag{3}$$

2.3 The change of measure

We fix one $\mathcal{Z} = (z_1, \ldots, z_m)$. We define m blocs $(J_k)_{k=0}^{m-1}$ and their union $J_{\mathcal{Z}}$:

$$J_k := \left\{(s,y) \in \mathbb{R}_+ \times \mathbb{Z}^2,\ kn \leq s < (k+1)n \text{ and } |y - z_{k-1}\sqrt{n}| < C_1\sqrt{n}\right\} \tag{4}$$

where C_1 will be a big constant.



the boxes J_k

Let $\delta > 0$ be a small real number (we will fix its value in (6)), we define a new measure $\widetilde{Q}_{\mathcal{Z}}$ on the environment :

$$\frac{d\widetilde{Q}_{\mathcal{Z}}}{dQ} = \prod_{k=0}^{m-1} \prod_{|y-z_k|<C_1\sqrt{n}} e^{-\delta(B^y_{(k+1)n} - B^y_{kn})} = \prod_{y\in\mathbb{Z}} \exp\left(-\delta \int_0^T 1_{(s,y)\in J_{\mathcal{Z}}}\, dB_s^y\right).$$

By Cameron-Martin-Grisanov theorem, under this new measure, the processes $(B^x)_{x\in\mathbb{Z}}$ are still independent and verify the following equations :

$$dB_t^x = d\widetilde{B}_t^x - \delta 1_{(t,x)\in J_{\mathcal{Z}}}\, dt,$$

where the $(\widetilde{B}_t^x)_{x\in\mathbb{Z}}$ are independent Brownian Motions under $\widetilde{Q}_{\mathcal{Z}}$.

We use Hölder's inequality to get an upper bound on $Q[\overline{Z}_{\mathcal{Z}}^\theta]$:

$$\begin{aligned} Q[\overline{Z}_{\mathcal{Z}}^\theta] &= \widetilde{Q}_{\mathcal{Z}}\left[\left(\tfrac{d\widetilde{Q}_{\mathcal{Z}}}{dQ}\right)^{-1}\overline{Z}_{\mathcal{Z}}^\theta\right] \\ &\leq \widetilde{Q}_{\mathcal{Z}}\left[\left(\tfrac{d\widetilde{Q}_{\mathcal{Z}}}{dQ}\right)^{\frac{-1}{1-\theta}}\right]^{1-\theta}\widetilde{Q}_{\mathcal{Z}}\left[\overline{Z}_{\mathcal{Z}}\right]^\theta \end{aligned} \tag{5}$$

we can compute the first term of (5), it is standard Gaussian calculus :

$$\widetilde{Q}_{\mathcal{Z}}\left[\left(\frac{d\widetilde{Q}_{\mathcal{Z}}}{dQ}\right)^{\frac{-1}{1-\theta}}\right]^{1-\theta} = Q\left[\left(\frac{d\widetilde{Q}_{\mathcal{Z}}}{dQ}\right)^{\frac{-\theta}{1-\theta}}\right]^{1-\theta} = \exp\left(\frac{\delta^2\theta^2}{2(1-\theta)}|J_{\mathcal{Z}}|\right).$$

We know that the size of $J_{\mathcal{Z}}$ is approximately $2C_1 n\sqrt{n}m$. We will fix the value of δ such that this quantity is small (recall that $\theta = 1/2$) :

$$\text{let } \delta = C_1^{-1/2}n^{-3/4}, \text{ then } \widetilde{Q}_{\mathcal{Z}}\left[\left(\frac{d\widetilde{Q}_{\mathcal{Z}}}{dQ}\right)^{\frac{-1}{1-\theta}}\right]^{1-\theta} \leq e^m \tag{6}$$

2.4 Upper bound on the second term

Using (3) with (5) and (6), we obtain

$$Q[Z_T^\theta] \leq e^m \sum_{\mathcal{Z}\in\mathbb{Z}^m} \widetilde{Q}_{\mathcal{Z}}[\overline{Z}_{\mathcal{Z}}]^\theta. \tag{7}$$

