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On generalized Cramér-Rao inequalities and characterizations of generalized q-Gaussian distributions: the multidimensional case ‡

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Abstract.
Extended Cramér-Rao inequalities of estimation theory, together with related Cramér-Rao inequalities saturated by generalized q-Gaussian distributions, were presented in a recent paper. These results introduced an extended version of Fisher information and a new characterization of generalized q-Gaussian distributions which are important in several areas of physics and mathematics. In the present work, we extend these results to the mutidimensional case. We show how the generalized Cramér-Rao inequality for the estimation of a parameter can be extended to the mutidimensional case, with a formulation that involves a general norm on $\mathbb{R}^n$ and its dual norm. As a particular case, we obtain a new multidimensional Cramér-Rao inequality which is saturated by generalized q-Gaussian distributions. We give another related Cramér-Rao inequality, for a general norm, which is also saturated by these distributions. These results yield a new information theoretic characterization of generalized q-Gaussian distributions.

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1. Introduction

It is well known that the Gaussian distribution is a central distribution with respect to classical information measures and inequalities. In particular, the Gaussian distribution is both a maximum entropy and a minimum Fisher information distribution over all distributions with the same variance. We will show that the same kind of result holds for the family of generalized q-Gaussians, for Rényi or Tsallis entropy and a suitable extension of the Fisher information.

These generalized q-Gaussians appear in statistical physics, where they are the maximum entropy distributions of the nonextensive thermostatistics [1]. The generalized q-Gaussian distributions define a versatile family that can describe problems with compact support as well as problems with heavy tailed distributions. They are also analytical solutions of actual physical problems, see [2], [3] [4], [5], and are sometimes known as Barenblatt-Pattle functions, following their identification by [6], [7]. We shall also mention that the Generalized q-Gaussian distributions appear in other fields, namely as the solution of non-linear diffusion equations, or as the distributions that saturate some sharp inequalities in functional analysis [8], [9], [10], [11].

In the literature, and in particular within nonextensive thermostatistics, several extensions of the Fisher information and of the Cramér-Rao inequality have been proposed, e.g. [12], [13], [14], [15], [16]. In information theory, the remarkable work by Lutwak et al. [17], [18] also defines an extended Fisher information and a Cramér-Rao inequality saturated by q-Gaussian distributions. However the Fisher information is originally defined in a broader context as the information about a parameter of a parametric family of distributions. It is only in the special case of a location parameter that it reduces to the Fisher information of the distribution. The Fisher information is especially important for the formulation of the Cramér-Rao inequality. This well-known inequality appears in the context of estimation theory, where it defines a lower bound on the variance of any estimator of a parameter.

In our recent work [19], we have thrown a bridge between concepts in estimation theory and tools of nonextensive thermostatistics. Using the notion of escort distribution, we have established an extended version of the Cramér-Rao inequality for the estimation of a parameter. This new Cramér-Rao inequality includes the standard one, as well as Barankin-Vajda versions [20, Corollary 5.1],[21] as particular cases. Furthermore, in the case of a location parameter, we have obtained extended versions of the standard Cramér-Rao inequality, which are saturated by the generalized q-Gaussians. This means that among all distributions with a given moment, the generalized q-Gaussians are also the minimizers of extended versions of the Fisher information, just as the standard Gaussian minimizes Fisher information over all distributions with a given variance. This result yields a new information-theoretic characterization of these generalized Gaussian distributions.

However, a quite frustrating point is that these results were limited to the univariate case, while the multidimensional case is obviously of high importance. This restriction is overcomed in the present paper where we show that previous results can be extended to the multidimensional case. More than that, we consider an even wider context where moments of the error are computed with respect to two different probability distributions. In addition, giving our results for general norms will be hardly more difficult than for Euclidean norms, so we consider this general case from the beginning. Let us now give a brief overview of the results, together with the organization of the paper.

Let $\theta \in \Theta \subseteq \mathbb{R}^n$ be a multidimensional parameter that we wish to estimate using data x. We show that for $\hat{\theta}(x)$ an estimator of θ, if $f(x;\theta)$ and $g(x;\theta)$ are two probability densities, and if α

and β are Hölder conjugates of each other, then

$$E\left[\left\|\hat{\theta}(x)-\theta\right\|^{\alpha}\right]^{\frac{1}{\alpha}} I_{\beta}[f|g;\theta]^{\frac{1}{\beta}} \geq \left|n+\nabla_{\theta}.\, B_f(\theta)\right|, \tag{1}$$

where $\|.\|$ is a general norm on $\mathbb{R}^n$, $\nabla_{\theta}.\, B_f(\theta)$ represents the divergence of the bias between $\hat{\theta}(x)$ and θ, and $I_{\beta}[f|g;\theta]$ stands for a generalized Fisher information that measures the information in f about θ, and is taken with respect to g. This general result is established in section 3. Then we discuss in subsection 3.2 some special cases of this general inequality. In particular, if f and g is a pair of q-escort distributions, we obtain

$$E\left[\left\|\hat{\theta}(x)-\theta\right\|^{\alpha}\right]^{\frac{1}{\alpha}} I_{\beta,q}\left[f|g;\theta\right]^{\frac{1}{\beta}} \geq \left|n+\nabla_{\theta}.E_q\left[\hat{\theta}(x)-\theta\right]\right| \tag{2}$$

$$E_{\bar{q}}\left[\left\|\hat{\theta}(x)-\theta\right\|^{\alpha}\right]^{\frac{1}{\alpha}} I_{\beta,q}\left[f|g;\theta\right]^{\frac{1}{\beta}} \geq \left|n+\nabla_{\theta}.E\left[\hat{\theta}(x)-\theta\right]\right|, \tag{3}$$

where $I_{\beta,q}\left[f|g;\theta\right]$ is the generalized (β,q)-Fisher information, and $E_q[.]$ denotes the q-expectation which is used in nonextensive statistics. These results are the mutidimensional extensions, with an arbitrary norm, of our previous q-Cramér-Rao inequalities [19]. In the monodimensional case and $q=1$, these inequalities reduce to the Barankin-Vajda Cramér-Rao inequality, and to the standard Cramér-Rao inequality for $\alpha=\beta=2$. In addition, in the case of a location parameter, we show that

$$E\left[\|x\|^{\alpha}\right]^{\frac{1}{\alpha}} I_{\beta,q}\left[g\right]^{\frac{1}{\beta}} \geq n \tag{4}$$

which reduces again to our previous results in the univariate case. Examining carefully the cases for equality in (4), we exhibit that the lower bound is attained by generalized q-Gaussian distributions, and we prove that these generalized Gaussian are the unique extremal functions, provided that the dual norm is strictly convex. For a random vector x in $\mathbb{R}^n$, these generalized q-Gaussian have the probability density

