Nonlinear analysis with resurgent functions

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Abstract

We provide estimates for the convolution product of an arbitrary number of resurgent functions, more precisely of Ω-continuable germs, where Ω is a closed discrete subset of the complex plane which is stable under addition. Such estimates are needed to perform nonlinear operations like substitution in a convergent series, composition or functional inversion with resurgent functions; they also yield implicitly defined resurgent functions.

1 Introduction

In the 1980s, to deal with local analytic problems of classification of dynamical systems, J. Écalle started to develop his theory of resurgent functions [Eca81], [Eca84], [Eca93], which is an efficient tool for dealing with divergent series arising from complex dynamical systems or WKB expansions, analytic invariants of differential or difference equations, linear and nonlinear Stokes phenomena [Mal82], [Mal85], [Eca92], [CNP93], [DP99], [GS01], [OSS03], [Sau06], [Cos09], [Sau10], [LRR11], [FS11], [Ram12], [DS12].

The starting point is the definition of certain subalgebras of the algebra of formal power series by means of the formal Borel transform

$$\mathcal{B}\colon\ \tilde\varphi(z)=\sum_{n=0}^{\infty}a_n z^{-n-1}\in z^{-1}\mathbb{C}[[z^{-1}]]\mapsto\hat\varphi(\zeta)=\sum_{n=0}^{\infty}a_n\frac{\zeta^n}{n!}\in\mathbb{C}[[\zeta]] \tag{1}$$

(using negative power expansions in the left-hand side and changing the name of the indeterminate from z to ζ are just convenient conventions); it turns out that, for a lot of interesting functional equations, one can find divergent formal solutions whose Borel transforms have positive radius of convergence and define germs of holomorphic function at 0 with particular properties of analytic continuation. The simplest examples are the Euler series [CNP93], [Ram12], which can be written $\tilde\varphi^{\mathrm{E}}(z)=\sum_{n=0}^{\infty}n!z^{-n-1}$ and solves a first-order linear non-homogeneous differential equation, and the Stirling series [Eca81, Vol. 3] $\tilde\varphi^{\mathrm{S}}(z)=\sum_{k=1}^{\infty}\frac{B_{2k}}{2k(2k-1)}z^{-2k+1}$, solution of a linear non-homogeneous difference equation derived from the functional equation for Euler's Gamma function by taking logarithms. In both examples the Borel transform gives rise to convergent series with a meromorphic extension to the ζ-plane, namely $(1-\zeta)^{-1}$ for the Euler series and $\zeta^{-2}\left(\frac{\zeta}{2}\coth\frac{\zeta}{2}-1\right)$ for the Stirling series, but in more complicated situations the analytic continuation is no longer single-valued (due to nonlinearities); the property which persists in many interesting situations is rather the possibility of following the analytic continuation without encountering natural barriers.

One is thus led to distinguish certain subspaces $\hat{\mathscr{R}}$ of $\mathbb{C}\{\zeta\}$, characterized by properties which ensure a locally discrete set of singularities for each of its members (and which do not exclude

multiple-valuedness of the analytic continuation), and to consider

$$\tilde{\mathscr{R}} := \mathbb{C} \oplus \mathcal{B}^{-1}(\hat{\mathscr{R}}) \subset \mathbb{C}[[z^{-1}]].$$

Typically one has the strict inclusion $\mathbb{C}\{z^{-1}\} \subsetneq \tilde{\mathscr{R}}$ but the divergent series in $\tilde{\mathscr{R}}$ can be "summed" by means of Borel-Laplace summation. The formal series in $\tilde{\mathscr{R}}$ as well as the holomorphic functions whose germ at 0 belongs to $\hat{\mathscr{R}}$ are termed "resurgent".

Later we shall be more specific about the definition of $\hat{\mathscr{R}}$. This article is concerned with the convolution of resurgent functions: the convolution in $\mathbb{C}\{\zeta\}$ is the commutative associative product defined by

$$\hat{\varphi}_1 * \hat{\varphi}_2(\zeta) = \int_0^\zeta \hat{\varphi}_1(\zeta_1)\hat{\varphi}_2(\zeta - \zeta_1)\,\mathrm{d}\zeta_1 \qquad \text{for } |\zeta| \text{ small enough,} \tag{2}$$

for any $\hat{\varphi}_1, \hat{\varphi}_2 \in \mathbb{C}\{\zeta\}$, which reflects the Cauchy product of formal series via the formal Borel transform:

$$\mathcal{B}\tilde{\varphi}_1 = \hat{\varphi}_1 \text{ and } \mathcal{B}\tilde{\varphi}_2 = \hat{\varphi}_2 \quad \Longrightarrow \quad \mathcal{B}(\tilde{\varphi}_1\tilde{\varphi}_2) = \hat{\varphi}_1 * \hat{\varphi}_2.$$

Since the theory was designed to deal with nonlinear problems, it is of fundamental importance to control the convolution product of resurgent functions; however, this requires to follow the analytic continuation of the function defined by (2), which turns out not to be an easy task. In fact, probably the greatest difficulties in understanding and applying resurgence theory are connected with the problem of controlling the analytic continuation of functions defined by such integrals or by analogous multiple integrals. Even the mere stability under convolution of the spaces $\hat{\mathscr{R}}$ is not obvious [Eca81], [CNP93], [Ou10], [Sau12].

We thus need to estimate the convolution product of two or more resurgent functions, both for concrete manipulations of resurgent functions in nonlinear contexts and for the foundations of the resurgence theory. For instance, such estimates will allow us to check that, when we come back to the resurgent series via $\mathcal{B}$, the exponential of a resurgent series is resurgent and that more generally one can substitute resurgent series in convergent power expansions, or define implicitly a resurgent series. They will also show that the group of "formal tangent-to-identity diffeomorphisms at ∞", *i.e.* the group (for the composition law) $z + \mathbb{C}[[z^{-1}]]$, admits $z + \tilde{\mathscr{R}}$ as a subgroup, which is particularly useful for the study of holomorphic tangent-to-identity diffeomorphisms f (in this problem of local holomorphic dynamics, the Fatou coordinates have the same resurgent asymptotic expansion, the so-called direct iterator $f^* \in z + \tilde{\mathscr{R}}$ of [Eca81]; thus its inverse, the inverse iterator, also belongs to $z + \tilde{\mathscr{R}}$, and its exponential appears in the Bridge equation which is connected with the "horn maps"—see [Sau06] or [DS12]).

Such results of stability of the algebra of resurgent series under nonlinear operations are mentioned in Écalle's works, however the arguments there are quite sketchy and it seemed desirable to fill the gaps. This was one of the tasks undertaken in [CNP93] but, despite its merits, one cannot say that this book clearly settled this particular issue: the proof of the estimates for the convolution is obscure and certainly contains at least one mistake (see Remark 1.1).

In this article, we shall deal with a particular case of resurgence called Ω-continuability or Ω-resurgence, which means that we fix in advance a discrete subset Ω of $\mathbb{C}$ and restrict ourselves to those resurgent functions whose analytic continuation has no singular points outside of Ω. An outline is as follows:

– In Section 2, we recall the precise definition of the corresponding algebras of resurgent functions, denoted by $\hat{\mathscr{R}}_\Omega$, and state Theorem 1, which is our main result on the control of the convolution product of an arbitrary number of Ω-continuable functions.

– In Section 3, we give applications to the construction of a Fréchet algebra structure on $\tilde{\mathscr{R}}_\Omega$ (Theorem 2) and to the stability of Ω-resurgent series under substitution (Theorem 3), implicit function (Theorem 4) and composition (Theorem 5); we also mention other possible applications.

– The proof of Theorem 1 is given in Sections 4–7.

– Finally, there is an appendix on a few facts of the theory of currents which are used in the proof of the main theorem.

Our method arose as attempt to understand a rather sketchy appendix of the seminal book [CNP93]. The point is to represent the analytic continuation of a convolution product as the integral of a holomorphic n-form on a singular n-simplex obtained as a suitable deformation of the standard n-simplex; we explain in Sections 4–5 what kind of deformations ("adapted origin-fixing isotopies" of the identity) are licit in order to provide the analytic continuation and how to produce them. We found the theory of currents very convenient to deal with our integrals of holomorphic forms, because it allowed us to content ourselves with little regularity: the deformations we use are only Lipschitz continuous, because they are built from the flow of non-autonomous Lipschitz vector fields—see Section 6. Section 7 contains the last part of the proof, which consists in providing appropriate estimates.

Remark 1.1. As far as we understand, there is a mistake in [CNP93], in the final argument given to bound a determinant analogous to our formula (32): roughly speaking, these authors produce a deformation of the standard n-simplex through the flow of an autonomous vector field in $\mathbb{C}^n$ (the definition of which is not clear to us) and then use the linear differential equation satisfied by the Jacobian determinant of the flow; however, they overlook the fact that, since their vector field is not holomorphic, the Jacobian determinant which can be controlled this way is the real one, corresponding to the identification $\mathbb{C}^n \simeq \mathbb{R}^{2n}$, whereas the determinant which appears when computing the integral and that one needs to bound is a complex linear combination of the $n \times n$ minors of the $2n \times 2n$ real Jacobian matrix.

2 The convolution of Ω-continuable germs

Notation 2.1. *For any $R > 0$ and $\zeta_0 \in \mathbb{C}$ we use the notations $D(\zeta_0, R) \coloneqq \{\, \zeta \in \mathbb{C} \mid |\zeta - \zeta_0| < R \,\}$ and*

$$\mathbb{D}_R \coloneqq D(0, R), \qquad \mathbb{D}_R^* \coloneqq \mathbb{D}_R \setminus \{0\}.$$

We call "path" any continuous piecewise C^1 function $\gamma\colon\ J \to \mathbb{C}$, where $J = [a, b]$ is a compact interval of $\mathbb{R}$.

Let Ω be a closed, discrete subset of $\mathbb{C}$ containing 0. We set

$$\rho(\Omega) \coloneqq \min\big\{|\omega|,\ \omega \in \Omega \setminus \{0\}\big\}.$$

Recall [Sau12] that the space $\hat{\mathscr{R}}_\Omega$ of all Ω-continuable germs is the subspace of $\mathbb{C}\{\zeta\}$ which can be defined by the fact that, for arbitrary $\zeta_0 \in \mathbb{D}_{\rho(\Omega)}$,

$$\hat{\varphi} \in \hat{\mathscr{R}}_\Omega \quad \Longleftrightarrow \quad \left| \begin{array}{l} \hat{\varphi} \text{ germ of holomorphic function of } \mathbb{D}_{\rho(\Omega)} \text{ admitting analytic continuation} \\ \text{along any path } \gamma\colon\ [0,1] \to \mathbb{C} \text{ such that } \gamma(0) = \zeta_0 \text{ and } \gamma\big((0,1]\big) \subset \mathbb{C} \setminus \Omega. \end{array} \right.$$

For example, for the Euler series, resp. the Stirling series, the Borel transform belongs to $\hat{\mathscr{R}}_\Omega$ as soon as $1 \in \Omega$, resp. $2\pi\mathrm{i}\mathbb{Z}^* \subset \Omega$.

It is convenient to rephrase the property of Ω-continuability as holomorphy on a certain Riemann surface.

Definition 2.2. *Consider the set $\mathscr{P}_\Omega$ of all paths $\gamma\colon [0,1] \to \mathbb{C}$ such that either $\gamma\big([0,1]\big) = \{0\}$ or $\gamma(0) = 0$ and $\gamma\big((0,1]\big) \subset \mathbb{C} \setminus \Omega$. We denote by $\mathscr{S}_\Omega$ the set of all equivalence classes of $\mathscr{P}_\Omega$ for the relation of homotopy with fixed endpoints. The map $\gamma \in \mathscr{P}_\Omega \mapsto \gamma(1) \in \{0\} \cup \mathbb{C} \setminus \Omega$ passes to the quotient and defines the "projection"*

$$\pi_\Omega\colon\ \zeta \in \mathscr{S}_\Omega \to \overset{\bullet}{\zeta} \in \{0\} \cup \mathbb{C} \setminus \Omega. \tag{3}$$

We equip $\mathscr{S}_\Omega$ with the unique structure of Riemann surface which turns π_Ω into a local biholomorphism. The equivalence class of the trivial path $\gamma(t) \equiv 0$ is denoted by 0_Ω and called the origin of $\mathscr{S}_\Omega$.

We obtain a connected, simply connected Riemann surface $\mathscr{S}_\Omega$, which is somewhat analogous to the universal cover of $\mathbb{C} \setminus \Omega$ except for the special role played by 0 and 0_Ω: since we assumed $0 \in \Omega$, the origin of $\mathscr{S}_\Omega$ is the only point which projects onto 0. It belongs to the *principal sheet of* $\mathscr{S}_\Omega$, which is defined as the set of all points ζ which can be represented by a line segment, *i.e.* such that the path $t \in [0,1] \mapsto t\overset{\bullet}{\zeta}$ belongs to $\mathscr{P}_\Omega$ and represents ζ.

Any holomorphic function of $\mathscr{S}_\Omega$ identifies itself with a convergent germ at the origin of $\mathbb{C}$ which admits analytic continuation along all the paths of $\mathscr{P}_\Omega$, so that

$$\hat{\mathscr{R}}_\Omega \simeq \mathscr{O}(\mathscr{S}_\Omega)$$

(see [Eca81], [Sau06]). Usually, we shall use the same symbol $\hat{\varphi}$ for a function of $\mathscr{O}(\mathscr{S}_\Omega)$ or the corresponding germ of holomorphic function at 0 (*i.e.* its Taylor series).

From now on we assume that Ω is stable under addition. According to [Sau12], this ensures that $\hat{\mathscr{R}}_\Omega$ is stable under convolution. Our aim is to provide explicit bounds for the analytic continuation of a convolution product of two or more factors belonging to $\hat{\mathscr{R}}_\Omega$.

It is well-known that, if $U \subset \{0\} \cup (\mathbb{C} \setminus \Omega)$ is open and star-shaped with respect to 0 and two function $\hat{\varphi}_1, \hat{\varphi}_2$ are holomorphic in U, then their convolution product has an analytic continuation to U which is given by the very same formula (2); by induction, one gets a representation of a product of n factors $\hat{\varphi}_j \in \mathscr{O}(U)$ as an iterated integral, which eventually leads to

$$|\hat{\varphi}_1 * \cdots * \hat{\varphi}_n(\zeta)| \leq \frac{|\zeta|^n}{n!} \max_{[0,\zeta]} |\hat{\varphi}_1| \cdots \max_{[0,\zeta]} |\hat{\varphi}_n|, \qquad \zeta \in U. \tag{4}$$

This allows one to control convolution products in the principal sheet of $\mathscr{S}_\Omega$ but, to reach the other sheets, formula (2) must be replaced by something else, as explained *e.g.* in [Sau12]. What about the bounds a product of n factors then? To state our main result, we introduce

Notation 2.3. *The function $R_\Omega\colon \mathscr{S}_\Omega \to (0, +\infty)$ is defined by*

$$\zeta \in \mathscr{S}_\Omega \ \mapsto\ R_\Omega(\zeta) \coloneqq \begin{cases} \operatorname{dist}\big(\overset{\bullet}{\zeta}, \Omega \setminus \{0\}\big) & \text{if } \zeta \text{ belongs to the principal sheet of } \mathscr{S}_\Omega \\ \operatorname{dist}\big(\overset{\bullet}{\zeta}, \Omega\big) & \text{if not} \end{cases} \tag{5}$$

(where $\dot{\zeta}$ is the shorthand for $\pi_\Omega(\zeta)$ defined by (3)). For $\delta, L > 0$, we set

$$\mathcal{K}_{\delta,L}(\Omega) \coloneqq \big\{\, \zeta \in \mathscr{S}_\Omega \mid \exists \text{ a path } \gamma \text{ of } \mathscr{S}_\Omega \text{ with endpoints } 0_\Omega \text{ and } \zeta, \text{ of length} \le L, \text{ such that } R_\Omega(\gamma(t)) \ge \delta \text{ for all } t \,\big\}. \quad (6)$$

Informally, $\mathcal{K}_{\delta,L}(\Omega)$ consists of the points of $\mathscr{S}_\Omega$ which can be joined to 0_Ω by a path of length $\le L$ "staying at distance $\ge \delta$ from the boundary". Observe that $\big(\mathcal{K}_{\delta,L}(\Omega)\big)_{\delta,L>0}$ is an exhaustion of $\mathscr{S}_\Omega$ by compact subsets. If $L+\delta < \rho(\Omega)$, then $\mathcal{K}_{\delta,L}(\Omega)$ is just the lift of the closed disc $\overline{\mathbb{D}}_L$ in the principal sheet of $\mathscr{S}_\Omega$.