Under the measure $\widetilde{Q}_{\mathcal{Z}}$, all the brownian motion in the boxes J_k have a negative drift $-\delta$, so *if X_t stays in the boxes*, $H_T(X)$ is a gaussian variable of mean $-T\delta$ and variance T, and

$$\widetilde{Q}_{\mathcal{Z}}[\exp(\beta H_T(X) - T\beta^2/2)] = e^{-\beta\delta T}.$$

Now recall the value of δ from (6), we can take n big enough such that this quantity is really small (C_2 will be a big constant):

$$\text{let } n = C_1^2 C_2 \beta^{-4}, \text{ then } e^{-\beta\delta n} = (e^{-\beta\delta T})^{1/m} \leq e^{-C_2^{1/4}} \text{ is really small.} \tag{8}$$

Now we have to consider that the trajectory may exit the box at some time. Let $\varepsilon > 0$. We note $T_{J_{\mathcal{Z}}} = T_{J_{\mathcal{Z}}}(X)$ the time spent in $J_{\mathcal{Z}}$ by the trajectory X, and T_{J_k} the time spent in the box J_k.

$$\begin{aligned}
\widetilde{Q}_{\mathcal{Z}}[\overline{Z}_{\mathcal{Z}}] &= \widetilde{Q}_{\mathcal{Z}} P[\exp(\beta H_T(X) - T\beta^2/2) 1_{\{X_{in} \in I_{z_i}, i=1,\dots,m\}}] \\
&= P[\exp(-\beta\delta T_{J_{\mathcal{Z}}}) 1_{\{X_{in} \in I_{z_i}, i=1,\dots,m\}}] \\
&= P\left[\prod_{k=0}^{m-1} \exp(-\beta\delta T_{J_k}) 1_{\{X_{in} \in I_{z_i}, i=1,\dots,m\}}\right]
\end{aligned}$$

A trajectory verifying $X_{in} \in I_{z_i}$ for $i = 1, \dots, m$ can be seen as m slices of trajectory, respectively starting from one point of $I_{z_{i-1}}$ and finishing at a point of I_{z_i}, which are similar to slices starting from somewhere in I_0 and finishing in $I_{z_i - z_{i-1}}$. So we can bound the contribution in the above expectation by maximizing over the starting point x in I_0, and we get the following upper bound

$$\widetilde{Q}_{\mathcal{Z}}[\overline{Z}_{\mathcal{Z}}] \leq \prod_{k=0}^{m-1} \max_{x \in I_0} P_x \left[\exp(-\beta\delta T_{J_0}) 1_{\{X_n \in I_{z_k - z_{k-1}}\}}\right]$$

where P_x is the probability under which the trajectory starts from x.

Now that each term of the sum (7) is split in m independent terms :

$$\begin{aligned}
\sum_{\mathcal{Z} \in \mathbb{Z}^m} \widetilde{Q}_{\mathcal{Z}}[\overline{Z}_{\mathcal{Z}}]^\theta &\leq \sum_{\mathcal{Z} \in \mathbb{Z}^m} \prod_{k=0}^{m-1} \max_{x \in I_0} P_x \left[\exp(-\beta\delta T_{J_0}) 1_{\{X_n \in I_{z_k - z_{k-1}}\}}\right]^\theta \\
&= \left(\sum_{z \in \mathbb{Z}} \max_{x \in I_0} P_x \left[\exp(-\beta\delta T_{J_0}) 1_{\{X_n \in I_z\}}\right]^\theta\right)^m .
\end{aligned}$$

Basic properties of the random walk on $\mathbb{Z}$ tell us that the probability $P[X_n \in I_z]$ decreases like $\exp(-z^2/2)$ when $z \to \pm\infty$, so we can find a R big enough such that

$$\sum_{|z|>R} \max_{x \in I_0} P_x \left[\exp(-\beta\delta T_{J_0}) 1_{\{X_n \in I_z\}}\right]^\theta < \varepsilon. \tag{9}$$