$$G_{\gamma}(x)=\begin{cases}\frac{1}{Z(\gamma)}\left(1-(q-1)\,\gamma\|x\|^{\alpha}\right)_{+}^{\frac{1}{q-1}} & \text{for } q\neq 1\\ \frac{1}{Z(\gamma)}\exp\left(-\gamma\|x\|^{\alpha}\right) & \text{if } q=1\end{cases} \tag{5}$$

for $\alpha\in(0,\infty)$, γ a real positive parameter and $q>(n-\alpha)/n$, where we use the notation $(x)_{+}=\max\{x,0\}$, and where $Z(\gamma)$ is the partition function §. For $q>1$, the density has a compact support, while for $q\leq 1$ it is defined on the whole $\mathbb{R}^n$ and behaves as a power distribution for $\|x\|\to\infty$. A shorthand notation for the expression of the generalized q-Gaussian density is

$$G_{\gamma}(x)=\frac{1}{Z(\gamma)}\exp_{q^*}\left(-\gamma\|x\|^{\alpha}\right), \tag{6}$$

with $q^*=2-q$, and where the so-called q-exponential function is defined by

$$\exp_q(x):=\left(1+(1-q)x\right)_{+}^{\frac{1}{1-q}},\ \text{for } q\neq 1 \text{ and } \exp_{q=1}(x):=\exp(x). \tag{7}$$

§ In the case of the Euclidean norm, the general expressions of the main information measures attached to the generalized Gaussians are derived in Appendix A of [22]. Similar expressions can readily be obtained in the case of a general norm, using the change of variable in polar coordinates $x=r\,u$, with $u=x/||x||$ and the representation of the Lebesgue measure $\mathrm{d}x=r^{n-1}\mathrm{d}r\,\mathrm{d}\sigma(u)$, c.f. [23, p. 87], where $\mathrm{d}\sigma(u)$ denotes the surface element on the unit sphere. By this remark, the expressions in [22, Appendix A] are valid, with the proviso that ω_n will denote the volume of the n-dimensional unit ball $\mathcal{B}=\{x\in\mathbb{R}^n:\ \|x\|\leq 1\}$.

In section 4, we present another Cramér-Rao type inequality which is also saturated by the generalized q-Gaussian. This inequality has been originally established by [17], and extended to the multidimensional case in [18] and independently in [22] in the case of an Euclidean norm. We show here that this last inequality can readily be stated and proved in the case of an arbitrary norm.

In order to derive these different inequalities, we will need some preliminary results, in particular concerning some properties of general norms on $\mathbb{R}^n$. This is the objective of section 2 where we first define the notion of dual norms and prove a result on the gradient of a general norm. Then, we establish, together with its equality conditions, a general Hölder-type inequality. This inequality will be an essential ingredient in the derivation of the general Cramér-Rao inequality (1).

2. Preliminary results

As mentioned above, we will consider here general norms on $\mathbb{R}^n$. Let us first simply recall that a norm on $\mathbb{R}^n$ is a function $\|.\|$: $\mathbb{R}^n \to \mathbb{R}_+$ such that for any x, $y \in \mathbb{R}^n$ and $\gamma \in \mathbb{R}$, then

$$\text{(a) } \|\gamma x\| = |\gamma| \|x\|, \text{ (b) } \|x+y\| \leq \|x\| + \|y\|, \text{ and (c) } \|x\| = 0 \quad \text{iff } x = 0. \tag{8}$$

A large class of norms is the class of L_p-norms, $p \geq 1$, given by $\|x\|_p = \left(\sum_{i=1}^n |x|_i^p\right)^{\frac{1}{p}}$. As important particular cases, we have the L_1-norm, $\|x\|_1 = \sum_{i=1}^n |x|_i$, the max-norm or L_∞-norm $\|x\|_\infty = \max\left(|x_1|, \ldots |x_n|\right)$, and of course the Euclidean L_2-norm $\|x\|_2 = \left(\sum_{i=1}^n x_i^2\right)^{\frac{1}{2}}$. We shall mention that it is possible to use weightened versions of the norms above, e.g. $\|x\|_{w,p} = \left(\sum_{i=1}^n w_i |x|_i^p\right)^{\frac{1}{p}}$, with $w_i > 0$, and that any injective linear transformation A leads to a new norm, such as $\|x\|_A = \|Ax\|$. Finally, it is also possible to construct new norms by combining different norms defined on subvectors of x.

A related important notion is the notion of dual-norm. Let $E = (\mathbb{R}^n, \|.\|)$ a n-dimensional normed space, where $\|.\|$ is an arbitrary norm, and let us denote $E^* = (\mathbb{R}^n, \|.\|_*)$ its dual space. For $Y \in E^*$, the dual norm $\|.\|_*$ is defined by

$$\|Y\|_* = \sup_{\|X\| \leq 1} X.Y, \tag{9}$$

where $X.Y$ is the standard scalar product $X.Y = \sum_{i=1}^n X_i Y_i$. In particular, it is well known that if $\|.\|$ is a L_p-norm, then $\|.\|_*$ is the L_q-norm, where p and q are Hölder conjugates of each other, i.e. $p^{-1} + q^{-1} = 1$, see e.g. [24, chapter 5]. By a direct consequence of the definition of the dual norm, we always have

$$X.Y \leq \|X\| \, \|Y\|_* . \tag{10}$$

Note that when the dot product $X.Y$ is negative, we can always take the minus of one of the elements to get $|X.Y| = X.(-Y) \leq \|X\| \|-Y\|_* = \|X\| \|Y\|_*$. Hence, we see that we actually have an extension of Hölder's inequality for vectors:

$$|X.Y| \leq \|X\| \, \|Y\|_* . \tag{11}$$

Obviously, we recover here the Cauchy-Schwarz inequality if $\|.\| = \|.\|_2$ and the standard Hölder inequality for vectors if $\|.\| = \|.\|_p$, and thus $\|.\|_* = \|.\|_q$.

In the following, we will need several facts on the gradient of a norm. These facts are stated in the next Lemma.

Lemma 1. *Let $\|.\|$ be differentiable at $x \in E$, and denote $x^* = \nabla_x \|.\|(x) \in E^*$ the gradient of the norm at x. The gradient of $\|x\|$ satisfies (a) $x.x^* = \|x\|$ and (b) $\|x^*\|_* = 1$. Furthermore, when the dual norm $\|.\|_*$ is strictly convex, then the gradient x^* is the unique vector that satisfies (a) and (b).*

Proof. We begin by equality (a). Let x and v two vectors of E and λ a real parameter. By the triangle inequality, we have $\|x + \lambda v\| \leq \|x\| + \lambda \|v\|$, so that

$$\lim_{\lambda \to 0} \frac{\|x + \lambda v\| - \|x\|}{\lambda} \leq \|v\|. \tag{12}$$

In the other hand, the chain rule for derivation $\frac{\mathrm{d}\|Z\|}{\mathrm{d}\lambda} = \frac{\mathrm{d}Z}{\mathrm{d}\lambda}.\nabla_Z \|Z\|$, with $Z = x + \lambda v$, gives

$$\left.\frac{\mathrm{d}\|x + \lambda v\|}{\mathrm{d}\lambda}\right|_{\lambda=0} = v.\nabla_x \|x\| \leq \|v\|, \tag{13}$$

where the right inequality follows from (12). Of course, taking $v = x$ in (12) gives the equality sign, and (13) becomes $x.\nabla_x \|x\| = \|x\|$, that is (a).