Theorem 1. *Let $\Omega \subset \mathbb{C}$ be closed, discrete, stable under addition, with $0 \in \Omega$. Let $\delta, L > 0$ with $\delta < \rho(\Omega)$ and*

$$C \coloneqq \rho(\Omega)\, \mathrm{e}^{3+6L/\delta}, \qquad \delta' \coloneqq \frac{1}{2}\rho(\Omega)\, \mathrm{e}^{-2-4L/\delta}, \qquad L' \coloneqq L + \frac{\delta}{2}. \quad (7)$$

Then, for any $n \ge 1$ and $\hat{\varphi}_1, \dots, \hat{\varphi}_n \in \hat{\mathscr{R}}_\Omega$,

$$\max_{\mathcal{K}_{\delta,L}(\Omega)} |\hat{\varphi}_1 * \cdots * \hat{\varphi}_n| \le \frac{2}{\delta} \cdot \frac{C^n}{n!} \cdot \max_{\mathcal{K}_{\delta',L'}(\Omega)} |\hat{\varphi}_1| \cdots \max_{\mathcal{K}_{\delta',L'}(\Omega)} |\hat{\varphi}_n|. \quad (8)$$

The proof of Theorem 1 will start in Section 4. We emphasize that $\delta, \delta', L, L', C$ do not depend on n, which is important in applications.

3 Application to nonlinear operations with Ω-resurgent series

3.1 Fréchet algebra structure on $\tilde{\mathscr{R}}_\Omega$

Recall that Ω is a closed discrete subset of $\mathbb{C}$ which contains 0 and is stable under addition. The space of Ω-resurgent series is

$$\tilde{\mathscr{R}}_\Omega = \mathbb{C} \oplus \mathcal{B}^{-1}(\hat{\mathscr{R}}_\Omega).$$

As a vector space, it is isomorphic to $\mathbb{C} \times \mathscr{O}(\mathscr{S}_\Omega)$. We now define seminorms on $\tilde{\mathscr{R}}_\Omega$ which will ease the exposition.

Definition 3.1. *Let $K \subset \mathscr{S}_\Omega$ be compact. We define the seminorm $\|\cdot\|_K \colon \tilde{\mathscr{R}}_\Omega \to \mathbb{R}^+$ by*

$$\tilde{\phi} \in \tilde{\mathscr{R}}_\Omega \mapsto \|\tilde{\phi}\|_K \coloneqq \max\big\{|c|, \max_K |\hat{\varphi}|\big\},$$

where $\tilde{\phi} = c + \mathcal{B}^{-1}\hat{\varphi}$, $c \in \mathbb{C}$, $\hat{\varphi} \in \hat{\mathscr{R}}_\Omega$.

Choosing $K_N = \mathcal{K}_{\delta_N, L_N}(\Omega)$, $N \in \mathbb{N}^*$, with any pair of sequences $\delta_N \downarrow 0$ and $L_N \uparrow \infty$ (so that $\mathscr{S}_\Omega$ is the increasing union of the compact sets K_N), we get a countable family of seminorms which defines a structure of Fréchet space on $\tilde{\mathscr{R}}_\Omega$. A direct consequence of Theorem 1 is the continuity of the Cauchy product for this Fréchet structure. More precisely:

Theorem 2. *For any K there exist $K' \supset K$ and $C > 0$ such that, for any $n \ge r \ge 0$,*

$$\|\tilde{\phi}_1 \cdots \tilde{\phi}_n\|_K \le \frac{C^n}{r!} \|\tilde{\phi}_1\|_{K'} \cdots \|\tilde{\phi}_n\|_{K'} \quad (9)$$

for every sequence $(\tilde{\phi}_1, \dots, \tilde{\phi}_n)$ of Ω-resurgent series, r of which have no constant term.

In particular, $\tilde{\mathscr{R}}_\Omega$ is a Fréchet algebra.

Proof. Let us fix K compact and choose $\delta, L > 0$ so that $K \subset \mathcal{K}_{\delta,L}(\Omega)$. Let δ', L' be as in (7) and $K' \coloneqq \mathcal{K}_{\delta',L'}(\Omega)$. According to Theorem 2, we can choose $C_0 > 0$ large enough so that for any $m \geq 1$ and $\tilde{\varphi}_1, \dots, \tilde{\varphi}_m \in \mathcal{B}^{-1}(\hat{\mathscr{R}}_\Omega)$,

$$\|\tilde{\varphi}_1 \cdots \tilde{\varphi}_m\|_K \leq \frac{C_0^m}{m!} \|\tilde{\varphi}_1\|_{K'} \cdots \|\tilde{\varphi}_m\|_{K'}. \tag{10}$$

Let $n \geq r$ and and $s \coloneqq n - r$. Given n resurgent series among which r have no constant term, we can label them so that

$$\tilde{\phi}_1 = c_1 + \tilde{\varphi}_1, \ \dots, \ \tilde{\phi}_s = c_s + \tilde{\varphi}_s, \ \ \tilde{\phi}_{s+1} = \tilde{\varphi}_{s+1}, \ \dots, \ \tilde{\phi}_n = \tilde{\varphi}_n,$$

with $c_1, \dots, c_s \in \mathbb{C}$ and $\tilde{\varphi}_1, \dots, \tilde{\varphi}_n \in \mathcal{B}^{-1}(\hat{\mathscr{R}}_\Omega)$. Then

$$\tilde{\phi}_1 \cdots \tilde{\phi}_n = \sum_I c_{i_1} \cdots c_{i_p} \tilde{\varphi}_{j_1} \cdots \tilde{\varphi}_{j_q} \tilde{\varphi}_{s+1} \cdots \tilde{\varphi}_n,$$

where the summation is over all subsets $I = \{i_1, \dots, i_p\}$ of $\{1, \dots, s\}$ (of any cardinality p), with $\{j_1, \dots, j_q\} \coloneqq \{1, \dots, s\} \setminus I$. Using inequality (10), we get $\|\tilde{\phi}_1 \cdots \tilde{\phi}_n\|_K \leq$

$$\sum_I \frac{C_0^{q+r}}{(q+r)!} |c_{i_1} \cdots c_{i_p}| \, \|\tilde{\varphi}_{j_1}\|_{K'} \cdots \|\tilde{\varphi}_{j_q}\|_{K'} \|\tilde{\varphi}_{s+1}\|_{K'} \cdots \|\tilde{\varphi}_n\|_{K'} \leq A \, \|\tilde{\phi}_1\|_{K'} \cdots \|\tilde{\phi}_n\|_{K'},$$

with

$$A = \sum_{q=0}^{s} \frac{C_0^{q+r}}{(q+r)!} \binom{s}{q} \leq \frac{C_0^r}{r!} \sum_{q=0}^{s} \frac{C_0^q}{q!} \binom{s}{q} \leq \frac{C_0^r}{r!} (C_0 + 1)^s,$$

whence (9) follows with $C \coloneqq C_0 + 1$.

The continuity of the multiplication in $\tilde{\mathscr{R}}_\Omega$ follows, as a particular case when $n = 2$. □

Remark 3.2. $\tilde{\mathscr{R}}_\Omega$ is even a differential Fréchet algebra since $\frac{\mathrm{d}}{\mathrm{d}z}$ induces a continuous derivation of $\tilde{\mathscr{R}}_\Omega$. Indeed, the very definition of $\mathcal{B}$ in (1) shows that

$$\tilde{\phi} = c + \mathcal{B}^{-1}\hat{\varphi} \quad \Longrightarrow \quad \frac{\mathrm{d}\tilde{\phi}}{\mathrm{d}z} = \mathcal{B}^{-1}\hat{\psi} \ \text{ with } \hat{\psi}(\zeta) = -\zeta\hat{\varphi}(\zeta),$$

whence $\|\frac{\mathrm{d}\tilde{\phi}}{\mathrm{d}z}\|_K \leq D(K)\|\tilde{\phi}\|_K$ with $D(K) = \max_{\zeta \in K} |\zeta|$.

3.2 Substitution and implicit resurgent functions

Definition 3.3. *For any $r \in \mathbb{N}^*$, we define $\tilde{\mathscr{R}}_\Omega\{w_1, \dots, w_r\}$ as the subspace of $\tilde{\mathscr{R}}_\Omega[[w_1, \dots, w_r]]$ consisting of all formal power series*

$$\tilde{H} = \sum_{\boldsymbol{k}=(k_1,\dots,k_r) \in \mathbb{N}^r} \tilde{H}_{\boldsymbol{k}}(z) \, w_1^{k_1} \cdots w_r^{k_r}$$

with coefficients $\tilde{H}_{\boldsymbol{k}} = \tilde{H}_{\boldsymbol{k}}(z) \in \tilde{\mathscr{R}}_\Omega$ such that, for every compact $K \subset \mathscr{S}_\Omega$, there exist positive numbers A_K, B_K such that

$$\|\tilde{H}_{\boldsymbol{k}}\|_K \leq A_K \, B_K^{|\boldsymbol{k}|} \tag{11}$$

for all $\boldsymbol{k} \in \mathbb{N}^r$ (with the notation $|\boldsymbol{k}| = k_1 + \cdots + k_r$).

The idea is to consider formal series "resurgent in z and convergent in $w_1, \dots, w_r$". We now show that one can substitute resurgent series in such a convergent series. Observe that $\tilde{\mathscr{R}}_\Omega\{w_1, \dots, w_r\}$ can be considered as a subspace of $\mathbb{C}[[z^{-1}, w_1, \dots, w_r]]$.

Theorem 3. *The space $\tilde{\mathscr{R}}_\Omega\{w_1, \dots, w_r\}$ is a subalgebra of $\mathbb{C}[[z^{-1}, w_1, \dots, w_r]]$.*

Suppose that $\tilde\varphi_1, \dots, \tilde\varphi_r \in \tilde{\mathscr{R}}_\Omega$ have no constant term. Then for any $\tilde H = \sum \tilde H_{\boldsymbol{k}}\, w_1^{k_1} \cdots w_r^{k_r} \in \tilde{\mathscr{R}}_\Omega\{w_1, \dots, w_r\}$, the series

$$\tilde H(\tilde\varphi_1, \dots, \tilde\varphi_r) \coloneqq \sum_{\boldsymbol{k}\in\mathbb{N}^r} \tilde H_{\boldsymbol{k}}\, \tilde\varphi_1^{k_1} \cdots \tilde\varphi_r^{k_r}$$

is convergent in $\tilde{\mathscr{R}}_\Omega$ and, for every compact $K \subset \mathscr{S}_\Omega$, there exist a compact $K' \supset K$ and a constant $C > 0$ so that

$$\|\tilde H(\tilde\varphi_1, \dots, \tilde\varphi_r)\|_K \le C A_{K'}\, \mathrm{e}^{C B_{K'}\left(\|\tilde\varphi_1\|_{K'} + \cdots + \|\tilde\varphi_r\|_{K'}\right)}$$

(with notations similar to those of Definition 3.3 for $A_{K'}, B_{K'}$).

Moreover, the map $\tilde H \in \tilde{\mathscr{R}}_\Omega\{w_1, \dots, w_r\} \mapsto \tilde H(\tilde\varphi_1, \dots, \tilde\varphi_r) \in \tilde{\mathscr{R}}_\Omega$ is an algebra homomorphism.

Proof. The proof of the first statement is left as an exercise. Observe that the series of formal series

$$\tilde\chi = \sum_{\boldsymbol{k}\in\mathbb{N}^r} \tilde H_{\boldsymbol{k}}\, \tilde\varphi_1^{k_1} \cdots \tilde\varphi_r^{k_r}$$

is formally convergent[1] in $\mathbb{C}[[z^{-1}]]$, because $\tilde H_{\boldsymbol{k}}\, \tilde\varphi_1^{k_1} \cdots \tilde\varphi_r^{k_r}$ has order $\ge |\boldsymbol{k}|$; this is in fact a particular case of composition of formal series and the fact that the map

$$\tilde H \in \tilde{\mathbb{C}}[[z^{-1}, w_1, \dots, w_r]] \mapsto \tilde H(\tilde\varphi_1, \dots, \tilde\varphi_r) \in \mathbb{C}[[z^{-1}]]$$

is an algebra homomorphism is well-known. The last statement will thus follow from the second one.

Let us fix $K \subset \mathscr{S}_\Omega$ compact. We first choose K' and C as in Theorem 2, and then $A = A_{K'}$, $B = B_{K'}$ so that (11) holds relatively to K'. For each $\boldsymbol{k} \in \mathbb{N}^r$, inequality (9) yields

$$\|\tilde H_{\boldsymbol{k}}\, \tilde\varphi_1^{k_1} \cdots \tilde\varphi_r^{k_r}\|_K \le \frac{C^{|\boldsymbol{k}|+1}}{|\boldsymbol{k}|!} \|\tilde H_{\boldsymbol{k}}\|_{K'} \|\tilde\varphi_1\|_{K'}^{k_1} \cdots \|\tilde\varphi_r\|_{K'}^{k_r} \le C A \frac{(CB)^{|\boldsymbol{k}|}}{|\boldsymbol{k}|!} \|\tilde\varphi_1\|_{K'}^{k_1} \cdots \|\tilde\varphi_r\|_{K'}^{k_r}$$

and the conclusion follows easily. □

As an illustration, for $\tilde\phi = c + \tilde\varphi$ with $c \in \mathbb{C}$ and $\tilde\varphi \in \mathcal{B}^{-1}(\hat{\mathscr{R}}_\Omega)$, we have

$$\exp(\tilde\phi) = \mathrm{e}^c \sum_{n\ge 0} \frac{1}{n!} \tilde\varphi^n \in \tilde{\mathscr{R}}_\Omega$$

and, if moreover $c \ne 0$,

$$1/\tilde\phi = \sum_{n\ge 0} (-1)^n c^{-n-1} \tilde\varphi^n \in \tilde{\mathscr{R}}_\Omega.$$

[1] A family of formal series in $\mathbb{C}[[z^{-1}]]$ is formally summable if it has only finitely many members of order $\le N$ for every $N \in \mathbb{N}$. Notice that if a formally summable family is made up of Ω-resurgent series and is summable for the semi-norms $\|\cdot\|_K$, then the formal sum in $\mathbb{C}[[z^{-1}]]$ and the sum in $\tilde{\mathscr{R}}_\Omega$ coincide (because the Borel transform of the formal sum is nothing but the Taylor series at 0 of the Borel transform of the sum in $\tilde{\mathscr{R}}_\Omega$).

For instance, one can take the exponential of the Stirling series $\tilde{\varphi}^{\mathrm{S}}$ mentioned in the introduction: we obtain the $2\pi\mathrm{i}\mathbb{Z}$-resurgence of the divergent series $\exp(\tilde{\varphi}^{\mathrm{S}})$ which, according to the refined Stirling formula, is the asymptotic expansion of $\frac{1}{\sqrt{2\pi}} z^{\frac{1}{2}-z} \mathrm{e}^z\, \Gamma(z)$.