Now we use the following trivial bound for the remainig terms :

$$\begin{aligned}
&\sum_{|z| \leq R} \max_{x \in I_0} P_x \left[\exp(-\beta\delta T_{J_0}) 1_{\{X_n \in I_z\}}\right]^\theta \\
&\leq (2R+1) \max_{x \in I_0} P_x \left[\exp(-\beta\delta T_{J_0})\right]^\theta .
\end{aligned}$$

We now estimate roughly the time T_{J_0} spent in the box : $T_{J_0} = n$ if the trajectory never leaves the box, and is bounded from below by 0 if it does :

$$P_x \left[\exp(-\beta\delta T_{J_0})\right] \leq \exp(-\beta\delta n) + P_x[\text{the trajectory leaves the box}]$$

Then we take C_1 big enough such that

$$P\left[\max_{t\in[0,n]}|X_t| > (C_1-1)\sqrt{n}\right] < \frac{\varepsilon}{2R+1}, \tag{10}$$

and we just have to inject (9) and (10) in (7) and we get :

$$Q[Z_t^\theta] \leq e^m\left((2R+1)e^{-C_2^{1/4}} + 2\varepsilon\right)^m,$$

and taking C_2 big enough, this is smaller than e^{-m}. To conclude, we put this result in (1) and (5) :

$$\frac{1}{T}Q[\log Z_T] \leq \frac{-m}{\theta T} = \frac{-2}{n} = \frac{-2\beta^4}{C_1^2 C_2}.$$

3 Proof in dimension 2

Once again we sketch the proof before starting :

1. We reduce the problem to the exponential decay of $Q[Z_t^\theta]$ for some $\theta \in (0,1)$.
2. We use a decomposition of Z_t by splitting it into different contributions that corresponds to trajectories that stays in a large corridor.
3. To estimate the fractional moment terms appearing in the decomposition, we will add a term that will handicap the environments that have a lot of positive correlations around the corridors corresponding to each contribution. We define this handicapping term in such a way that the new measure is not very different from the original one.
4. We use some basic properties of a Random Walk in $\mathbb{Z}^2$ to compute the expectation with the new term.

The steps 1 and 2 are exactly the same than in the 1-dimensional case, refer to the sections 2.1 and 2.2. We define the interval I_z as follows :

$$\text{if } \ z=(a,b)\in\mathbb{Z}^2, \ \text{ then } \ I_z = [a\sqrt{n},(a+1)\sqrt{n}) \times [b\sqrt{n},(b+1)\sqrt{n}).$$

We obtain the following bound :

$$Q[Z_T^\theta] \leq \sum_{\mathcal{Z}\in(\mathbb{Z}^2)^m} Q[\overline{Z}_{\mathcal{Z}}^\theta]. \tag{11}$$

3.1 The tweaking of the measure

We fix one $\mathcal{Z} = (z_1, \dots, z_m)$. We define m blocs $(J_k)_{k=0}^{m-1}$ and their union $J_{\mathcal{Z}}$:

$$J_k := \left\{(s,y) \in \mathbb{R}_+ \times \mathbb{Z}^2,\ kn \leq s < (k+1)n \text{ and } |y - z_{k-1}\sqrt{n}| < C_3\sqrt{n}\right\} \tag{12}$$

where C_3 will be a big constant. In the following, I will abuse this notation and say that $y \in J_k$ if $|y - z_k\sqrt{n}| < C_3\sqrt{n}$ (*ie* y is in the vertical component of J_k).