By (13), we also get that

$$\nabla_x \|x\| . \frac{v}{\|v\|} \leq 1, \tag{14}$$

with equality if $v = x$. Therefore, in the definition of the dual norm $\|\nabla_x \|x\|\|_* = \sup_{\|w\| \leq 1} w.\nabla_x \|x\|$, the supremum is equal to one and is attained for $w = x/\|x\|$. This proves (b).

In functional analysis, the existence of a solution to a system analogue to (a),(b) is granted by a consequence of the Hahn-Banach theorem, see e.g. the background material in [25]. In this context, the uniqueness of extension in the Hahn-Banach theorem, therefore the uniqueness of x^*, is guaranteed if the primal space E is smooth, which in turn is equivalent to the strict convexity of the dual norm [26, Chapter 2]. In our setting, it is easy to check that we have uniqueness of the solution to the system (a),(b) provided that the dual norm is strictly convex. Indeed, if x_1^* and x_2^* are two solutions to (a),(b), we have by (b) $\frac{x}{\|x\|}.(x_1^* + x_2^*) = 2$. Accordingly, the dual norm $\|x_1^* + x_2^*\|_* = \sup_{\|w\| \leq 1} w.(x_1^* + x_2^*)$ is necessarily greater than 2: $\|x_1^* + x_2^*\|_* \geq 2$. In the other hand, if the dual norm is strictly convex and using (a), we have $\|x_1^* + x_2^*\|_* \leq \|x_1^*\|_* + \|x_2^*\|_* = 2$, with equality if and only if $x_1^* = x_2^*$. Combining the two inequalities, we see that the two solutions are necessarily equal. Finally, since we have already identified that the gradient of the norm satisfies (a),(b), we get the last item in the Lemma. □

The standard Hölder's inequality works for functions and relates the L_1 norm of the product of two functions to the product of their L_p and L_q norms: $\|fg\|_1 \leq \|f\|_p \|g\|_q$, with $1 \leq p, q \leq \infty$ and $1/p + 1/q = 1$. For vectors and an arbitrary norm $\|.\|$, the inequality (11) gives another kind of Hölder inequality (actually, this inequality is also true in a broader context, see e.g. [27]). By combining these two inequalities, we obtain another Hölder-type inequality for vector-valued functions, which involves arbitrary norms. This inequality will be a key in the derivation of the new multidimensional Cramér-Rao inequality. It is given, with its equality condition, in the following Lemma.

Lemma 2. *Let $E = (\mathbb{R}^n, \|.\|)$ be a n-dimensional normed space and $E^* = (\mathbb{R}^n, \|.\|_*)$ its dual space. If $X(t)$ and $Y(t)$ are two functions taking values respectively in E and E^*, and if $w(t)$ is a weight*

function, then

$$\left(\int \|X(t)\|^{\alpha}\, w(t)\mathrm{d}t\right)^{\frac{1}{\alpha}} \left(\int \|Y(t)\|_*^{\beta}\, w(t)\mathrm{d}t\right)^{\frac{1}{\beta}} \geq \int |X(t).Y(t)|\; w(t)\mathrm{d}t, \tag{15}$$

$$\geq \left|\int X(t).Y(t)\, w(t)\mathrm{d}t\right| \tag{16}$$

with α and β Hölder conjugates of each other, i.e. $\alpha^{-1}+\beta^{-1}=1$, $\alpha \geq 1$. The equality is obtained if

$$Y(t) = K\|X(t)\|^{\alpha-1}\nabla_{X(t)}\|X(t)\|, \tag{17}$$

with $K \in \mathbb{R}$ for inequality (15), and with $K \in \mathbb{R}_+$ for the lower bound (16). If the dual norm is strictly convex, then the function $Y(t)$ in (17) above is the unique function which saturates the inequalities (15-16).

Proof. By inequality (11), we have $|X(t).Y(t)| \leq \|X(t)\|\|Y(t)\|_*$. Integrating this inequality with respect to t, we obtain

$$\int |X(t).Y(t)|\; w(t)\mathrm{d}t \leq \int \|X(t)\|\|Y(t)\|_* w(t)\mathrm{d}t. \tag{18}$$

Obviously, we always have

$$\left|\int X(t).Y(t)\, w(t)\mathrm{d}t\right| \leq \int |X(t).Y(t)|\; w(t)\mathrm{d}t, \tag{19}$$

with equality if $X(t).Y(t) \geq 0$ everywhere. Then, it only remains to apply the standard Hölder inequality to right hand side of (18):

$$\int \|X(t)\|\|Y(t)\|_* w(t)\mathrm{d}t \leq \left(\int \|X(t)\|^{\alpha}\, w(t)\mathrm{d}t\right)^{\frac{1}{\alpha}} \left(\int \|Y(t)\|_*^{\beta}\, w(t)\mathrm{d}t\right)^{\frac{1}{\beta}} \tag{20}$$

to obtain (15). The inequality (16) then follows by (19).

As far as the cases of equality are concerned, we know that in the Hölder inequality (20), the equality is obtained if and only if $\|Y(t)\|_*^{\beta} = K\|X(t)\|^{\alpha}$, with K a positive constant. Using the fact that $\alpha/\beta = \alpha - 1$, the condition becomes $\|Y(t)\|_* = K\|X(t)\|^{\alpha-1}$. This condition implies that $Y(t)$ must be of the form

$$Y(t) = K\|X(t)\|^{\alpha-1}u, \tag{21}$$

where u is a vector of E^* with unit norm: $\|u\|_* = 1$. By inequality (11) we see that the integrand in the left side of (18) is always less or equal the integrand on the right. Therefore, we will only get equality in (18) if the integrands are equal. Then if we plug (21) in the inequality (11), we obtain the condition $\|X(t)\|^{\alpha-1}\,|X(t).u| = \|X(t)\|^{\alpha}$, that is finally $|X(t).u| = \|X(t)\|$. Since we know by Lemma 1 that $v = \nabla_{X(t)}\|X(t)\|$ is a unit vector that satisfies $X(t).v = \|X(t)\|$, and is unique if the dual norm is strictly convex, we see that $u = \pm v = \pm\nabla_{X(t)}\|X(t)\|$ and this concludes the proof of the first inequality. For equality to hold in the lower bound (16), the integrand must be positive, which in turn implies that $X(t).u = \|X(t)\|$ and $u = \nabla_{X(t)}\|X(t)\|$. □

3. The generalized Cramér-Rao inequality

In this section, we first derive a main Cramér-Rao inequality for the estimation of a multidimensional parameter and introduce a generalized version of Fisher information. Next, we examine the particular case of a pair of escort distributions, and then the case of a location parameter. So doing, we obtain multidimensional versions of the q-Cramér-Rao inequality and a Cramér-Rao inequality characterizing generalized q-Gaussian distributions.