Theorem 4. *Let $F(x,y) \in \mathbb{C}[[x,y]]$ be such that $F(0,0) = 0$ and $\partial_y F(0,0) \neq 0$, and call $\varphi(x)$ the unique solution in $x\mathbb{C}[[x]]$ of the equation*

$$F\big(x, \varphi(x)\big) = 0. \tag{12}$$

Let $\tilde{F}(z,y) \coloneqq F(z^{-1},y) \in \mathbb{C}[[z^{-1},y]]$ and $\tilde{\varphi}(z) \coloneqq \varphi(z^{-1}) \in z^{-1}\mathbb{C}[[z^{-1}]]$, so that $\tilde{\varphi}$ is implicitly defined by the equation $\tilde{F}\big(z, \tilde{\varphi}(z)\big) = 0$. Then

$$\tilde{F}(z,y) \in \tilde{\mathscr{R}}_\Omega\{y\} \quad \Longrightarrow \quad \tilde{\varphi}(z) \in \tilde{\mathscr{R}}_\Omega.$$

Proof. Without loss of generality we can assume $\partial_y F(0,0) = -1$ and write

$$F(x,y) = -y + f(x) + R(x,y)$$

with $f(x) = F(x,0) \in x\mathbb{C}[[x]]$ and a quadratic remainder

$$R(x,y) = \sum_{n\geq 1} R_n(x) y^n, \qquad R_n(x) \in \mathbb{C}[[x]], \quad R_1(0) = 0.$$

When viewed as formal transformation in y, the formal series $\theta(x,y) \coloneqq y - R(x,y)$ is invertible, with inverse given by the Lagrange reversion formula: the series

$$H(x,y) \coloneqq y + \sum_{k\geq 1} \frac{1}{k!} \partial_y^{k-1}(R^k)(x,y)$$

is formally convergent (the order of $\partial_y^{k-1}(R^k)$ is at least $k+1$ because $\mathrm{ord}(R) \geq 2$) and satisfies $\theta\big(x, H(x,y)\big) = y$. Rewriting (12) as $\theta\big(x, \varphi(x)\big) = f(x)$, we get $\varphi(x) = H\big(x, f(x)\big)$.

Now, the y-expansion of H can be written

$$H(x,y) = \sum_{m\geq 1} H_m(x) y^m, \qquad H_m = \sum_{k\geq 1} \frac{(m+k-1)!}{m!\,k!} \sum_{\boldsymbol{n}} R_{n_1} \cdots R_{n_k},$$

where the last summation is over all k-tuples of integers $\boldsymbol{n} = (n_1, \dots, n_k)$ such that $n_1, \dots, n_k \geq 1$ and $n_1 + \cdots + n_k = m + k - 1$. If we group together the indices i such that $n_i = 1$, we get an expression of H_m as a formally convergent series in $\mathbb{C}[[x]]$:

$$H_m = \sum_{r\geq 0} \sum_{\substack{s\geq 0 \\ r+s\geq 1}} \frac{(m+r+s-1)!}{m!\,r!\,s!} \sum_{\boldsymbol{j}} R_1^r R_{j_1} \cdots R_{j_s}, \tag{13}$$

where the last summation is over all s-tuples of integers $\boldsymbol{j} = (j_1, \dots, j_s)$ such that $j_1, \dots, j_s \geq 2$ and $j_1 + \cdots + j_s = m + s - 1$, with an empty summation giving rise to a factor 1 when $m = 1$ (then we simply get $H_1 = (1 - R_1)^{-1}$). Observe that, if $m \geq 2$, one must restrict oneself to $s \leq m-1$ and that there are $\frac{(m-2)!}{(s-1)!(m-s)!}$ summands in the $\boldsymbol{j}$-summation.

Replacing x by z^{-1}, we get

$$\tilde{\varphi}(z) = \tilde{H}\big(z, \tilde{f}(z)\big)$$

with $\tilde{f}(z) \coloneqq f(z^{-1}) \in \tilde{\mathscr{R}}_\Omega$ without constant term and $\tilde{H}(z,y) \coloneqq \sum \tilde{H}_m(z)y^n$, $\tilde{H}_m(z) \coloneqq H_m(z^{-1})$. In view of Theorem 3 it is thus sufficient to check that $\tilde{H} \in \tilde{\mathscr{R}}_\Omega\{y\}$.

Let $K \subset \mathscr{S}_\Omega$ be compact. Setting $\tilde{R}_n(z) \coloneqq R(z^{-1})$ for all $n \geq 1$, by Theorem 2 we can find $K' \supset K$ compact and $C > 0$ such that $\|\tilde{R}_1^r \tilde{R}_{j_1} \cdots \tilde{R}_{j_s}\|_K \leq \frac{C^{r+s}}{r!} \|\tilde{R}_1\|_{K'}^r \|\tilde{R}_{j_1}\|_{K'} \cdots \|\tilde{R}_{j_s}\|_{K'}$. Assuming $\tilde{F}(z,y) \in \tilde{\mathscr{R}}_\Omega\{y\}$, we can find $A, B > 0$ such that $\|\tilde{R}_n\|_{K'} \leq AB^n$ for all $n \geq 1$. Enlarging A if necessary, we can assume $ABC \geq 1/4$. We then see that the series (13) is convergent in $\tilde{\mathscr{R}}_\Omega$:

$$\begin{aligned}
\|\tilde{H}_m\|_K &\leq \sum_{r\geq 0} \sum_{0\leq s\leq m-1} \frac{(m+r+s-1)!}{m!\,r!\,s!} \frac{(m-2)!}{(s-1)!(m-s)!} \frac{C^{r+s}}{r!} A^{r+s} B^{m+r+s-1} \\
&\leq \frac{1}{m} \sum_{r\geq 0} \frac{1}{r!} \sum_{0\leq s\leq m-1} 4^{m+r+s-1} (CA)^{r+s} B^{m+r+s-1} \\
&\leq (4B)^{m-1} \sum_{r\geq 0} \frac{1}{r!} (4ABC)^{r+m-1} \leq \alpha\beta^m,
\end{aligned}$$

with $\alpha = \exp(4ABC)$ and $\beta = 16AB^2C$, hence $\tilde{H} \in \tilde{\mathscr{R}}_\Omega\{y\}$ as desired. □

3.3 The group of resurgent tangent-to-identity diffeomorphisms

One of the first applications by J. Écalle of his resurgence theory was the iteration theory for tangent-to-identity local analytic diffeomorphisms [Eca81, Vol. 2]. In the language of holomorphic dynamics, this corresponds to a parabolic fixed point in one complex variable, for which, classically, one introduces the Fatou coordinates to describe the dynamics and to define the "horn map" [Mil06]. In the resurgent approach, one places the variable at infinity and deals with formal diffeomorphisms: starting from $F(w) = w + O(w^2) \in \mathbb{C}\{w\}$ or $\mathbb{C}[[w]]$, one gets $f(z) \coloneqq 1/F(1/z) = z + \sum_{m=0}^\infty a_m z^{-m} \in z + \mathbb{C}\{z^{-1}\}$ or $z + \mathbb{C}[[z^{-1}]]$. The set

$$\tilde{\mathscr{G}} \coloneqq z + \mathbb{C}[[z^{-1}]]$$

is a group for the composition law: this is the group of formal tangent-to-identity diffeomorphisms.

Convergent diffeomorphisms form a subgroup $z + \mathbb{C}\{z^{-1}\}$. In the simplest case, one is given a specific dynamical system $z \mapsto f(z) = z + \alpha + O(z^{-1}) \in z + \mathbb{C}\{z^{-1}\}$ with $\alpha \in \mathbb{C}^*$ and there is a formal conjugacy between f and the trivial dynamics $z \mapsto z + \alpha$, *i.e.* the equation $\tilde{v} \circ f = \tilde{v} + \alpha$ admits a solution $\tilde{v} \in \tilde{\mathscr{G}}$ (strictly speaking, an assumption is needed for this to be true, without which one must enlarge slightly the theory to accept a logarithmic term in $\tilde{v}(z)$; we omit the details here—see [Eca81], [Sau06]). One can give a direct proof [DS12] that $\tilde{v}(z) - z$ is Ω-resurgent with $\Omega = 2\pi\mathrm{i}\alpha^{-1}\mathbb{Z}$. The inverse of $\tilde{v}$ is a solution $\tilde{u}$ of the difference equation $\tilde{u}(z+\alpha) = f\big(\tilde{u}(z)\big)$ and the exponential of $\tilde{v}$ plays a role in Écalle's "bridge equation", which is related to the Écalle-Voronin classification theorem and to the horn map (again, we refrain from giving more details here).

This may serve as a motivation for the following

Theorem 5. *Assume that Ω is a closed discrete subset of $\mathbb{C}$ which contains 0 and is stable under addition. Then the Ω-resurgent tangent-to-identity diffeomorphisms make up a subgroup*

$$\tilde{\mathscr{G}}_\Omega \coloneqq z + \tilde{\mathscr{R}}_\Omega \subset \tilde{\mathscr{G}},$$

which contains $z + \mathbb{C}\{z^{-1}\}$.

Proof. We must prove that, for arbitrary $\tilde{f}(z) = z + \tilde{\phi}(z), \tilde{g}(z) = z + \tilde{\psi}(z) \in \tilde{\mathscr{G}}_\Omega$, both $\tilde{f} \circ \tilde{g}$ and $\tilde{h} \coloneqq \tilde{f}^{\circ(-1)}$ belong to $\tilde{\mathscr{R}}_\Omega$.

We have $\tilde{f} \circ \tilde{g} = \tilde{g} + \tilde{\phi} \circ \tilde{g}$, where the last term can be defined by the formally convergent series

$$\tilde{\phi} \circ \tilde{g} = \tilde{\phi} + \sum_{n \geq 1} \frac{1}{n!} \tilde{\psi}^n \Big(\frac{\mathrm{d}}{\mathrm{d}z}\Big)^n \tilde{\phi}. \tag{14}$$

Let $K \subset \mathscr{S}_\Omega$ be compact, and let $K' \supset K$ and $C > 0$ be as in Theorem 2. We have

$$\|\tilde{\psi}^n \Big(\frac{\mathrm{d}}{\mathrm{d}z}\Big)^n \tilde{\phi}\|_K \leq C^{n+1} \|\tilde{\psi}\|_{K'}^n \, \|\Big(\frac{\mathrm{d}}{\mathrm{d}z}\Big)^n \tilde{\phi}\|_{K'} \leq C^{n+1} D(K')^n \|\tilde{\psi}\|_{K'}^n \, \|\tilde{\phi}\|_{K'},$$

where $D(K') \coloneqq \max_{\zeta \in K'} |\zeta|$ (by Remark 3.2), hence the series (14) is convergent in $\tilde{\mathscr{R}}_\Omega$, and $\|\tilde{\phi} \circ \tilde{g}\|_K \leq C \|\tilde{\phi}\|_{K'} \exp\big(C D(K') \|\tilde{\psi}\|_{K'}\big)$.

As for $\tilde{h}$, the Lagrange reversion formula yields it in the form of a formally convergent series

$$\tilde{h} = z + \sum_{k=1}^{\infty} \frac{(-1)^k}{k!} \Big(\frac{\mathrm{d}}{\mathrm{d}z}\Big)^{k-1} (\tilde{\phi}^k). \tag{15}$$

We have

$$\|\Big(\frac{\mathrm{d}}{\mathrm{d}z}\Big)^{k-1} (\tilde{\phi}^k)\|_K \leq D(K)^{k-1} \|\tilde{\phi}^k\|_K \leq D(K)^{k-1} C^k \|\tilde{\phi}\|_{K'}^k$$

(again by Remark 3.2 and Theorem 2), hence the series (15) is convergent in $\tilde{\mathscr{R}}_\Omega$, and $\|\tilde{h} - z\|_K \leq C \|\tilde{\phi}\|_{K'} \exp\big(C D(K) \|\tilde{\phi}\|_{K'}\big)$. □

Remark 3.4. One can easily deduce from the estimates obtained in the above proof that $\tilde{\mathscr{G}}_\Omega$ is a topological group: composition and inversion are continuous if we transport the topology of $\tilde{\mathscr{R}}_\Omega$ onto $\tilde{\mathscr{G}}_\Omega$ by the bijection $\tilde{\phi} \mapsto z + \tilde{\phi}$.

3.4 Other possible applications

In this article, we stick to the simplest case which presents itself in resurgence theory: formal expansions in negative integer powers of z, whose Borel transforms converge and extend analytically outside a set Ω fixed in advance, but

– the theory of "resurgent singularities" was developed by J. Écalle to deal with much more general formal objects,

– the condition of Ω-continuability can be substituted with "continuability without a cut" or "endless continuability" which allow for Riemann surfaces much more general than $\mathscr{S}_\Omega$ [Eca81, Vol. 3], [CNP93].

The extension to more general Rieman surfaces is necessary in certain problems, particularly those involving parametric resurgence or quantum resurgence (in relation with semi-classical asymptotics). To make our method accomodate the notion of continuability without a cut, one could for instance imitate the way [Ou12] deals with "discrete filtered sets". The point is that, when convolving germs in the ζ-plane, the singular points of the analytic continuation of each factor may produce a singularity located at the sum of these singular points, but being continuable without a cut means that the set of singular points is locally finite, thus one can explore sequentially the Riemann surface of the convolution product, considering longer and

longer paths of analytic continuation and saturating the corresponding Riemann surface by removing at each step the (finitely many) sums of singular points already encountered.

The formalism of general resurgent singularities also can be accomodated. The reader is referred to [Eca81] and [Sau06] for the corresponding extension of the definition of convolution. In short, the formal Borel transform (1), which must be considered as a termwise inverse Laplace transform, can be generalized by considering the action of the Laplace transform on monomials like $\zeta^\alpha(\log\zeta)^m$ with $m\in\mathbb{N}$ and $\alpha\in\mathbb{C}$ for instance. One is thus led to deal with holomorphic functions of ζ defined for arbitrarily small nonzero $|\zeta|$ but not holomorphic at the origin: one must rather work in subsets of the Riemann surface of the logarithm before considering their analytic continuation for large values of $|\zeta|$. If one restricts oneself to functions which are integrable at 0, like the convergent expansions involving monomials $\zeta^\alpha(\log\zeta)^m$ with $\Re e\,\alpha>-1$, then formula (2) may still be used to define the convolution. To deal with general resurgent singularities, one must replace it with the so-called convolution of majors. This should be the subject of another article.

As another possible application, it would be worth trying to adapt our method to the weighted convolution products which appear in [Eca94]. Their definition is as follows: given a sequence of pairs $B_1=(\omega_1,b_1)$, $B_2=(\omega_2,b_2)$, etc. with $\omega_n\in\mathbb{C}$ and $b_n\in\mathbb{C}\{\zeta\}$ and assuming that

$$\breve{\omega}_n=\omega_1+\cdots+\omega_n\neq 0,\qquad n\in\mathbb{N}^*,$$

one defines a sequence $\hat{S}^{B_1},\hat{S}^{B_1,B_2},\ldots\in\mathbb{C}\{\zeta\}$ by the formulas

$$\hat{S}^{B_1}(\zeta)\coloneqq\frac{1}{\omega_1}b_1\Big(\frac{\zeta}{\omega_1}\Big),\qquad \hat{S}^{B_1,B_2}(\zeta)\coloneqq\frac{1}{\omega_1}\int_0^{\zeta/\breve{\omega}_2}b_1\Big(\frac{\zeta-\omega_2\xi_2}{\omega_1}\Big)b_2(\xi_2)\,\mathrm{d}\xi_2,$$

$$\hat{S}^{B_1,B_2,B_3}(\zeta)\coloneqq\frac{1}{\omega_1}\int_0^{\zeta/\breve{\omega}_3}\mathrm{d}\xi_3\int_{\xi_3}^{(\zeta-\omega_3\xi_3)/\breve{\omega}_2}\mathrm{d}\xi_2\,b_1\Big(\frac{\zeta-\omega_2\xi_2-\omega_3\xi_3}{\omega_1}\Big)b_2(\xi_2)b_3(\xi_3),\qquad\text{etc.}$$

The general formula is $\hat{S}^{B_1,\ldots,B_n}(\zeta)\coloneqq\dfrac{1}{\omega_1}\displaystyle\int\mathrm{d}\xi_n\cdots\mathrm{d}\xi_2\,b_1(\xi_1)b_2(\xi_2)\cdots b_n(\xi_n)$, where the integral is taken over

$$\xi_n\in\Big[0,\frac{\zeta}{\breve{\omega}_n}\Big],\qquad \xi_i\in\Big[\xi_{i+1},\frac{\zeta-(\omega_{i+1}\xi_{i+1}+\cdots+\omega_n\xi_n)}{\breve{\omega}_i}\Big]\ \text{ for } i=n-1,n-2,\ldots,2$$

and $\xi_1\coloneqq\dfrac{\zeta-(\omega_2\xi_2+\cdots+\omega_n\xi_n)}{\breve{\omega}_1}$. There is a relation with the ordinary convolution called symmetrality: if $\boldsymbol{B'}=B^{i_1}\cdots B^{i_n}$ and $\boldsymbol{B''}=B^{j_1}\cdots B^{j_m}$, then $\hat{S}^{\boldsymbol{B'}}*\hat{S}^{\boldsymbol{B''}}$ is the sum $\sum\hat{S}^{\boldsymbol{B}}$ over all words $\boldsymbol{B}$ belonging to the shuffle of $\boldsymbol{B'}$ and $\boldsymbol{B''}$, *e.g.*

$$\hat{S}^{B_1}*\hat{S}^{B_2}=\hat{S}^{B_1,B_2}+\hat{S}^{B_2,B_1},\qquad \hat{S}^{B_1,B_2}*\hat{S}^{B_3}=\hat{S}^{B_1,B_2,B_3}+\hat{S}^{B_1,B_3,B_2}+\hat{S}^{B_3,B_1,B_2},\qquad\text{etc.}$$

It is argued in [Eca94] that the weighted convolutions $\hat{S}^{B_1,\ldots,B_n}$ associated with endlessly continuable germs $b_1,b_2,\ldots$ are themselves endlessly continuable and constitute the "building blocks" of the resurgent functions which appear in parametric resurgence or quantum resurgence problems (see [Sau95] for an example with $\omega_i=1$ for all i). It would thus be interesting and natural (because the weighted convolution products present themselves as multiple integrals not so different from the n-fold integrals (18) below) to try to deform the integration simplex, in a manner similar to the one that will be employed for convolution products in Sections 4–7, in order to control the analytic continuation of $\hat{S}^{B_1,\ldots,B_r}$.

