Spectral monodromy of non selfadjoint operators

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SPECTRAL MONODROMY OF NON SELFADJOINT OPERATORS

QUANG SANG PHAN

ABSTRACT. We propose to build in this paper a combinatorial invariant, called the "spectral monodromy" from the spectrum of a single (non-selfadjoint) h-pseudodifferential operator with two degrees of freedom in the semi-classical limit.
Our inspiration comes from the quantum monodromy defined for the joint spectrum of an integrable system of n commuting selfadjoint h-pseudodifferential operators, given by S. Vu Ngoc.
The first simple case that we treat in this work is a normal operator. In this case, the discrete spectrum can be identified with the joint spectrum of an integrable quantum system.
The second more complex case we propose is a small perturbation of a selfadjoint operator with a classical integrability property. We show that the discrete spectrum (in a small band around the real axis) also has a combinatorial monodromy. The difficulty here is that we do not know the description of the spectrum everywhere, but only in a Cantor type set. In addition, we also show that the monodromy can be identified with the classical monodromy (which is defined by J. Duistermaat). These are the main results of this article.

CONTENTS

1. INTRODUCTION

1.1. **General framework.** This paper aims at understanding the structure of the spectrum of some classes of non-selfadjoint operators in the semi classical limit. It is a quantum problem that we treat with the help of semi-classical techniques combined with the general spectral theory of pseudo-differential operators.

We will also make the link with classical results that illuminate the initial quantum problem.

1.2. **Monodromy of h-pseudo-differential non-selfadjoint operators.** More explicitly, in this paper, we propose to build a new characteristic objet of the structure of the spectrum of non-selfadjoint h-pseudo-differential operators in the semi-classical limit.

Our inspiration comes from quantum monodromy, which is defined for the joint spectrum (see the definition 4) of a system of n $h-$ pseudo-differential operators that commute (i.e a completely integrable quantum system). This is a quantum invariant given by San Vu Ngoc [45](or [47]).

Under certain conditions, the joint spectrum on a domain U of regular values of the map of principal symbols is an asymptotic affine lattice in the sense that there is an invertible symbol, denoted f_α, from any small ball $B_\alpha \subset U$ in $\mathbb{R}^n$ that sends the joint spectrum to $\mathbb{Z}^n$ modulo $\mathcal{O}(h^\infty)$ (a result of Charbonnel [12]).

These (f_α, B_α) are considered as local charts of U whose transition functions, denoted by $A_{\alpha\beta}$, are in the integer affine group $GA(n,\mathbb{Z})$. The quantum monodromy is defined as the 1-cocycle $\{A_{\alpha\beta}\}$ modulo-coboundary in the Čech cohomology $\check{H}^1(U, GA(n,\mathbb{Z}))$.

For details of this monodromy, we can see the article [45] or section 2.2 of this paper.

Since this work, a mysterious question remains open: Can we define (and detect) such an invariant for a single h-pseudo-differential operator? If this happens, we will call it the "spectral monodromy".

From a geometrical point of view, since the joint spectrum of a quantum integrable system is an asymptotic affine lattice, if one realizes the parallel transport on the lattice of a basic rectangle with a vertex c along a some closed path γ_c (of base c) and returns to the starting point, then the initial rectangle can become a different rectangle (see figure below). It is the existence of quantum monodromy. Contrariwise, by the spectrum of a selfadjoint operator being contained in a straight line (real axis), it seems impossible to define such a parallel transport for a single operator. It is not known how to define a monodromy in this case.

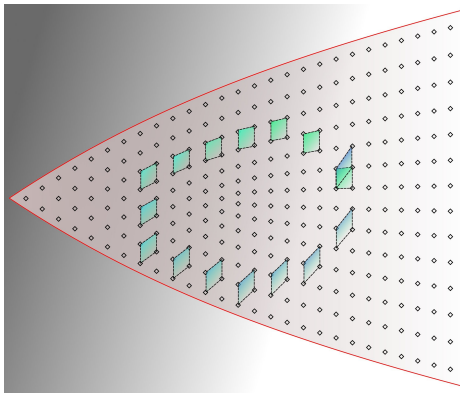

Figure 1. Joint spectrum of the spherical pendulum with monodromy
(Image by S. Vu Ngoc)

However, if we perturb a self-adjoint operator by a non-symmetric term, the spectrum becomes complex, and we may hope to find a geometric structure (lattice, monodromy ...). We propose in this paper to apply this idea to certain classes of h-pseudo-differential operators of two degrees of freedom.

1.2.1. *The first case.* The first simple case that we propose is a h-pseudo-differential operator of form $P_1 + iP_2$ with two self-adjoint operators P_1, P_2 that commute. This is the form of a normal operator. We show in section 3 that the discrete spectrum of $P_1 + iP_2$ is identified with the joint spectrum of the integrable quantum system (P_1, P_2) (see Theorems 2.4 and 2.6). One can simply define the "affine spectral monodromy" of operator $P_1 + iP_2$ as the quantum monodromy of the joint spectrum. For details, see section 2.4.

1.2.2. *The second case.* The second case that we study is more complex. For the quantum monodromy and thus the affine spectral monodromy (in the first case), the quantum integrability condition of P_1, P_2 is necessary but seems a bit heavy to have a monodromy because the quantum monodromy has a relationship with the classical monodromy (which is given by J.Duistermaat [18]) of the integrable classical system (p_1, p_2), the corresponding principal symbols of P_1, P_2. It is a result in article [45].

For this reason, keeping the property of classical integrability, we will propose to consider a small perturbation of a self-adjoint operator of the form $P_\varepsilon := P_1 + i\varepsilon P_2$ assuming that the principal symbols p_1, p_2 commute, $\varepsilon \to 0$ and in the regime $h \ll \varepsilon = \mathcal{O}(h^\delta)$ for $0 < \delta < 1$.

Here there is no joint spectrum, so we can not apply the construction of the quantum monodromy.

However with the help of the results of asymptotic spectral theory of M.Hitrik, J.Sjöstrand, S. Vu Ngoc ([23], [24], [25] and especially [27]) (under conditions detailed in the section 4) by revisiting the procedure of Birkhoff normal form (section (3.3)), the spectrum of P_ε is located in a horizontal band of height $\mathcal{O}(\varepsilon)$ and in this band, we can give the asymptotic expansion of eigenvalues of P_ε in some "good rectangles" $R(\chi_a, \varepsilon, h)$ (see definition 3.5) of size $\mathcal{O}^\delta \times \mathcal{O}(\varepsilon h^\delta)$ which are associated with Diophantine torus Λ_a.

There is a correspondence between $\lambda \in \sigma(P_\varepsilon) \cap R(\chi_a, \varepsilon, h)$ and hk in a part of $h\mathbb{Z}^2$ by a diffeomorphism (a micro-chart) of form (see the formula (81)):

$$\begin{aligned} f : R(\chi_a, \varepsilon, h) &\to E(a, \varepsilon, h) \\ \sigma(P_\varepsilon) \cap R(\chi_a, \varepsilon, h) \ni \lambda &\mapsto f(\lambda, \varepsilon; h) \in h\mathbb{Z}^2 + \mathcal{O}(h^\infty). \end{aligned} \tag{1}$$

For more details of this idea, see the section 3.4.

Nevertheless all Diophantine tori do not quite fill the phase space (see [8], [36]) and despite its density, is not yet known whether such a expansion holds globally on any small domain of spectral band. However, we will prove the global existence (for such rectangles) of the first term of this expansion on any small area in the set of regular values of the application $(p, \varepsilon q)$.

The spectrum of P_ε is the model of a particular set $\Sigma(\varepsilon, h)$ on a domain $U(\varepsilon)$ that we define in the section 4 and call "asymptotic pseudo-lattice" (see definition 3.26) whose differential transition functions between the adjacent "pseudo-locals charts" are in the group $GL(2, \mathbb{Z})$ modulo $\mathcal{O}(\varepsilon, \frac{h}{\varepsilon})$.

This allows us to treat the inverse problem: define a combinatorial invariant (the spectral monodromy) from the spectrum of P_ε.

This is the main result of this paper, presented in section 3.

It would be very interesting to extend these results to the case where p is a perturbation an integrable system, using the work of Broer, Cusham, Fassò et Takens [9].

2. Affine Spectral Monodromy

The quantum monodromy that is defined for the discrete joint spectrum of a integrable quantum system of n commuting $h-$pseudo-differential operators is completely given by S. Vu Ngoc [45].

We propose to define the monodromy for a single $h-$pseudo-differential operator and in this section, we will treat the simple case of a normal operator. To do this, we will give an identification between the discrete spectrum of a normal operator and the joint spectrum of an integrable quantum system (theorems 2.4 and 2.6).

First we briefly recall the standard class of h-(Weyl-)pseudo-differential operators which is used through in this article. Then we give the results of spectral theory that allow us to define the "affine spectral monodromy" of a pseudo-differential normal operator.

2.1. **Pseudo-differential operators.** We will work throughout this article with pseudo-differentials operators obtained by the $h-$Weyl-quantization with standard classes of symbols on $M = T^*\mathbb{R}^n =$

$\mathbb{R}^{2n}_{(x,\xi)}$. These operators admit the standard properties of pseudo differential operators. For more details, see the references [17], [38], [39].

Definition 2.1. *A function $m : \mathbb{R}^{2n} \to (0, +\infty)$ is called an order function (or tempered weight in the book of D. Robert* [38]*) if there are constants $C, N > 0$ such that*

$$m(X) \leqslant C\langle X - Y\rangle^N m(Y), \forall X, Y \in \mathbb{R}^{2n},$$

with notation $\langle Z\rangle = (1 + |Z|^2)^{1/2}$ for $Z \in \mathbb{R}^{2n}$.

One use often the order function $m(Z) \equiv 1$ or

$$m(Z) = \langle Z\rangle^{l/2} = (1 + |Z|^2)^{l/2},$$

with a given constant $l \in \mathbb{R}$.

Definition 2.2. *Let m be an order function and $k \in \mathbb{R}$, we define classes of symbols of h-order k, $S^k(m)$ (families of functions) of $(a(\cdot; h))_{h\in(0,1]}$ on $\mathbb{R}^{2n}_{(x,\xi)}$ by*

$$S^k(m) = \{a \in C^\infty(\mathbb{R}^{2n}) \mid \forall \alpha \in \mathbb{N}^{2n}, \quad |\partial^\alpha a| \leqslant C_\alpha h^k m\}, \tag{2}$$

for some constant $C_\alpha > 0$, uniformly in $h \in (0, 1]$.

A symbol a is called $\mathcal{O}(h^\infty)$ if it's in $\cap_{k\in\mathbb{R}} S^k(m) := S^\infty(m)$.

Then $\Psi^k(m)(M)$ denotes the set of all (in general unbounded) linear operators A_h on $L^2(\mathbb{R}^n)$, obtained from the $h-$Weyl-quantization of symbols $a(\cdot; h) \in S^k(m)$ by the integral:

$$(A_h u)(x) = (Op^w_h(a)u)(x) = \frac{1}{(2\pi h)^n} \int_{\mathbb{R}^{2n}} e^{\frac{i}{h}(x-y)\xi} a(\frac{x+y}{2}, \xi; h)u(y)dyd\xi. \tag{3}$$

In this paper, we always assume that the symbols admit a classical asymptotic expansion in integer powers of h. The leading term in this expansion is called the principal symbol of the operator.

2.2. **Quantum monodromy of Integrable quantum systems.** If an integrable quantum system $(P_1(h), \ldots, P_n(h))$ with joint principal symbol p is proper, then near a regular value of p, the joint spectrum of the system locally has the structure of an affine integral lattice [12], [44]. By S. Vu Ngoc, this leads to the construction of quantum monodromy- a natural invariant defined from the spectrum, see [45]. The non-triviality of this invariant obstructs the global existence of lattice structure of the joint spectrum. More explicit:

Let X a compact manifold of dimension n or $X = \mathbb{R}^n$ and let $M := T^*X$ the tangent bundle of X. Let an integrable quantum system $(P_1(h), \ldots, P_n(h))$ of n commuting selfadjoint $h-$Weyl pseudo-differential operators on $L^2(X)$: $[P_i(h), P_j(h)] = 0$. We will assume that these $P_j(h)$ are in $\Psi^0(M)$, classical and of order zero. In any coordinate chart their Weyl symbols $p_j(h)$ have an asymptotic expansion of the form:

$$p_j(x, \xi; h) = p^j_0(x, \xi) + hp^j_1(x, \xi) + h^2 p^j_2(x, \xi) + \cdots .$$

Assume that the differentials of the principal symbols p^j_0 are linearly independent almost everywhere on M. The map of joint principal symbols $p = (p^1_0, \ldots, p^n_0)$ is a momentum map with respect to the symplectic Poisson bracket on T^*X ($\{p^i_0, p^j_0\} = 0$). We will assume that p is proper.

Let U_r be an open subset of regular values of p and let U be a certain open subset with compact closure $K := \overline{U}$ in U_r.

We define the joint spectrum of the system in K, noted $\sigma_{conj}(P_1, \ldots, P_n)$ by:

$$\sigma_{conj}(P_1(h), \ldots, P_n(h)) = \{(E_1(h), \ldots, E_n(h)) \in K | \cap^n_{j=1} Ker(P_j(h) - E_j(h)) \neq \emptyset\}. \tag{4}$$

Let $\Sigma(h) = \sigma_{conj}(P_1(h), \ldots, P_n(h)) \cap U$. Is is known from the work of Colin de Verdière [44] and Charbonnel [12], $\Sigma(h)$ is discrete and for small h is composed of simple eigenvalues. Moreover, $\Sigma(h)$ is "an asymptotic affine lattice" on U in the sense: there are locally invertible

symbols of order zero, denoted $f_\alpha(\cdot;h)$ from any small ball $B_\alpha \subset U$ in $\mathbb{R}^n$, sending $\Sigma(h)$ in $h\mathbb{Z}^n$ modulo $\mathcal{O}(h^\infty)$. These (f_α, B_α) are considered as locals charts of $\Sigma(h)$ on U whose transition functions, denoted by $A_{\alpha\beta}$ are in the integer affine group $GA(n,\mathbb{Z})$. The quantum monodromy is defined as the 1-cocycle $\{A_{\alpha\beta}\}$ modulo-coboundary in the Čech cohomology $\check{H}^1(U, GA(n,\mathbb{Z}))$ (see following definition).

We denote

$$[\mathcal{M}_{qu}] \in \check{H}^1(U, GA(n,\mathbb{Z})).$$

Remark 2.3 We recall here the definition of Čech cohomology that is use often in this paper. Let M be a manifold and $(G,.)$ be a group. Assume that $\{U_\alpha\}_{\alpha\in I}$ is a locally finite cover of open sets of M such that the intersection of a finite number of U_α is either contractible or empty. We denote $\mathcal{C}^0(M,G)$ the set of 0-cochains and $\mathcal{C}^1(M,G)$ the set of 1-cochains with values in G by:

$$\mathcal{C}^0(M,G) = \{(c_\alpha)_{\alpha\in I} \mid c_\alpha \in G\},$$

$$\mathcal{C}^1(M,G) = \{(c_{\alpha,\beta}) \mid U_\alpha \cap U_\beta \neq \emptyset, c_{\alpha,\beta} \in G\}.$$

We denote $\check{Z}^1(M,G)$ the set of 1-cochains satisfying the cocycle condition

$$c_{\alpha,\beta}.c_{\beta,\gamma} = c_{\alpha,\gamma}$$

if $U_\alpha \cap U_\beta \cap U_\gamma \neq \emptyset$.
We define an equivalence relation, denoted " $\sim$ " on $\check{Z}^1(M,G)$: two cocycles $(c_{\alpha,\beta})$ and $(c'_{\alpha,\beta})$ are equivalent $(c_{\alpha,\beta}) \sim (c'_{\alpha,\beta})$ if there exists a 0-cochain $(d_\alpha) \in \mathcal{C}^0(M,G)$ such that $c'_{\alpha,\beta} = d_\alpha.c_{\alpha,\beta}.d_\beta^{-1}$ for any $U_\alpha \cap U_\beta \neq \emptyset$.
Then the Čech cohomology of M with values in G is the quotient set

$$\check{H}^1(M,G) = \check{Z}^1(M,G)/\sim .$$

Note that it does not depends on choice of cover $\{U_\alpha\}_{\alpha\in I}$. △

The quantum monodromy can be considered as a group morphism (holonomy):

$$\mu : \pi_1(U) \to GA(n,\mathbb{Z})/\{\sim\} \tag{5}$$

which is the product of transition functions along a closed loop modulo by conjugation " $\sim$ ".

For more details and discussion of this monodromy, we can see [45], [47].

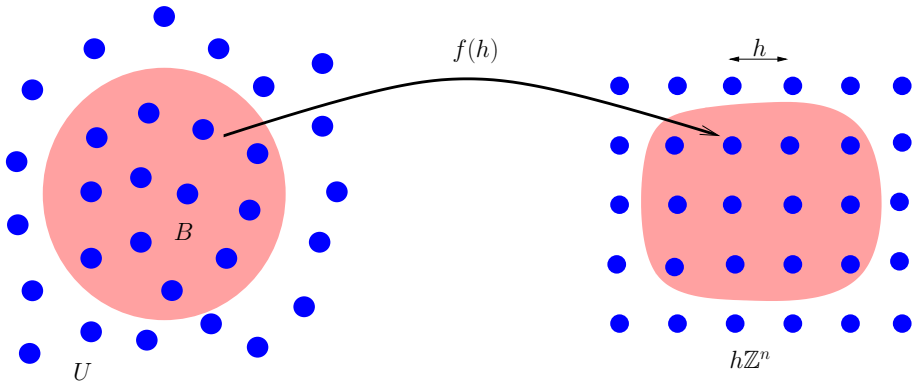

Figure 2. Asymptotic affine lattice

2.3. **Normal operators.** In this section, we will show the natural statement that: the discrete spectrum of a unbounded normal operator A can be identified with the joint spectrum of an integrable system which consists of the real part and the imaginary part of A. This allows us to define the monodromy of normal operator as an application of quantum monodromy.