First of all, let me go on a little tangent. If B and $\widehat{B}$ are two brownian motions, let us define what I will call *the correlation between* B *and* $\widehat{B}$ *in* $[0,T]$ $\rho_T(B,\widehat{B})$:

$$\rho_T(B,\widehat{B}) = \int_0^T \left(\int_0^s dB_u\right) d\widehat{B}_s + \int_0^T \left(\int_0^s d\widehat{B}_u\right) dB_s.$$

This quantity is easier to understand if we consider that the Brownian motions are constituted of a sum of tiny discrete increments $\triangle_n B_s = B_{s+\frac{1}{n}} - B_s$. Then the quantity becomes

$$\rho_T(B,\widehat{B}) = \lim_{n\to\infty} \sum_{\substack{0 \leq s,u < T,\, s \neq u \\ s,u \in \frac{1}{n}\mathbb{Z}}} \triangle_n B_s \triangle_n \widehat{B}_u.$$

So if the quantity $\rho_T(B,\widehat{B})$ is big, it means that there are a lot of discrete increments $(\triangle_n B_s, \triangle_n \widehat{B}_u)$ that are the same sign, or in rough term that B and $\widehat{B}$ are both "mostly increasing" or both "mostly decreasing" (actually, it is easy to compute that $\rho_T(B,\widehat{B}) = B_T \cdot \widehat{B}_T$, but I wanted to stress the "tiny increments" vision).

The quantity that will measure the correlations of the environment in one box J_k is a more complicated version of what we just did :

$$R_k = R_k(\omega) = \sum_{x\in J_k}\sum_{y\in J_k} \int_{kn}^{(k+1)n} \left(\int_{kn}^{s} V_{(s,x)(u,y)} dB_u^y\right) dB_s^x \tag{13}$$

where

$$V_{(s,x)(u,y)} := \frac{1_{\{|x-y| \leq C_4\sqrt{|s-u|}\}}}{100 C_3 C_4 n \sqrt{\log n}(|s-u|+1)}. \tag{14}$$

The quantity R_k has mean zero, and, roughly speaking, it is positive when there is a lot of close brownian motions which are "mostly going the same way". To compute the mean and variance of R_k we simply use that the B_t^x are independent Brownian motions, on the same filtration $\mathcal{G}_t$, and then

$$Q\left[\left(\int_0^n f(s)dB_s^x\right)\left(\int_0^n g(u)dB_u^y\right)\right] = \int_0^n f(s)g(s)ds \text{ if } x = y \text{ and } 0 \text{ otherwise.}$$

$$Var_Q(R_k) = \sum_{x\in J_k}\sum_{y\in J_k} \int_{kn}^{(k+1)n} \int_{kn}^{s} V_{(s,x)(u,y)}^2 du ds.$$

Let us compute this term :

$$\begin{aligned}
Var_Q(R_0) &= \sum_{x\in J_0}\sum_{y\in J_0}\int_0^n\int_0^s \frac{1_{\{|x-y|\le C_4\sqrt{|s-u|}\}}}{10^4C_3^2C_4^2n^2\log n(|s-u|+1)^2}duds \\
&\le \sum_{x\in J_0}\int_0^n\int_0^s \frac{4C_4^2|s-u|}{10^4C_3^2C_4^2n^2\log n(|s-u|+1)^2}duds \\
&\le 4C_3^2n\int_0^n\int_0^s \frac{4C_4^2|s-u|}{10^4C_3^2C_4^2n^2\log n(|s-u|+1)^2}duds \\
&\le 4C_4 31^2 n\int_0^n \frac{4C_4^2}{10^4C_3^2C_4^2n^2\log n}\log(n)ds \\
&\le \frac{16C_3^2C_4^2n^2\log n}{10^4C_3^2C_4^2n^2\log n}\le 1
\end{aligned}$$

Let K be a large constant. One defines the function f_K on $\mathbb{R}$ to be :

$$f_K(x) := -K1_{\{x>\exp K^2\}}.$$

We define $g_{\mathcal{Z}}$ function of the environment as

$$g_{\mathcal{Z}}(\omega) := \exp\left(\sum_k f_K\left(R_k\right)\right)$$

So if in a box J_k the quantity R_k is too big, we affect a weight e^{-K}.