3.1. The main Cramér-Rao inequality for the estimation of a parameter

The problem of estimation is to determine a function $\hat{\theta}(x)$ in order to estimate an unknown parameter θ. Let $f(x;\theta)$ and $g(x;\theta)$ be two probability density functions, with $x \in X \subseteq \mathbb{R}^k$ and θ a parameter of these densities, $\theta \in \mathbb{R}^n$. An underlying idea in the statement of the new Cramér-Rao inequality is that it is possible to evaluate the moments of the error with respect to different probability distributions. For instance, in the estimation setting the estimation error is $\hat{\theta}(x) - \theta$. The bias can be evaluated with respect to f according to

$$B_f(\theta) = \int_X \left(\hat{\theta}(x) - \theta\right) f(x;\theta)\,\mathrm{d}x = \mathrm{E}_f\left[\hat{\theta}(x) - \theta\right] \tag{22}$$

while a general moment of a norm of the error can be computed with respect to another distribution, $g(x;\theta)$, as in

$$\mathrm{E}_g\left[\left\|\hat{\theta}(x) - \theta\right\|^\beta\right] = \int_X \left\|\hat{\theta}(x) - \theta\right\|^\beta g(x;\theta)\,\mathrm{d}x \tag{23}$$

The distributions $f(x;\theta)$ and $g(x,\theta)$ can be chosen arbitrarily and are not necessarily directly related. However, $g(x;\theta)$ can be designed as a transformation of $f(x;\theta)$ that highlights, or perhaps scores out, some characteristics of $f(x;\theta)$. Typically, $g(x;\theta)$ can be a weightened version of $f(x;\theta)$, i.e. $g(x;\theta) = h(x;\theta)f(x;\theta)$. The distribution $g(x;\theta)$ can also be a quantized version of $f(x;\theta)$, such as $g(x;\theta) = [f(x;\theta)]$, where $[.]$ denotes the integer part. Another important special case is when $g(x;\theta)$ is defined as the escort distribution of order q of $f(x;\theta)$, where q plays the role of a tuning parameter. We will see that this special case, which is particularly important in the context of nonextensive statistical physics, will lead to generalized q-Gaussians as the extremal functions. We are now in position to state and prove an extended version of the Cramér-Rao inequality.

Theorem 1. *Let $f(x;\theta)$ be a multivariate probability density function defined over a subset $X \subseteq \mathbb{R}^n$, and $\theta \in \Theta \subseteq \mathbb{R}^k$ a parameter of the density. The set Θ is equipped with a norm $\|.\|$, and the corresponding dual norm is denoted $\|.\|_*$. Let $g(x;\theta)$ denote another probability density function also defined on $(X;\Theta)$. Assume that $f(x;\theta)$ is a jointly measurable function of x and θ, is integrable with respect to x, is absolutely continuous with respect to θ, and that the derivatives with respect to each component of θ are locally integrable. For any estimator $\hat{\theta}(x)$ of θ, we have*

$$E\left[\left\|\hat{\theta}(x) - \theta\right\|^\alpha\right]^{\frac{1}{\alpha}} I_\beta[f|g;\theta]^{\frac{1}{\beta}} \geq |n + \nabla_\theta . B_f(\theta)| \tag{24}$$

with α and β Hölder conjugates of each other, i.e. $\alpha^{-1} + \beta^{-1} = 1$, $\alpha \geq 1$, and where the (β, g)-Fisher information

$$I_\beta[f|g;\theta] = \int_X \left\|\frac{\nabla_\theta f(x;\theta)}{g(x;\theta)}\right\|_*^\beta g(x;\theta)\ \ dx \tag{25}$$

is the generalized Fisher information of order β on the parameter θ contained in the distribution f and taken with respect to g. The equality case is obtained if

$$\frac{\nabla_\theta f(x;\theta)}{g(x;\theta)} = K \left\|\hat{\theta}(x) - \theta\right\|^{\alpha-1} \nabla_{\hat{\theta}(x)-\theta} \|\hat{\theta}(x) - \theta\|, \tag{26}$$

with $K > 0$.

Proof. The bias in (22) is a n-dimensional vector. Let us consider its divergence with respect to variations of θ:

$$\operatorname{div} B_f(\theta) = \nabla_\theta . B_f(\theta). \tag{27}$$

The regularity conditions in the statement of the theorem enable to interchange integration with respect to x and differentiation with respect to θ, and

$$\nabla_\theta . B_f(\theta) = \int_X \nabla_\theta . \left(\hat{\theta}(x) - \theta\right) f(x;\theta)\, \mathrm{d}x + \int_X \nabla_\theta f(x;\theta) . \left(\hat{\theta}(x) - \theta\right) \mathrm{d}x. \tag{28}$$

In the first term on the right, we have $\nabla_\theta . \theta = n$, and the integral reduces to $-n \int_X f(x;\theta)\, \mathrm{d}x = -n$, since $f(x;\theta)$ is a probability density on X. The second term can be rearranged so as to obtain an integration with respect to the density $g(x;\theta)$, assuming that the derivatives with respect to each component of θ are absolutely continuous with respect to $g(x;\theta)$, i.e. $g(x;\theta) \gg \nabla_\theta f(x;\theta)$. This gives

$$n + \nabla_\theta . B_f(\theta) = \int_X \frac{\nabla_\theta f(x;\theta)}{g(x;\theta)} . \left(\hat{\theta}(x) - \theta\right) g(x;\theta)\, \mathrm{d}x. \tag{29}$$