Consider a normal operator (usually unbounded) A with dense domain $D(A) = D$ on a Hilbert space H. It is known that the adjoint operator A^* has the same domain $D(A^*) = D(A) = D$. We denote the real part and imaginary part of A by $A = A_1 + iA_2$ with

$$A_1 = \frac{A + A^*}{2}, A_2 = \frac{A - A^*}{2i}, D(A_1) = D(A_2) = D. \tag{6}$$

It is true that A_1 and A_2 defined by the formula (6) are self-adjoint. Moreover, the commutativity of A and A^* is equivalent to the commutativity of A_1, A_2 and therefore $A_1A_2 = A_2A_1$.

In this article, one say that two selfadjoint operators (usually unbounded) $(A_1, D(A_1))$ and $(A_2, D(A_2))$ commute if $A_1A_2 = A_2A_1$ and this definition includes the requirement of domain:

$$Ran(A_2) \subseteq D(A_1), Ran(A_1) \subseteq D(A_2), D(A_1A_2) = D(A_2A_1).$$

Conversely, let two commuting self-adjoint operators A_1 and A_2, $D(A_1) = D(A_2) = D$ dense in H. Then the operator defined by $A := A_1 + iA_2$, $D(A) = D$ is a well-defined normal operator (and hence closed) on H.

In the literature, there are alternative definitions of discrete spectrum (see[15], [17], [34], [5]) which coincide in the self-adjoint case. In this article, we use the following general definition of discrete spectrum, see [37], [31].

Definition 2.4. *For a closed operator $(A, D(A))$, let λ be an isolated point of $\sigma(A)$: there is $\varepsilon > 0$ such that $\{\mu \in \mathbb{C}, |\ |z - \lambda| < \varepsilon\} \cap \sigma(A) = \{\lambda\}$. For all $0 < r < \varepsilon$, we can define the projection P_λ (not necessarily orthogonal) by*

$$P_\lambda = \frac{1}{2\pi i} \int_{|z-\lambda|=r} (z - A)^{-1}. \tag{7}$$

We say that $\lambda \in \sigma(A)$ is in the discrete spectrum of A, denoted $\sigma_{disc}(A)$ if λ is isolated in $\sigma(A)$ and P_λ has finite rank.

We define the essential spectrum of A, denoted $\sigma_{ess}(A)$ as the complement of the discrete spectrum

$$\sigma_{ess}(A) = \sigma(A) \setminus \sigma_{disc}(A).$$

Proposition 2.5. *Let A_1, A_2 two commuting self-adjoint operators on a Hilbert space H with dense domain $D(A) = D(B) = D$, $A_1A_2 = A_2A_1$. Then we have:*

1. *If $\lambda \in \sigma(A_1 + iA_2)$, then $Re(\lambda) \in \sigma(A_1)$ and $Im(\lambda) \in \sigma(A_2)$.*
2. *If $\lambda \in \sigma_p(A_1 + iA_2)$, then $Re(\lambda) \in \sigma_p(A_1)$ and $Im(\lambda) \in \sigma_p(A_2)$.*
   Moreover if λ is an eigenvalue of infinite multiplicity of $A_1 + iA_2$, then $Re(\lambda), Im(\lambda)$ are the eigenvalues of infinite multiplicity corresponding of A_1, A_2.
3. *If $\lambda \in \sigma_{ess}(A_1 + iA_2)$, then $Re(\lambda) \in \sigma_{ess}(A_1)$ and $Im(\lambda) \in \sigma_{ess}(A_2)$.*

Proof. Let $A = A_1 + iA_2$, $D(A) = D$. As we explained earlier at the beginning of this section, A is a normal operator with adjoint operator $A^* = A_1 - iA_2$. For any complex number $\lambda \in \mathbb{C}$, $A - \lambda$ is still a normal operator. Then for any $u \in D$, the equality $\|(A - \lambda)u\| = \|(A^* - \overline{\lambda})u\|$ gives us

$$\|(A_1 + iA_2 - \lambda)u\|^2 = \|(A_1 - Re(\lambda))u\|^2 + \|(A_2 - Im(\lambda))u\|^2 \tag{8}$$

This equation implies that:

$$Ker(A - \lambda) = Ker(A_1 - Re(\lambda)) \cap Ker(A_2 - Im(\lambda)). \tag{9}$$

If $\lambda \in \sigma(A_1 + iA_2)$, by the Weyl theorem (see [5], page 102)), there exists a Weyl sequence for A and λ: a sequence $u_n \in D$, $\|u_n\| = 1$ such that $\lim_{n\to\infty} \|(A - \lambda)u_n\| = 0$. By the equation (8),

it is still a Weyl sequence for A_1 and $Re(\lambda)$, for A_2 and $Im(\lambda)$. Again by the Weyl theorem, we have $Re(\lambda) \in \sigma(A_1)$ and $Im(\lambda) \in \sigma(A_2)$.

If λ is an eigenvalue of $A_1 + iA_2$, by the equation (9) we have $Re(\lambda) \in \sigma_p(A_1)$ and $Im(\lambda) \in \sigma_p(A_2)$. Moreover, it is obvious that if λ is an eigenvalue of infinite multiplicity of $A_1 + iA_2$, then $Re(\lambda), Im(\lambda)$ are also eigenvalues of infinite multiplicity corresponding to A_1, A_2.

We also note that if u is an eigenvector corresponding to λ of $A_1 + iA_2$, then u is also simultaneous eigenvector of A_1, A_2.

If $\lambda \in \sigma_{ess}(A_1 + iA_2)$, then there exists a orthogonal Weyl sequence for $A_1 + iA_2$ in λ such that: $u_n \in D$, $\|u_n\| = 1$ and $\lim_{n\to\infty} \|(A - \lambda)u_n\| = 0$. By the equation (8) and by the Weyl theorem for a self-adjoint operator (see [17], [34] page 287), [5] page 173, [15]...), we obtain $Re(\lambda) \in \sigma_{ess}(A_1)$ and $Im(\lambda) \in \sigma_{ess}(A_2)$. The proposition is shown. □

One can easily show that the reverse implications are false in general.

From this proposition, by identifying $\mathbb{C} \cong \mathbb{R}^2$ we have the following result:

Theorem 2.6. *Let A_1, A_2 two commuting self-adjoint operators on a Hilbert space H with dense domain $D(A) = D(B) = D$. Let I_1, I_2 be two intervals of $\mathbb{R}$ such that the corresponding spectra of A_1,A_2 in I_1, I_2 are discrete.*

Then the spectrum of $A_1 + iA_2$ in $I_1 + iI_2 \cong I_1 \times I_2$ is discrete and

$$\sigma(A_1 + iA_2) \cap (I_1 + iI_2) \cong \sigma_{conj}(A_1, A_2) \cap (I_1 \times I_2).$$

Proof. We always have inclusion:

$$\sigma_{conj}(A_1, A_2) \cap I_1 \times I_2 \subseteq \sigma_p(A_1 + iA_2) \cap I_1 + iI_2 \subseteq \sigma(A_1 + iA_2) \cap I_1 + iI_2.$$

Let us consider the inverse inclusion. For any $\lambda \in \sigma(A_1 + iA_2) \cap I_1 + iI_2$, the previous proposition says that: $Re(\lambda) \in \sigma(A_1+) \cap I_1$ and $Im(\lambda) \in \sigma(A_2) \cap I_2$.
Because the corresponding spectra of A_1, A_2 in I_1, I_2 are discrete, $Re(\lambda)$ and $Im(\lambda)$ are respectively isolated eigenvalues of finite multiplicity of A_1, A_2.
From the equation (9), λ must be an eigenvalue of finite multiplicity of $A_1 + iA_2$ and there exists a common eigenvalue for A_1, A_2: $u \in D, \|u\| = 1, A_1u = Re(\lambda)u, A_2u = Im(\lambda)u$. Therefore

$$\lambda = (Re(\lambda), Im(\lambda)) \in \sigma_{conj}(A_1, A_2) \cap I_1 \times I_2.$$

□

We will give a version of this theorem for a normal operator with spectrum discrete in a rectangle area of $\mathbb{C}$.

Theorem 2.7. *Let A a normal operator and I_1, I_2 two intervals of $\mathbb{R}$ such that the spectrum of A in $I_1 + iI_2$ is discrete. We denote the real part and the imaginary part of A by A_1 and A_2. Then we have:*

$$\sigma(A) \cap (I_1 + iI_2) \cong \sigma_{conj}(A_1, A_2) \cap (I_1 \times I_2).$$

Proof. It is obvious that

$$\sigma_{conj}(A_1, A_2) \cap (I_1 \times I_2) \subseteq \sigma(A) \cap (I_1 + iI_2).$$

For the inverse inclusion: if $\lambda \in \sigma(A) \cap (I_1 + iI_2)$, then λ is an eigenvalue of A because $\sigma(A) \cap (I_1 + iI_2)$ is discrete. The proposition 2.5 implies that $Re(\lambda) \in \sigma_p(A_1) \cap I_1$ and $Im(\lambda) \in \sigma_p(A_2) \cap I_2$ with a nonzero common eigenvector (by equation (9)) and we get the inverse inclusion. □

This theorem allows us to define the monodromy of a normal pseudo-differential operator returning the quantum monodromy of the joint spectrum as below.

2.4. **Monodromy of Normal pseudo-differential operators.** In this section, we work with a space of dimension $n = 2$. Let $P(h)$ a h-pseudo-differential operator on $L^2(X)$.

We assume that $P(h)$ is normal and classical of order zero, $P(h) \in \Psi^0(M)$. As in the previous section, we can write $P(h) = P_1(h) + iP_2(h)$ where $P_1(h), P_2(h)$ are the real part and imaginary part of $P(h)$.
The commutativity of $P_1(h), P_2(h)$ gives us the integrable quantum system $(P_1(h), P_2(h))$ and thanks to its joint spectrum, we can define its quantum monodromy $[\mathcal{M}_{qu}] \in \check{H}^1(U, GA(n, \mathbb{Z}))$ as in the previous section. Here U is some open subset with compact closure in the set of regular values of the momentum map p_0 of principal symbols of $P_1(h), P_2(h)$, $p_0 = (Re(p), Im(p))$ where p is the principal symbol of $P(h)$.

We assume moreover that the spectrum of $P(h)$ in U is discrete. The previous theorem gives us an identification on U between this spectrum and the joint spectrum. We have therefore the following definition of a combinatorial invariant from the discrete spectrum.

Definition 2.8. *With the above hypothesis, the monodromy of a normal h-Weyl-pseudo-differential $P(h)$ on U is defined as the quantum monodromy of the integrable quantum system $(P_1(h), P_2(h))$ on U.*

We call it the affine spectral monodromy.

3. Linear Spectral Monodromy

3.1. **Introduction.** In this section, we propose to define the monodromy of a particular class of non-self-adjoint h-pseudo-differentials operator two degree of freedom which are small perturbations of selfadjoint operators, of the form $P_\varepsilon := P + i\varepsilon Q$ (P is selfadjoint) with principal symbols p, q that commute for the Poisson bracket and in the regime $h \ll \varepsilon = \mathcal{O}(h^\delta)$ for some $0 < \delta < 1$.

The asymptotic spectral theory by M.Hitrik-J.Sjöstrand- S. Vu Ngoc ([23], [24], [25]...) allows us to concretely give the asymptotic expansion of eigenvalues of P_ε in a adapted complex window.

The spectrum of P_ε is the model of a particular discrete set which we will define in this section and call "pseudo-asymptotic lattice" (see definition 3.26). By calculating the transition functions between the "pseudo-local cards" that are in the group $GL(2, \mathbb{Z})$, we can define a combinatorial invariant (the monodromy) of this lattice.

This allows us to treat the inverse problem: define the monodromy from the spectrum of P_ε.

We first recall some important results and analyze the general asymptotic spectral theory ([27], [23], [24]...). Then we will detail these results in our particular case by restating the Birkhoff procedure of normal form. Next, we give some necessary steps for the construction of the monodromy of a pseudo-asymptotic lattice and then apply it to the spectrum of P_ε.

Finally, noting that with the property of integrability, the classical monodromy (given by J.Duitermaat, [18]) is well defined, we also give the relationship between two monodromy types.

3.2. Spectral Asymptotic.

3.2.1. *Assumptions.* We will first give the general assumptions of our operator as in the articles [27], [22], [23], [24], [25] and the assumptions on the classical flow of the principal symbol of the non-perturbed operator and some associated spectral results: the discrete spectrum, the localization of the spectrum, the expansion of asymptotic eigenvalues ...

M denotes $\mathbb{R}^2$ or a connected compact analytic real (riemannian) manifold of dimension 2 and we denote by $\widetilde{M}$ the canonical complexification of M, which is either $\mathbb{C}^2$ in the Euclidean case or a Grauert tube in the case of manifold (see [10], [30]).

We consider a non-selfadjoint h-pseudo-differential operator P_ε on M and suppose that

$$P_{\varepsilon=0} := P \quad \text{is formally self-adjoint.} \tag{10}$$

Note that if $M = \mathbb{R}^2$, the volume form $\mu(dx)$ is naturally induced by the Lebesgue measure on $\mathbb{R}^2$, contrariwise in the case M is compact riemannian manifold, the volume form $\mu(dx)$ is

induced by the given riemannian structure of M. So, in all cases, the operator P_ε is seen as an (unbounded) operator on $L^2(M, \mu(dx))$.

We always denote the principal symbol of P_ε by p_ε which is defined on T^*M as we discussed in previous section.

We'll assume the ellipticity condition at infinity for P_ε as follows:

When $M = \mathbb{R}^2$, let

$$P_\varepsilon = P(x, hD_x, \varepsilon; h) \tag{11}$$

be the Weyl quantification of a total symbol $P(x, \xi, \varepsilon; h)$ depending smoothly on ε in a neighborhood of $(0, \mathbb{R})$ and taking values in the space of holomorphic functions of (x, ξ) in a tubular neighborhood of $\mathbb{R}^4$ in $\mathbb{C}^4$ on which we assume that:

$$|P(x, \xi, \varepsilon; h)| \leqslant \mathcal{O}(1)m(Re(x, \xi)). \tag{12}$$

Here m is an order function in the sense of definition 2.1.

We assume moreover that $m > 1$ and P_ε is classical

$$P(x, \xi, \varepsilon; h) \sim \sum_{j=0}^{\infty} p_{j,\varepsilon}(x, \xi)h^j, h \to 0, \tag{13}$$

in the selected space of symbols.

In this case, the main symbol is the first term of the above expansion, $p_\varepsilon = p_{0,\varepsilon}$ and the ellipticity condition at infinity is

$$|p_{0,\varepsilon}(x, \xi)| \geqslant \frac{1}{C} m(Re(x, \xi)), | \ (x, \xi) \ | \geqslant C, \tag{14}$$

for some $C > 0$ large enough.

When M is a manifold, we consider P_ε a differential operator on M such that in local coordinates x of M, it is of the form:

$$P_\varepsilon = \sum_{|\alpha| \leqslant m} a_{\alpha,\varepsilon}(x; h)(hD_x)^\alpha, \tag{15}$$

Where $D_x = \frac{1}{i}\frac{\partial}{\partial x}$ and $a_{\alpha,\varepsilon}$ are smooth functions of ε in a neighborhood of 0 with values in the space of holomorphic functions on a complex neighborhood of $x = 0$.

We assume that these $a_{\alpha,\varepsilon}$ are classic

$$a_{\alpha,\varepsilon}(x; h) \sim \sum_{j=0}^{\infty} a_{\alpha,\varepsilon,j}(x)h^j, h \to 0, \tag{16}$$

in the selected space of symbols.

In this case, the principal symbol p_ε in the local canonical coordinates associated (x, ξ) on T^*M is

$$p_\varepsilon(x, \xi) = \sum_{|\alpha| \leqslant m} a_{\alpha,\varepsilon,0}(x)\xi^\alpha \tag{17}$$

and the elipticity condition at infinity is

$$|p_\varepsilon(x, \xi)| \geqslant \frac{1}{C}\langle \xi \rangle^m, (x, \xi) \in T^*M, | \ \xi \ | \geqslant C, \tag{18}$$

for some $C > 0$ large enough.

Note here that M has a riemannian metric, then $| \ \xi \ |$ and $\langle \xi \rangle = (1+ | \ \xi \ |^2)^{1/2}$ is well defined.

It is known from articles [27], [23] that with the above conditions, the spectrum of P_ε in a small but fixed neighborhood of $0 \in \mathbb{C}$ is discrete, when $h > 0, \varepsilon \geqslant 0$ are small enough. Moreover, this spectrum is contained in a band of size ε:

$$|Im(z)| \leqslant \mathcal{O}(\varepsilon). \tag{19}$$

This gives the first location of the spectrum of P_ε.

Let $p = p_{\varepsilon=0}$, it is principal symbol of the selfadjoint unperturbed operator P and therefore real.

We assume that

$$p^{-1}(0) \cap T^*M \qquad \text{is connected} \tag{20}$$

and the energy level $E = 0$ is regular for p, i.e $dp \neq 0$ on $p^{-1}(0) \cap T^*M$.

Let $q = \frac{1}{i}(\frac{\partial}{\partial \varepsilon})_{\varepsilon=0} p_\varepsilon$, so

$$p_\varepsilon = p + i\varepsilon q + \mathcal{O}(\varepsilon^2) \tag{21}$$

in the neighborhood of $p^{-1}(0) \cap T^*M$.