4 The initial n-dimensional integration current

We now begin the proof of Theorem 1. Notice that convolution with the constant germ 1 amounts to integration from 0, according to (2), thus $\frac{\mathrm{d}}{\mathrm{d}\zeta}(1 * \hat{\varphi}) = \hat{\varphi}$ and, by associativity of the convolution,

$$\hat{\varphi}_1 * \cdots * \hat{\varphi}_n = \frac{\mathrm{d}}{\mathrm{d}\zeta}\big(1 * \hat{\varphi}_1 * \cdots * \hat{\varphi}_n\big) \tag{16}$$

for any $\hat{\varphi}_1, \ldots, \hat{\varphi}_n \in \mathbb{C}\{\zeta\}$.

We shall now dedicate ourselves to the proof of a statement similar to Theorem 1 for convolution products of the form $1 * \hat{\varphi}_1 * \cdots * \hat{\varphi}_n$, with $\hat{\varphi}_1, \ldots, \hat{\varphi}_n \in \mathscr{O}(\mathscr{S}_\Omega)$; this will be Theorem 1' of Section 7. The proof of Theorem 1 itself will then follow by the Cauchy inequalities.

It turns out that, for $\zeta \in \mathscr{S}_\Omega$ close to 0_Ω, there is a natural way of representing $1 * \hat{\varphi}_1 * \cdots * \hat{\varphi}_n(\zeta)$ as the integral of a holomorphic n-form over an n-dimensional chain of the complex manifold $\mathscr{S}_\Omega^n$; this is formula (17) of Proposition 4.3, which will be our starting point for the proof of Theorem 1'.

Notation 4.1. *Given $\zeta \in \mathscr{S}_\Omega$, we denote by $\mathcal{L}_\zeta \colon \mathbb{D}_{R_\Omega(\zeta)} \to \mathscr{S}_\Omega$ the holomorphic map defined by*

$$\mathcal{L}_\zeta(\xi) \coloneqq \textit{endpoint of the lift which starts at } \zeta \textit{ of the path } t \in [0,1] \mapsto \dot{\zeta} + t\xi$$

(so that $\mathcal{L}_\zeta(\xi)$ can be thought of as "the lift of $\dot{\zeta} + \xi$ wich sits on the same sheet of $\mathscr{S}_\Omega$ as ζ"). We shall often use the shorthand

$$\zeta + \xi \coloneqq \mathcal{L}_\zeta(\xi)$$

(beware that, in the latter formula, $\xi \in \mathbb{D}_{R_\Omega(\zeta)}$ is a complex number but not ζ and $\zeta + \xi$, which are points of $\mathscr{S}_\Omega$). If $n \geq 1$ and $\underline{\zeta} = (\zeta_1, \ldots, \zeta_n) \in \mathscr{S}_\Omega^n$, we also set

$$S_n(\underline{\zeta}) \coloneqq \dot{\zeta}_1 + \cdots + \dot{\zeta}_n \in \mathbb{C},$$

$$\mathcal{L}_{\underline{\zeta}}(\underline{\xi}) \coloneqq \underline{\zeta} + \underline{\xi} \coloneqq \big(\mathcal{L}_{\zeta_1}(\xi_1), \ldots, \mathcal{L}_{\zeta_n}(\xi_n)\big) \in \mathscr{S}_\Omega^n$$

for $\underline{\xi} = (\xi_1, \ldots, \xi_n) \in \mathbb{C}^n$ close enough to 0 (it suffices that $|\xi_j| < R_\Omega(\zeta_j)$; observe that $S_n(\underline{\zeta}+\underline{\xi}) = S_n(\underline{\zeta}) + \xi_1 + \cdots + \xi_n$).

Notation 4.2. *For any $n \geq 1$, we denote by Δ_n the n-dimensional simplex*

$$\Delta_n \coloneqq \{\, (s_1, \ldots, s_n) \in \mathbb{R}^n \mid s_1, \ldots, s_n \geq 0 \textit{ and } s_1 + \cdots + s_n \leq 1 \,\}$$

with the standard orientation, and by $[\Delta_n] \in \mathscr{E}_n(\mathbb{R}^n)$ the corresponding integration current:

$$[\Delta_n] \colon \alpha \textit{ complex-valued smooth } n\textit{-form on } \mathbb{R}^n \mapsto \int_{\Delta_n} \alpha \in \mathbb{C}.$$

For every $\zeta \in \mathbb{D}_{\rho(\Omega)}$, we consider the map

$$\underline{\mathscr{Q}}(\zeta) \colon \underline{s} = (s_1, \ldots, s_n) \mapsto \underline{\mathscr{Q}}(\zeta, \underline{s}) \coloneqq 0_\Omega + (s_1\zeta, \ldots, s_n\zeta) \in \mathscr{S}_\Omega^n,$$

defined in a neighbourhood of Δ_n in $\mathbb{R}^n$, and denote by $\underline{\mathscr{Q}}(\zeta)_\#[\Delta_n] \in \mathscr{E}_n(\mathscr{S}_\Omega^n)$ the push-forward of $[\Delta_n]$ by $\underline{\mathscr{Q}}(\zeta)$:

$$\underline{\mathscr{Q}}(\zeta)_\#[\Delta_n] \colon \beta \textit{ smooth } n\textit{-form on } \mathscr{S}_\Omega^n \mapsto [\Delta_n]\big(\underline{\mathscr{Q}}(\zeta)^\# \beta\big).$$

See Appendix A for our notations in relation with currents. Notice that the last formula makes sense because $\underline{\mathscr{Q}}(\zeta)$ is a smooth map, thus the pullback form $\underline{\mathscr{Q}}(\zeta)^{\#}\beta$ is well-defined in a neighbourhood of Δ_n. The reason for using the language of currents and Geometric Measure Theory is that later we shall require the push-forward of integration currents by Lipschitz maps which are not smooth everywhere. The reader is referred to Appendix A for a survey of a few facts of the theory which will be useful for us.

Proposition 4.3. *For $\hat{\varphi}_1, \dots, \hat{\varphi}_n \in \hat{\mathscr{R}}_\Omega$ and $\zeta \in \mathbb{D}_{\rho(\Omega)}$, one has*

$$1 * \hat{\varphi}_1 * \cdots * \hat{\varphi}_n(\zeta) = \underline{\mathscr{Q}}(\zeta)_{\#}[\Delta_n](\beta) \quad \text{with } \beta = \hat{\varphi}_1(\zeta_1) \cdots \hat{\varphi}_n(\zeta_n)\, \mathrm{d}\zeta_1 \wedge \cdots \wedge \mathrm{d}\zeta_n, \tag{17}$$

where we denote by $\mathrm{d}\zeta_1 \wedge \cdots \wedge \mathrm{d}\zeta_n$ the pullback in $\mathscr{S}_\Omega^n$ by $\pi_\Omega^{\otimes n}\colon \underline{\zeta} \in \mathscr{S}_\Omega^n \mapsto \underline{\xi} = \big(\overset{\bullet}{\zeta}_1, \dots, \overset{\bullet}{\zeta}_n\big)$ of the n-form $\mathrm{d}\xi_1 \wedge \cdots \wedge \mathrm{d}\xi_n$ of $\mathbb{C}^n$.

Proof. Since $(\zeta_1, \dots, \zeta_n) \in \mathbb{D}^n_{\rho(\Omega)} \mapsto 0_\Omega + (\zeta_1, \dots, \zeta_n) \in \mathscr{S}_\Omega^n$ is an analytic chart which covers a neighbourhood of $\underline{\mathscr{Q}}(\zeta)(\Delta_n)$, we can write $\underline{\mathscr{Q}}(\zeta)^{\#}\beta = \hat{\varphi}_1(s_1\zeta) \cdots \hat{\varphi}_n(s_n\zeta)\mathrm{d}s_1 \wedge \cdots \wedge \mathrm{d}s_n$. Since

$$\Delta_n = \{\, (s_1, \dots, s_n) \in \mathbb{R}^n \mid s_1 \in [0,1], s_2 \in [0, 1-s_1], \dots, s_n \in [0, 1-(s_1 + \cdots + s_{n-1})] \,\}$$

with the standard orientation, the right-hand side of the identity stated in (17) can be rewritten

$$\zeta^n \int_0^1 \mathrm{d}s_1 \int_0^{1-s_1} \mathrm{d}s_2 \dots \int_0^{1-(s_1+\cdots+s_{n-1})} \mathrm{d}s_n\, \hat{\varphi}_1(s_1\zeta) \cdots \hat{\varphi}_n(s_n\zeta)$$

or

$$\int_0^\zeta \mathrm{d}\zeta_1 \int_0^{\zeta-\zeta_1} \mathrm{d}\zeta_2 \dots \int_0^{\zeta-(\zeta_1+\cdots+\zeta_{n-1})} \mathrm{d}\zeta_n\, \hat{\varphi}_1(\zeta_1) \cdots \hat{\varphi}_n(\zeta_n). \tag{18}$$

When $n = 1$, formula (17) is thus the very definition of $1 * \hat{\varphi}_1(\zeta)$. Writing

$$1 * \hat{\varphi}_1 * \cdots * \hat{\varphi}_n(\zeta) = \int_0^\zeta \mathrm{d}\zeta_1\, \hat{\varphi}_1(\zeta_1)\big(1 * \hat{\varphi}_2 * \cdots * \hat{\varphi}_n\big)(\zeta - \zeta_1),$$

we get the general case by induction. □

5 Deformation of the n-dimensional integration current in $\mathscr{S}_\Omega^n$

In this section, we fix an interval $J = [a, b]$ and a path $\gamma\colon J \mapsto \mathbb{C} \setminus \Omega$ such that $\gamma(a) \in \mathbb{D}^*_{\rho(\Omega)}$; we denote by $\tilde{\gamma}$ the lift of γ which starts in the principal sheet of $\mathscr{S}_\Omega$. In order to obtain the analytic continuation of formula (17), we shall deform the n-dimensional integration current $\underline{\mathscr{Q}}(\zeta)_{\#}[\Delta_n]$ as indicated in Proposition 5.2 below.

Definition 5.1. *Given $n \geq 1$, for $\zeta \in \mathbb{C}$ and $j = 1, \dots, n$, we set*

$$\mathcal{N}(\zeta) \coloneqq \{\, \underline{\zeta} \in \mathscr{S}_\Omega^n \mid S_n(\underline{\zeta}) = \zeta \,\}, \qquad \mathcal{N}_j \coloneqq \{\, \underline{\zeta} = (\zeta_1, \dots, \zeta_n) \in \mathscr{S}_\Omega^n \mid \zeta_j = 0_\Omega \,\}.$$

We call γ-adapted origin-fixing isotopy in $\mathscr{S}_\Omega^n$ any family $(\Psi_t)_{t\in J}$ of homeomorphisms of $\mathscr{S}_\Omega^n$ such that $\Psi_a = \mathrm{Id}$, the map $(t, \underline{\zeta}) \in J \times \mathscr{S}_\Omega^n \mapsto \Psi_t(\underline{\zeta}) \in \mathscr{S}_\Omega^n$ is locally Lipschitz,[2] and for any $t \in J$ and $j = 1, \dots, n$,

$$\begin{aligned} \underline{\zeta} \in \mathcal{N}\big(\gamma(a)\big) \quad &\Rightarrow \quad \Psi_t(\underline{\zeta}) \in \mathcal{N}\big(\gamma(t)\big), \\ \underline{\zeta} \in \mathcal{N}_j \quad &\Rightarrow \quad \Psi_t(\underline{\zeta}) \in \mathcal{N}_j. \end{aligned}$$

[2] By that, we mean that each point of $\mathscr{S}_\Omega^n$ admits an open neighbourhood $\mathscr{U}$ on which $\pi_\Omega^{\otimes n}\colon \mathscr{S}_\Omega^n \to \mathbb{C}^n$ induces a biholomorphism and such that the map $(t, \underline{\xi}) \in J \times \pi_\Omega^{\otimes n}(\mathscr{U}) \mapsto \pi_\Omega^{\otimes n} \circ \Psi_t \circ \big((\pi_\Omega^{\otimes n})_{|\mathscr{U}}\big)^{-1}(\underline{\xi}) \in \mathbb{C}^n$ is Lipschitz.

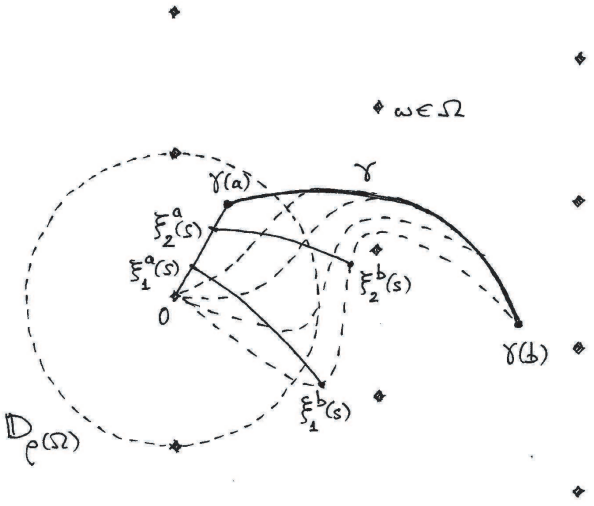

Figure 1: Projections of $\big(\xi_1^t(\underline{s}),\dots,\xi_n^t(\underline{s})\big) \coloneqq \Psi_t\big(s_1\gamma(a),\dots,s_n\gamma(a)\big) = \Psi_t \circ \underline{\mathscr{Q}}\big(\gamma(a)\big)(\underline{s})$.

Proposition 5.2. *Suppose that $(\Psi_t)_{t\in J}$ is a γ-adapted origin-fixing isotopy in $\mathscr{S}_\Omega^n$. Then, for any $\hat\varphi_1,\dots,\hat\varphi_n \in \hat{\mathscr{R}}_\Omega$, the analytic continuation of $1 * \hat\varphi_1 * \cdots * \hat\varphi_n$ along γ is given by*

$$\big(1 * \hat\varphi_1 * \cdots * \hat\varphi_n\big)\big(\tilde\gamma(t)\big) = \big(\Psi_t \circ \underline{\mathscr{Q}}(\gamma(a))\big)_\#[\Delta_n](\beta), \qquad t \in J, \tag{19}$$

with $\beta = \hat\varphi_1(\zeta_1)\cdots\hat\varphi_n(\zeta_n)\,\mathrm{d}\zeta_1\wedge\cdots\wedge\mathrm{d}\zeta_n$.