Now, it only remains to apply the generalized Hölder-type inequality (16) in Lemma 2 to the integral on the right side, with $X(x) = \hat{\theta}(x) - \theta$, $Y(x) = \frac{\nabla_\theta f(x;\theta)}{g(x;\theta)}$, and $w(x) = g(x;\theta)$. This yields in all generality

$$\left(\int_X \left\|\hat{\theta}(x) - \theta\right\|^\alpha g(x;\theta)\ \mathrm{d}x\right)^{\frac{1}{\alpha}} \left(\int_X \left\|\frac{\nabla_\theta f(x;\theta)}{g(x;\theta)}\right\|_*^\beta g(x;\theta)\ \mathrm{d}x\right)^{\frac{1}{\beta}} \geq |n + \nabla_\theta . B_f(\theta)| \tag{30}$$

which is (24). By Lemma 2 again, we know that the case of equality occurs if $Y(t) = K\|X(t)\|^{\alpha-1}\nabla_{X(t)}\|X(t)\|$, $K > 0$, which gives (26). □

3.2. Main consequences of the general result

3.2.1. Case of a q-escort distribution Let $f(x;\theta)$ and $g(x;\theta)$ be a pair of of q-escort distributions linked by

$$f(x;\theta) = \frac{g(x;\theta)^q}{M_q\,[g;\theta]} \quad \text{and} \quad g(x;\theta) = \frac{f(x;\theta)^{\bar{q}}}{M_{\bar{q}}\,[f;\theta]}, \tag{31}$$

with $q > 0$, $\bar{q} = 1/q$, and the information generating function $M_q\,[g;\theta]$ defined by

$$M_q\,[g;\theta] = \int_X g(x;\theta)\, \mathrm{d}x. \tag{32}$$

As usual, we will denote by $E_q\,[.]$ the q-expectation, which is the expectation taken with respect to an escort distribution of order q. Here we see that the expectation with respect to $f(x;\theta)$ is also the

q-expectation with respect to $g(x;\theta)$. Let us also recall that the inverse function of the deformed q-exponential (7), the so-called q-logarithm, is defined by

$$\ln_q(x) := \frac{x^{1-q}-1}{1-q}. \tag{33}$$

With these notations, we have the following corollary of the general Cramér-Rao inequality.

Corollary 1. *For the pair of escort distributions (31), the equivalent Cramér-Rao inequalities*

$$E\left[\left\|\hat{\theta}(x)-\theta\right\|^{\alpha}\right]^{\frac{1}{\alpha}} I_{\beta,q}\left[f|g;\theta\right]^{\frac{1}{\beta}} \geq \left|n+\nabla_\theta.E_q\left[\hat{\theta}(x)-\theta\right]\right| \tag{34}$$

$$E_{\bar{q}}\left[\left\|\hat{\theta}(x)-\theta\right\|^{\alpha}\right]^{\frac{1}{\alpha}} I_{\beta,q}\left[f|g;\theta\right]^{\frac{1}{\beta}} \geq \left|n+\nabla_\theta.E\left[\hat{\theta}(x)-\theta\right]\right|, \tag{35}$$

hold, where the generalized (β,q)-Fisher information is given by

$$I_{\beta,q}\left[f|g;\theta\right] = \frac{1}{M_q\left[g;\theta\right]^{\beta}} E\left[g(x;\theta)^{\beta(q-1)}\left\|\nabla_\theta \ln\frac{g(x;\theta)^q}{M_q\left[g;\theta\right]}\right\|_*^{\beta}\right] \tag{36}$$

$$= M_{\bar{q}}\left[f;\theta\right]^{\beta}\ E_{\bar{q}}\left[f(x;\theta)^{\beta(1-\bar{q})}\left\|\nabla_\theta \ln f(x;\theta)\right\|_*^{\beta}\right], \tag{37}$$

and where equality occurs if

$$\nabla_\theta \ln_{\bar{q}} f(x;\theta) = K\left\|\hat{\theta}(x)-\theta\right\|^{\alpha-1}\nabla_{\hat{\theta}(x)-\theta}\|\hat{\theta}(x)-\theta\|, \tag{38}$$

with $K>0$.

Proof. The Cramér-Rao inequalities (34) and (35) directly follow from the general Cramér-Rao inequality (24), the relations (31) between $f(x;\theta)$ and $g(x;\theta)$, and the notation of q-expectations. The expressions of the generalized (β,q)-Fisher information also follow by direct calculation. Finally, the equality condition yields

$$f(x;\theta)^{(1-\bar{q})}\nabla_\theta \ln f(x;\theta) = K\left\|\hat{\theta}(x)-\theta\right\|^{\alpha-1}\nabla_{\hat{\theta}(x)-\theta}\|\hat{\theta}(x)-\theta\|. \tag{39}$$

Noticing that the term on the left is nothing but the gradient of the deformed q-logarithm, $f(x;\theta)^{(1-\bar{q})}\nabla_\theta \ln f(x;\theta) = \nabla_\theta \ln_{\bar{q}}\ f(x;\theta)$, we immediately obtain (38). □

3.2.2. Case of a translation family In the particular case of a translation parameter, the generalized Cramér-Rao inequality induces a new class of inequalities.

Let $\theta\in\mathbb{R}^n$ be a location parameter, $x\in X\subseteq\mathbb{R}^n$, and define by $f(x;\theta)$ the family of density $f(x;\theta)=f(x-\theta)$. In this case, we have $\nabla_\theta f(x;\theta) = -\nabla_x f(x-\theta)$, provided that f is differentiable at $x-\theta$, and the Fisher information becomes a characteristic of the information in the distribution. If X is a bounded subset, we will assume that $f(x)$ vanishes and is differentiable on the boundary ∂X (otherwise the Fisher information defined for the function extended to $\mathbb{R}^n$ is not defined).

Let us denote by μ_f the mean of $f(x)$. We immediately get that the mean of $f(x;\theta)$ is $(\mu_f+\theta)$, so that an unbiased estimator of θ could be $\hat{\theta}(x)=x-\mu_f$. If we choose $\hat{\theta}(x)=x$, the estimator

will be biased, $B_f(\theta) = E_q\left[\hat{\theta}(x) - \theta\right] = \mu_f$, but independent of θ, so that the gradient of the bias with respect to θ is zero. In these conditions, the generalized Cramér-Rao inequality becomes

$$\left(\int \|x-\theta\|^{\alpha} g(x;\theta)\ \mathrm{d}x\right)^{\frac{1}{\alpha}} \left(\int \left\|\frac{\nabla_x f(x-\theta)}{g(x;\theta)}\right\|_*^{\beta} g(x;\theta)\ \mathrm{d}x\right)^{\frac{1}{\beta}} \geq n. \tag{40}$$

Furthermore, we can also choose $\theta = 0$, and obtain, as a corollary, the following interesting functional inequality.