For $T > 0$, we introduce the symmetric average time T of q along the flow of p, defined near $p^{-1}(0) \cap T^*M$:

$$\langle q \rangle_T = \frac{1}{T} \int_{-T/2}^{T/2} q \circ exp(tH_p) dt, \tag{22}$$

where $H_p = \frac{\partial p}{\partial \xi} \cdot \frac{\partial}{\partial x} - \frac{\partial p}{\partial x} \cdot \frac{\partial}{\partial \xi}$ is the hamiltonian vector field of p.

Note that $H_p(\langle q \rangle_T) = \{p, \langle q \rangle_T\} = \mathcal{O}(1/T)$.

As explained in [27], by introducing a Fourier integral operator (which is defined microlocally close to $p^{-1}(0) \cap T^*M$), we can reduce our operator to a new operator, denoted again by P_ε, with principal symbol $p + i\varepsilon \langle q \rangle_T + \mathcal{O}(\varepsilon^2)$ and $P_{\varepsilon=0}$ is still the original unperturbed operator. So we can assume that our operator P_ε is microlocally defined in the neighborhood of $p^{-1}(0) \cap T^*M$ with h- principal symbol

$$p + i\varepsilon \langle q \rangle_T + \mathcal{O}(\varepsilon^2). \tag{23}$$

Consequently, with the help of the sharp Garding inequality the spectrum of P_ε in the domain $\{z \in \mathbb{C} : |z| < \delta\}$, when $\varepsilon, h, \delta \to 0$ is confined in the band (voir [40], [41]):

$$]-\delta, \delta[+ i\varepsilon \big[\lim_{T\to\infty} \inf_{p^{-1}(0)} Re\langle q \rangle_T - o(1), \lim_{T\to\infty} \sup_{p^{-1}(0)} Re\langle q \rangle_T + o(1) \big]. \tag{24}$$

With more assumptions about the dynamics of classical flow of the first term of the unperturbed symbol (in a certain energy level), one can obtain more detailed results on the asymptotic distribution of eigenvalues in such a band.

Let a given value $F_0 \in \big[\lim_{T\to\infty} \inf_{p^{-1}(0)} Re\langle q \rangle_T, \lim_{T\to\infty} \sup_{p^{-1}(0)} Re\langle q \rangle_T\big]$, we want to determine all the eigenvalues of P_ε in a rectangle of center $(0, \varepsilon F_0)$ and of size $\mathcal{O}(h^\delta) \times \mathcal{O}(\varepsilon h^\delta)$ (which is included in the previous band) for

$$h \ll \varepsilon \leqslant \mathcal{O}(h^\delta),$$

where $\delta > 0$ is some number small enough but fixed.

Remark 3.1 The problem of determining asymptotically the eigenvalues of P_ε in such a rectangle of spectral domain has been proposed in the literature with different assumptions on the Hamiltonian flow of p: this flow can be periodic on an energy E near 0, completely integrable or almost integrable.

The force of the perturbation ε is treated with several regimes: $h^M \leqslant \varepsilon \leqslant \mathcal{O}(h^\delta)$, for M fixed large enough, $h \ll \varepsilon \leqslant \mathcal{O}(h^\delta)$, $h^{1/3-\delta} < \varepsilon \leqslant \varepsilon_0$,... and the size of the rectangle: depends on h or does not depend on h.

One can read the articles [23], [24], [27], [25], [26], [42].

Here, we present the completely integrable case in the regime $h \ll \varepsilon \leqslant \mathcal{O}(h^\delta)$. $\triangle$

Now, assume that p is completely integrable in a neighborhood of $p^{-1}(0) \cap T^*M$, i.e there exists a smooth real function f, independent of p such that $\{p, f\} = 0$. As explained in ([27], page 21-22 and 55), the energy space $p^{-1}(0)$ is decomposed into a singular foliation:

$$p^{-1}(0) \cap T^*M = \bigcup_{a \in J} \Lambda_a, \tag{25}$$

where J is assumed to be a connected graph with a finite number of vertices and of edges. We denote by S the set of vertices.

For each $a \in J$, Λ_a is a connected compact subset invariant with respect to H_p. Moreover, if $a \in J\backslash S$, Λ_a is a invariant Lagrangian torus depending analytically on a. Each edge of J can be identified with a bounded interval of $\mathbb{R}$.

Next, we assume the continuity of tori: let $\Lambda_{a_0}, a_0 \in J\backslash S$, for all $\mu > 0, \exists \gamma > 0$, such that if $dist(a, a_0) < \gamma$, then $\Lambda_a \subset \{\rho \in p^{-1}(0) \cap T^*M : dist(\rho, \Lambda_0) < \mu\}$. Note that this hypothesis holds for integrable systems with non-degenerate singularities.

For each torus $\Lambda_a, a \in J\backslash S$, there are action-angle local coordinates (x, ξ) near Λ_a such that $\Lambda_a \simeq \{\xi = 0\}$ and that p becomes a function of ξ, $p = p(\xi) = p(\xi_1, \xi_2)$. The frequency of Λ_a can be defined as an element of the real projective line by

$$\omega(a) = [p'_{\xi_1}(0) : p'_{\xi_2}(0)]. \tag{26}$$

Sometimes $\omega(a)$ is seen as an element of $\mathbb{R}$.

Moreover, by the action-angle theorem, we know that $\omega(a)$ depends analytically of $a \in J\backslash S$. We will assume that the function $a \mapsto \omega(a)$ is not identically constant on any connected component of $J\backslash S$.

For each $a \in J$, we define a compact interval in $\mathbb{R}$:

$$Q_\infty(a) = \big[\lim_{T\to\infty} \inf_{\Lambda_a} Re\langle q\rangle_T, \lim_{T\to\infty} \sup_{\Lambda_a} Re\langle q\rangle_T\big]. \tag{27}$$

Then the spectral localization (24) becomes

$$Im(\sigma(P_\varepsilon) \cap \{z \in \mathbb{C} : |Rez| \leqslant \delta\}) \subset \varepsilon\big[\inf \bigcup_{a\in J} Q_\infty(a) - o(1), \sup \bigcup_{a\in J} Q_\infty(a) + o(1)\big], \tag{28}$$

when $\varepsilon, h, \delta \to 0$.

From now, for simplicity, we will assume that q is real.

For each torus $\Lambda_a, a \in J\backslash S$, one defines $\langle q\rangle_{\Lambda_a}$ the average of q with respect to the natural Liouville measure on Λ_a

$$\langle q\rangle_{\Lambda_a} = \int_{\Lambda_a} q \tag{29}$$

Remark 3.2

In action-angle coordinates (x, ξ) near Λ_a such that $\Lambda_a \simeq \{\xi = 0\}$, we have

$$\langle q\rangle(\xi) = \frac{1}{(2\pi)^2} \int_{\mathbb{T}^2} q(x, \xi)dx. \tag{30}$$

In particular, $\langle q\rangle_{\Lambda_a} = \langle q\rangle(0)$. △

Remark 3.3 [([27], page 56-57)] For $a \in J\backslash S$:

- if $\omega(a) \notin \mathbb{Q}$, then $Q_\infty(a) = \{\langle q\rangle_{\Lambda_a}\}$.
- if $\omega(a) = \frac{m}{n} \in \mathbb{Q}$ ($m \in \mathbb{Z}, n \in \mathbb{N}$), then

$$Q_\infty(a) = \langle q\rangle_{\Lambda_a} + \mathcal{O}\big(\frac{1}{k(\omega(a))^\infty}\big)[-1, 1], \quad k(\omega(a)) := |m| + |n|.$$

In particular

$$\sum_{a:\omega(a)\in\mathbb{Q}} |Q_\infty(a)| < \infty.$$

△

$\langle q\rangle_{\Lambda_a}$ depends analytically of $a \in J\backslash S$ and we assume it can be extended continuously on J. Furthermore, we assume that the function $a \mapsto \langle q\rangle(a) = \langle q\rangle_{\Lambda_a}$ is not identically constant on any connected component of $J\backslash S$.

Note that p and $\langle q\rangle$ commute in neighborhood of $p^{-1}(0) \cap T^*M$.

3.2.2. *Asymptotic eigenvalues.*

Definition 3.4. *For a torus Λ_a, $a \in J\backslash S$ and $\omega(a)$ defined as (26) and let $\alpha > 0$, $d > 0$, we say that Λ_a is $(\alpha, d)-$Diophantine if:*

$$\big|\omega(a) - \frac{m}{n}\big| \geqslant \frac{\alpha}{n^{1+d}}, \quad \forall m \in \mathbb{Z}, n \in \mathbb{N}^*, \tag{31}$$

here $\omega(a)$ is seen as an element of $\mathbb{R}$.

Note also that when $d > 0$ is fixed, the Diophantine property (for some α) of Λ_a is independent of the choice of action-angle coordinates.

Definition 3.5. *For $\alpha > 0$ and $d > 0$, we define the set of "good values" $\mathcal{G}(\alpha, d)$ obtained from $\cup_{a\in J} Q_\infty(a)$ by removing the following set of "bad values" $\mathcal{B}(\alpha, d)$:*

$$\mathcal{B}(\alpha,d) = \Big(\bigcup_{dist(a,S)<\alpha} Q_\infty(a)\Big) \bigcup \Big(\bigcup_{a\in J\backslash S:|\omega'(a)|<\alpha} Q_\infty(a)\Big) \bigcup \Big(\bigcup_{a\in J\backslash S:|d\langle q\rangle_{\Lambda_a}|<\alpha} Q_\infty(a)\Big)$$
$$\bigcup \Big(\bigcup_{a\in J\backslash S:\omega(a) \text{ is not } (\alpha,d)-\text{Diophantine}} Q_\infty(a)\Big).$$

Remark 3.6

- The measure of the set of bad values $\mathcal{B}(\alpha, d)$ in $\cup_{a\in J} Q_\infty(a)$ is small ($\mathcal{O}(\alpha)$) when $\alpha > 0$ is small and $d > 0$ is fixed, provided that the measure of
$$\Big(\bigcup_{a\in J\backslash S:\omega(a)|\in\mathbb{Q}} Q_\infty(a)\Big) \bigcup \Big(\bigcup_{a\in S} Q_\infty(a)\Big) \tag{32}$$
is sufficiently small, depending on α (see [27]).
- If $F_0 \in \mathcal{G}(\alpha, d)$ is a good value , then by definition of $\mathcal{B}(\alpha, d)$ and remark (3.3), the pre-image $\langle q\rangle^{-1}(F_0)$ is a finite set
$$\langle q\rangle^{-1}(F_0) = \{a_1, \ldots, a_L\} \subset J \setminus S.$$
The corresponding tori $\Lambda_{a_1}, \ldots, \Lambda_{a_L}$ are tori (α, d)-Diophantine of $p^{-1}(0) \cap T^*M$. By this way, when F_0 varies in $\mathcal{G}(\alpha, d)$, we obtain a Cantor family of invariant tori (α, d)-Diophantine in the energy space $p^{-1}(0) \cap T^*M$.

△

Definition 3.7 (([43], page 16, 42), [2], [11])**.** *Let E is a symplectic space and his Lagrangian Grassmannian $\Lambda(E)$ (which is set of all Lagrangian subspaces of E). We consider a bundle B in E over the circle or a compact interval provided with a Lagrangian subbundle called vertical. Let $\lambda(t)$ a section of $\Lambda(B)$ which is transverse to the vertical edges of the interval in the case where the base is an interval.*

The Maslov index of $\lambda(t)$ is the intersection number of this curve with the singular cycle of Lagrangians which do not cut transversely the vertical subbundle.

Theorem 3.8 ([27])**.** *Suppose that P_ε is an operator with principal symbol (21) and satisfying the above conditions.*

Let $F_0 \in \mathcal{G}(\alpha, d)$ a good value. As in the remark (3.6), we write $\langle q\rangle^{-1}(F_0) = \{a_1, \ldots, a_L\} \subset J \setminus S$ and the corresponding tori $\Lambda_{a_1}, \ldots, \Lambda_{a_L}$ in $p^{-1}(0) \cap T^*M$.

For each $j = 1, \ldots, L$, note $S_j \in \mathbb{R}^2$ the action and $k_j \in \mathbb{Z}^2$ the Maslov index of the fundamental cycles $(\gamma_{1,j}, \gamma_{2,j})$ of Λ_{a_j} which are defined by

$$\kappa_j(\gamma_{l,j}) = \{x \in \mathbb{T}^2 : x_l = 0\}, l = 1, 2,$$

where κ_j is a action-angle coordinates in neighborhood of torus Λ_{a_j},

$$\kappa_j : (\Lambda_{a_j}, T^*M) \to (\xi = 0, T^*\mathbb{T}^2) \tag{33}$$

We assume that $h \ll \varepsilon = \mathcal{O}(h^\delta)$ for $0 < \delta < 1$.

Then the eigenvalues of P_ε with multiplicity in a rectangle of form

$$R(\varepsilon, h) = \Big[-\frac{h^\delta}{\mathcal{O}(1)}, \frac{h^\delta}{\mathcal{O}(1)}\Big] + i\varepsilon\Big[F_0 - \frac{h^\delta}{\mathcal{O}(1)}, F_0 + \frac{h^\delta}{\mathcal{O}(1)}\Big] \tag{34}$$

are given by

$$P_j^{(\infty)}\Big(h(k - \frac{k_j}{4}) - \frac{S_j}{2\pi}, \varepsilon; h\Big) + \mathcal{O}(h^\infty), k \in \mathbb{Z}^2, 1 \leqslant j \leqslant L. \tag{35}$$

Here $P_j^{(\infty)}(\xi, \varepsilon; h)$ is a smooth function of ξ in a neighborhood of $(0, \mathbb{R}^2)$ and ε in a neighborhood of $(0, \mathbb{R})$, real valued for $\varepsilon = 0$ and admits an asymptotic expansion in the space of symbols.

$$1 \leqslant j \leqslant L, \quad P_j^{(\infty)}(\xi, \varepsilon; h) \sim \sum_{k=0}^{\infty} h^k p_{j,k}^{(\infty)}(\xi, \epsilon) \tag{36}$$

whose principal symbol is

$$p_{j,0}^{(\infty)}(\xi, \varepsilon) = p_j(\xi) + i\varepsilon\langle q_j\rangle(\xi) + \mathcal{O}(\varepsilon^2). \tag{37}$$

Here p_j, q_j are the expressions of p, q in action-angle variables near of Λ_j, given by (33) and $\langle q_j\rangle$is the average of q_j on tori, defined in (30).

Remark 3.9

In the case of the above theorem that for every $j = 1, \ldots, L$, the eigenvalues form a deformed spectrum lattice in the rectangle (34) of size $(h^\delta \times \varepsilon h^\delta)$. Therefore the spectrum of P_ε in the rectangle therefore is the union of L such lattices.

Note that this is not valid for every rectangle. However, it is valid for a "good rectangle" whose center $(0, \varepsilon F_0)$ with F_0 is a good value. However, as we said in the remark 3.6, with the condition (32), the complement of the set of good values is a small measure (see [36]), then there are many such good rectangles in the band (28). This signifies that one can give asymptotically "almost all" eigenvalues of P_ε in this band.

△

Remark 3.10 In the case where p is nearly integrable, the result of the theorem is still true thanks to the existence invariant KAM tori which allows us to realize microlocally the construction of the quantum normal form of P_ε (see section 7.3 in [27]).

For the KAM theory (Kolmogorov-Arnold-Moser), one may consult the references [36], [9], [6], [7], [16].

△

Main idea of the proof of theorem(3.8). For a detailed proof of the theorem, one can consult [27], [22]. We will give here some important ideas of the proof of theorem.

The principle is the formal construction of the Birkhoff quantum normal form for P_ε, microlocally near a fixed Diophantine torus in $p^{-1}(0) \cap T^*M$, say $\Lambda_1 \in \{\Lambda_{a_1}, \ldots, \Lambda_{a_L}\}$. The Diophantine condition is necessary for this construction. For this method, see also [1], [35],[4], [19].

In this procedure we first use (formally) a canonical (symplectic) transformation for the total symbol of P_ε in order to reduce it to the normal form (36),(37) modulo $\mathcal{O}(h^\infty)$ which is independent of x and homogeneous in (h, ξ, ε) in all orders. Then, the operator P_ε is conjugated by a Fourier integral operator with complex phase to a new operator with such a total symbol.

Indeed, by introducing action-angle coordinates near Λ_1, P_ε is microlocally defined around the section $\xi = 0$ in $T^*\mathbb{T}^2$ and its principal symbol (21) has the form:

$$p_\varepsilon(x,\xi) = p(\xi) + i\varepsilon q(x,\xi) + \mathcal{O}(\varepsilon^2) \tag{38}$$

with $p(\xi) = \widetilde{\omega}\cdot\xi + \mathcal{O}(\xi^2)$, where $\widetilde{\omega} = (p'_{\xi_1}(0), p'_{\xi_2}(0))$ and the frequency $\omega(a) = [p'_{\xi_1}(0) : p'_{\xi_2}(0)]$, defined in (26) satisfies the condition (31).