See Figure 1. Observe that, for each $t \in J$, the map $\Psi_t \circ \underline{\mathscr{Q}}\big(\gamma(a)\big)\colon \Delta_n \to \mathscr{S}_\Omega^n$ is Lipschitz, so that the push-forward $\big(\Psi_t \circ \underline{\mathscr{Q}}(\gamma(a))\big)_\#[\Delta_n]$ is a well-defined n-dimensional current of $\mathscr{S}_\Omega^n$ (see Appendix A). The proof of Proposition 5.2 relies on the following more general statement:

Notation 5.3. *Given a map $\underline{C} = (C_1,\dots,C_n)\colon J\times\Delta_n \to \mathscr{S}_\Omega^n$, for each $t\in J$ we denote by $\underline{C}_t\colon \Delta_n \to \mathscr{S}_\Omega^n$ the partial map defined by*

$$\underline{s} \in \Delta_n \mapsto \underline{C}_t(\underline{s}) \coloneqq \underline{C}(t,\underline{s})$$

(not to be confused with the components $C_j\colon J\times\Delta_n\to\mathscr{S}_\Omega$, $j=1,\dots,n$).

Proposition 5.4. *Let β be a holomorphic n-form on $\mathscr{S}_\Omega^n$ and*

$$F\colon \zeta \in \mathbb{D}_{\rho(\Omega)} \mapsto F(\zeta) \coloneqq \underline{\mathscr{Q}}(\zeta)_\#[\Delta_n](\beta).$$

Then F is a holomorphic function in $\mathbb{D}_{\rho(\Omega)}$.

Let $\underline{C}\colon J\times\Delta_n\to\mathscr{S}_\Omega^n$ be a Lipschitz map[3] *such that the partial map corresponding to $t=a$ satisfies*

$$\underline{C}_a = \underline{\mathscr{Q}}\big(\gamma(a)\big)$$

[3]in the sense that $\pi_\Omega^{\otimes n}\circ\underline{C}\colon J\times\Delta_n\to\mathbb{C}^n$ is Lipschitz

and that, for every $t \in J$, $\underline{s} = (s_1, \dots, s_n) \in \Delta_n$ *and* $j = 1, \dots, n$,

$$\begin{aligned} s_1 + \cdots + s_n = 1 \quad &\Rightarrow \quad \underline{C}(t, \underline{s}) \in \mathcal{N}\big(\gamma(t)\big) \\ s_j = 0 \quad &\Rightarrow \quad \underline{C}(t, \underline{s}) \in \mathcal{N}_j. \end{aligned}$$

Then F *admits analytic continuation along* γ *and, for each* $t \in J$,

$$F\big(\tilde{\gamma}(t)\big) = (\underline{C}_t)_{\#}[\Delta_n](\beta). \tag{20}$$

The proof of Proposition 5.4 requires the following consequence of the Cauchy-Poincaré Theorem:

Lemma 5.5. *Let* M *be a complex analytic manifold of dimension* n *and let* $N_0, N_1, \dots, N_n$ *be complex analytic hypersurfaces of* M. *Let* $\underline{H}\colon [0,1] \times \Delta_n \to M$ *be a Lipschitz map such that, for every* $\tau \in [0,1]$, $\underline{s} = (s_1, \dots, s_n) \in \Delta_n$ *and* $j = 1, \dots, n$,

$$\begin{aligned} s_1 + \cdots + s_n = 1 \quad &\Rightarrow \quad \underline{H}(\tau, s) \in N_0 \\ s_j = 0 \quad &\Rightarrow \quad \underline{H}(\tau, s) \in N_j. \end{aligned}$$

Then the partial maps $\underline{H}_0$ *and* $\underline{H}_1$ *corresponding to* $\tau = 0$ *and* $\tau = 1$ *satisfy*

$$(\underline{H}_0)_{\#}[\Delta_n](\beta) = (\underline{H}_1)_{\#}[\Delta_n](\beta) \tag{21}$$

for any holomorphic n*-form* β *on* M.

Proof of Lemma 5.5. Let β be a holomorphic n-form on M. Let us consider $P \coloneqq [0,1] \times \Delta_n$ and the corresponding $(n+1)$-dimensional integration current $[P] \in \mathscr{E}_{n+1}(\mathbb{R}^{n+1})$. Its boundary can be written

$$\partial[P] = Q_1 - Q_0 + B_0 + \cdots + B_n,$$

where each summand is an n-dimensional current with compact support:

$$\operatorname{spt} Q_i = \{i\} \times \Delta_n, \qquad \operatorname{spt} B_j = [0,1] \times F_j$$

with $F_j \coloneqq$ the face of $\partial\Delta_n$ defined by $s_j = 0$ if $j \geq 1$ or $s_1 + \cdots + s_n = 1$ if $j = 0$. This is a simple adaptation of formula (45) of Appendix A; in fact, $Q_i = [A_i(\Delta_n)]$ with an affine map $A_i\colon \underline{x} \in \mathbb{R}^n \mapsto (i, \underline{x}) \in \mathbb{R}^{n+1}$ and $B_j = \pm[A_j^*(\Delta_n)]$ with some other injective affine maps $A_j^*\colon \mathbb{R}^n \to \mathbb{R}^{n+1}$ mapping Δ_n to $[0,1] \times F_j$. In this situation, according to Lemma A.3 and formula (42), we have

$$\partial \underline{H}_{\#}[P] = \underline{H}_{\#}\partial[P], \qquad \underline{H}_{\#}Q_i = (\underline{H} \circ A_i)_{\#}[\Delta_n], \qquad \underline{H}_{\#}B_j = (\underline{H} \circ A_j^*)_{\#}[\Delta_n].$$

On the one hand, the Cauchy-Poincaré Theorem tells us that $\partial \underline{H}_{\#}[P](\beta) = 0$ (because $\mathrm{d}\beta = 0$), and $H \circ A_i = H_i$, thus

$$(\underline{H}_0)_{\#}[\Delta_n](\beta) - (\underline{H}_1)_{\#}[\Delta_n](\beta) = \underline{H}_{\#}B_0(\beta) + \cdots + \underline{H}_{\#}B_n(\beta).$$

On the other hand $\operatorname{spt} \underline{H}_{\#}B_j \subset N_j$ and the restriction of β to any complex hypersurface vanishes identically (because it is a holomorphic form of maximal degree), thus $\underline{H}_{\#}B_j(\beta) = 0$, and (21) is proved. □

Proof of Proposition 5.4. Observe that the function R_Ω defined by (5) is continuous, thus we can define a positive number

$$R^* \coloneqq \min\big\{R_\Omega\big(C_j(t,\underline{s})\big) \mid t \in J,\ \underline{s} \in \Delta_n,\ j = 1,\dots,n\big\}$$

and, for each $t \in J$ and $\zeta \in D\big(\gamma(t), R^*\big)$, a Lipschitz map and a complex number

$$\underline{\mathscr{Q}}_t(\zeta)\colon\ \underline{s} \in \Delta_n \mapsto \underline{C}(t,\underline{s}) + \big(\zeta - \gamma(t)\big)\underline{s} \in \mathscr{S}^n_\Omega, \qquad G_t(\zeta) \coloneqq \underline{\mathscr{Q}}_t(\zeta)_\#[\Delta_n](\beta).$$

For $\zeta \in D\big(\gamma(a), R^*\big)$, we have $\underline{\mathscr{Q}}_a(\zeta) = \underline{\mathscr{Q}}(\zeta)$, hence $G_a(\zeta) = F(\zeta)$. For $t \in J$, we have $\underline{\mathscr{Q}}_t\big(\gamma(t)\big) = \underline{C}_t$, hence

$$G_t\big(\gamma(t)\big) = (\underline{C}_t)_\#[\Delta_n](\beta).$$

Therefore it suffices to show that, for each $t \in J$,

i) the function G_t is holomorphic in $D\big(\gamma(t), R^*\big)$ (and $G_a = F$ is even holomorphic in $\mathbb{D}_{\rho(\Omega)}$);

ii) for any $t' \in J$ close enough to t, the functions G_t and $G_{t'}$ coincide in a neighbourhood of $\gamma(t)$.

i) The case of $G_a = F$ is easier because, for $\zeta \in \mathbb{D}_{\rho(\Omega)} \cup D\big(\gamma(a), R^*\big)$, the range of $\underline{\mathscr{Q}}_a(\zeta) = \underline{\mathscr{Q}}(\zeta)$ entirely lies in a domain $\mathscr{U} = \mathscr{U}_1 \times \cdots \mathscr{U}_n$, where each $\mathscr{U}_j$ is an open subset of $\mathscr{S}_\Omega$ in restriction to which π_Ω is injective, so that

$$\chi = \pi_\Omega^{\otimes n}\colon\ (\zeta_1,\dots,\zeta_n) \in \mathscr{U} \mapsto (\xi_1,\dots,\xi_n) = \big(\overset{\bullet}{\zeta}_1,\dots,\overset{\bullet}{\zeta}_n\big) \tag{22}$$

is an analytic chart of $\mathscr{S}^n_\Omega$; we can write $\chi^\# \beta = f(\xi_1,\dots,\xi_n)\,\mathrm{d}\xi_1 \wedge \cdots \wedge \mathrm{d}\xi_n$ with a holomorphic function f and $\chi \circ \underline{\mathscr{Q}}_a(\zeta)(\underline{s}) = (s_1\zeta,\dots,s_n\zeta)$, therefore

$$G_a(\zeta) = F(\zeta) = \zeta^n \int_{\Delta_n} f(s_1\zeta,\dots,s_n\zeta)\,\mathrm{d}s_1 \cdots \mathrm{d}s_n$$

is holomorphic.

Given $t \in J$, by compactness, we can cover Δ_n by simplices $Q^{[m]}$, $1 \le m \le M$, so that any intersection $Q^{[m]} \cap Q^{[m']}$ is contained in an affine hyperplane of $\mathbb{R}^n$ and each $Q^{[m]}$ is small enough for $\bigcup_{\zeta \in D(\gamma(t),R^*)} \underline{\mathscr{Q}}_t(\zeta)\big(Q^{[m]}\big)$ to be contained in the domain $\mathscr{U}^{[m]}$ of an analytic chart $\chi^{[m]}$ similar to (22) (*i.e.* $\mathscr{U}^{[m]}$ is a product of factors on which π_Ω is injective and $\chi^{[m]}$ is defined by the same formula as χ but on $\mathscr{U}^{[m]}$). For each m, we can write $\big(\chi^{[m]}\big)^\# \beta = f^{[m]}(\xi_1,\dots,\xi_n)\,\mathrm{d}\xi_1 \wedge \cdots \wedge \mathrm{d}\xi_n$ with a holomorphic function $f^{[m]}$ and $\chi^{[m]} \circ \underline{\mathscr{Q}}_t(\zeta) = \big(\xi_1^{[m]}(\zeta,\,\cdot\,),\dots,\xi_n^{[m]}(\zeta,\,\cdot\,)\big)$ with, for each $j = 1,\dots,n$,

$$(\zeta,\underline{s}) \in D\big(\gamma(t), R^*\big) \times Q^{[m]} \mapsto \xi_j^{[m]}(\zeta,\underline{s}) = \pi_\Omega \circ C_j(t,\underline{s}) + s_j\big(\zeta - \gamma(t)\big).$$

These functions $\xi_j^{[m]}$ are holomorphic in ζ; applying Rademacher's theorem to $\underline{s} \mapsto \pi_\Omega \circ C_j(t,\underline{s})$ (recall that t is fixed), we see that, for almost every $\underline{s}$, the partial derivatives of $\xi_j^{[m]}$ exist and are holomorphic in ζ, therefore

$$G_t(\zeta) = \sum_{m=1}^{M} \int_{Q^{[m]}} f^{[m]}\big(\xi_1^{[m]}(\zeta,\underline{s}),\dots,\xi_n^{[m]}(\zeta,\underline{s})\big) \det\left[\frac{\partial \xi_i^{[m]}}{\partial s_j}(\zeta,\underline{s})\right]_{1\le i,j\le n} \mathrm{d}s_1 \cdots \mathrm{d}s_n$$

is holomorphic for $\zeta \in D\big(\gamma(t), R^*\big)$.

ii) We now fix $t \in J$. By compactness, for $t' \in J$ close enough to t, we can write

$$\underline{C}(t', \underline{s}) = \underline{C}(t, \underline{s}) + \underline{\delta}(\underline{s})$$

for all $\underline{s} \in \Delta_n$, with

$$\delta_j(\underline{s}) \coloneqq \pi_\Omega\big(C_j(t', \underline{s}) - C_j(t, \underline{s})\big) \in \mathbb{D}_{\frac{R^*}{2n}}, \qquad j = 1, \dots, n.$$

Then $\gamma(t') \in D\big(\gamma(t), R^*/2\big)$ (because $s_1 + \cdots + s_n = 1$ implies $S_n \circ \delta(\underline{s}) = \gamma(t') - \gamma(t)$) and, for $\zeta \in D\big(\gamma(t'), R^*/2\big)$, we have

$$G_t(\zeta) \coloneqq \underline{\mathscr{Q}}_t(\zeta)_\#[\Delta_n](\beta), \qquad G_{t'}(\zeta) \coloneqq \underline{\mathscr{Q}}_{t'}(\zeta)_\#[\Delta_n](\beta)$$

with

$$\underline{\mathscr{Q}}_t(\zeta)(\underline{s}) = \underline{C}(t, \underline{s}) + \big(\zeta - \gamma(t)\big)\underline{s}, \qquad \underline{\mathscr{Q}}_{t'}(\zeta)(\underline{s}) = \underline{C}(t, \underline{s}) + \underline{\delta}(s) + \big(\zeta - \gamma(t')\big)\underline{s}.$$

Let us define a Lipschitz map $\underline{H}\colon [0,1] \times \Delta_n \to \mathscr{S}^n_\Omega$ by

$$\underline{H}(\tau, \underline{s}) \coloneqq \underline{C}(t, \underline{s}) + (1-\tau)\big(\zeta - \gamma(t)\big)\underline{s} + \tau\big(\underline{\delta}(s) + \big(\zeta - \gamma(t')\big)\underline{s}\big).$$

An easy computation yields

$$\begin{aligned} s_1 + \cdots + s_n = 1 \quad &\Rightarrow \quad S_n \circ \underline{H}(\tau, \underline{s}) = \zeta \\ s_j = 0 \quad &\Rightarrow \quad H_j(\tau, \underline{s}) = 0_\Omega. \end{aligned}$$

We can thus apply Lemma 5.5 with $N_0 = \mathcal{N}(\zeta)$ and $N_j = \mathcal{N}_j$, and conclude that $G_t \equiv G_{t'}$ on $D\big(\gamma(t'), R^*/2\big)$. □

Proof of Proposition 5.4. In view of Proposition 4.3, we can apply Proposition 5.4 with $\beta = \hat\varphi_1(\zeta_1) \cdots \hat\varphi_n(\zeta_n)\, \mathrm{d}\zeta_1 \wedge \cdots \wedge \mathrm{d}\zeta_n$ and $\underline{C}_t = \Psi_t \circ \underline{\mathscr{Q}}\big(\gamma(a)\big)$. □

6 Construction of an adapted origin-fixing isotopy in $\mathscr{S}^n_\Omega$

To prove Theorem 1, formula (16) tells us that it is sufficient to deal with the analytic continuation of products of the form $1 * \hat\varphi_1 * \cdots * \hat\varphi_n$ instead of $\hat\varphi_1 * \cdots * \hat\varphi_n$ itself, and Proposition 5.2 tells us that, to do so, we only need to construct an explicit γ-adapted origin-fixing isotopy (Ψ_t) and to provide estimates.

This section aims at constructing (Ψ_t) (estimates are postponed to Section 7). Our method is inspired by an appendix of [CNP93] and is a generalization of Section 6.2 of [Sau12].