Corollary 2. *Let $f(x)$ and $g(x)$ be two multivariate probability density functions defined over a subset X of $\mathbb{R}^n$. Assume that $f(x)$ is a measurable differentiable function of x, which vanishes and is differentiable on the boundary ∂X, that $\nabla_x f(x)$ is absolutely continuous with respect to $g(x)$, and finally that the involved integrals exist and are finite. Then, the following inequality holds*

$$\left(\int_X \|x\|^{\alpha} g(x)\ \ dx\right)^{\frac{1}{\alpha}} \left(\int_X \left\|\frac{\nabla_x f(x)}{g(x)}\right\|_*^{\beta} g(x)\ \ dx\right)^{\frac{1}{\beta}} \geq n, \tag{41}$$

with equality if (and only if when the dual norm is strictly convex)

$$\nabla_x f(x) = -K\, g(x)\|x\|^{\alpha-1}\nabla_x\|x\|, \tag{42}$$

with $K > 0$.

As an elementary application, let us consider the univariate case, with $X = [0,1]$. Let us take for $g(x)$ the uniform distribution on the interval. Finally, let us choose for $f(x)$ a β-distribution: $f(x) = x^{a-1}(1-x)^{b-1}/B(a,b)$, with $B(a,b)$ the β-function. Firstly, we obviously have $\int_0^1 |x|^{\alpha}\mathrm{d}x = 1$. Secondly, $f'(x) = \left((a-1)x^{a-2}(1-x)^{b-1} + (b-1)x^{a-1}(1-x)^{b-2}\right)$, so that the inequality is

$$\left(\int_0^1 \left|(a-1)x^{a-2}(1-x)^{b-1} + (b-1)x^{a-1}(1-x)^{b-2}\right|^{\beta}\ \mathrm{d}x\right)^{\frac{1}{\beta}} \geq B(a,b). \tag{43}$$

Taking now $\beta = 1$ and $a > 1$, $b > 1$, we obtain the following inequality for β-functions:

$$(a-1)B(a-1,b) + (b-1)B(a,b-1) \geq B(a,b) \tag{44}$$

3.2.3. Case of a location parameter within a pair of escort distributions By combining the two aspects presented above, namely the case of a pair of escort distributions and the case of a location parameter, we will obtain a new Cramér-Rao inequality saturated by multivariate generalized q-Gaussians. This provides a new information theoretic characterization of generalized q-Gaussian and extend our previous results to the multivariate case and arbitrary norms. As in Corollary 1, we use a pair of of q-escort distributions:

$$f(x) = \frac{g(x)^q}{M_q\,[g]} \quad \text{and} \quad g(x) = \frac{f(x)^{\bar{q}}}{M_{\bar{q}}\,[f]}, \tag{45}$$

with $\bar{q} = 1/q$, and we denote by $E\,[.]$ the standard expectation with respect to $g(x)$, and by $E_{\bar{q}}\,[.]$ the $\bar{q}$-expectation with respect to $f(x)$, which is simply the standard expectation taken with respect to the escort $f(x)^{\bar{q}}/M_{\bar{q}}\,[f]$. In the statement of the following corollary, we will use the deformed exponential and logarithm defined in (7),(33). We will also use the notation $q_* = 2-q$ that changes the quantities $(1-q_*)$ into $(q-1)$.

Corollary 3. *Let $g(x)$ be a multivariate probability density function defined over a subset $X \subseteq \mathbb{R}^n$. Assume that $g(x)$ is a measurable differentiable function of x, which vanishes and is differentiable on the boundary ∂X, and finally that the involved integrals exist and are finite. Then, for the pair of escort distributions (45), the following q-Cramér-Rao inequality holds*

$$m_\alpha[g]^{\frac{1}{\alpha}}\, I_{\beta,q}\,[g]^{\frac{1}{\beta}} \geq n \tag{46}$$

with

$$\begin{cases} m_\alpha[g] = E\left[\|x\|^\alpha\right] \\ I_{\beta,q}\,[g] = (q/M_q\,[g])^\beta\ E\left[g(x)^{\beta(q-1)}\,\|\nabla_x \ln g(x)\|_*^\beta\right] \\ \qquad\quad = (q/M_q\,[g])^\beta\ E\left[\|\nabla_x \ln_{q*} g(x)\|_*^\beta\right], \end{cases} \tag{47}$$

where α and β are Hölder conjugates of each other, i.e. $\alpha^{-1}+\beta^{-1}=1$, $\alpha \geq 1$, and where $I_{\beta,q}\,[g]$ denotes the generalized (β,q)-Fisher information.

In terms of $\bar{q}$-expectations with respect to $f(x)$, it can also be written

$$m_{\alpha,\bar{q}}[f]^{\frac{1}{\alpha}}\, \bar{I}_{\beta,q}\,[f]^{\frac{1}{\beta}} \geq n \tag{48}$$

with

$$\begin{cases} m_{\alpha,\bar{q}}[f] = E_{\bar{q}}\left[\|x\|^\alpha\right] \\ \bar{I}_{\beta,q}\,[f] = M_{\bar{q}}\,[f]^\beta\ E_{\bar{q}}\left[f(x)^{\beta(1-\bar{q})}\,\|\nabla_x \ln f(x)\|_*^\beta\right] \\ \qquad\quad = M_{\bar{q}}\,[f]^\beta\ E_{\bar{q}}\left[\|\nabla_x \ln_{\bar{q}} f(x)\|_*^\beta\right]. \end{cases} \tag{49}$$

In both cases, equality occurs if

$$f(x) \propto \exp_{\bar{q}}\left(-\gamma\,\|x\|^\alpha\right), \quad \textit{or equivalently}\ \ g(x) \propto \exp_{q_*}\left(-\gamma\,\|x\|^\alpha\right), \text{ with } \gamma > 0. \tag{50}$$

If the dual norm is strictly convex, then this generalized q-Gaussian is the unique probability density function that achieves the equality in the extended Cramér-Rao inequalities.

Proof. As indicated above, the result is a direct consequence of Theorem 1 in the case of a pair of escort distributions and of the estimation of a location parameter, with $\hat{\theta}(x) = x$ and $\theta = 0$. Using (45), the condition for equality (42) becomes

$$g(x)^{q-1}\nabla_x g(x) = -K\,\|x\|^{\alpha-1}\nabla_x\|x\|\,g(x). \tag{51}$$

From this equation, we see that $g(x)$ will only be a function of the norm of x, and therefore will be radially symmetric. Furthermore, we see that the gradient of $g(x)$ behaves as the negative of the gradient of $\|x\|$. This means that $g(x)$, which is a function of $\|x\|$, is non increasing with $\|x\|$.

For $g(x) \neq 0$, the equality condition can be written

$$g(x)^{q-2}\nabla_x g(x) = \frac{1}{q-1}\nabla_x g(x)^{q-1} = -\frac{K}{\alpha}\,\nabla_x\|x\|^\alpha, \tag{52}$$

which, after integration of the two sides, gives

$$g(x)^{q-1} = -\frac{K}{\alpha}(q-1)\|x\|^\alpha + C, \tag{53}$$

where C is a constant of integration. Since $g(x)$ is a probability density function, the solution is restricted to the domain where the right hand side is non negative, and $g(x) = 0$ elsewhere. In particular, when C is negative, we see that $g(x)$ vanishes around the origin and presents a singularity at $\|x\| = C\alpha/K(q-1)$. Since we assumed $g(x)$ differentiable everywhere, this solution must be discarded.