Then, by the Birkhoff normal form procedure, for any arbitrary fixed N large enough, we can construct a holomorphic canonical transformation $\kappa_\varepsilon^{(N)}$ defined in a complex neighborhood of $\xi = 0$ in $T^*\mathbb{T}^2$ such that the total symbol P of P_ε is reduced to a new symbol:

$$P \circ \kappa_\varepsilon^{(N)}(x,\xi,\varepsilon;h) = p_0 + hp_1 + h^2 p_2 + \cdots, \tag{39}$$

where every $p_j = p_j(x,\xi,\varepsilon), j \geqslant 1$ holomorphic near $\xi = 0$ in $T^*\mathbb{T}^2$, depending smoothly in $\varepsilon \in (0,\mathbb{R})$, independent of x to order N and it is important that the principal symbol p_0 satisfies

$$p_0 = p_\varepsilon \circ \kappa_\varepsilon^{(N)}(x,\xi) = p^{(N)}(\xi,\varepsilon) + r_{N+1}(x,\xi,\varepsilon), \tag{40}$$

where $p^{(N)}(\xi,\varepsilon) = p(\xi) + i\varepsilon\langle q\rangle(\xi) + \mathcal{O}(\varepsilon^2)$, $\langle q\rangle(\xi)$ given by (30), $r_{N+1}(x,\xi,\varepsilon) = \mathcal{O}((\xi,\varepsilon)^{N+1})$.

Thus, p_0 has the same form as (37).

On operator level, P_ε is conjugated to a new operator of the form

$$P^{(N)}(hD_x,\varepsilon;h) + R_{N+1}(x,hD_x,\varepsilon;h), \tag{41}$$

where $P^{(N)}(hD_x,\varepsilon;h)$ has a total symbol independent of x whose principal symbol is $p^{(N)}$ and $R_{N+1}(x,\xi,\varepsilon;h) = \mathcal{O}((h,\xi,\varepsilon)^{N+1})$.

The operator (41) acts on the space $L^2_\theta(\mathbb{T}^2)$ of Floquet periodic functions microlocally defined over $\mathbb{T}^2$ whose an element u satisfies

$$u(x-\nu) = e^{i\theta\cdot\nu}u(x), \quad \theta = \frac{S_1}{2\pi h} + \frac{k_1}{4}, \quad \nu \in 2\pi\mathbb{Z}^2.$$

An orthonormal basis of this space is

$$\Big\{x \in \mathbb{T}^2, e_k(x) = e^{ix(k-\theta)} = e^{\frac{i}{h}x.\left(h(k-\frac{k_1}{4})-\frac{S_1}{2\pi}\right)}, k \in \mathbb{Z}^2\Big\}.$$

Consequently, the eigenvalues of P_ε modulo $\mathcal{O}(h^\infty)$ are given by (35). □

Remark 3.11 For all $j = 1,\ldots,L$, from (37), at $\xi = 0$ we have $p_{j,0}^{(\infty)}(0,\varepsilon) = i\varepsilon F_0 + \mathcal{O}(\varepsilon^2)$ and therefore $P_j^{(\infty)}(0,\varepsilon;h) = i\varepsilon F_0 + \mathcal{O}(\varepsilon^2) + \mathcal{O}(h)$. Consequently, $p_{j,0}^{(\infty)}(0,\varepsilon) \sim i\varepsilon F_0$ when $\varepsilon \to 0$ and $P_j^{(\infty)}(0,\varepsilon;h) \sim i\varepsilon F_0$ when $\varepsilon, h \to 0, h \ll \varepsilon$.

Moreover, we have also $d(p_j)_{|\xi=0} =: a_j = (a_{1,j}, a_{2,j}) \in \mathbb{R}^2$ and $d(\langle q_j\rangle)_{|\xi=0} =: b_j = (b_{1,j}, b_{2,j}) \in \mathbb{R}^2$ are $\mathbb{R}$-linearly independent. We can rewrite the principal symbol (37) in the form

$$p_{j,0}^{(\infty)}(\xi,\epsilon) = i\varepsilon F_0 + (a_j + i\varepsilon b_j)\cdot\xi + \mathcal{O}(\xi^2) + \mathcal{O}(\varepsilon^2). \tag{42}$$

△

Proposition 3.12. *Let $\lambda = P(\xi;\varepsilon,h)$ a complex-valued smooth function of ξ near $0 \in \mathbb{R}^2$ and of small parameters h,ε near $0 \in \mathbb{R}$. Suppose that we can write P in the form*

$$P(\xi;\varepsilon,h) = P_0 + \mathcal{O}(h)$$

with

$$P_0 = P_0(\xi;\varepsilon) = g_1(\xi) + i\varepsilon g_2(\xi) + \mathcal{O}(\varepsilon^2)$$

such that $dg_1(0) \wedge dg_2(0) \neq 0$.

If we assume that $h \ll \varepsilon$, then for h et ε small enough, there are $\rho, r > 0$ small enough such that P is a local diffeomorphism near $\xi = 0$ from $B(0,\rho)$ to its image, denoted $B(\varepsilon)$.

Proof. First, seeing P as a function of $\mathbb{R}^2$, we set $\widehat{P} := \chi^{-1} \circ P$. Then we can write

$$\widehat{P} = g_1(\xi) + ig_2(\xi) + \mathcal{O}(\varepsilon) + \mathcal{O}(\frac{h}{\varepsilon}).$$

Let $a = (a_1, a_2) = dg_1(0), b = (b_1, b_2) = dg_2(0)$ and $M =\mid a_1b_2 - a_2b_1 \mid > 0$.

The differential of $\widehat{P}$ in $\xi = 0$ is

$$\frac{\partial \widehat{P}}{\partial \xi}(0) = a + i\varepsilon b + \mathcal{O}(\varepsilon) + \mathcal{O}(\frac{h}{\varepsilon}) = \begin{pmatrix} a_1 + \mathcal{O}(\varepsilon) + \mathcal{O}(\frac{h}{\varepsilon}) & a_2 + \mathcal{O}(\varepsilon) + \mathcal{O}(\frac{h}{\varepsilon}) \\ b_1 + \mathcal{O}(\varepsilon) + \mathcal{O}(\frac{h}{\varepsilon}) & b_2 + \mathcal{O}(\varepsilon) + \mathcal{O}(\frac{h}{\varepsilon}) \end{pmatrix}$$

and thus

$$\mid det(\frac{\partial \widehat{P}}{\partial \xi}(0)) \mid = M + \mathcal{O}(\varepsilon) + \mathcal{O}(\frac{h}{\varepsilon}).$$

Then for h, ε small enough and $h \ll \varepsilon$, it's clear that $\mid det(\frac{\partial \widehat{P}}{\partial \xi}(0)) \mid \simeq M$ is nonzero. Therefore, the local inverse function theorem ensures that $\widehat{P}$ is local diffeomorphism in $\xi = 0$. Hence we get the desired result for P. □

Let us return to the spectral problem of P_ε discussed in the theorem (3.8). For each $j = 1, \ldots, L$, as an application of the previous lemma with $P = P_j^{(\infty)}$, then $P_j^{(\infty)}$ is a smooth local diffeomorphism in $\xi = 0 \in \mathbb{R}^2$ from a neighborhood of 0 to its image, noted by $B_j(\varepsilon)$. Note that for h small enough, the good rectangle $R(\varepsilon, h)$ is always included in $B_j(\varepsilon)$.

We denote $\Sigma_j(\varepsilon, h) \subset R(\varepsilon, h)$ the quasi-eigenvalues of P_ε in $R(\varepsilon, h)$, given by the image by $P_j^{(\infty)}$ of $\xi = h(k - \frac{k_j}{4}) - \frac{S_j}{2\pi}, k \in \mathbb{Z}^2$.

Writing $hk = \xi + h\frac{k_j}{4} + \frac{S_j}{2\pi}$ and letting

$$f_j := (P_j^{(\infty)})^{-1} + h\frac{k_j}{4} + \frac{S_j}{2\pi}, \tag{43}$$

then $f_j = f_j(\lambda, \varepsilon; h)$ is a local diffeomorphism from $B_j(\varepsilon)$ to its image. Denote $E_j(\varepsilon, h) = f_j(R(\varepsilon, h))$ which is close to $\frac{S_j}{2\pi}$ and $\Gamma_j(\varepsilon, h) := f_j(\Sigma_j(\varepsilon, h))$, then we have $\Gamma_j(\varepsilon, h) = h\mathbb{Z}^2 \cap E_j(\varepsilon, h)$.

In summary, we have:

$$f_j : \qquad R(\varepsilon, h) \to E_j(\varepsilon, h) \tag{44}$$

$$f_j \mid_{\Sigma_j(\varepsilon,h)}: \quad \Sigma_j(\varepsilon, h) \to \Gamma_j(\varepsilon, h) \subset h\mathbb{Z}^2 \tag{45}$$

Remark 3.13 On the other hand, if we assume that $L = 1$, the theorem (3.8) asserts that in $R(\varepsilon, h)$, the quasi-eigenvalues are equal to the real eigenvalues of P_ε modulo $\mathcal{O}(h^\infty)$:

$$\sigma(P_\varepsilon) \cap R(\varepsilon, h) = \Sigma_1(\varepsilon, h) + \mathcal{O}(h^\infty), \tag{46}$$

in the sense that there is a bijection

$$\chi : \Sigma_1(\varepsilon, h) \to \sigma(P_\varepsilon) \cap R(\varepsilon, h) \tag{47}$$

such that $\chi = Id + \mathcal{O}(h^\infty)$. The diffeomorphism $f := f_1$ in (44) thus satisfies

$$\begin{array}{rcl} f : R(\varepsilon, h) & \to & E_1(\varepsilon, h) \\ \sigma(P_\varepsilon) \cap R(\varepsilon, h) \ni \lambda & \mapsto & f(\lambda, \varepsilon; h) \in h\mathbb{Z}^2 + \mathcal{O}(h^\infty). \end{array} \tag{48}$$

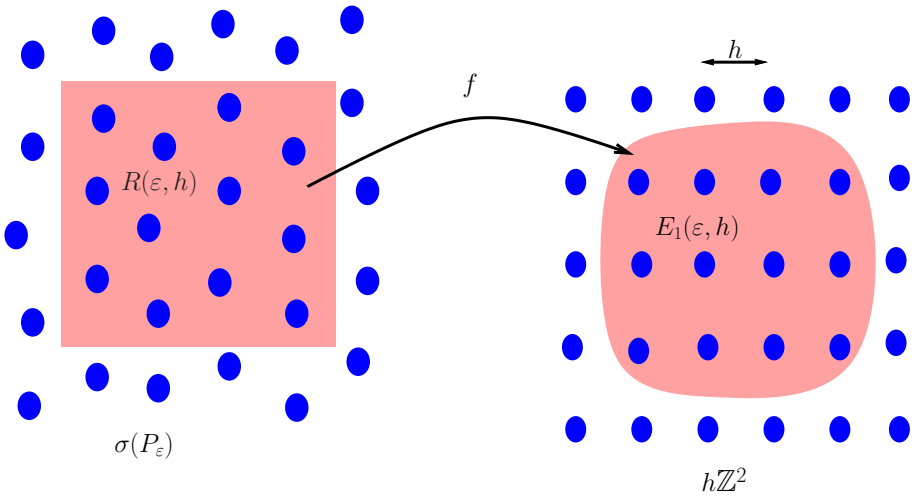

Figure 3. A micro-chart of spectrum of P_ε

In particular, we have a bijection between the sets

$$\sigma(P_\varepsilon) \cap R(\varepsilon, h) \simeq \Sigma_1(\varepsilon, h) \simeq \Gamma_1(\varepsilon, h) \subset h\mathbb{Z}^2 \tag{49}$$

△

3.2.3. *What is the size of $E(\varepsilon, h)$?* As we know, the surface $R(\varepsilon, h)$ is of size $\mathcal{O}(h^\delta)\cdot\mathcal{O}(\varepsilon h^\delta)$, Now we are interested the size of $E := E_j(\varepsilon, h)$ which is the image of $R(\varepsilon, h)$ by the diffeomorphism f_j (44). Let P be still one of $P_j^{(\infty)}$. By the proposition (3.12), in the regime $h \ll \varepsilon$, the differential of $\lambda = P(\xi, \varepsilon; h)$ in $\xi = 0$ is a determinant of size $\mathcal{O}(\varepsilon)$:

$$\mid det(\frac{\partial P}{\partial \xi}(0, \varepsilon; h)) \mid = M\varepsilon,$$

with $M > 0$. By writing the Taylor expansion of $\xi = P^{-1} = g(\lambda, \varepsilon; h)$ in $\lambda_0 = P(0, \varepsilon; h)$, we have:

$$|\xi| \leqslant \frac{1}{M\varepsilon}|\lambda - \lambda_0| + \mathcal{O}(|\lambda - \lambda_0|^2).$$

Hence, if $\lambda \in R(\varepsilon, h)$, then $|\lambda - \lambda_0| \leqslant \mathcal{O}(h^\delta)$ and so

$$|\xi| \leqslant \frac{1}{M\varepsilon}\mathcal{O}(h^\delta) + (\mathcal{O}(h^\delta))^2 \leqslant \mathcal{O}(\frac{h^\delta}{\varepsilon}).$$

Consequently, from the formula (43), we have that $E := E_j(\varepsilon, h)$ is contained in a ball of radius $\mathcal{O}(\frac{h^\delta}{\varepsilon})$.

3.2.4. *How is the lattice of quasi-eigenvalues and the lattice of spectrum?* For each $j = 1, \ldots, L$, from the remark 3.11 we can express $P_j^{(\infty)}$ in the form

$$\lambda = P_j^{(\infty)}(\xi, \varepsilon; h) = i\varepsilon F_0 + (a_j + i\varepsilon b_j)\cdot\xi + \mathcal{O}(\xi^2) + \mathcal{O}(\varepsilon^2) + \mathcal{O}(h)$$

and thus

$$\lambda_1 = Re(\lambda) = a_j \cdot \xi + \mathcal{O}(\xi^2) + \mathcal{O}(\varepsilon^2) + \mathcal{O}(h),$$

$$\lambda_2 = Im(\lambda) = i\varepsilon F_0 + \varepsilon b_j \cdot \xi + \mathcal{O}(\xi^2) + \mathcal{O}(\varepsilon^2) + \mathcal{O}(h).$$

Note that we work in the regime $h \ll \varepsilon$ and that

$$\frac{\partial \lambda_1}{\partial \xi}|_{\xi=0} = a_j + \mathcal{O}(\varepsilon^2) + \mathcal{O}(h) \sim a_j$$

and

$$\frac{\partial \lambda_2}{\partial \xi}|_{\xi=0} = \varepsilon b_j + \mathcal{O}(\varepsilon^2) + \mathcal{O}(h) \sim b_j \varepsilon.$$

Thus the variations of the image with respect to the reference value are

$$|\Delta\lambda_1| = |a_j||\Delta\xi| + \mathcal{O}(|\Delta\xi|^2)$$

and

$$|\Delta\lambda_2| = \varepsilon|b_j||\Delta\xi| + \mathcal{O}(|\Delta\xi|^2).$$

Hence, if $|\Delta\xi| \sim Ch$ ($C > 0$), then

$$|\Delta\lambda_1| \sim |a_j| \cdot Ch + \mathcal{O}(h^2) \sim C_1 h,$$

with $C_1 > 0$ and

$$|\Delta\lambda_2| \sim \varepsilon|b_j| \cdot Ch + \mathcal{O}(h^2) \sim C_2 \varepsilon h(1 + \mathcal{O}(\frac{h}{\varepsilon})) \sim C_2 \varepsilon h,$$

with $C_2 > 0$. Note that $\xi = h(k - \frac{k_j}{4}) - \frac{S_j}{2\pi}, k \in \mathbb{Z}^2$, then $|\Delta\xi| = h|\Delta k|$ and we can assert that the spectrum of P_ε in a good rectangle $R(\varepsilon, h)$ of the form (34) is the union of L deformed lattices, with a horizontal spacing h and vertical spacing εh.

Of course, the lattices $\Sigma_j(\varepsilon, h)$ are all described the same way.

As a corollary, we have:

Remark 3.14 The cardinal of such a spectral network in $R(\varepsilon, h)$ is $\mathcal{O}(\frac{h^\delta . \varepsilon h^\delta}{h.\varepsilon h}) = \mathcal{O}(h^{2(\delta-1)})$ which converges to ∞ when $h \to 0$. This means that the asymptotic expansion is applied to many eigenvalues of P_ε.

Moreover, a recent work of M.Hitrik-J.Sjöstrand allows us to calculate the cardinal of eigenvalues of P_ε in the rectangle $R(\varepsilon, h)$. △

3.3. **Birkhoff normal form.**

3.3.1. *Motivation.* In this section, we will discuss the procedure of Birkhoff normal form of a perturbed pseudo-differential operator P_ε which depends on small positive parameters h, ε around a Diophantine torus Λ and treat it explicitly in a particular case (but important for our work).

For the Birkhoff normal form, we can consult [49], [13], [1], [35].

We assume that Λ is equal to the section $\{\xi = 0\}$ in $T^*\mathbb{T}^n$ and that P_ε is microlocally defined near $\{\xi = 0\} \in T^*\mathbb{T}^n$, with h-Weyl (total) symbol $P = P(x, \xi, \varepsilon, h)$ which is holomorphic in (x, ξ) near a complex neighborhood of $\xi = 0 \in T^*\mathbb{T}^n$ and C^∞ in (h, ε) near 0.

In the article [27] (section 3) one realized the Birkhoff normal form of P_ε whose h-principal symbol is of the form (38)

$$p_\varepsilon(x, \xi) = p(\xi) + i\varepsilon q(x, \xi) + \mathcal{O}(\varepsilon^2)$$

and the principal symbol of the obtained normal form is of the form (like 37))

$$P_0^{(\infty)} = P_0^{(\infty)}(\xi, \varepsilon) = p(\xi) + i\varepsilon\langle q\rangle(\xi) + \mathcal{O}(\varepsilon^2).$$

Our work requires treat a more specific case when the principal symbol of P_ε already does not depend in x and with no term $\mathcal{O}(\varepsilon^2)$:

$$p_\varepsilon = P_0(\xi, \varepsilon) = p(\xi) + i\varepsilon q(\xi).$$

In the above formula for $P_0^{(\infty)}$, can we remove the term $\mathcal{O}(\varepsilon^2)$ and is $P_0^{(\infty)}$ equal to P_0 ? This is an important result that we want. However, this is not obvious: the proof in [27] is not enough to explain it because one used transformations depending also on ε.