Proposition 6.1. *Let $\gamma\colon J = [a,b] \to \mathbb{C} \setminus \Omega$ be a path such that $\gamma(a) \in \mathbb{D}^*_{\rho(\Omega)}$, and let $\eta\colon \mathbb{C} \to [0, +\infty)$ be a locally Lipschitz function such that*

$$\{\, \xi \in \mathbb{C} \mid \eta(\xi) = 0 \,\} = \Omega.$$

Then the function

$$(t, \underline{\zeta}) \in J \times \mathscr{S}^n_\Omega \mapsto D(t, \underline{\zeta}) \coloneqq \eta(\overset{\bullet}{\zeta}_1) + \cdots + \eta(\overset{\bullet}{\zeta}_n) + \eta\big(\gamma(t) - S_n(\underline{\zeta})\big) \tag{23}$$

is everywhere positive and the formula

$$X(t,\underline{\zeta}) \left| \begin{array}{l} X_1 \coloneqq \dfrac{\eta(\dot{\zeta}_1)}{D(t,\underline{\zeta})}\gamma'(t) \\ \qquad \vdots \\ X_n \coloneqq \dfrac{\eta(\dot{\zeta}_n)}{D(t,\underline{\zeta})}\gamma'(t) \end{array} \right. \tag{24}$$

defines a non-autonomous vector field $X(t,\underline{\zeta}) \in T_{\underline{\zeta}}\big(\mathscr{S}_\Omega^n\big) \simeq \mathbb{C}^n$ *(using the canonical identification between the tangent space of* $\mathscr{S}_\Omega$ *at any point and* $\mathbb{C}$ *provided by the tangent map of the local biholomorphism* π_Ω*) which admits a flow map* Ψ_t *between time* a *and time* t *for every* $t \in J$ *and induces a* γ*-adapted origin-fixing isotopy* $(\Psi_t)_{t\in J}$ *in* $\mathscr{S}_\Omega^n$*.*

An example of function which satisfies the assumptions of Proposition 6.1 is

$$\eta(\xi) \coloneqq \operatorname{dist}(\xi,\Omega), \qquad \xi \in \mathbb{C}.$$

Proof of Proposition 6.1. Let us denote by $J_k = [t_k, t_{k+1}]$, $k = 0,\dots,N-1$, subintervals forming a subdivision of J so that γ is C^1 on each J_k.

a) Observe that $D\big(t,(\zeta_1,\dots,\zeta_n)\big) = \tilde{D}\big(t,(\dot{\zeta}_1,\dots,\dot{\zeta}_n)\big)$ with

$$(t,\underline{\xi}) \in J \times \mathbb{C}^n \mapsto \tilde{D}(t,\underline{\xi}) \coloneqq \eta(\xi_1) + \cdots + \eta(\xi_n) + \eta\big(\gamma(t) - \tilde{S}_n(\underline{\xi})\big) \tag{25}$$

and $\tilde{S}_n(\underline{\xi}) \coloneqq \xi_1 + \cdots + \xi_n$ for any $\underline{\xi} \in \mathbb{C}^n$. The function $\tilde{D}$ is everywhere positive: suppose indeed $\tilde{D}(t,\underline{\xi}) = 0$ with $t \in J$ and $\underline{\xi} \in \mathbb{C}^n$, we would have

$$\xi_1,\dots,\xi_n,\gamma(t) - \tilde{S}(\underline{\xi}) \in \Omega,$$

whence $\gamma(t) \in \Omega$ by the stability under addition of Ω, but this is contrary to the hypothesis on γ.

Therefore $D > 0$, the vector field X is well-defined and in fact

$$X\big(t,(\zeta_1,\dots,\zeta_n)\big) = \tilde{X}\big(t,(\dot{\zeta}_1,\dots,\dot{\zeta}_n)\big)$$

with a non-autonomous vector field $\tilde{X}$ defined in $J \times \mathbb{C}^n$, the components of which are

$$\tilde{X}_j(t,\underline{\xi}) \coloneqq \frac{\eta(\xi_j)}{\tilde{D}(t,\underline{\xi})}\gamma'(t), \qquad j = 1,\dots,n. \tag{26}$$

These functions are locally Lipschitz on $J_k \times \mathbb{C}^n$ for each k. Therefore, the Cauchy-Lipschitz theorem on the existence and uniqueness of solutions to differential equations applies to $\mathrm{d}\underline{\xi}/\mathrm{d}t = \tilde{X}(t,\underline{\xi})$ on each interval J_k: for every $t^* \in J_k$ and $\underline{\xi} \in \mathbb{C}^n$, there is a unique maximal solution $t \mapsto \tilde{\Phi}^{t^*,t}(\underline{\xi})$ such that $\tilde{\Phi}^{t^*,t^*}(\underline{\xi}) = \underline{\xi}$. The fact that the vector field $\tilde{X}$ is bounded implies that $\tilde{\Phi}^{t^*,t}(\underline{\xi})$ is defined for all $t \in J_k$ and the classical theory guarantees that $(t^*,t,\underline{\xi}) \mapsto \tilde{\Phi}^{t^*,t}(\underline{\xi})$ is locally Lipschitz on $J_k \times J_k \times \mathbb{C}^n$.

b) For each $\omega \in \Omega$ and $j = 1,\dots,n$, we set

$$\tilde{\mathcal{N}}_j(\omega) \coloneqq \{\, \underline{\xi} = (\xi_1,\dots,\xi_n) \in \mathbb{C}^n \mid \xi_j = \omega \,\}.$$

We have $\tilde{X}_j \equiv 0$ on $J_k \times \tilde{\mathcal{N}}_j(\omega)$, thus $\tilde{\Phi}^{t^*,t}$ leaves $\tilde{\mathcal{N}}_j(\omega)$ invariant for every $(t^*,t) \in J_k \times J_k$. In particular, since $0 \in \Omega$,

$$\underline{\xi} \in \tilde{\mathcal{N}}_j(0) \quad \Rightarrow \quad \tilde{\Phi}^{t^*,t}(\underline{\xi}) \in \tilde{\mathcal{N}}_j(0). \tag{27}$$

The non-autonomous flow property $\tilde{\Phi}^{t,t^*} \circ \tilde{\Phi}^{t^*,t} = \tilde{\Phi}^{t^*,t} \circ \tilde{\Phi}^{t,t^*} = \mathrm{Id}$ implies that, for each $(t^*,t) \in J_k \times J_k$, $\tilde{\Phi}^{t^*,t}$ is a homeomorphism the inverse of which is $\tilde{\Phi}^{t,t^*}$, which leaves $\tilde{\mathcal{N}}_j(\omega)$ invariant, hence also

$$\underline{\xi} \in \mathbb{C}^n \setminus \tilde{\mathcal{N}}_j(\omega) \quad \Rightarrow \quad \tilde{\Phi}^{t^*,t}(\underline{\xi}) \in \mathbb{C}^n \setminus \tilde{\mathcal{N}}_j(\omega). \tag{28}$$

Properties (27) and (28) show that the flow map between times t^* and t for X is well-defined in $\mathscr{S}^n_\Omega$: for $\underline{\zeta} \in \mathscr{S}^n_\Omega$, the solution $t \mapsto \Phi^{t^*,t}(\underline{\zeta})$ can be obtained as the lift starting at $\underline{\zeta}$ of the path $t \mapsto \tilde{\Phi}^{t^*,t}\big(\overset{\bullet}{\zeta}_1, \dots, \overset{\bullet}{\zeta}_n\big)$ (indeed, each component of this path has its range either reduced to $\{0\}$ or contained in $\mathbb{C} \setminus \Omega$). Moreover, if for each $t \in J$ we define a homeomorphism of $\mathscr{S}^n_\Omega$ by

$$\Psi_t \coloneqq \Phi^{t_k,t} \circ \Phi^{t_{k-1},t_k} \circ \dots \circ \Phi^{t_0,t_1},$$

where k is determined by the condition $t \in J_k$, we get

$$\Psi_t(\mathcal{N}_j) \subset \mathcal{N}_j.$$

Observe that $\Psi_a = \mathrm{Id}$ and $(t,\underline{\zeta}) \mapsto \Psi_t(\underline{\zeta})$ is locally Lipschitz on $J \times \mathscr{S}^n_\Omega$.

c) It only remains to be proved that

$$\Psi_t\big(\mathcal{N}(\gamma(a))\big) \subset \Psi_t\big(\mathcal{N}(\gamma(t))\big) \tag{29}$$

for every $t \in J$.

For $\underline{\zeta} \in \mathscr{S}^n_\Omega$ fixed and $k = 0, \dots, N-1$, let

$$\xi_0 \colon\ t \in J_k \mapsto \gamma(t) - S_n \circ \Psi_t(\underline{\zeta}).$$

This function is C^1 on J_k and an easy computation yields its derivative in the form $\xi_0'(t) = h(t)\gamma'(t)/d(t)$, with Lipschitz functions

$$h(t) \coloneqq \eta\big(\xi_0(t)\big), \qquad d(t) \coloneqq D\big(t, \Psi_t(\underline{\zeta})\big).$$

Since η is Lipschitz on the range of ξ_0, say with Lipschitz constant K, the function $h = \eta \circ \xi_0$ is Lipschitz on J_k, hence its derivative h' exists almost everywhere on J_k; writing $|h(t') - h(t)| \le K|\xi_0(t') - \xi_0(t)|$, we see that $\big|h'(t)\big| \le K\big|\xi_0'(t)\big| \le Kh(t) \max\limits_{J_k} |\frac{\gamma'}{d}|$ a.e., hence

$$g(t) \coloneqq \frac{h'(t)}{h(t)} \text{ exists a.e. and defines } g \in L^\infty(J_k).$$

By the fundamental theorem of Lebesgue integral calculus, $t \mapsto \int_{t_k}^t g(\tau)\,\mathrm{d}\tau$ is differentiable a.e. and

$$h(t) = h(t_k)\, \exp\Big(\int_{t_k}^t g(\tau)\,\mathrm{d}\tau \Big), \qquad t \in J_k.$$

It follows that, for each $t \in J$, $h(t)$ is the product of $h(a)$ and a positive factor. Now, if $\underline{\zeta} \in \mathcal{N}\big(\gamma(a)\big)$, then $\xi_0(a) = 0$, thus $h(a) = 0$, thus $h \equiv 0$ on J, thus $\xi_0(t)$ stays in Ω for all $t \in J$, thus $\xi_0 \equiv 0$ on J, *i.e.* $\Psi_t(\underline{\zeta}) \in \mathcal{N}\big(\gamma(t)\big)$. □

7 Estimates

We are now ready to prove

Theorem 1'. *Let $\delta, L > 0$ with $\delta < \rho(\Omega)/2$ and*

$$\delta' \coloneqq \frac{1}{2}\rho(\Omega)\,\mathrm{e}^{-2L/\delta}. \tag{30}$$

Then, for any $n \geq 1$ and $\hat\varphi_1, \dots, \hat\varphi_n \in \hat{\mathscr{R}}_\Omega$,

$$\max_{\mathcal{K}_{\delta,L}(\Omega)} |1 * \hat\varphi_1 * \cdots * \hat\varphi_n| \leq \frac{1}{n!}\big(\rho(\Omega)\,\mathrm{e}^{3L/\delta}\big)^n \max_{\mathcal{K}_{\delta',L}(\Omega)} |\hat\varphi_1| \cdots \max_{\mathcal{K}_{\delta',L}(\Omega)} |\hat\varphi_n|. \tag{31}$$

The proof of Theorem 1' will follow from

Proposition 7.1. *Let $\delta, L > 0$. Let $\gamma\colon J = [a,b] \to \mathbb{C} \setminus \Omega$ be a path such that $\gamma(a) \in \mathbb{D}^*_{\rho(\Omega)/2}$, $|\gamma(a)| + b - a \leq L$ and*

$$\big|\gamma'(t)\big| = 1 \ \text{ and } \ \operatorname{dist}\big(\gamma(t), \Omega\big) \geq \delta, \qquad t \in J.$$

Consider the γ-adapted origin-fixing isotopy $(\Psi_t)_{t\in J}$ defined as in Proposition 6.1 by the flow of the vector field (24) with the choice $\eta(\xi) = \operatorname{dist}(\xi, \Omega)$. Then, for each $t \in J$,

- *the Lipschitz map $\Psi_t \circ \underline{\mathscr{Q}}\big(\gamma(a)\big) = (\xi_1^t, \dots, \xi_1^t)$ maps Δ_n in $\big(\mathcal{K}_{\delta',L}(\Omega)\big)^n$, with δ' as in (30),*
- *the almost everywhere defined partial derivatives $\frac{\partial \overset{\bullet}{\xi}{}_i^t}{\partial s_j}\colon \Delta_n \to \mathbb{C}$ satisfy*

$$\left|\det\left[\frac{\partial \overset{\bullet}{\xi}{}_i^t}{\partial s_j}(\underline{s})\right]_{1\leq i,j\leq n}\right| \leq \big(\rho(\Omega)\,\mathrm{e}^{3L/\delta}\big)^n \quad \text{for a.e. } \underline{s} \in \Delta_n. \tag{32}$$

Proof of Proposition 7.1. We first fix $\underline{s} \in \Delta_n$, omitting it in the notations, and study the solution

$$t \in J \mapsto \underline{\xi}^t \coloneqq (\xi_1^t, \dots, \xi_n^t) \coloneqq \Psi_t\big(\underline{\mathscr{Q}}\big(\gamma(a)\big)(\underline{s})\big)$$

of the vector field X defined by (24), the components of the initial condition being $\xi_i^0 = 0_\Omega + s_i\gamma(a)$.

a) We observe that $\mathrm{d}\overset{\bullet}{\xi}{}_i^t/\mathrm{d}t = X_i(t, \underline{\xi}^t)$ has modulus ≤ 1 for each $i = 1, \dots, n$, thus the path $t \in J \mapsto \overset{\bullet}{\xi}{}_i^t$ has length $\leq b - a$ and stays in $\mathbb{D}_L$.

b) The denominator (23) is

$$d(t) \coloneqq D(t, \underline{\xi}^t) \geq \delta, \qquad t \in J.$$

Indeed, we can write $d(t) = \eta\big(\overset{\bullet}{\xi}{}_0^t\big) + \eta\big(\overset{\bullet}{\xi}{}_1^t\big) + \cdots + \eta\big(\overset{\bullet}{\xi}{}_n^t\big)$ with $\overset{\bullet}{\xi}{}_0^t \coloneqq \gamma(t) - S_n(\underline{\xi}^t)$, and, since Ω is stable under addition and $\overset{\bullet}{\xi}{}_0^t + \overset{\bullet}{\xi}{}_1^t + \cdots + \overset{\bullet}{\xi}{}_n^t = \gamma(t)$, the triangle inequality yields

$$d(t) = \sum_{i=0}^n \operatorname{dist}\big(\overset{\bullet}{\xi}{}_i^t, \Omega\big) \geq \operatorname{dist}\big(\gamma(t), \Omega\big),$$

which is $\geq \delta$ by assumption.

c) We now check that for $t \in J$ and $i = 1, \dots, n$,

$$\mathrm{e}^{-L/\delta}\, \eta\big(\overset{\bullet}{\xi}{}_i^a\big) \le \eta\big(\overset{\bullet}{\xi}{}_i^t\big) \le \mathrm{e}^{L/\delta}\, \eta\big(\overset{\bullet}{\xi}{}_i^a\big). \tag{33}$$

Since η is 1-Lipschitz, the function $h_i := \eta \circ \overset{\bullet}{\xi}{}_i^t$ is Lipschitz on J and its derivative exists a.e.; writing $|h_i(t') - h_i(t)| \le |\overset{\bullet}{\xi}_i(t') - \overset{\bullet}{\xi}_i(t)|$, we see that a.e. $|h_i'(t)| \le |\overset{\bullet}{\xi}{}_i'(t)| = h_i(t)/d(t)$ hence $g_i(t) := \frac{h_i'(t)}{h_i(t)}$ exists a.e. and defines $g_i \in L^\infty(J)$ with

$$|g_i(t)| \le 1/\delta \quad \text{for a.e. } t \in J. \tag{34}$$

By the fundamental theorem of Lebesgue integral calculus, $t \mapsto \int_a^t g_i(\tau)\,\mathrm{d}\tau$ is differentiable a.e. and

$$h_i(t) = h_i(a)\, \exp\Big(\int_a^t g_i(\tau)\,\mathrm{d}\tau \Big), \qquad t \in J,$$

whence (33) follows in view of (34).

d) Now, the fact that $\overset{\bullet}{\xi}{}_i^a = s_i\gamma(a) \in \mathbb{D}_{\rho(\Omega)/2}$ implies that $\operatorname{dist}\big(\overset{\bullet}{\xi}{}_i^a, \Omega \setminus \{0\}\big) \ge \rho(\Omega)/2$, whence

$$\eta(\overset{\bullet}{\xi}{}_i^a) = \operatorname{dist}\big(\overset{\bullet}{\xi}{}_i^a, \Omega\big) = |\overset{\bullet}{\xi}{}_i^a| \le \rho(\Omega)/2.$$

If $|\overset{\bullet}{\xi}{}_i^a| < \frac{1}{2}\rho(\Omega)\,\mathrm{e}^{-L/\delta}$, then the second inequality in (33) shows that $\eta\big(\overset{\bullet}{\xi}{}_i^t\big)$ stays $< \frac{1}{2}\rho(\Omega)$, hence ξ_i^t stays in the lift of $\mathbb{D}_{\rho(\Omega)/2}$ in the principal sheet and $R_\Omega(\xi_i^t)$ stays $\ge \frac{1}{2}\rho(\Omega) > \delta'$.