Therefore, the constant of integration C must be positive, and

$$g(x) \propto \left(1 - \frac{K}{C\alpha}(q-1)\|x\|^{\alpha}\right)^{\frac{1}{q-1}} \propto \exp_{q_*}\left(-\gamma\,\|x\|^{\alpha}\right) \tag{54}$$

which is (50). The expression of $f(x)$ simply follows from the fact that $f(x)$ is the escort distribution of $g(x)$. □

4. Another Cramér-Rao inequality saturated by generalized Gaussians

We finish this paper with another Cramér-Rao type inequality, which involves a variant $\phi_{\beta,q}[g]$ of the generalized Fisher information $I_{\beta,q}[g]$ above, and which is also saturated by generalized q-Gaussian distributions. However, this inequality is less directly related to estimation results than the inequality (46) which is just a special case of the general Cramér-Rao inequality. The monodimensional version of this inequality has been established by [17], and extended to the multidimensional case in [18] and in [22] in the case of an Euclidean norm. Actually, the inequality can readily be stated and proved for a general norm.

Theorem 2. *For $n \geq 1$, β and α Hölder conjugates of each other, $\alpha > 1$, $q > \max\{(n-1)/n,\, n/(n+\alpha)\}$ then for any probability density g on $\mathbb{R}^n$, supposed continuously differentiable and such that the involved information measures are finite,*

$$m_{\alpha}[g]^{\frac{1}{\alpha}}\ \phi_{\beta,q}[g]^{\frac{1}{\beta\lambda}} \geq m_{\alpha}[G]^{\frac{1}{\alpha}}\ \phi_{\beta,q}[G]^{\frac{1}{\beta\lambda}} \tag{55}$$

with $\lambda = n(q-1)+1$ and where the general Fisher information is given by

$$\phi_{\beta,q}[g] = \left(M_q[g]/q\right)^{\beta} I_{\beta,q}[g] = E\left[g(x)^{\beta(q-1)}\,\|\nabla_x \ln g(x)\|_*^{\beta}\right], \tag{56}$$

and where the equality holds iff g is a generalized Gaussian $g = G_{\gamma}$.

For the proof of this inequality, we will use two general inequalities relating the moment $m_{\alpha}[g]$, the generalized Fisher information $\phi_{\beta,q}[g]$ and the information generating function $M_q[g] = \int g(x)^q \mathrm{d}x$. We will also use the notation $N_q[g] = M_q[g]^{\frac{1}{1-q}}$,which is known as the "Rényi entropy power".

Lemma 3. *For $n \geq 1$, $\alpha \in (0,\infty)$, $q > n/(n+\alpha)$, and if g is a probability density on random vectors of $\mathbb{R}^n$ with $m_{\alpha}[g] = E[\|x\|^{\alpha}] < \infty$, $N_q[g] < \infty$, then*

$$\frac{m_{\alpha}[g]^{\frac{1}{\alpha}}}{N_q[g]^{\frac{1}{n}}} \geq \frac{m_{\alpha}[G]^{\frac{1}{\alpha}}}{N_q[G]^{\frac{1}{n}}}, \tag{57}$$

with equality if and only if g is a generalized Gaussian.

Proof. The inequality (57) has been stated and proved in [28] in the case of an Euclidean norm. We simply note here that the proof in [28] works as well in the case of a general norm. □

We will also use a generalized Stam inequality derived from a general sharp Gagliardo-Nirenberg inequality proved in the remarkable paper of Cordero et al. [10].

Lemma 4. *For $n \geq 1$, β and α Hölder conjugates of each other, $\alpha > 1$, and $q > \max\{(n-1)/n, n/(n+\alpha)\}$, then for any probability density on $\mathbb{R}^n$, supposed continuously differentiable, the following generalized Stam inequality holds*

$$\phi_{\beta,q}[g]^{\frac{1}{\beta\lambda}} \; N_q[g]^{\frac{1}{n}} \geq \phi_{\beta,q}[G]^{\frac{1}{\beta\lambda}} \; N_q[G]^{\frac{1}{n}}, \tag{58}$$

with $\lambda = n(q-1)+1$ and with equality if and only if g is any generalized q-Gaussian (6).

Proof. In our notations, the sharp Gagliardo-Nirenberg inequality [10, Eq. (34) p. 320], with $a > 1$ and $\|\nabla u\|_\beta = \left(\int \|\nabla u\|_*^\beta \mathrm{d}x\right)^{\frac{1}{\beta}}$, is

$$\frac{\|\nabla u\|_\beta \|u\|_{a(\beta-1)+1}^{\frac{1}{\theta}-1}}{\|u\|_{a\beta}^{\frac{1}{\theta}}} \geq K \tag{59}$$

where K is a sharp constant which is attained if and only if u is a generalized Gaussian with exponent $1/(1-a)$, and where θ is given by $\theta = n(a-1)/a(n\beta-(a\beta+1-a)(n-\beta))$. The idea is to take $u = g^t$, g being a probability density function, with $a\beta t = 1$, and to note $q = [a(\beta-1)+1]\,t$. With these notations, we get that $\beta t = \beta(q-1)+1$, and (59) becomes

$$\phi_{\beta,q}[g]^{\frac{1}{\beta}} \; N_q[g]^{\frac{(\theta-1)(1-q)t}{\theta q}} \geq K. \tag{60}$$

Simplifying the expression of θ and the exponent in (60), we finally obtain, with $q < 1$, the generalized Stam inequality (58), with equality if and only if g is any generalized q-Gaussian (6). Actually, this generalized Stam inequality is also valid in the case $q > 1$, as it can be checked from [10]'s results using similar steps as above. The conditions on q simply ensure the existence of the information measures for the generalized Gaussian. □

We end with the proof of theorem 2, which is now an easy task.