We will prove this result here, by providing a normalization of the total symbol $P(x, \xi, \varepsilon, h)$ in all three variables (ξ, ε, h). The approach that we propose is different from [27].

3.3.2. *Formal series and operators.* Let $\mathcal{E} = Hol(\mathbb{T}^n_x)[[\xi, \varepsilon, h]]$ denote the space of formal series in (ξ, ε, h) with holomorphic coefficients in $x \in \mathbb{T}^n$,

$$\mathcal{E} = \big\{ a^{(\mathcal{E})} = \sum_{k,m,l=0}^{\infty} a_{k,m,l}(x)\xi^k \varepsilon^m h^l \quad \text{such that } a_{k,m,l}(x) \text{are analytic in } x \ \big\}.$$

There is a correspondence between an h-Weyl pseudo-differential operator and an element of $\mathcal{E}$: if we denote A h-Weyl pseudo-differential operator, $a := \sigma_w(A)$ its Weyl total symbol and $\sigma^{(\mathcal{E})}(A) \in \mathcal{E}$ the formal Taylor series of a in (ξ, ε, h), then this correspondence is given by a map, denoted by $\sigma^{(\mathcal{E})}$ from the algebra of (Weyl) pseudo-differential operators Ψ to the space of formal series $\mathcal{E}$:

$$\begin{aligned} \sigma^{(\mathcal{E})} : \Psi &\to \mathcal{E} \\ A &\mapsto \sigma^{(\mathcal{E})}(A), \end{aligned} \tag{50}$$

$$\sigma^{(\mathcal{E})}(A) = \sum_{|\alpha|,j,l=0}^{\infty} \frac{1}{(|\alpha| + j + l)!} \big(\partial_\xi^\alpha \partial_\varepsilon^j \partial_h^l a(x, \xi, \varepsilon, h)\big|_{\xi=\varepsilon=h=0}\big) \xi^\alpha \varepsilon^j h^l.$$

The Moyal formula (see [33], [21], [50]) for the composition of two operators from the Weyl semi-classical calculation say that if $a := \sigma_w(A), b := \sigma_w(B)$, then $A \circ B$ is still a h-pseudo-differential whose Weyl symbol satisfies

$$\begin{aligned} \sigma_w(A \circ B) &= (a \sharp^w b)(x, \xi, \varepsilon, h) \\ &= e^{ih[D_\eta D_x - D_y D_\xi]} a(x, \xi, \varepsilon, h) b(x, \xi, \varepsilon, h)\big|_{y=x, \eta=\xi} \\ &\sim \sum_{\alpha,\beta} \frac{h^{|\alpha|+|\beta|}(-1)^{|\alpha|}}{(2i)^{|\alpha|+|\beta|}\alpha!\beta!} (\partial_x^\alpha \partial_\xi^\beta a(x, \xi, \varepsilon, h))(\partial_\xi^\alpha \partial_x^\beta b(x, \xi, \varepsilon, h)) \\ &= a(x, \xi, \varepsilon, h) b(x, \xi, \varepsilon, h) + \frac{h}{2i}\{a(x, \xi, \varepsilon, h), b(x, \xi, \varepsilon, h)\} + \cdots \end{aligned} \tag{51}$$

On the other hand, by the Borel theorem, any formal series $a_{\mathcal{E}} \in \mathcal{E}$ can be seen as Taylor series of a smooth function $a = a(x, \xi, \varepsilon, h)$ (which is not unique) and we can associate to it a h-pseudo-differential operator by asking $A = \mathrm{Op}_h^w(a)$. We have then $a^{(\mathcal{E})} = \sigma^{(\mathcal{E})}(A)$. In this way and from Moyal formula (51), we can define a product on $\mathcal{E}$, denoted by $\star$: let $a^{(\mathcal{E})}, b^{(\mathcal{E})} \in \mathcal{E}$, then

$$a^{(\mathcal{E})} \star b^{(\mathcal{E})} = \sigma^{(\mathcal{E})}(A \circ B)$$

is the Taylor series of $a \sharp^w b$. Thus $\mathcal{E}$ becomes an algebra with this product. The associative bracket thus is well defined and is called by the Moyal bracket.

$$[a^{(\mathcal{E})}, b^{(\mathcal{E})}] = a^{(\mathcal{E})} \star b^{(\mathcal{E})} - b^{(\mathcal{E})} \star a^{(\mathcal{E})}.$$

Consequently, let $a^{(\mathcal{E})} \in \mathcal{E}$, we can define on $\mathcal{E}$ the adjoint operator:

$$\mathrm{ad}_{a^{(\mathcal{E})}} := [a^{(\mathcal{E})}, \cdot].$$

For any $p^{(\mathcal{E})} \in \mathcal{E}$, the formal series

$$e^{\mathrm{ad}_{a^{(\mathcal{E})}}} p^{(\mathcal{E})} = \sum_{k=0}^{\infty} \frac{1}{k!} (\mathrm{ad}_{(\mathcal{E})})^k p^{(\mathcal{E})}$$

is well defined in $\mathcal{E}$ because it contain only a finite number of terms of fixed degree. Thus the exponential operator $\exp(\mathrm{ad}_{a^{(\mathcal{E})}})$ is well defined on $\mathcal{E}$.

Moreover, for two h-pseudo-differential operators A et P, we have

$$\sigma^{(\mathcal{E})}([A, P]) = [\sigma^{(\mathcal{E})}(A), \sigma^{(\mathcal{E})}(P)]$$

and $\sigma^{(\mathcal{E})}$ becomes an (associative) algebra morphism.

Similarly, we have also

$$\exp(\mathrm{ad}_{\sigma^{(\varepsilon)}(A)})(\sigma^{(\mathcal{E})}(P)) = \sigma^{(\mathcal{E})}(\exp(\mathrm{ad}_A)P),$$

where the exponential operator $\exp(\mathrm{ad}_A)$ is defined below.

3.3.3. *Action by conjugation.* Let A be a bounded operator on a Hilbert space H. We define in $\mathcal{B}(H)$ the exponential operator $e^A \in \mathcal{B}(H)$ by the absolutely convergent series

$$e^A = \sum_{k\geqslant 0} \frac{1}{k!} A^k. \tag{52}$$

Next, we associate with A a bounded operator $\mathrm{ad}_A := [A, \cdot] \in \mathcal{B}(\mathcal{B}(H))$ and by (52) the operator $\exp(\mathrm{ad}_A)$ (or e^{ad_A}) is well defined in $\mathcal{B}(\mathcal{B}(H))$. In addition, we have the following result:

Lemma 3.15. *Let A et P be two bounded operators of $\mathcal{B}(H)$, we have*

$$e^A P e^{-A} = exp(ad_A)P. \tag{53}$$

Proof. For $t \in \mathbb{R}$, let $f(t) = e^{tA}Pe^{-tA}$ and $g(t) = \exp(\mathrm{ad}_{tA})P$ which are analytic functions from $\mathbb{R}$ to $\mathcal{B}(H)$. We will calculate its derivatives. First note that $\frac{d}{dt}(e^{tA}) = Ae^{tA} = e^{tA}A$ and so

$$\frac{d}{dt}(e^{tA})P = Ae^{tA}P = e^{tA}AP,$$

we have

$$f'(t) = Ae^{tA}Pe^{-tA} - e^{tA}Pe^{-tA}A = Af(t) - f(t)A = [A, f(t)]$$

and

$$g'(t) = \frac{d}{dt}(e^{t\cdot\mathrm{ad}_A}P) = \mathrm{ad}_A \circ e^{t\cdot\mathrm{ad}_A}P = \mathrm{ad}_A g(t) = [A, g(t)].$$

Then $f(t), g(t)$ satisfy the same linear differential equation of first order

$$u' = [A, u].$$

But the initial value in $t = 0$, $f(0) = g(0) = P$ and consequently we have $f(t) = g(t)$ for all $t \in \mathbb{R}$. Especially for $t = 1$, we well have $e^A P e^{-A} = \exp(\mathrm{ad}_A)P$. □

As an application, we have:

3.3.4. *Idea of the construction of Birkhoff normal form.* The main idea of this construction is to find a pseudo-differential operator A such that the associated Fourier integral operator $U(h) := e^{\frac{i}{h}A}$ reduce the initial operator P_ε to its conjugate operator

$$e^{\frac{i}{h}A}P_\varepsilon e^{-\frac{i}{h}A} = e^{\frac{i}{h}\mathrm{ad}_A}(P_\varepsilon) := \widetilde{P_\varepsilon} \tag{54}$$

the formal series of whose total symbol $\sigma^{(\mathcal{E})}(\widetilde{P_\varepsilon})$ in $\mathcal{E}$ does not depend on x. Noticing that $\sigma^{(\mathcal{E})}(\widetilde{P_\varepsilon}) = \exp(\mathrm{ad}_{\sigma^{(\varepsilon)}(A)})(\sigma^{(\mathcal{E})}(P_\varepsilon))$ as in the previous section, the work is concentrated to seek $\sigma_{\mathcal{E}}$ as a series of homogenous terms in (ξ, ε, h). In search of this series, the Diophantine condition is essential, see (65).

Remark 3.16

(1) The above operator $\widetilde{P_\varepsilon}$ is still a pseudo-differential by the Egorov theorem (see for example [20],[48], [29]). In the special case when $\widehat{A} = h\widehat{B}$, then the operator $U(h) := e^{\frac{i}{h}\widehat{A}} = e^{i\widehat{B}}$ is really a pseudo-differential operator and $\widetilde{P_\varepsilon}$ is simply the composition of pseudo-differential operators

(2) The conjugation at the operator level is replaced by the adjoint action on the space of formal series.

△

3.3.5. *Construction of the Birkhoff normal form.* In this part, by convention, for a h-pseudo-differential operator P, we identify it with its total (Weyl) symbol $P = P(x, \xi, \varepsilon, h)$ and its formal series $\sigma_{\mathcal{E}}(P)$.

We use a particular order for (ξ, ε, h) by counting the power in ξ plus twice the power in ε and h. The associated filtration is denoted by the symbol $\mathcal{O}(j)$.

Let us denote also $\mathcal{D}(j)$ the subspace of $\mathcal{E}$ of homogenous polynomials of degree j with respect to (ξ, ε, h) in this order. Thus, we have:

$$\mathcal{D}(j) = Vect\{\xi^k \varepsilon^m h^l | k + 2(m + l) = j\} \otimes Hol(\mathbb{T}^n_x)$$

and

$$\mathcal{O}(j) = \bigoplus_{n \geqslant j} \mathcal{D}(j).$$

We have of course $\mathcal{D}(j) \subset \mathcal{O}(j)$ and $\mathcal{O}(j + 1) \subset \mathcal{O}(j)$. As usual, we allow the notation $A = B + \mathcal{O}(j)$ to say that $A - B \in \mathcal{O}(j)$.

If $K_j = K_j(x, \xi, \varepsilon, h) \in \mathcal{O}(j)$, then it is obvious that the Poisson bracket satisfies $\{K_j, K_l\} = \mathcal{O}(j + l - 1)$.

For the Moyal bracket $i[K_j, K_l]$, from (51), it can be computed as a series in $(\frac{h}{i}\frac{\partial}{\partial \xi}, \frac{\partial}{\partial x})$ and is well of order $j + l + 1$ because every time we lose a degree in ξ we win also a degree in h. Moreover, we have

$$\begin{aligned} i[K_j, K_l] &= h\{K_j, K_l\} + h\mathcal{O}(j + l) && (55)\\ &= h\mathcal{O}(j + l - 1) = \mathcal{O}(j + l + 1). && (56)\end{aligned}$$

Consequently, we have also

$$\begin{aligned} [\mathcal{O}(j_n), [\dots, [\mathcal{O}(j_2), \mathcal{O}(j_1)] \dots] &= h^{n-1}\mathcal{O}\big(j_1 + \cdots + j_n - (n - 1)\big) \\ &= \mathcal{O}(j_1 + \cdots + j_n + n - 1) && (57)\\ \text{and} \quad [\dots [\mathcal{O}(j_1), \mathcal{O}(j_2)], \dots], \mathcal{O}(j_n)] &= h^{n-1}\mathcal{O}\big(j_1 + \cdots + j_n - (n - 1)\big) \\ &= \mathcal{O}(j_1 + \cdots + j_n + n - 1). && (58)\end{aligned}$$

Theorem 3.17. *Suppose that $P = P(x, \xi, \varepsilon, h)$ is an analytic Weyl h-pseudo-differential operator on $\mathbb{T}^n$ (microlocally defined close to $\xi = 0$) with principal h-symbol $P_0 = p(\xi) + i\varepsilon q(\xi)$ such that $p(\xi) = \langle a, \xi\rangle + \mathcal{O}(\xi^2)$ and a is Diophantine as in the definition 3.4. Then for any integer $N \geqslant 1$, there exists a function $G^{(N)} = \sum_{j=2}^{N} G_j$ ($G^{(1)} = 0$) where $G_j = G_j(x, \xi, \varepsilon, h) \in \mathcal{D}(j - 2)$ (for $j \geqslant 2$) is analytic in x, homogenous in (ξ, ε, h) such that*

$$exp\big(iad_{G^{(N)}}\big)P = P_0 + hP_1^{(N)} + hR_{N-1}, \tag{59}$$

where $P_1^{(N)} = P_1^{(N)}(\xi, \varepsilon, h) \in \mathcal{E}$ is independent of x and $R_{N-1} = \mathcal{O}(N - 1)$.

Proof. We can write P in the form

$$P = P(x, \xi, \varepsilon, h) = P_0 + hP_1,$$

with $P_1 = P_1(x, \xi, \varepsilon, h)$ holomorphic in (x, ξ) and analytic in (h, ε) close to 0. We will show the property (59) by induction on N.

For $N = 1$, we take $G^{(1)} = P_1^{(1)} = 0$, $R_0 = P_1(x, \xi, \varepsilon, h)$ and the property (59) is valid.

Assume that it is valid for N with a found function $G^{(N)}$. We now seek a function $G_{N+1} \in \mathcal{D}(N)$ such that $G^{(N+1)} = G^{(N)} + G_{N+1}$ satisfies the equation (59). By developing the exponential and using (55) with the attention that $P = \mathcal{O}(1)$ and $G^{(N)} \in \mathcal{O}(0)$, we can write

$$\exp\big(i\mathrm{ad}_{G^{(N+1)}}\big)P = \exp\big(i\mathrm{ad}_{G^{(N)}}\big)P + i\mathrm{ad}_{G_{N+1}}P + h\mathcal{O}(N). \tag{60}$$

Indeed, denoting $A_j := i\mathrm{ad}_{G_j}$ and $A^{(N)} = \sum_{j=2}^{N} A_j$, we have the expansion

$$\begin{aligned}
\exp\big(i\mathrm{ad}_{G^{(N+1)}}\big)P &= \exp\big(A^{(N)} + A_{N+1}\big)P \\
&= \sum_{k\geqslant 0}\frac{1}{k!}\big(A^{(N)} + A_{N+1}\big)^k P \\
&= \sum_{k\geqslant 0}\frac{1}{k!}\sum_{l=0}^{k}(A^{(N)})^{k-l} * (A_{N+1})^l P \\
&= \sum_{k\geqslant 0}\frac{1}{k!}(A^{(N)})^k P + \sum_{k\geqslant 1}\underbrace{\frac{1}{k!}\sum_{l=1}^{k}(A^{(N)})^{k-l} * (A_{N+1})^l P}_{B_k} \\
&= \exp\big(A^{(N)}\big)P + \sum_{k\geqslant 1} B_k. \qquad (61)
\end{aligned}$$

In the above formulas, we used the symbol " $*$ " which means that $A^m * B^n$ is the sum of all compositions containing m times the operator A and n times the operator B.

Particularly, for $k = 1$ we have

$$B_1 = A_{N+1}P = i\mathrm{ad}_{G_{N+1}}P = i\mathrm{ad}_{G_{(N+1)}}P.$$

For all $k \geqslant 2$, by using the formula (57) with remarks $P = \mathcal{O}(1), G^{(N)} = \mathcal{O}(0), G_{(N+1)} = \mathcal{O}(N-1)$, we obtain that

$$B_k = \mathcal{O}(N+k) = h\mathcal{O}(N+k-2)$$

because for $l = 1, \ldots, k$ all the terms $A^{(N)})^{k-l} * (A_{N+1})^l P = \mathcal{O}\big((k-l)\times 0 + l(N-1) + 1 + k\big) = \mathcal{O}\big(l(N-1)+1+k\big) = \mathcal{O}(h)$ and the inequality $l(N-1)+1+k \geqslant N+k$ is always true. Hence the formula (60) is shown.