We use Hölder's inequality to get an upper bound on $Q[\overline{Z}_{\mathcal{Z}}^{\theta}]$:

$$\begin{aligned}
Q[\overline{Z}_{\mathcal{Z}}^{\theta}] &= Q[g_{\mathcal{Z}}(\omega)^{-\theta}(g_{\mathcal{Z}}(\omega)\overline{Z}_{\mathcal{Z}})^{\theta}] \\
&\le Q[g_{\mathcal{Z}}(\omega)^{-\frac{\theta}{1-\theta}}]^{1-\theta}Q[g_{\mathcal{Z}}(\omega)\overline{Z}_{\mathcal{Z}}]^{\theta}
\end{aligned} \tag{15}$$

The block structure of $g_{\mathcal{Z}}$ allows to express the first term as a power of m.

$$Q\left[g_{\mathcal{Z}}(\omega)^{-\frac{\theta}{1-\theta}}\right] = Q\left[\exp\left(-\frac{\theta}{1-\theta}f_K\left(R_0\right)\right)\right]^m \tag{16}$$

We have already seen that $Q[R_0]=0$ and $Var_Q(R_0)\le 1$. In consequence,

$$Q\left[R_0 > \exp K^2\right] \le \exp(-2K^2),$$

hence

$$Q\left[g_{\mathcal{Z}}(\omega)^{-\frac{\theta}{1-\theta}}\right] \le \left(1+\exp\left(\frac{\theta}{1-\theta}K-2K^2\right)\right)^m \le 2^m \tag{17}$$

if K is large enough. We are left with estimating the second term

$$Q\left[g_{\mathcal{Z}}(\omega)\overline{Z}_{\mathcal{Z}}\right] = QP\left[g_{\mathcal{Z}}(\omega)\exp(\beta H_T - T\beta^2/2)1_{\{X_{in}\in I_{z_i}\ ,\ i=1\dots m\}}\right] \tag{18}$$

For a fixed trajectory of the random walk X, we consider $\widehat{Q}_X$ the modified measure on the environment with density

$$\frac{\mathrm{d}\widehat{Q}_X}{\mathrm{d}Q} := \exp(\beta H_T(X) - T\beta^2/2). \tag{19}$$

Under the measure $\widehat{Q}_X$, the B_t^x are still independent processes on the filtration $\mathcal{G}_t$, and

$$dB_t^x = d\widehat{B}_t^x + \beta 1_{\{X_t = x\}} dt,$$

where the $\widehat{B}_t^x$ are independent brownian motions under $\widehat{Q}_X$.

With the measure $\widehat{Q}_X$, (18) becomes

$$Q\left[g_{\mathcal{Z}}(\omega)\overline{Z}_{\mathcal{Z}}\right] = P\widehat{Q}_X[g_{\mathcal{Z}}(\omega)1_{\{X_{in}\in I_{z_i},\ i=1\ldots m\}}].$$

Now we try to bound this term by something that could be decomposed on the blocks J_k.

As we did in dimension 1, we slice the trajectory $X_{[0,T]}$ in m bits of trajectories starting from somewhere in I_0 and finishing in $I_{z_i - z_{i-1}}$. So we can bound the contribution in the above expectation by maximizing over the starting point x in I_0, and we get the following upper bound

$$Q\left[g_{\mathcal{Z}}(\omega)\overline{Z}_{\mathcal{Z}}\right] \leq \prod_{i=1}^{m} \max_{x\in I_0} P_x\widehat{Q}_X\left[\exp\left(f_K\left(R_0\right)\right)1_{\{X_n\in I_{z_i - z_{i-1}}\}}\right]. \tag{20}$$

Using this with (17), (15) and (11) we get the inequality

$$P[Z_t^\theta] \leq 2^{m(1-\theta)}\left(\sum_{z\in\mathbb{Z}^2}\max_{x\in I_0} P_x\widehat{Q}_\omega\left[\exp\left(f_K\left(R_0\right)\right)1_{\{X_n\in I_z\}}\right]^\theta\right)^m.$$

Therefore to prove the exponential decay of $Q[W_t^\theta]$ it is sufficient to prove that

$$\sum_{z\in\mathbb{Z}^2}\max_{x\in I_0} P_x\widehat{Q}_X\left[\exp\left(f_K\left(R_0\right)\right)1_{\{X_n\in I_z\}}\right]^\theta$$

is small.