If $|\overset{\bullet}{\xi}{}_i^a| \ge \frac{1}{2}\rho(\Omega)\,\mathrm{e}^{-L/\delta}$, then the first inequality in (33) shows that $\eta\big(\overset{\bullet}{\xi}{}_i^t\big) \ge \frac{1}{2}\rho(\Omega)\,\mathrm{e}^{-2L/\delta}$ which equals δ', hence $R_\Omega(\xi_i^t)$ stays $\ge \delta'$.

We infer that $\xi_i^t \in \mathcal{K}_{\delta',L}(\Omega)$ for all $t \in J$ in both cases (in view of point **a**, since $\xi_i^t \in \mathscr{S}_\Omega$ can be represented by the the path $\Gamma_s|t \in \mathscr{P}_\Omega$ which is obtained by concatenation of $[0, s_i\gamma(a)]$ and $\tau \in [a,t] \mapsto \overset{\bullet}{\xi}{}_i^\tau$ and has length $\le |\gamma(a)| + b - a \le L$).

e) It only remains to study the partial derivatives $\frac{\partial \xi_i^t}{\partial s_j}(\underline{s})$ which, given $t \in J$, exist for almost every $\underline{s} \in \Delta_n$ by virtue of Rademacher's theorem. We first prove that for every $t \in J$, $\underline{s}, \underline{s}' \in \Delta_n$,

$$\sum_{i=1}^n \Big| \overset{\bullet}{\xi}{}_i^t(\underline{s}') - \overset{\bullet}{\xi}{}_i^t(\underline{s}) \Big| \le \mathrm{e}^{3L/\delta} |\gamma(a)| \sum_{i=1}^n |s_i' - s_i|. \tag{35}$$

Lemma 7.2. *Whenever the function η is 1-Lipschitz on $\mathbb{C}$ and $|\gamma'(\tau)| \le 1$ for all $\tau \in J$, the vector field X defined by (23)–(24) satisfies*

$$\sum_{i=1}^n \big| X_i(\tau, \underline{\zeta}') - X_i(\tau, \underline{\zeta}) \big| \le \frac{3}{D(\tau, \underline{\zeta}')} \sum_{i=1}^n \Big| \overset{\bullet}{\zeta}{}_i' - \overset{\bullet}{\zeta}_i \Big| \tag{36}$$

for any $\tau \in J$ and $\underline{\zeta}, \underline{\zeta}' \in \mathscr{S}_\Omega^n$.

Proof of Lemma 7.2. Let $\tau \in J$ and $\underline{\zeta}, \underline{\zeta}' \in \mathscr{S}_\Omega^n$. For $i = 1, \dots, n$, we can write

$$X_i(\tau, \underline{\zeta}') - X_i(\tau, \underline{\zeta}) = \bigg(\eta\big(\overset{\bullet}{\zeta}{}_i'\big) - \eta\big(\overset{\bullet}{\zeta}_i\big) - \big(D(\tau, \underline{\zeta}') - D(\tau, \underline{\zeta})\big) \frac{\eta\big(\overset{\bullet}{\zeta}_i\big)}{D(\tau, \underline{\zeta})} \bigg) \frac{\gamma'(\tau)}{D(\tau, \underline{\zeta}')},$$

with $D(\tau,\underline{\zeta}') - D(\tau,\underline{\zeta}) = \sum_{j=0}^n \big(\eta\big(\overset{\bullet}{\zeta}{}'_j\big) - \eta\big(\overset{\bullet}{\zeta}_j\big)\big)$, using the notations $\overset{\bullet}{\zeta}_0 = \gamma(\tau) - S_n(\underline{\zeta})$, $\overset{\bullet}{\zeta}{}'_0 = \gamma(\tau) - S_n(\underline{\zeta}')$. Since η is 1-Lipschitz, we have $|\eta\big(\overset{\bullet}{\zeta}{}'_j\big) - \eta\big(\overset{\bullet}{\zeta}_j\big)| \le |\overset{\bullet}{\zeta}{}'_j - \overset{\bullet}{\zeta}_j|$ for $j = 0,\dots,n$ and $|\overset{\bullet}{\zeta}{}'_0 - \overset{\bullet}{\zeta}_0| \le \sum_{j=1}^n |\overset{\bullet}{\zeta}{}'_j - \overset{\bullet}{\zeta}_j|$, whence $|D(\tau,\underline{\zeta}') - D(\tau,\underline{\zeta})| \le \sum_{j=0}^n |\overset{\bullet}{\zeta}{}'_j - \overset{\bullet}{\zeta}_j| \le 2\sum_{j=1}^n |\overset{\bullet}{\zeta}{}'_j - \overset{\bullet}{\zeta}_j|$. The result follows because $\sum_{i=1}^n \eta\big(\overset{\bullet}{\zeta}_i\big) \le D(\tau,\underline{\zeta})$. □

Proof of inequality (35). Let us fix $\underline{s},\underline{s}' \in \Delta_n$ and denote by $\Delta(t)$ the left-hand side of (35), *i.e.*

$$\Delta(t) = \sum_{i=1}^n |\Delta_i(t)|, \qquad \Delta_i(t) \coloneqq \overset{\bullet}{\xi}{}^t_i(\underline{s}') - \overset{\bullet}{\xi}{}^t_i(\underline{s}).$$

For every $t \in J$, we have

$$\Delta_i(t) = \Delta_i(a) + \int_a^t \Big(X_i\big(\tau,\underline{\xi}^\tau(\underline{s}')\big) - X_i\big(\tau,\underline{\xi}^\tau(\underline{s})\big)\Big)\,\mathrm{d}\tau, \qquad i = 1,\dots,n.$$

By Lemma (7.2), we get

$$|\Delta(t) - \Delta(a)| \le \sum_{i=1}^n |\Delta_i(t) - \Delta_i(a)| \le \int_a^t \frac{3}{D\big(\tau,\underline{\xi}^\tau(\underline{s}')\big)}\Delta(\tau)\,\mathrm{d}\tau.$$

We have seen that $D\big(\tau,\underline{\xi}^\tau(\underline{s}')\big)$ stays $\ge \delta$ (this was point **b**), thus $|\Delta(t) - \Delta(a)| \le \frac{3}{\delta}\int_a^t \Delta(\tau)\,\mathrm{d}\tau$ for all $t \in J$. Gronwall's lemma yields

$$|\Delta(t)| \le \Delta(a)\,\mathrm{e}^{3(t-a)/\delta}, \qquad t \in J,$$

and, in view of the initial conditions $\Delta_i(a) = (s'_i - s_i)\gamma(a)$, (35) is proved. □

f) Let us fix $t \in J$. For any $\underline{s} \in \Delta_n$ at which $(\overset{\bullet}{\xi}{}^t_1,\dots,\overset{\bullet}{\xi}{}^t_n)$ is differentiable, because of (35), the entries of the matrix $\mathscr{J} \coloneqq \Big[\frac{\partial\overset{\bullet}{\xi}{}^t_i}{\partial s_j}(\underline{s})\Big]_{1\le i,j\le n}$ satisfy

$$\sum_{i=1}^n \left|\frac{\partial\overset{\bullet}{\xi}{}^t_i}{\partial s_j}(\underline{s})\right| \le \mathrm{e}^{3L/\delta}|\gamma(a)|, \qquad j = 1,\dots,n.$$

We conclude by observing that

$$|\det(\mathscr{J})| \le \Big(\sum_{i=1}^n |\mathscr{J}_{i,1}|\Big)\cdots\Big(\sum_{i=1}^n |\mathscr{J}_{i,n}|\Big) \le \big(\mathrm{e}^{3L/\delta}|\gamma(a)|\big)^n$$

(because the left-hand side is bounded by the sum of the products $\big|\mathscr{J}_{\sigma(1),1}\cdots\mathscr{J}_{\sigma(n),n}\big|$ over all bijective maps $\sigma\colon [1,n] \to [1,n]$, while the middle expression is equal to the sum of the same products over *all* maps $\sigma\colon [1,n] \to [1,n]$). □

Proof of Theorem 1'. Let $\delta, L > 0$ with $\delta < \rho(\Omega)/2$ and $\zeta \in \mathcal{K}_{\delta,L}(\Omega)$. We want to prove

$$|1 * \hat\varphi_1 * \cdots * \hat\varphi_n(\zeta)| \le \frac{1}{n!}\big(\rho(\Omega)\,\mathrm{e}^{3L/\delta}\big)^n \max_{\mathcal{K}_{\delta',L}(\Omega)}|\hat\varphi_1|\cdots\max_{\mathcal{K}_{\delta',L}(\Omega)}|\hat\varphi_n|$$

for any $n \geq 1$ and $\hat{\varphi}_1, \dots, \hat{\varphi}_n \in \hat{\mathscr{R}}_\Omega$.

We may assume $\zeta \notin \mathcal{L}_{0_\Omega}(\mathbb{D}_{\rho(\Omega)})$ (since the behaviour of convolution products on the principal sheet is already settled by (4) and $\zeta \in \mathcal{L}_{0_\Omega}(\mathbb{D}_{\rho(\Omega)})$ would imply $\frac{|\zeta|}{n+1} < \rho(\Omega)\,\mathrm{e}^{3L/\delta}$). We can then choose a representative path of ζ in $\mathscr{P}_\Omega$, the initial part of which is a line segment ending in $\mathbb{D}_{\rho(\Omega)/2} \setminus \mathbb{D}_\delta$; since we prefer to parametrize our paths by arc-length, we take $\tilde{\gamma}\colon [\tilde{a}, b] \to \mathbb{C}$ piecewise C^1 such that $\tilde{\gamma}'(t) \equiv 1$ and $\mathrm{length}(\tilde{\gamma}) = b - \tilde{a} \leq L$, and $a \in (\tilde{a}, b)$ such that

- $\tilde{\gamma}(a) \in \mathbb{D}_{\rho(\Omega)/2}$,
- $\tilde{\gamma}(t) = \frac{t-\tilde{a}}{a-\tilde{a}}\tilde{\gamma}(a)$ for all $t \in [\tilde{a}, a]$,
- $\mathrm{dist}\big(\tilde{\gamma}(t), \Omega\big) \geq \delta$ for all $t \in [a, b]$.

Now the restriction γ of $\tilde{\gamma}$ to $[a, b]$ satisfies all the assumptions of Proposition 7.1, while formula (19) of Proposition 5.2 for $t = b$ can be interpreted as

$$1 * \hat{\varphi}_1 * \cdots * \hat{\varphi}_n(\zeta) = \int_{\Delta_n} \hat{\varphi}_1\big(\xi_1^b(\underline{s})\big) \cdots \hat{\varphi}_n\big(\xi_n^b(\underline{s})\big) \det\left[\frac{\partial \overset{\bullet}{\xi}{}_i^b}{\partial s_j}(\underline{s})\right]_{1 \leq i,j \leq n} \mathrm{d}s_1 \cdots \mathrm{d}s_n. \tag{37}$$

The conclusion follows immediately, since the Lebesgue measure of Δ_n is $1/n!$. □

We can now prove the main result which was announced in Section 2.

Proof of Theorem 1. Let $\delta, L > 0$ with $\delta < \rho(\Omega)$, $n \geq 1$ and $\hat{\varphi}_1, \dots, \hat{\varphi}_n \in \hat{\mathscr{R}}_\Omega$. Let $\zeta \in \mathcal{K}_{\delta,L}(\Omega)$. We must prove

$$|\hat{\varphi}_1 * \cdots * \hat{\varphi}_n(\zeta)| \leq \frac{2}{\delta} \cdot \frac{C^n}{n!} \cdot \max_{\mathcal{K}_{\delta',L'}(\Omega)} |\hat{\varphi}_1| \cdots \max_{\mathcal{K}_{\delta',L'}(\Omega)} |\hat{\varphi}_n|.$$

One can check that any $\zeta' \in \mathcal{L}_\zeta(\mathbb{D}_{\delta/2}) = \{\, \zeta + w \mid |w| < \delta/2 \,\}$ satisfies

$$\zeta' \in \mathcal{K}_{\delta/2,L'}(\Omega), \qquad \text{where } L' \coloneqq L + \delta/2. \tag{38}$$

Indeed, ζ is the endpoint of a path γ starting from 0_Ω, of length $\leq L$, which has $R_\Omega\big(\gamma(t)\big) \geq \delta$. In particular $R_\Omega(\zeta) \geq \delta$ thus the path $t \in [0,1] \mapsto \sigma(t) \coloneqq \zeta + t\big(\overset{\bullet}{\zeta}{}' - \overset{\bullet}{\zeta}\big)$ is well-defined. Either ζ does not lie in the principal sheet of $\mathscr{S}_\Omega$, then $\mathrm{dist}(\overset{\bullet}{\zeta}, \Omega) \geq \delta$ implies $\mathrm{dist}\big(\sigma(t), \Omega\big) \geq \delta/2$ and, by concatenating γ and σ, we see that (38) holds; or ζ is in the principal sheet and then we can choose γ contained in the principal sheet and we have at least $\mathrm{dist}\big(\sigma(t), \Omega \setminus \{0\}\big) \geq \delta/2$; if moreover $\overset{\bullet}{\zeta} \in \mathbb{D}_{\rho(\Omega)}$ then also σ is contained in the principal sheet, with $R_\Omega\big(\sigma(t)\big) \geq \delta/2$, whereas if $\overset{\bullet}{\zeta} \notin \mathbb{D}_{\rho(\Omega)}$ then $\mathrm{dist}\big(\sigma(t), \{0\}\big) \geq \rho(\Omega) - \delta/2 \geq \delta/2$, hence again $R_\Omega\big(\sigma(t)\big) \geq \delta/2$, thus (38) holds in all cases.

Thus, by Theorem 1',

$$\max_{\mathcal{L}_\zeta(\mathbb{D}_{\delta/2})} |1 * \hat{\varphi}_1 * \cdots * \hat{\varphi}_n| \leq \frac{C^n}{n!} \max_{\mathcal{K}_{\delta',L'}(\Omega)} |\hat{\varphi}_1| \cdots \max_{\mathcal{K}_{\delta',L'}(\Omega)} |\hat{\varphi}_n|$$

with $\delta' \coloneqq \frac{1}{2}\rho(\Omega)\,\mathrm{e}^{-4L'/\delta}$ and $C \coloneqq \rho(\Omega)\,\mathrm{e}^{6L'/\delta}$, which are precisely the values indicated in (7). The conclusion follows from the Cauchy inequalities. □

A Appendix: a class of rectifiable currents and their Lipschitz push-forwards

In this appendix, we single out a few facts from Geometric Measure Theory which are useful in the proof of our main result. Among the standard references on the subject one can quote [Fed69], [Sim83], [AK00], [Mor09].