From (60), the induction hypothesis (59) and the formulas (55) by noting that one can write

$$P = \langle a, \xi\rangle + \mathcal{O}(\xi^2) + i\varepsilon q(\xi) + \mathcal{O}(h) = \langle a, \xi\rangle + \mathcal{O}(2),$$

we therefore have:

$$\begin{aligned}
\exp\big(i\mathrm{ad}_{G^{(N+1)}}\big)P &= P_0 + hP_1^{(N)} + hR_{N-1} + i\mathrm{ad}_{G_{N+1}}P + h\mathcal{O}(N) \\
&= P_0 + hP_1^{(N)} + hR_{N-1} + i[G_{N+1}, \langle a, \xi\rangle + \mathcal{O}(2)] \\
&= P_0 + hP_1^{(N)} + hR_{N-1} + i[G_{N+1}, \langle a, \xi\rangle] + i[G_{N+1}, \mathcal{O}(2)] \\
&= P_0 + hP_1^{(N)} + hR_{N-1} + h\{G_{N+1}, \langle a, \xi\rangle\} + h\mathcal{O}(N-1+1) \\
&\quad + h\mathcal{O}(N-1+2-1) \\
&= P_0 + h\{G_{N+1}, \langle a, \xi\rangle\} + hR_{N-1} + hP_1^{(N)} + h\mathcal{O}(N) \qquad (62)
\end{aligned}$$

Then, the equation for G_{N+1} becomes

$$P_0 + h\{G_{N+1}, \langle a, \xi\rangle\} + hR_{N-1} + hP_1^{(N)} + h\mathcal{O}(N) = P_0 + hP_1^{(N+1)} + h\mathcal{O}(N)$$

and it is equivalent to the following cohomological equation

$$\{G_{N+1}, \langle a, \xi\rangle\} + R_N = K_N + \mathcal{O}(N), \qquad (63)$$

where $K_N := P_1^{(N+1)} - P_1^{(N)}$ should not depend on x. Or, this equation is well solvable.
Indeed: in the above equation, as the rest is of order N, we can replace R_{N-1} by its homogenous part of order $N-1$, denoted by $\overline{R}_{N-1}$ and we will solve the equation:

$$\{G_{N+1}, \langle a, \xi\rangle\} + \overline{R}_{N-1} = K_N. \qquad (64)$$

Develop G_{N+1} and $\overline{R}_{N-1}$ in Fourier series of $x \in \mathbb{T}^n$

$$\begin{aligned} G_{N+1} &= \sum_{k\in\mathbb{Z}^n} \widehat{G}_{N+1}(k)e^{ik\cdot x} \\ \overline{R}_{N-1} &= \sum_{k\in\mathbb{Z}^n} \widehat{\overline{R}}_{N-1}(k)e^{ik\cdot x}, \end{aligned}$$

where $\widehat{G}_{N+1}(k), \widehat{\overline{R}}_{N-1}(k)$ are polynomials in $\mathbb{R}[\xi, \varepsilon, h]$.

Note that the crochet $\{G_{N+1}, \langle a, \xi\rangle\} = -(a\cdot\partial x)G_{N+1}$, we can write the equation (64) in the form

$$-\sum_{k\in\mathbb{Z}^n} i(a\cdot k)\widehat{G}_{N+1}(k)e^{ik\cdot x} + \sum_{k\in\mathbb{Z}^n} \widehat{\overline{R}}_{N-1}(k)e^{ik\cdot x} = K_N$$

or

$$\widehat{\overline{R}}_{N-1}(0) + \sum_{k\in\mathbb{Z}^n\setminus\{0\}} \big(\widehat{\overline{R}}_{N-1}(k) - i(a\cdot k)\widehat{G}_{N+1}(k)\big)e^{ik\cdot x} = K_N.$$

This equation is well resolved by posing

$$K_N = \widehat{\overline{R}}_{N-1}(0)$$

(this is also equal to x-average $\langle \overline{R}_{N-1}\rangle$) and for $k \in \mathbb{Z}^n\setminus\{0\}$,

$$\widehat{G}_{N+1}(k) = -i\frac{\widehat{\overline{R}}_{N-1}(k)}{(a\cdot k)},$$

(here $(a\cdot k) \neq 0$ by the Diophantine condition of a).

In addition, the Diophantine condition of a (see (31)), there exist two constants $C_0, N_0 > 0$ such that for all $k \neq 0$ we have the estimate:

$$\mid \widehat{G}_{N+1}(k) \mid = \frac{\mid \widehat{\overline{R}}_{N-1}(k) \mid}{\mid (a\cdot k) \mid} \leqslant C_0 \mid k \mid^{N_0} \mid \widehat{\overline{R}}_{N-1}(k) \mid \tag{65}$$

that ensures convergence and analyticity of G_{N+1} en x because $\overline{R}_{N-1}$ is. □

Remark 3.18

(1) In the above theorem, by taking N converge to infinity and by posing $[A] := G_1 + G_2 + \cdots$, then$[A]$ is the desired formal series discussed in the last section and the Birkhoff normal form of $P = P(x, \xi, \varepsilon, h)$ is the limit in $\mathcal{E}$ of $P_0 + hP_1^{(N)}$ as $N \to \infty$. On the other hand, there exits a C^∞ function, denoted often by $P^{(\infty)}$ which admits this limit as its asymptotic expansion.

(2) We see an important thing that in the case of theorem, the first term (or yet the h- principal term) along the procedure of Birkhoff normal form of P is always

$$P_0 = p(\xi) + i\varepsilon q(\xi).$$

△

Proposition 3.19. *Let $\widehat{P} = \widehat{P}(\xi; X)$ a complex-valued smooth function of ξ near $0 \in \mathbb{R}^2$ and X near $0 \in \mathbb{R}^n$. Assume that $\widehat{P}$ admits an asymptotic expansion in X near 0 of the form*

$$\widehat{P}(\xi; X) \sim \sum_\alpha C_\alpha(\xi)X^\alpha$$

with $C_\alpha(\xi)$ are smooth functions and $C_0(\xi) := \widehat{P}_0(\xi)$ is local diffeomorphism in $\xi = 0$.

Then, for $\mid X \mid$ small enough, $\widehat{P}$ is too a local diffeomorphism in $\xi = 0$ and its inverse admits an asymptotic expansion in X near 0 whose the first term is $(\widehat{P}_0)^{-1}$.

Proof. One can write $\widehat{P}$ in the form

$$\widehat{P}(\xi; X) = \widehat{P}_0(\xi) + \mathcal{O}(X).$$

The determinant

$$\mid det(\frac{\partial \widehat{P}}{\partial \xi}(0)) \mid = \mid det(\frac{\partial \widehat{P}_0}{\partial \xi}(0)) + \mathcal{O}(\mid X \mid) \mid$$

is nonzero for $\mid X \mid$ small enough and it ensures that $\widehat{P}$ is a local diffeomorphism in $\xi = 0$.

Then, by induction, we can show that $\widehat{P}^{-1}$ admits an asymptotic expansion in X near $0 \in \mathbb{R}^n$. □

3.4. **Operator $P_\varepsilon = P + i\varepsilon Q$, the case $\{p, q\} = 0$.** In this section, we will work on a particular case of the operator P_ε considered in the previous section when the principal symbols p, q commute. This is a simple, but very interesting case. More explicitly:

We now assume that P_ε if of the form

$$P_\varepsilon = P + i\varepsilon Q, \tag{66}$$

with P, Q two h-pseudo-differential operators and $P = P_{\varepsilon=0}$ is selfadjoint (Q is not necessarily selfadjoint).

Suppose that p, q are associated principal symbols of P, Q. Note that p is real-valued, q is not necessarily real-valued but for more simple, we will assume also that q is real-valued (if not just replace q by $Re(q)$).

Then the principal symbol of P_ε is

$$p_\varepsilon = p + i\varepsilon q.$$

Note that the principal symbol in (21) becomes the same form as last but without term $\mathcal{O}(\varepsilon^2)$.

We assume further that p, q commute i.e. $\{p, q\} = H_p(q) = 0$ with respect to the Poisson bracket on T^*M and that dp, dq are linearly independent almost everywhere.

Remark 3.20 On the operator level, in this case, P, Q are not necessarily in involution, contrariwise their commutant is power of order 2 of h,

$$[P, Q] = \mathcal{O}(h^2).$$

△

3.4.1. *Asymptotic spectrum of $P_\varepsilon = P + i\varepsilon Q$, the case $\{p, q\} = 0$.* We know that by the commutativity, q is invariant under the flow of p, the function $\langle q \rangle_T$ (for all $T > 0$) in (22) is so still q and by the action-angle theorem, q is constant on any invariant torus Λ_a, the average of q on Λ_a (definition in (29)) is still q.

Consequently, we can replace $\langle q \rangle_T$ and $\langle q \rangle$ in all definitions, assumptions and assertions of the last section by q.

Particulary, for each $a \in J$, the compact interval of $Q_\infty(a)$ defined in (27) becomes a single point:

$$Q_\infty(a) = \{q_{|\Lambda_a}\} \tag{67}$$

and locations of the spectrum of P_ε given in (24), (28) become:

$$\begin{aligned} Im(\sigma(P_\varepsilon) \cap \{z \in \mathbb{C} : |Rez| \leqslant \delta\}) \subset & \quad \varepsilon\big[\inf_{p^{-1}(0)} q - o(1), \sup_{p^{-1}(0)} q + o(1)\big] \\ \subset & \quad \varepsilon\big[\inf \bigcup_{a\in J} q_{|\Lambda_a} - o(1), \sup \bigcup_{a\in J} q_{|\Lambda_a} + o(1)\big], \end{aligned} \tag{68}$$

as $\varepsilon, h, \delta \to 0$.

In action-angle coordinates (x, ξ) near Λ_a such that $\Lambda_a \simeq \{\xi = 0\}$, we have $p = p(\xi)$ and the formula (30) becomes $\langle q \rangle(\xi) = q(\xi)$. Then microlocally, the principal symbol becomes

$$p_\varepsilon = p(\xi) + i\varepsilon q(\xi). \tag{69}$$

As an important application of section 3.3.2 (the theorem 3.17), the microlocal construction of Birkhoff quantic normal form of P_ε in neighborhood of a Diophantine torus does not change the principal symbol. Ie in the coordinates (x,ξ) near the section $\xi=0$ in $T^*\mathbb{T}^2$ such that P_ε has the normal form, its h-principal symbol is still of the form $p(\xi)+i\varepsilon q(\xi)$.

So in this case, concerning the theorem (3.8), the eigenvalues of P_ε in the rectangle $R(\varepsilon,h)$ (34) modulo $\mathcal{O}(h^\infty)$ are given by asymptotic expansion of a smooth function $P_j^{(\infty)}$, $j=1,\dots,L$ in (ε,h) and in

$$\xi = h(k-\frac{k_j}{4})-\frac{S_j}{2\pi}, k\in\mathbb{Z}^2$$

whose the first term (the principal symbol (37) in the case of the theorem (3.8)) is

$$p_{j,0}(\xi,\varepsilon)=p_j(\xi)+i\varepsilon q_j(\xi), \tag{70}$$

where p_j, q_j are expressions of p,q in action-angle variables near Λ_j of p,q.

In reduced form, we can write

$$\begin{aligned}\sigma(P_\varepsilon)\cap R(\varepsilon,h)\ni\lambda &= P_j^{(\infty)}\Big(h(k-\frac{k_j}{4})-\frac{S_j}{2\pi},\varepsilon;h\Big)+\mathcal{O}(h^\infty)\\ &= p_j\Big(h(k-\frac{k_j}{4})-\frac{S_j}{2\pi}\Big)+i\varepsilon q_j\Big(h(k-\frac{k_j}{4})-\frac{S_j}{2\pi}\Big)\\ &+ \mathcal{O}(h),\quad k\in\mathbb{Z}^2,\end{aligned} \tag{71}$$

uniformly for h,ε small.

Remark 3.21 We just give the asymptotic expansion of eigenvalues of P_ε in a good rectangle in the neighborhood of $0\in\mathbb{C}$.

However, if we assume the same assumptions on the energy space $p^{-1}(E)\cap T^*M$ ($E\in\mathbb{R}$) as $p^{-1}(0)\cap T^*M$ and by introducing the same definition of set of good values, we can build the same result for the eigenvalues of P_ε in any good rectangle of center of form $E+iF_0$. $\triangle$

3.4.2. *The detailed spectral formula.* In this paragraph, We consider the operator P_ε as in the preceding paragraph and more we will assume all the same assumptions on the energy space $p^{-1}(E)\cap T^*M$ que $p^{-1}(0)\cap T^*M$ and introduce the definition of good values, similarly the definition (3.5). Here we take E in a bounded interval of $\mathbb{R}$ because we want uniform estimates with respect to E.

As we said in the previous remark and with the help of the coordinates action-angle, we will explicitly give the asymptotic expansion of eigenvalues of P_ε in an arbitrary good rectangle of size $\mathcal{O}(h^\delta)\times\mathcal{O}(\varepsilon h^\delta)$ with a good center $E+i\varepsilon F$.

Moreover, it is interesting that one can construct a such expansion whose principal symbol is globally well defined for all good rectangle near a regular value of $(p,\varepsilon q)$.
Indeed, for more simple, we assume that the momentum map $\Phi:=(p,q)$ is proper and of connected fibre (each compact regular fibre is thus a Liouville lagrangian invariant torus. **In this case** $L=1$.

Denoted by U_r the set of regular values of $\Phi=(p,q)$ and let a point $c\in U_r$.
We recall that by the action-angle theorem 4.2, we have action-angle coordinates in a neighborhood of torus $\Lambda_c:=\Phi^{-1}(c)$ in M: there exists $r>0$, a neighborhood $\Omega:=\Phi^{-1}(B(c,r))$ of Λ_c, un small open $D\subset\mathbb{R}^2$ of center 0, a symplectomorphism $\kappa:\Omega\to\mathbb{T}^2\times D$ and a diffeomorphism $\varphi: D\to\varphi(D)=B(c,r)$ such that: $\kappa(\Lambda_c)=\{\xi=0\}$, $\Phi\circ\kappa^{-1}(x,\xi)=\varphi(\xi)$, for all $x\in\mathbb{T}^2,\xi\in D$ and $\varphi(0)=c$.

We introduce the function

$$\chi:\mathbb{R}^2\ni u=(u_1,u_2)\mapsto\chi_u=(u_1,\varepsilon u_2)\cong u_1+i\varepsilon u_2 \tag{72}$$

and denote

$$B(\chi_u,r,\varepsilon):=\chi(B(u,r))$$

for a certain ball $B(u,r)$ ($r>0$), $U_r(\varepsilon):=\chi(U_r)$.

For any point $\chi_a \in B(\chi_c, r, \varepsilon)$ such that $F := a_2$ is a good value, $E := a_1$ and $p^{-1}(E) \cap T^*M$ satisfies the same assumptions as $p^{-1}(0) \cap T^*M$ in the section 3.2.1. We will construct the asymptotic expansion of eigenvalues of P_ε in a "good rectangle" of "good center" χ_a (which is just a translation of rectangle $R(\varepsilon, h)$ in (34)):

$$R(\chi_a, \varepsilon, h) = \chi_a + R(0, \varepsilon, h), \tag{73}$$

where

$$R(0, \varepsilon, h) = \Big[-\frac{h^\delta}{\mathcal{O}(1)}, \frac{h^\delta}{\mathcal{O}(1)} \Big] + i\varepsilon \Big[-\frac{h^\delta}{\mathcal{O}(1)}, +\frac{h^\delta}{\mathcal{O}(1)} \Big] = R(\varepsilon, h) - i\varepsilon a_2. \tag{74}$$

Let $\Lambda_1 = \Phi^{-1}(a)$, this is an invariant torus of type (α, d)-Diophantine, defined in (31) and suppose that its action-angle coordinates are ξ_a, ie $\{\xi = \xi_a\} = \kappa(\Lambda_1)$ in $T^*\mathbb{T}^2$ or $\xi_a = \varphi^{-1}(a) \in D$.

After, let $\widetilde{P}_\varepsilon := P_\varepsilon - \chi_a$ to reduce the spectrum of P_ε near χ_a to the spectrum of $\widetilde{P}_\varepsilon$ near 0 by noting that:

$$\sigma(P_\varepsilon) = \sigma(\widetilde{P}_\varepsilon) + \chi_a. \tag{75}$$

The principal symbol of $\widetilde{P}_\varepsilon$ is $\widetilde{p} + i\varepsilon\widetilde{q}$, with $\widetilde{p} := p - a_1, \widetilde{q} := q - a_2$. Note that we have still an integrable system $(\widetilde{p}, \widetilde{q})$ and if let $\widetilde{\xi} = \xi - \xi_a$, then $\widetilde{\xi}$ is the new action variable for this system in which $\Lambda_a \simeq \{\widetilde{\xi} = 0\}$, as the standard case of theorem 3.8.

The principal symbol of $\widetilde{P}_\varepsilon$ is microlocally reduced to

$$\varphi_1(\xi) + i\varepsilon\varphi_2(\xi) - \chi_a = \varphi_1(\xi_a + \widetilde{\xi}) + i\varepsilon\varphi_2(\xi_a + \widetilde{\xi}) - \chi_a.$$

Applying the theorem (3.8) in the case of last section for $\widetilde{P}_\varepsilon$ and from the formula (75), we have: all the eigenvalues of P_ε in the good rectangle $R(\chi_a, \varepsilon, h)$, defined by (73) modulo $\mathcal{O}(h^\infty)$ are given by asymptotic expansion of a smooth function $P(\xi, \varepsilon; h)$ in (ε, h) and in ξ in a neighborhood of ξ_a such that in reduced form (it's the same as (71)),

$$\begin{aligned} \sigma(P_\varepsilon) \cap R(\chi_a, \varepsilon, h) \ni \lambda &= P\Big(\xi_a + h(k - \frac{k_1}{4}) - \frac{S_1}{2\pi}, \varepsilon; h\Big) + \mathcal{O}(h^\infty) \\ &= \varphi_1\Big(\xi_a + h(k - \frac{k_1}{4}) - \frac{S_1}{2\pi}\Big) + i\varepsilon\varphi_2\Big(\xi_a + h(k - \frac{k_1}{4}) - \frac{S_1}{2\pi}\Big) \\ &\quad + \mathcal{O}(h), \quad k \in \mathbb{Z}^2, \end{aligned} \tag{76}$$

uniformly for h, ε small, where $S_1 \in \mathbb{R}^2$ is the action and $k_1 \in \mathbb{Z}^2$ is the Maslov index of fundamental cycles of Λ_1.