3.2 Computation of the second term

We fix some $\varepsilon > 0$. Asymptotic properties of the random walk guarantee that the probability $Q[X_n \in I_z]$ decays like $e^{-|z|^2/2}$ when $|z| \to \infty$. So we can find R such that

$$\sum_{|z|>R}\max_{x\in I_0} P_x\widehat{Q}_X\left[\exp\left(f_K\left(R_0\right)\right)1_{\{X_n\in I_z\}}\right]^\theta \leq \sum_{|z|>R}\max_{x\in I_0} P_x[X_n\in I_z]^\theta \leq \epsilon \tag{21}$$

To estimate the rest of the sum, we have the trivial bound

$$\sum_{|z|\leq R}\left[\max_{x\in I_0} P_x\widehat{Q}_X\left[\exp\left(f_K\left(R_0\right)\right)\right]1_{\{X_n\in I_{z_i-z_{i-1}}\}}\right]^\theta$$

$$\leq 4R^2\max_{x\in I_0} P_x\widehat{Q}_X\left[\exp\left(f_K\left(R_0\right)\right)\right]^\theta \tag{22}$$

Therefore it is sufficient for our purpose to check

$$\max_{x\in I_0} P_x\widehat{Q}_X\left[\exp\left(f_K\left(R_0\right)\right)\right] < \varepsilon' \tag{23}$$

for some small ε'.

Now we have to find an upper bound for

$$P_x\widehat{Q}_X\left[\exp\left(f_K(R_0)\right)\right]$$

that is uniform in $x\in I_0$. To study the left-hand term of the equation (17), we showed that under Q, R_0 had a mean 0 and a small variance, so $f_K(R_0)$ was 0 with very high Q-probability. Now we will show that under $\widehat{Q}_X$, R_0 has a really high mean and a small variance, so $f_K(R_0)=-K$ with high $\widehat{Q}_X$-probability.

We fix $\delta>0$ small. First, we force the random walk to stay in the zone where the environment is modified by writing

$$\max_{x\in I_0} P_x\widehat{Q}_X\left[\exp\left(f_K\left(R_0\right)\right)\right]\leq$$

$$P_0\left[\max_{s\in[0,n]}|X_s|\geq (C_3-1)\sqrt{n}\right]+\max_{x\in I_0} P_x\widehat{Q}_X\left[\exp\left(f_K\left(R_0\right)\right)\ 1_{\{X_{[0,n]}\subseteq J_0\}}\right] \tag{24}$$

and we can take C_3 big enough such that $P_0[\max_{[0,n]}|X_s|\geq (C_3-1)\sqrt{n}]\leq\delta$, so we can concentrate on the trajectories that are inside the box J_0.

Recall that under $\widehat{Q}_X$, $dB_t^x=d\widehat{B}_t^x+\beta 1_{X_t=x}dt$. Now we can compute the expectation of R_0:

$$\widehat{Q}_X\left[R_0\right]=\beta^2\int_0^n\int_0^s V_{(s,X_s)(u,X_u)}du\,ds \tag{25}$$

To study this term, let us define the quantity

$$D_n:=\int_0^n\int_0^s\frac{1}{n\sqrt{\log n}(|s-u|+1)}du\,ds \tag{26}$$

and the random variable (under P) Y :

$$Y:=\int_0^n\int_0^s\frac{1_{\{|X_s-X_u|\leq C_4\sqrt{|s-u|}\}}}{n\sqrt{\log n}(|s-u|+1)}du\,ds \tag{27}$$

Roughly speaking, D_n is what Y would be if $C_4 = \infty$. It is easy to check, with the central limit theorem, that we can take C_4 large enough such that $P[Y] \geq (1-\delta)D_n$, uniformly in n.