For a differentiable manifold M and an integer $m \geq 0$, we denote by $\mathscr{E}_m(M)$ the space of all m-dimensional currents with compact support, viewed as linear functionals on the space of all C^∞ differential m-forms (with complex-valued coefficients) which are continuous for the usual family of seminorms (defined by considering the partial derivatives of the coefficients of forms in compact subsets of charts). In fact, by taking real and imaginary parts, the situation is reduced to that of real-valued forms and real-valued currents. For us, $M = \mathbb{R}^N$ or $M = \mathscr{S}_\Omega^n$, but in the latter case, as far as currents are concerned, the local biholomorphism $\pi_\Omega^{\otimes n}$ makes the difference between $\mathscr{S}_\Omega^n$ and $\mathbb{C}^n$ immaterial, and the complex structure plays no role, so that one loses nothing when replacing M with $\mathbb{R}^{2n}$.

Integration currents associated with oriented compact rectifiable sets

Let $m, N \in \mathbb{N}^*$. We denote by $\mathscr{H}^m$ the m-dimensional Hausdorff measure in $\mathbb{R}^N$. A basic example of m-dimensional current in $\mathbb{R}^N$ is obtained as follows:

Definition A.1. *Let S be an oriented compact m-dimensional rectifiable subset of $\mathbb{R}^N$ (i.e. S is compact, $\mathscr{H}^m$-almost all of S is contained in the union of the images of countably many Lipschitz maps from $\mathbb{R}^m$ to $\mathbb{R}^N$ and we are given a measurable orientation of the approximate tangent m-planes[4] to S) and, for $\mathscr{H}^m$-a.e. $x \in S$, let $\tau(x)$ be a unit m-vector orienting the tangent m-plane at x; then the formula*

$$[S]\colon\ \alpha\ m\text{-form on } \mathbb{R}^N \mapsto \int_S \langle \tau(x), \alpha(x) \rangle \, \mathrm{d}\mathscr{H}^m(x) \tag{39}$$

defines a current $[S] \in \mathscr{E}_m(\mathbb{R}^N)$, the support of which is S.

This example belong to the class of *integer rectifiable currents*, for which the right-hand side of (39) more generally assumes the form

$$\int_S \langle \tau(x), \alpha(x) \rangle \mu(x) \, \mathrm{d}\mathscr{H}^m(x),$$

where μ is a *multiplicity function*, *i.e.* an $\mathscr{H}^m$-integrable function $\mu\colon S \to \mathbb{N}^*$.

One must keep in mind that a rectifiable current is determined by a triple (S, τ, μ) where the orienting m-vector τ is tangent to the support S (at $\mathscr{H}^m$-almost every point); this is of fundamental importance in what follows (taking an m-vector field τ which is not tangent to S almost everywhere would lead to very different behaviours when applying the boundary operator). In this appendix we shall content ourselves with the case $\mu \equiv 1$.

An elementary example is $[\Delta_N] \in \mathscr{E}_N(\mathbb{R}^N)$, with the standard N-dimensional simplex $\Delta_N \subset \mathbb{R}^N$ of Notation 4.2 oriented by $\tau = \frac{\partial}{\partial x_1} \wedge \cdots \wedge \frac{\partial}{\partial x_N}$.

[4] Recall that, at $\mathscr{H}^m$-almost every point of S, the cone of approximate tangent vectors is an m-plane [Fed69, 3.2.19], [Mor09, 3.12]).

Push-forward by smooth and Lipschitz maps

The push-forward of a current $T \in \mathscr{E}_m(\mathbb{R}^N)$ by a smooth map $\Phi\colon \mathbb{R}^N \to \mathbb{R}^{N'}$ is classically defined by dualizing the pullback of differential forms:

$$\Phi_\# T(\beta) \coloneqq T(\Phi^\# \beta), \qquad \beta \text{ any } m\text{-form on } \mathbb{R}^{N'},$$

which yields $\Phi_\# T \in \mathscr{E}_m(\mathbb{R}^{N'})$.

For an integration current $T = [S]$ as in (39), we observe that the smoothness of α is not necessary for the definition of $[S](\alpha)$ to make sense: it suffices that α be defined $\mathscr{H}^m$-almost everywhere on S, bounded and $\mathscr{H}^m$-measurable. Therefore, *in the top-dimensional case $m = N$, we can associate with the current $[S] \in \mathscr{E}_N(\mathbb{R}^N)$ a push-forward $\phi_\#[S] \in \mathscr{E}_N(\mathbb{R}^{N'})$ by any Lipschitz map $\phi\colon S \to \mathbb{R}^{N'}$, by means of the formula*

$$\phi_\#[S](\beta) \coloneqq [S](\phi^\# \beta), \qquad \beta \text{ any } N\text{-form on } \mathbb{R}^{N'}. \tag{40}$$

Indeed, Rademacher's theorem ensures that ϕ is differentiable $\mathscr{H}^N$-almost everywhere ($\mathscr{H}^N$ is the Lebesgue measure), with bounded partial derivatives, hence the pullback form $\phi^\#\beta$ is defined almost everywhere as

$$\beta = \sum_I g_I \,\mathrm{d}y^{I_1} \wedge \cdots \wedge \mathrm{d}y^{I_N} \quad \Longrightarrow$$

$$\phi^\#\beta = \sum_I (g_I \circ \phi)\,\mathrm{d}\phi^{I_1} \wedge \cdots \wedge \mathrm{d}\phi^{I_N} = \sum_I (g_I \circ \phi) \det\left[\frac{\partial \phi^{I_i}}{\partial x^j}\right]_{1\le i,j\le N} \mathrm{d}x^1 \wedge \cdots \wedge x^N,$$

where the sums are over all $I = \{1 \le I_1 < \cdots < I_N \le N'\}$, the coordinates in $\mathbb{R}^{N'}$ are denoted by $(y^1, \dots, y^{N'})$ and those in $\mathbb{R}^N$ by $(x^1, \dots, x^N)$. The pullback form $\alpha = \phi^\#\beta$ has its coefficients in $L^\infty(\mathbb{R}^N)$, hence we can define $\phi_\#[S](\beta) = [S](\alpha)$ by (39).

Having defined $\phi_\#[S] \in \mathscr{E}_N(\mathbb{R}^{N'})$ by formula (40), it is worth noticing that $\phi_\#[S]$ can also be obtained by a regularization process:

Lemma A.2. *Let S be an oriented compact N-dimensional rectifiable subset of $\mathbb{R}^N$ and let $\phi\colon S \to \mathbb{R}^{N'}$ be a Lipschitz map. Consider smooth Lipschitz maps $\Phi_\ell\colon \mathbb{R}^N \to \mathbb{R}^{N'}$, $\ell \in \mathbb{N}$, which have uniformly bounded Lipschitz constants and converge uniformly to ϕ on S as $\ell \to \infty$. Then*

$$(\Phi_\ell)_\#[S](\beta) \xrightarrow[\ell\to\infty]{} \phi_\#[S](\beta), \qquad \beta \text{ any } N\text{-form on } \mathbb{R}^{N'}. \tag{41}$$

The proof relies on equicontinuity estimates derived from Reshetnyak's theorem[5] which guarantees that in this situation, not only do we have the weak-$*$ convergence in $L^\infty(\mathbb{R}^N)$ for the partial derivatives $\frac{\partial \Phi_\ell^{I_k}}{\partial x_j} \mathrel{\overset{*}{\rightharpoonup}} \frac{\partial \phi^{I_k}}{\partial x_j}$, but also for the minors of the Jacobian matrix: $\det\left[\frac{\partial \phi_\ell^{I_i}}{\partial x^j}\right] \mathrel{\overset{*}{\rightharpoonup}} \det\left[\frac{\partial \phi^{I_i}}{\partial x^j}\right]$, whence $\Phi_\ell^\#\beta \mathrel{\overset{*}{\rightharpoonup}} \phi^\#\beta$ componentwise in $L^\infty(\mathbb{R}^N)$ and (41) follows.

Another case of interest is $T = [A(\Delta)] \in \mathscr{E}_m(\mathbb{R}^N)$ with $m \le N$, Δ an oriented compact m-dimensional rectifiable subset of $\mathbb{R}^m$ and $A\colon \mathbb{R}^m \to \mathbb{R}^N$ an injective affine map (the unit m-vector field orienting $A(\Delta)$ is chosen to be a positive multiple of the image of the unit m-vector field orienting Δ by the m-linear extension of the linear part of A to $\Lambda_m\mathbb{R}^m$). We have

[5] See [Eva98], § 8.2.4, Lemma on the weak continuity of determinants.

$[A(\Delta)] = A_\#[\Delta]$, thus the natural definition of the push-forward of $[A(\Delta)]$ by a Lipschitz map $\phi\colon\ A(\Delta) \to \mathbb{R}^{N'}$ is clearly

$$\phi_\#[A(\Delta)] \coloneqq (\phi_A)_\#[\Delta] \in \mathscr{E}_{N-1}(\mathbb{R}^{N'}), \qquad \text{with } \phi_A \coloneqq \phi \circ A\colon\ \Delta \to \mathbb{R}^{N'}. \tag{42}$$

Indeed, one easily checks that when ϕ is the restriction to $A(\Delta)$ of a smooth map $\Phi\colon\ \mathbb{R}^N \to \mathbb{R}^{N'}$, the above-defined push-forward $\phi_\#[A(\Delta)]$ coincides with the classical push-forward $\Phi_\#[A(\Delta)]$. Moreover, also in this case is the regularization process possible: for any sequence of smooth Lipschitz maps $\Phi_\ell\colon\ \mathbb{R}^N \to \mathbb{R}^{N'}$, $\ell \in \mathbb{N}$, which have uniformly bounded Lipschitz constants and converge uniformly to ϕ on $A(\Delta)$ as $\ell \to \infty$, we have

$$(\Phi_\ell)_\#[A(\Delta)](\beta) \xrightarrow[\ell\to\infty]{} \phi_\#[A(\Delta)](\beta), \qquad \beta \text{ any } N\text{-form on } \mathbb{R}^{N'} \tag{43}$$

(simply because the left-hand side is $(\Phi_\ell \circ A)_\#[\Delta](\beta)$ and we can apply (41) to the sequence $\Phi_\ell \circ A$ uniformly converging to $\phi \circ A$ on Δ).

The boundary operator and Stokes's theorem

The boundary operator is defined by duality on all currents $T \in \mathscr{E}_m(\mathbb{R}^N)$:

$$\partial T(\alpha) \coloneqq T(\mathrm{d}\alpha), \qquad \alpha \ m\text{-form on } \mathbb{R}^N. \tag{44}$$

The boundary of an integer rectifiable current T is not necessarily an integer rectifiable current; if it happens to be, then T is called an *integral current*. An example is provided by oriented smooth submanifolds M with boundary; Stokes's theorem then relates the action of the boundary operator ∂ on the corresponding integration currents with the action of the boundary operator ∂ of homology: $\partial[M] = [\partial M] \in \mathscr{E}_{m-1}(\mathbb{R}^N)$.

Another example is provided by the standard simplex $\Delta_N \subset \mathbb{R}^N$ of Notation 4.2; recall that the orienting unit n-vector field is $\tau \coloneqq \frac{\partial}{\partial x_1} \wedge \cdots \wedge \frac{\partial}{\partial x_N}$. Stokes's Theorem yields

$$\partial[\Delta_N] = [\Gamma_0] + \cdots + [\Gamma_N] \in \mathscr{E}_{N-1}(\mathbb{R}^N),$$

where

$$\Gamma_j = \begin{vmatrix} \Delta_N \cap \{x_1 + \cdots + x_N = 1\} & \text{if } j = 0, \\ \Delta_N \cap \{x_j = 0\} & \text{if } 1 \le j \le n, \end{vmatrix}$$

with orienting $(N-1)$-vectors τ_j defined by $\nu_j \wedge \tau_j = \tau$, where ν_j is the outward-pointing unit normal vector field for the piece Γ_j of $\partial\Delta_N$; with the notation $e_j = \frac{\partial}{\partial x_j}$, the result is $\tau_0 = \frac{1}{\sqrt{N}}(e_2 - e_1) \wedge (e_3 - e_1) \wedge \cdots \wedge (e_N - e_1)$ (because $\nu_0 = (e_1 + \cdots + e_N)/\sqrt{N}$) and $\tau_j = (-1)^j e_1 \wedge \cdots \wedge \overset{\wedge}{e_j} \wedge \cdots \wedge e_N$ for $j \ge 1$ (because $\nu_j = -e_j$).

Observe that one can write

$$\partial[\Delta_N] = [A_0(\Delta_{N-1})] - [A_1(\Delta_{N-1})] + \cdots + (-1)^N [A_N(\Delta_{N-1})] \tag{45}$$

with an injective affine map $A_j\colon\ \mathbb{R}^{N-1} \to \mathbb{R}^N$ for each $j = 0, \ldots, N$ (taking $A_0(x_1, \ldots, x_{N-1}) = (1 - x_1 - \cdots - x_{N-1}, x_1, \ldots, x_{N-1})$ and $A_j(x_1, \ldots, x_{N-1}) = (x_1, \ldots, x_{j-1}, 0, x_j, \ldots, x_{N-1})$ for $j \ge 1$).

The commutation formula $\phi_\# \partial[P] = \partial \phi_\#[P]$

For any $T \in \mathscr{E}_m(\mathbb{R}^N)$ and any smooth map $\Phi\colon \mathbb{R}^N \to \mathbb{R}^{N'}$, the formula

$$\Phi_\# \partial T = \partial \Phi_\# T \in \mathscr{E}_{m-1}(\mathbb{R}^{N'}) \tag{46}$$

is a simple consequence of the identity $\mathrm{d} \circ \Phi^\# = \Phi^\# \circ \mathrm{d}$ on differential forms. We can also try to deal with a Lipschitz map ϕ when restricting ourselves to integral currents. The following is used in the proof of the main result of this article:

Lemma A.3. *Let $N \geq 1$ and let $\phi\colon \Delta_N \to \mathbb{R}^{N'}$ be Lipschitz; define $\phi_\#[\Delta_N]$ by means of* (40) *and $\phi_\# \partial[\Delta_N]$ by means of* (45) *and* (42)*. Then*

$$\partial \phi_\#[\Delta_N] = \phi_\# \partial[\Delta_N].$$

In fact, it is with $P = [0,1] \times \Delta_n$ instead of Δ_N that this commutation formula is used in Section 5; moreover, the target space is $\mathscr{S}^n_\Omega$ instead of $\mathbb{R}^{N'}$ but, as mentioned above, this makes no difference (just take $N' = 2n$). We leave it to the reader to adapt the proof.

Proof of Lemma A.3. We shall use the notation $T = [\Delta_N] \in \mathscr{E}_N(\mathbb{R}^N)$. Let β be a smooth $(N-1)$-form on $\mathbb{R}^{N'}$ and let $(\Phi_\ell)_{\ell \in \mathbb{N}}$ be any sequence of smooth maps from $\mathbb{R}^N$ to $\mathbb{R}^{N'}$ with uniformly bounded Lipschitz constants which converges uniformly to ϕ on Δ_N as $\ell \to \infty$. Then the sequence

$$\partial (\Phi_\ell)_\# T(\beta) = (\Phi_\ell)_\# T(\mathrm{d}\beta) \xrightarrow[\ell \to \infty]{} \phi_\# T(\mathrm{d}\beta) = \partial \phi_\# T(\beta)$$

by (44) and (41). But, by (46), this sequence coincides with

$$(\Phi_\ell)_\# \partial T(\beta) = \sum_{j=0}^{N} (-1)^j (\Phi_\ell)_\# [A_j(\Delta_{N-1})](\beta) \xrightarrow[\ell \to \infty]{} \sum_{j=0}^{N} (-1)^j \phi_\# [A_j(\Delta_{N-1})](\beta) = \phi_\# \partial T(\beta)$$

by (45) and (43). □

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