With the below remark (3.25), there exists a function $\tau_c \in \mathbb{R}^2$, locally constant in $c \in U_r$ (depending on the choice of local action-angle coordinates near $c \in U$) such that $\frac{S_1}{2\pi} = \xi_a + \tau_c$. So the formula for λ becomes:

$$\lambda = \varphi_1\Big(-\tau_c + h(k - \frac{k_1}{4})\Big) + i\varepsilon\varphi_2\Big(-\tau_c + h(k - \frac{k_1}{4})\Big) + \mathcal{O}(h). \tag{77}$$

There is a bijective correspondence between $\lambda \in \sigma(P_\varepsilon) \cap R(\chi_a, \varepsilon, h)$ and hk in a part of $h\mathbb{Z}^2$ (by the proposition 3.12). Moreover, as in (43) and (48), there exists a smooth local diffeomorphism $f = f(\lambda, \varepsilon; h)$ which sends $R(\chi_a, \varepsilon, h)$ on its image, denoted by $E(a, \varepsilon, h)$ which is close to $\frac{S_1}{2\pi}$ such that it sends $\sigma(P_\varepsilon) \cap R(\chi_a, \varepsilon, h)$ on $h\mathbb{Z}^2$ modulo $\mathcal{O}(h^\infty)$:

$$f = f(\lambda, \varepsilon; h) = \tau_c + h\frac{k_1}{4} + P^{-1}(\lambda).$$

Let $\widetilde{f} = f \circ \chi$,

$$\widetilde{f} = \tau_c + h\frac{k_1}{4} + P^{-1} \circ \chi.$$

Remark 3.22 Let $\widehat{P} := \chi^{-1} \circ P$. Because P admits an asymptotic expansion in (ξ, ε, h), so it is obvious that $\widehat{P}$ admits an asymptotic expansion in $(\xi, \varepsilon, \frac{h}{\varepsilon})$ (here $h \ll \varepsilon$):

$$\begin{aligned} \widehat{P}(\xi, \varepsilon, h) &= \sum_{\alpha,\beta,\gamma} C_{\alpha\beta\gamma} \xi^{\alpha} \varepsilon^{\beta} (\frac{h}{\varepsilon})^{\gamma} \\ &= \widehat{P}_0(\xi) + \mathcal{O}(\varepsilon, \frac{h}{\varepsilon}), \end{aligned} \tag{78}$$

with

$$\widehat{P}_0(\xi) = \varphi_1(\xi) + i\varphi_2(\xi)$$

is a local diffeomorphism.

Moreover, by looking at the Birkhoff normal form in section 3.3, we can rewrite it as form:

$$\widehat{P}(\xi, \varepsilon, h) = \widehat{P}_0(\xi) + \mathcal{O}(\frac{h}{\varepsilon}).$$

According to the proposition (3.19), $(\widehat{P})^{-1} = P^{-1} \circ \chi$ also admits an asymptotic expansion in $(\varepsilon, \frac{h}{\varepsilon})$ whose first term is

$$(\widehat{P}_0)^{-1} = (\varphi)^{-1}.$$

$\triangle$

Consequently, $\widetilde{f}$ admits an asymptotic expansion in $(\varepsilon, \frac{h}{\varepsilon})$ and it can moreover be written as

$$\widetilde{f} = \widetilde{f}_0 + \mathcal{O}(\frac{h}{\varepsilon}) \tag{79}$$

whose first term is

$$\widetilde{f}_0 = \tau_c + (\varphi)^{-1}. \tag{80}$$

We have an important remark that the first term $\widetilde{f}_0$ is well defined globally on $B(c, r)$ in the sense that it does not depends on selected good rectangle $R(\chi_a, \varepsilon, h)$.

In summary, for any regular value $c \in U_r$, there is a small domain $B(\chi_c, r, \varepsilon) = \chi(B(c, r))$ and for any good value $a \in B(c, r)$ (which is outside a set of small measure), we have a good rectangle $R(\chi_a, \varepsilon, h)$ of good center χ_a and a smooth local diffeomorphism f which sends $R(\chi_a, \varepsilon, h)$ on its image, denoted by $E(a, \varepsilon, h)$ of the form:

$$\begin{aligned} f : R(\chi_a, \varepsilon, h) &\to E(a, \varepsilon, h) \\ \sigma(P_\varepsilon) \cap R(\chi_a, \varepsilon, h) \ni \lambda &\mapsto f(\lambda, \varepsilon; h) \in h\mathbb{Z}^2 + \mathcal{O}(h^\infty). \end{aligned} \tag{81}$$

such that $\widetilde{f} = f \circ \chi$ admits an asymptotic expansion in $(\varepsilon, \frac{h}{\varepsilon})$ of the form (79) with the first term (80) is a diffeomorphism, globally defined on $B(c, r)$.

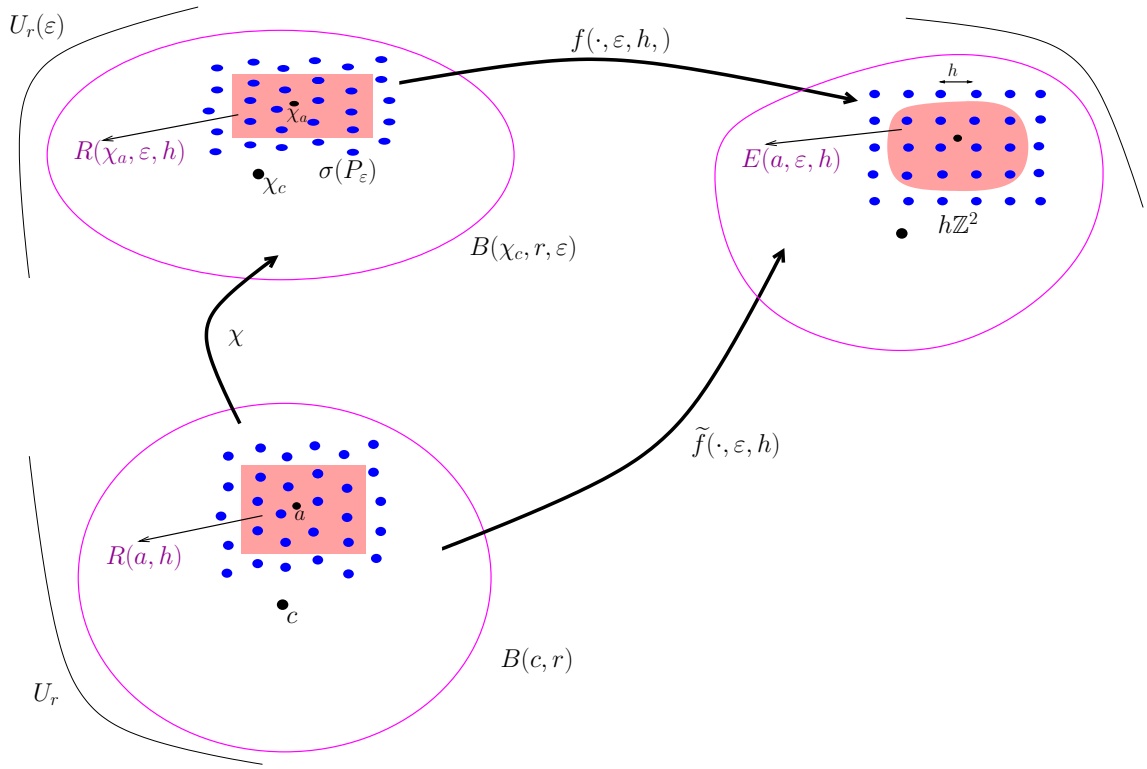

Figure 4. Pseudo-lattice spectrum of P_ε

Remark 3.23
We don't know if f admits an asymptotic expansion in $(\varepsilon, \frac{h}{\varepsilon})$ but we can express f in the form:

$$f(\lambda,\varepsilon,h)=\sum_{\alpha,\beta} C_{\alpha\beta}(\lambda_1,\frac{\lambda_2}{\varepsilon})\varepsilon^\alpha(\frac{h}{\varepsilon})^\beta$$

with

$$C_{00}(\lambda_1,\frac{\lambda_2}{\varepsilon})=\tau_c+\varphi^{-1}\circ\chi^{-1}(\lambda):=f_0 \tag{82}$$

which is well defined for all $\lambda\in B(\chi_c,r,\varepsilon)$.
Moreover, we can also write $f=f_0+\mathcal{O}(\varepsilon,\frac{h}{\varepsilon})$.

△

Remark 3.24 In terms of the definition 3.26, we say that couples $\big(f(\varepsilon;h),R(\chi_a,\varepsilon,h)\big)$ as above form a pseudo-chart $\sigma(P_\varepsilon)$ on the domain $U_r(\varepsilon)$. △

Remark 3.25 [Action integral] If $c\in U_r$ a regular value of Φ and (x,ξ) a coordinated action-angle, given by κ as previous. There is a Liouville 1-primitive form α of ω on $\Omega:=\Phi^{-1}(B(c,r))\subset M$ such that $d\alpha=\omega$.

Let $\widetilde{\omega}$ the canonical symplectic form and $\widetilde{\alpha}=\sum\xi_i dx_i$ a canonical form on $T^*\mathbb{T}^2$: $d\widetilde{\alpha}=\widetilde{\omega}$. As κ is symplectic, we have $\kappa^*\omega=\widetilde{\omega}$. This is equivalent to $d(\kappa^*\alpha-\widetilde{\alpha})=0$. I.e there is a 1-closed form β on $T^*\mathbb{T}^2$ such that

$$\kappa^*\alpha=\widetilde{\alpha}+\beta.$$

For any invariant torus $\Lambda_a\subset\Omega$, let (γ_1,γ_2) two fundamental cycles on Λ_a that are sent on the sides of the torus: $\kappa(\Lambda_a)=\{\xi=\xi_a\}$ by κ in $T^*\mathbb{T}^2$.

So the action of (γ_1, γ_2) is $S_1 = (S_{1,1}, S_{1,2})$, calculated by: for $j = 1, 2$,

$$\begin{aligned} S_{1,j} &= \int_{\gamma_j} \alpha = \int_{\kappa(\gamma_j)} \kappa^* \alpha = \int_{\kappa(\gamma_j)} (\widetilde{\alpha} + \beta) \\ &= \int_{\{x \in \mathbb{T}^2 : x_j = 0\}} (\sum \xi_i dx_i + \beta) = 2\pi(\xi_a + \tau_{c,j}), \end{aligned}$$

where

$$\tau_{c,j} := \int_{\{x \in \mathbb{T}^2 : x_j = 0\}} \beta$$

is a constant, independent of $\Lambda_a \subset \Omega$ (independent of $c \in U_r$) by the closure of β.

Then, there exists a function $\tau_c \in \mathbb{R}^2$, locally constant in $c \in U_r$ such that

$$\frac{S_1}{2\pi} = \xi_a + \tau_c. \tag{83}$$

$\triangle$

3.5. **Construction of the monodromy of asymptotic pseudo-lattice.** The spectrum of the operator P_ε considered in the previous section is a model of a more general lattice that we define and discuss below.

Let U a subset of $\mathbb{R}^2$ with compact closure and denote $U(\varepsilon) = \chi(U)$ où χ is the function defined as in previous section. Let $\Sigma(\varepsilon, h)$ (depends on small h and ε) a discrete set de $U(\varepsilon)$.

Definition 3.26. *For h, ε small enough and in regime $h \ll \varepsilon$, we say that $(\Sigma(\varepsilon, h), U(\varepsilon))$ is an asymptotic pseudo-lattice if: for any small parameter $\alpha > 0$, there exists a set of "good values" in $\mathbb{R}^2$, denoted by BV whose complement is of small measure in the sense*

$$\mid {}^C BV \cap I \mid \leqslant C\alpha \mid I \mid$$

for any domain $I \subset \mathbb{R}^2$ and $C > 0$ is a constant.
For all $c \in U$, there exists a ball $B(c, r) \subset U$ around c ($r > 0$) such that for every "good value" $a = (a_1, a_2) \in B(c, r)$ in the sense $a \in BV$, there is a good rectangle $R(\chi_a, \varepsilon, h) \subset \chi(B(c, r))$ of good center χ_a:

$$R(\chi_a, \varepsilon, h) = \chi(R(a, h))$$

where $R(a, h)$ is a rectangle of size $\mathcal{O}(h^\delta) \times \mathcal{O}(h^\delta)$, $0 < \delta < 1$ and a smooth local diffeomorphism (in χ_a) $f = f(\cdot; \varepsilon, h)$ which sends $R(\chi_a, \varepsilon, h)$ on its image, denoted by $E(a, \varepsilon, h)$ satisfying

$$\begin{aligned} f : R(\chi_a, \varepsilon, h) &\to E(a, \varepsilon, h) \\ \Sigma(\varepsilon, h) \cap R(\chi_a, \varepsilon, h) \ni \lambda &\mapsto f(\lambda; \varepsilon, h) \in h\mathbb{Z}^2 + \mathcal{O}(h^\infty) \end{aligned} \tag{84}$$

such that $\widetilde{f} := f \circ \chi$ admits an asymptotic expansion in $(\varepsilon, \frac{h}{\varepsilon})$ for the topology C^∞ for the variable u in a neighborhood of a and in the reduced form,

$$\widetilde{f}(u) = \widetilde{f}_0(u) + \mathcal{O}(\varepsilon, \frac{h}{\varepsilon}), \tag{85}$$

where the first term $\widetilde{f}_0$ is a diffeomorphism, independent of α, globally defined on $B(c, r)$ and independent of the chosen good value $a \in B(c, r)$.

We also say that the family of $(f(\cdot; \varepsilon, h), R(\chi_a, \varepsilon, h))$ is a local pseudo-chart on $B(\chi_c, r, \varepsilon) := \chi(B(c, r)$ and that a couple $(f(\cdot; \varepsilon, h), R(\chi_a, \varepsilon, h))$ is a micro-chart of $(\Sigma(\varepsilon, h), U(\varepsilon))$.

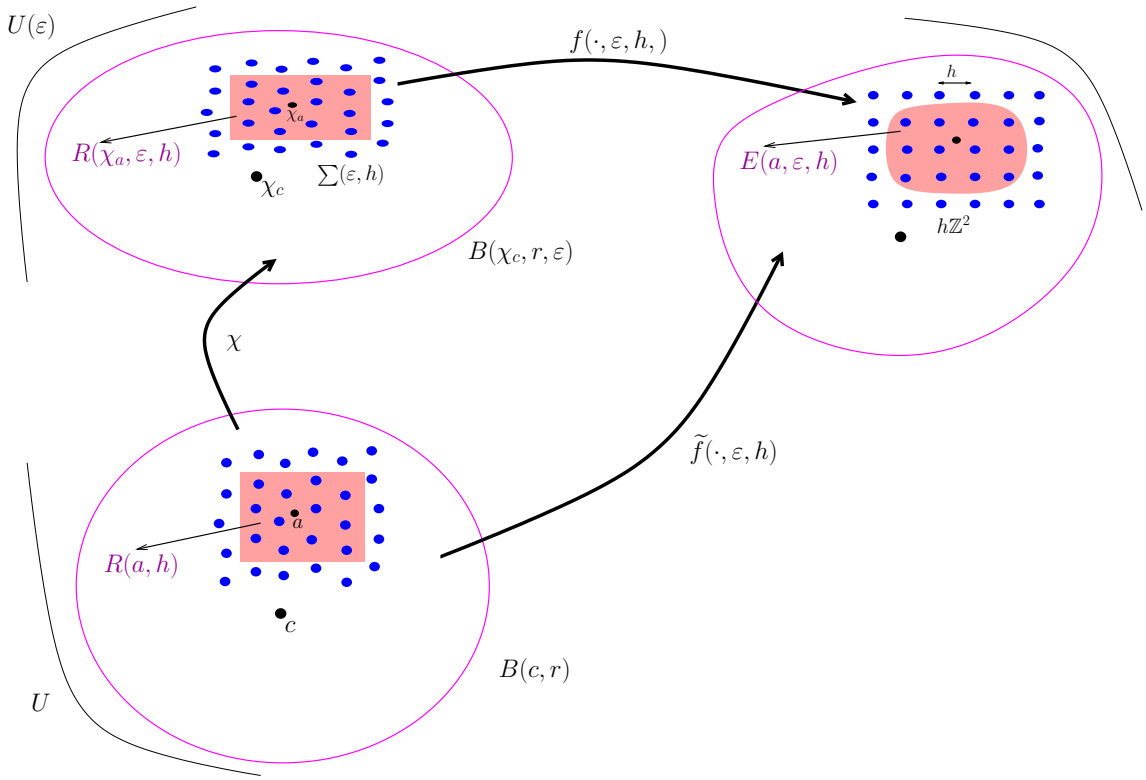

Figure 5. Asymptotic pseudo-lattice

Remark 3.27 It is clear that the spectrum of an operator $P_\varepsilon = P + i\varepsilon Q$ considered in the previous section is a good example of this definition. In this case, $\widetilde{f}_0$ is equal to coordinate actions.

We want to give a combinatorial invariant (spectral monodromy) of $\Sigma(\varepsilon,h) = \sigma(P_\varepsilon)$. As we know, in this case P, Q does not necessarily commute, it so can not have any joint spectrum as the integrable case that we discussed. Therefore, we do not yet define the monodromy for the spectrum of P_ε.

On the other hand, we are careful that the application f in (81) is not an affine chart of $U(\varepsilon)$ defined in previous section because it is only defined on a domain depending on h which will be reduced to a single point when $h \to 0$. Therefore, we can not apply the construction of the quantum monodromy for an affine asymptotic lattice as the article [45].