Then, as $Y \leq D_n$ a.s. it implies that

$$P_x\left[Y \leq \frac{D_n}{2}\right] \leq 2\delta \tag{28}$$

and a quick computation of the value of D_n gives us :

$$D_n \geq \sqrt{\log n}.$$

If we inject this result in the term in (25), we have that with large P_x-probability,

$$\widehat{Q}_X\left[R_0\right] \geq \frac{\beta^2\sqrt{\log n}}{200C_3C_4} \tag{29}$$

Now we can use the only free hand we have left : the value of n

$$\text{let } \; n = \exp\left(\frac{C_5}{\beta^4}\right), \;\; \text{then } \; \widehat{Q}_X\left[R_0\right] \geq \frac{\sqrt{C_5}}{200C_3C_4} \geq 2\exp K^2 \tag{30}$$

if we take C_5 large enough.

Now we have to bound the variance of R_0 under $\widehat{Q}_X$. We can decompose R_0,

$$\sum_{x,y\in J_0}\int_0^n\left(\int_0^s V_{(s,x)(u,y)}dB_u^y\right)dB_s^x = \beta^2\int_0^n\int_0^s V_{(s,X_s)(u,X_u)}du\,ds$$

$$+\beta\sum_{x\in J_0}\int_0^n\int_0^n V_{(s,x)(u,X_u)}du\,d\widehat{B}_s^x + \sum_{x,y\in J_0}\int_0^n\left(\int_0^s V_{(s,x)(u,y)}d\widehat{B}_u^y\right)d\widehat{B}_s^x$$

Now, using that $(x+y)^2 \leq 2x^2+2y^2$, we can bound the variance by

$$2\beta^2 n\sum_{x\in J_0}\left(\int_0^n V_{(s,x)(u,X_u)}du\right)^2 + 2\sum_{x\in J_0}\sum_{y\in J_0}\int_0^n\int_0^s V^2_{(s,x)(u,y)}du\,ds.$$

We already computed the right-hand term when we studied $Var_Q(R_0)$, it is ≤ 1. With similar techniques we find the other one :

$$\begin{aligned}
\sum_{x\in J_0}\int_0^n V_{(s,x)(u,X_u)}du &\leq \frac{C_4}{10C_3\sqrt{\log n}},\\
\sup_{x\in J_0}\int_0^n V_{(s,x)(u,X_u)}du &\leq \frac{\sqrt{\log n}}{10C_3C_4n},\\
\text{therefore } \; 2\beta^2 n\sum_{x\in J_0}\left(\int_0^n V_{(s,x)(u,X_u)}du\right)^2 &\leq \frac{2\beta^2}{100C_3^2} \leq 1
\end{aligned}$$

Now, using (24), (28) and (30), we can compute (23) :

$$P_x\widehat{Q}_X\left|\exp(f_K(R_0)) \leq P_0\left[\max_{s\in[0,n]}|X_s| \geq (C_3-1)\sqrt{n}\right]\right]$$

$$P_x\left[\widehat{Q}_X[R_0] \leq 2e^{K^2}1_{\{X_{[0,n]}\subseteq J_0\}}\right] + P_x\widehat{Q}_X\left[\exp(f_K(R_0))1_{\{\widehat{Q}_X[R_0]>2e^{K^2}\}}\right] \tag{31}$$

$$\leq 2\delta + e^{-K} + 3e^{-2K^2}$$

so that our result is proved provided that K and δ have been chosen respectively large and small enough.

In conclusion, we have proved (23), which with (11), (21) and (22) allow us to say

$$Q[Z_T^\theta] \leq e^{-m}$$

and with (1) and the value of n from (30) :

$$\frac{1}{T}Q[\log Z_T] \leq \frac{-m}{\theta T} = \frac{-2}{n} = \exp\left(\frac{-c}{\beta^4}\right).$$

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