Proof. The Cramér-Rao inequality (46) can also be written

$$m_\alpha[g]^{\frac{1}{\alpha}} \; \frac{\phi_{\beta,q}[g]^{\frac{1}{\beta}}}{\frac{n}{q} M_q[g]} \geq 1. \tag{61}$$

Eliminating $M_q[g]$ between this inequality and the moment-entropy inequality (57) with $q > 1$, we arrive at (55). Similarly, in the case $q < 1$, the elimination of $M_q[g]$ between the extended q-Cramér-Rao inequality for a location parameter (61) and the generalized Stam inequality (58) also leads to (55). The case of equality directly follows from the cases of equality in the initial inequalities. Alternatively, we can observe that (55) also follows at once by the combination of (57) and (58). □

5. Conclusions

This paper complements and improves our previous findings presented in [19]. We connect concepts in estimation theory to tools used in nonextensive thermostatistics and establish general Cramér-Rao type inequalities valid for estimation purposes. These results are given in the mutidimensional case, and a feature of our approach is that it works for arbitrary norms on $\mathbb{R}^n$. As a direct consequence, we obtain multidimensional versions of our q-Cramér-Rao inequalities, which includes the Barankin-Vajda as well as the standard Cramér-Rao inequality as particular cases. Furthermore, in the case of a translation family, we have shown that the corresponding Cramér-Rao type inequality is saturated by multidimensional q-Gaussian distributions. We have also presented a related general Cramér-Rao inequality which is saturated by the same q-Gaussian distributions. These results imply in particular that the generalized q-Gaussians are the minimizers of an extended version of the Fisher information, among all distributions with a given moment, just as the standard Gaussian minimizes Fisher information over all distributions with a given variance. Since these generalized Gaussian are already known to be the maximum entropy distributions for Rényi or Tsallis entropies, this yields a new, complementary, information theoretic characterization of these distributions.

As is well-known, the Weyl-Heisenberg uncertainty principle in statistical physics corresponds to the standard Cramér-Rao inequality for the location parameter. Thus it would certainly be of interest to investigate on the possible meanings of the uncertainty relationships that could be associated to the extended Cramér-Rao inequalities. Another open issue is the study of the potential convexity properties of the generalized Fisher information. Indeed, the standard Fisher information is a convex function of the density. If this were also true for any value of q, then it would be possible to associate to the generalized Fisher information a statistical mechanics with the standard Legendre structure and with the q-Gaussian as canonical distribution. Finally, in their recent work [18], Lutwak et al. have introduced an abstract, implicit, notion of Fisher information matrix attached to a probability density. It would be of interest to examine whether this notion could be extended and interpreted in the estimation theory framework.

References

[1] C. Tsallis. *Introduction to Nonextensive Statistical Mechanics*. Springer, April 2009.
[2] E. Lutz. Anomalous diffusion and Tsallis statistics in an optical lattice. *Physical Review A*, 67(5):051402, 2003.
[3] V. Schwämmle, F. D Nobre, and C. Tsallis. q-Gaussians in the porous-medium equation: stability and time evolution. *The European Physical Journal B-Condensed Matter and Complex Systems*, 66(4):537–546, 2008.
[4] C. Vignat and A. Plastino. Why is the detection of q-Gaussian behavior such a common occurrence? *Physica A*, 388(5):601–608, March 2009.
[5] A. Ohara and T. Wada. Information geometry of q-Gaussian densities and behaviors of solutions to related diffusion equations. *Journal of Physics A: Mathematical and Theoretical*, 43(3):035002, 2010.
[6] G. I. Barenblatt. On some unsteady motions of a liquid and a gas in a porous medium. *Prikladnaja Matematika i Mechanika*, 16:67–78, 1952.
[7] R. E. Pattle. Diffusion from an instantaneous point source with concentration dependent coefficient. *Quart. J. Mech. Appl. Math.*, 12:407–409, 1959.
[8] M. Del Pino and J. Dolbeault. Best constants for Gagliardo-Nirenberg inequalities and applications to nonlinear diffusions. *Journal de Mathématiques Pures et Appliquées*, 81(9):847–875, September 2002.
[9] M. Del Pino and J. Dolbeault. The optimal euclidean Lp-Sobolev logarithmic inequality. *Journal of Functional Analysis*, 197(1):151–161, January 2003.
[10] D. Cordero-Erausquin, B. Nazaret, and C. Villani. A mass-transportation approach to sharp sobolev and Gagliardo-Nirenberg inequalities. *Advances in Mathematics*, 182(2):307–332, March 2004.

[11] M. Agueh. Sharp Gagliardo-Nirenberg inequalities via p-Laplacian type equations. *Nonlinear Differential Equations and Applications NoDEA*, 15:457–472, December 2008.

[12] F. Pennini, A. Plastino, and G.L. Ferri. Semiclassical information from deformed and escort information measures. *Physica A*, 383(2):782–796, September 2007.

[13] S. Furuichi. On the maximum entropy principle and the minimization of the Fisher information in Tsallis statistics. *Journal of Mathematical Physics*, 50(1):013303–12, January 2009.

[14] S. Furuichi. On generalized Fisher informations and Cramér-Rao type inequalities. *Journal of Physics: Conference Series*, 201:012016, 2010.

[15] J. Naudts. Generalised exponential families and associated entropy functions. *Entropy*, 10(3):131–149, 2008.

[16] J. Naudts. The q-exponential family in statistical physics. *Central European Journal of Physics*, 7(3):405–413, 2009.

[17] E. Lutwak, D. Yang, and G. Zhang. Cramér-Rao and moment-entropy inequalities for Rényi entropy and generalized Fisher information. *IEEE Transactions on Information Theory*, 51(2):473–478, 2005.

[18] E. Lutwak, S. Lv, D. Yang, and G. Zhang. Extensions of Fisher information and Stam's inequality. *Information Theory, IEEE Transactions on*, 58(3):1319 –1327, March 2012.

[19] J.-F. Bercher. On generalized Cramér-Rao inequalities, generalized Fisher information and characterizations of generalized q-Gaussian distributions. *Journal of Physics A: Mathematical and Theoretical*, 45(25):255303, June 2012.

[20] E. W. Barankin. Locally best unbiased estimates. *The Annals of Mathematical Statistics*, 20(4):477–501, December 1949.

[21] I. Vajda. χ^α-divergence and generalized Fisher information. In *Transactions of the Sixth Prague Conference on Information Theory, Statistical Decision Functions and Random Processes*, page 223–234, 1973.

[22] J.-F. Bercher. On a (β, q)-generalized Fisher information and inequalities involving q-Gaussian distributions. *Journal of Mathematical Physics*, 53(6):063303, June 2012.

[23] D. W. Stroock. *A Concise Introduction to the Theory of Integration*. Springer, December 1998.

[24] R. A. Horn and C. R. Johnson. *Matrix Analysis*. Cambridge University Press, February 1990.

[25] J. F. Bonnans and A. Shapiro. *Perturbation Analysis of Optimization Problems*. Springer, May 2000.

[26] B. V. Limaye. *Functional Analysis*. New Age International, 1996.

[27] T. J. Morrison. *Functional Analysis: An Introduction to Banach Space Theory*. Wiley-Interscience, 1 edition, December 2000.

[28] E. Lutwak, D. Yang, and G. Zhang. Moment-Entropy inequalities for a random vector. *IEEE Transactions on Information Theory*, 53(4):1603–1607, April 2007.