However, we can successfully build this invariant for the discrete spectrum of P_ε due to the fact that the first term $\widetilde{f}_0$ is globally defined on a small ball $B(c,r)$.

△

Lemma 3.28. *Let $(\Sigma(\varepsilon,h), U(\varepsilon))$ an asymptotic pseudo-lattice as in the definition 3.26 and a point $\chi_a \in B(\chi_c,r,\varepsilon)$ with a a good value. Then, there is a family $\lambda(\varepsilon,h) \in \Sigma(\varepsilon,h) \cap R(\chi_a,\varepsilon,h)$ such that*

$$|\lambda_1(\varepsilon,h) - a_1| = \mathcal{O}(h) \tag{86}$$
$$|\lambda_2(\varepsilon,h) - \varepsilon a_2| = \mathcal{O}(\varepsilon.h), \tag{87}$$

uniformly for $h, \varepsilon \to 0$.

Proof. By the proposition 3.19, $u := \widetilde{f}^{-1}$ is also a local diffeomorphism in a fixed point ξ_a admitting an asymptotic expansion in $(\varepsilon, \frac{h}{\varepsilon})$ and by its main part, if ξ near ξ_a we have:

$$u(\xi) = \widetilde{f}^{-1}(\xi;\varepsilon,h) = (\widetilde{f}_0)^{-1}(\xi) + \mathcal{O}(\varepsilon, \frac{h}{\varepsilon}). \tag{88}$$

By calculating the differential of u in ξ_a (this is the same as the proof of proposition 3.19), we can assert that: if $\xi^{(1)}, \xi^{(2)}$ near ξ_a such that $|\xi^{(1)} - \xi^{(2)}| = \mathcal{O}(h)$, then

$$|u(\xi^{(1)}) - u(\xi^{(2)})| = \mathcal{O}(h)$$

uniformly for $h, \varepsilon \to 0$.
Let

$$\xi^{(1)} := \widetilde{f}(a, \varepsilon; h) \in E(a, \varepsilon, h).$$

On the other hand, one can find

$$\xi^{(2)} := hk = h.k(\varepsilon, h) \in h\mathbb{Z}^2 \cap E(a, \varepsilon, h)$$

such that $|\xi^{(1)} - \xi^{(2)}| \leqslant h$ by setting $k = k(\varepsilon, h)$ the integer part of $\frac{\xi^{(1)}}{h}$. In the end, with the remark $\chi(u(\xi^{(1)})) = \chi_a$ and there is $\lambda(\varepsilon, h) \in \Sigma(\varepsilon, h) \cap R(\chi_a, \varepsilon, h)$ such that $\lambda(\varepsilon, h) = \chi(u(\xi^{(2)})) + \mathcal{O}(h^\infty)$, we get the result of the lemma. □

Remark 3.29 The result of the lemma is still valid for the spectrum of an operator in general case of the theorem (3.8).

In the previous proof, we can choose k as the integer part of $\frac{\widetilde{f}_0(a)}{h}$ and then $k = k(h)$. △

3.5.1. *Transition function.* Let $(\Sigma(\varepsilon, h), U(\varepsilon))$ be an asymptotic pseudo-lattice.

Suppose that $B_\alpha := B(c, r)$ and $B_\beta := B(c', r')$ are two small balls in U with nonempty intersection $B_{\alpha\beta} := B_\alpha \cap B_\beta \neq \emptyset$ such that there are local two pseudo-charts on $B_\alpha(\varepsilon) := B(\chi_c, r, \varepsilon)$ and $B_\beta(\varepsilon) = B(\chi'_c, r', \varepsilon)$ of $(\Sigma(\varepsilon, h), U(\varepsilon))$.
Denote $B_{\alpha\beta}(\varepsilon) = B_\alpha(\varepsilon) \cap B_\beta(\varepsilon)$.
Because the good values in each B_α, B_β are outside the set of small measure $\mathcal{O}(\alpha)$, then the complement of good values in $B_{\alpha\beta}$ is still small measure $\mathcal{O}(\alpha)$.

Let $a \in B_{\alpha\beta}$ be a good value. Therefore, there is a associated good rectangle $R(\chi_a, \varepsilon, h) \subset B_{\alpha\beta}(\varepsilon)$ on which we have two micro-charts of $\Sigma(\varepsilon, h)$ in χ_a as in (84) of the previous definition:

$$\begin{aligned} f_\alpha(\varepsilon; h) &: R(\chi_a, \varepsilon, h) \to E_\alpha(a, \varepsilon, h) \\ f_\beta(\varepsilon; h) &: R(\chi_a, \varepsilon, h) \to E_\beta(a, \varepsilon, h). \end{aligned} \tag{89}$$

Theorem 3.30. *There exists a unique constant matrix $M_{\alpha\beta} \in GL(2, \mathbb{Z})$ such that*

$$d\widetilde{f}_{\alpha,0} = M_{\alpha\beta} d\widetilde{f}_{\beta,0}$$

in all good values $a \in B_{\alpha\beta}$.

Proof. In this demonstration, we keep the same notation of lemma 3.28. By this lemma 3.28 and the remark 3.29, let

$$\lambda(\varepsilon, h) = f_\alpha^{-1}(hk(h)) + \mathcal{O}(h^\infty) \tag{90}$$

(with $hk(h) \in h\mathbb{Z}^2 \cap E_\alpha(a, \varepsilon, h)$) is a family in $\Sigma(\varepsilon, h) \cap R(\chi_a, \varepsilon, h)$ such that

$$|\lambda_1(\varepsilon, h) - a_1| = \mathcal{O}(h)$$

and

$$|\lambda_2(\varepsilon, h) - \varepsilon a_2| = \mathcal{O}(\varepsilon.h),$$

uniformly for $h, \varepsilon \to 0$.

Let $k_0 \in \mathbb{Z}^2$ be arbitrary given. For h small enough, we can define

$$hk'(h) := hk(h) - hk_0 \in h\mathbb{Z}^2 \cap E_\alpha(a, \varepsilon, h). \tag{91}$$

Then there exists a family $\lambda'(\varepsilon, h) \in \Sigma(\varepsilon, h) \cap R(\chi_a, \varepsilon, h)$ such that

$$hk'(h) = f_\alpha(\lambda'(\varepsilon, h)) + \mathcal{O}(h^\infty), \tag{92}$$

uniformly for ε, h small.
We have also

$$hk(h) = f_\alpha(\lambda(\varepsilon, h)) + \mathcal{O}(h^\infty), \tag{93}$$

uniformly for ε, h small.
By replacing (92) and (93) in (91), we have:

$$f_\alpha(\lambda(\varepsilon,h)) - f_\alpha(\lambda'(\varepsilon,h)) = hk_0 + \mathcal{O}(h^\infty), \tag{94}$$

uniformly for ε, h small.
Note also that $|hk'(h) - hk(h)| = hk_0 = \mathcal{O}(h)$, then $|hk'(h) - \xi^{(1)}| = \mathcal{O}(h)$ and so the family $\lambda'(\varepsilon,h)$ has the same property as that of $\lambda(\varepsilon,h)$. That is

$$|\lambda_1'(\varepsilon,h) - a_1| = \mathcal{O}(h)$$

and

$$|\lambda_2'(\varepsilon,h) - \varepsilon a_2| = \mathcal{O}(\varepsilon.h),$$

uniformly for $h, \varepsilon \to 0$.

We recall the function

$$\begin{aligned} \chi : B_{\alpha\beta} &\to B_{\alpha\beta}(\varepsilon) \\ u = (u_1, u_2) &\mapsto \chi_u = (u_1, \varepsilon u_2) \end{aligned} \tag{95}$$

Let $u(\varepsilon,h) = \chi^{-1}(\lambda(\varepsilon,h))$ (i.e. $u_1 = \lambda_1, u_2 = \frac{\lambda_2}{\varepsilon}$) and in the same way $u'(\varepsilon,h) = \chi^{-1}(\lambda'(\varepsilon,h))$. We have

$$\begin{aligned} |u_i(\varepsilon,h) - a_i| &= \mathcal{O}(h), \\ |u_i'(\varepsilon,h) - a_i| &= \mathcal{O}(h), \end{aligned} \tag{96}$$

for $i = 1, 2$. The equation (94) so becomes

$$\widetilde{f}_\alpha(u(\varepsilon,h)) - \widetilde{f}_\alpha(u'(\varepsilon,h)) = hk_0 + \mathcal{O}(h^\infty),$$

or

$$\frac{\widetilde{f}_\alpha(u(\varepsilon,h)) - \widetilde{f}_\alpha(u'(\varepsilon,h))}{h} = k_0 + \mathcal{O}(h^\infty), \tag{97}$$

uniformly for ε, h small.
Because we can express

$$\widetilde{f}_\alpha(u(\varepsilon,h)) = \widetilde{f}_{\alpha,0}(u(\varepsilon,h)) + \mathcal{O}(\varepsilon, \frac{h}{\varepsilon})$$

and by wring the Taylor expansion of $\widetilde{f}_{\alpha,0}(u(\varepsilon,h))$ in a with the integral rest, by redoing the same work for $\widetilde{f}_\alpha(u'(\varepsilon,h))$ and more remark if $R(u,\varepsilon,h) = \mathcal{O}(\varepsilon, \frac{h}{\varepsilon})$, then

$$|R(u(\varepsilon,h),\varepsilon,h) - R(u'(\varepsilon,h),\varepsilon,h)| = \mathcal{O}(h) \times \mathcal{O}(\varepsilon, \frac{h}{\varepsilon}),$$

uniformly for h, ε petits and $h \ll \varepsilon$ as $|u(\varepsilon,h) - u'(\varepsilon,h)| = \mathcal{O}(h)$, The equation (97) gives us:

$$(d\widetilde{f}_{\alpha,0})(a)\frac{u(\varepsilon,h) - u'(\varepsilon,h)}{h} = k_0 + \mathcal{O}(\varepsilon, \frac{h}{\varepsilon})$$

Consequently, we have

$$\frac{u(\varepsilon,h) - u'(\varepsilon,h)}{h} = \big((d\widetilde{f}_{\alpha,0})(a)\big)^{-1}(k_0) + \big((d\widetilde{f}_{\alpha,0})(a)\big)^{-1}\big(\mathcal{O}(\varepsilon, \frac{h}{\varepsilon})\big). \tag{98}$$

On the other hand, as the norm of differential $\big((d\widetilde{f}_{\alpha,0})(a)\big)^{-1}$ is a independent constant of ε, h, the equation (98) allows us to write

$$\frac{u(\varepsilon,h) - u'(\varepsilon,h)}{h} = \big((d\widetilde{f}_{\alpha,0})(a)\big)^{-1}(k_0) + \mathcal{O}(\varepsilon, \frac{h}{\varepsilon}), \tag{99}$$

uniformly for h, ε small and $h \ll \varepsilon$.

Now we will work with $f_\beta(\varepsilon;h)$. Let also $\widetilde{f}_\beta = f_\beta \circ \chi$.
Because $\lambda(\varepsilon,h), \lambda'(\varepsilon,h)$ is in $\sigma(P_\varepsilon)\cap R(\chi_a,\varepsilon,h)$, then there exists a family, denoted by $k'(\varepsilon,h) \in \mathbb{Z}^2$ such that

$$\frac{\widetilde{f}_\beta(u(\varepsilon,h)) - \widetilde{f}_\beta(u'(\varepsilon,h))}{h} = k'(\varepsilon,h) + \mathcal{O}(h^\infty),$$

uniformly for ε, h small.
In the same way that before, we also get

$$\frac{u(\varepsilon,h) - u'(\varepsilon,h)}{h} = \big((d\widetilde{f}_{\beta,0})(a)\big)^{-1}(k'(\varepsilon,h)) + \mathcal{O}(\varepsilon, \frac{h}{\varepsilon}), \tag{100}$$

uniformly for ε, h small and $h \ll \varepsilon$.
Then the equation (99) and the equation (100) give us

$$\big((d\widetilde{f}_{\alpha,0})(a)\big)^{-1}(k_0) = \big((d\widetilde{f}_{\beta,0})(a)\big)^{-1}(k'(\varepsilon,h)) + \mathcal{O}(\varepsilon, \frac{h}{\varepsilon}),$$

uniformly for ε, h small, $h \ll \varepsilon$ and therefore

$$(d\widetilde{f}_{\beta,0})(a)\big((d\widetilde{f}_{\alpha,0})(a)\big)^{-1}(k_0) = k'(\varepsilon,h) + \mathcal{O}(\varepsilon, \frac{h}{\varepsilon}), \tag{101}$$

uniformly for ε, h small and $h \ll \varepsilon$.
As the left part of the last equation is a constant, $\mathcal{O}(\varepsilon, \frac{h}{\varepsilon})$ is small for ε, h small, $h \ll \varepsilon$ and $k'(\varepsilon,h) \in \mathbb{Z}^2$, it is necessary that $k'(\varepsilon,h) \in \mathbb{Z}^2$ is a constant $k' \in \mathbb{Z}^2$ and that we have

$$(d\widetilde{f}_{\beta,0})(a)\big((d\widetilde{f}_{\alpha,0})(a)\big)^{-1}(k_0) = k' \in \mathbb{Z}^2 \tag{102}$$

for all $k_0 \in \mathbb{Z}^2$ given.
This means that $(d\widetilde{f}_{\beta,0})(a)\big((d\widetilde{f}_{\alpha,0})(a)\big)^{-1} \in GL(2,\mathbb{Z})$.
On the other hand, $d\widetilde{f}_{\beta,0} \circ \big(d\widetilde{f}_{\alpha,0}\big)^{-1}$ is uniformly continuous on $B_{\alpha\beta}$ and the measure of complementary of good values in $B_{\alpha\beta}$ is $\mathcal{O}(\alpha)$. By taking α small enough and with the fact that the group $GL(2,\mathbb{Z})$ is discrete, the uniform continuity implies that therefore there is a constant matrix $M_{\beta\alpha} \in GL(2,\mathbb{Z})$ such that

$$(d\widetilde{f}_{\beta,0})(a) = M_{\beta\alpha}(d\widetilde{f}_{\alpha,0})(a),$$

for all good values $a \in B_{\alpha\beta}$. □

Because on intersection, the function $f_{\alpha,0} \circ (f_{\beta,0})^{-1}$ is well defined, smooth. Outside a set of measure $\mathcal{O}(\alpha)$, the value of $d(f_{\alpha,0} \circ (f_{\beta,0})^{-1})$ is a constant matrix $M_{\alpha\beta} \in GL(2,\mathbb{Z})$. Note that $d(f_{\alpha,0} \circ (f_{\beta,0})^{-1})$ is independent of α. By taking $\alpha \to 0$, we get that $d(f_{\alpha,0} \circ (f_{\beta,0})^{-1})$ is equal to the constant $M_{\alpha\beta}$ outside a set of measure 0. By the continuity, it must have $d(f_{\alpha,0} \circ (f_{\beta,0})^{-1}) = M_{\alpha\beta}$ everywhere.

Then we have:

Theorem 3.31. *There exists a unique constant matrix $M_{\alpha\beta} \in GL(2,\mathbb{Z})$ such that*

$$d\widetilde{f}_{\alpha,0} = M_{\alpha\beta} d\widetilde{f}_{\beta,0}$$

on $B_{\alpha\beta}$.

Remark 3.32 For the spectrum of the operator $P_\varepsilon = P + i\varepsilon Q$ considered in the previous section, we can see that the result of the theorem can be found independently from classical results. Indeed: We remember from (80) that the leading terms are

$$\begin{aligned} \widetilde{f}_{\alpha,0} &= \tau_c + \varphi_\alpha^{-1}, \\ \widetilde{f}_{\beta,0} &= \tau_{c'} + \varphi_\beta^{-1}. \end{aligned} \tag{103}$$

On the other hand, as an application of action-angle theorem (see the next section and the equation (108)), on $B_{\alpha\beta}$ nonempty, there exists an affine map $A_{\alpha\beta} \in GA_{\mathbb{R}}(2,\mathbb{Z})$ of the form

$$A_{\alpha\beta} := M^{cl}_{\alpha\beta} + C_{\alpha\beta},$$

with a matrix $M^{cl}_{\alpha\beta} \in GL(2,\mathbb{Z}), C_{\alpha\beta} \in \mathbb{R}^2$ such that

$$\varphi_\alpha^{-1} \circ \varphi_\beta = A_{\alpha\beta}. \tag{104}$$

Therefore the action coordinates ξ_a and ξ'_a of Λ_a in two action-angle coordinates, associated with $\varphi_\alpha, \varphi_\beta$ satisfy

$$\xi_a = A_{\alpha\beta}\xi'_a.$$

We have also the corresponding integral actions on Λ_a satisfying the relation

$$S_1 = M^{cl}_{\alpha\beta} S'_1.$$

Two last equations and the equation (83) give us

$$\tau_c = M^{cl}_{\alpha\beta}\tau'_c - C_{\alpha\beta}.$$

Consequently, we obtain

$$\widetilde{f}_{\alpha,0} = (M^{cl}_{\alpha\beta})\widetilde{f}_{\beta,0}. \tag{105}$$

It means that we recover the result of theorem with help of action-angle theorem.
Moreover, if we denote $M^{sp}_{\alpha\beta}$ the matrix $M_{\alpha\beta}$ defined by the theorem 3.31, we have so:

$$M^{sp}_{\alpha\beta} = (M^{cl}_{\alpha\beta}).$$

△

Remark 3.33 For an asymptotic pseudo-lattice, we would like to show a stronger result: there exists a unique constant matrix $M_{\alpha\beta} \in GL(2,\mathbb{Z})$ such that

$$\widetilde{f}_{\alpha,0} = M_{\alpha\beta}\widetilde{f}_{\beta,0}$$

on $B_{\alpha\beta}$.

By the remark 3.32, we have just seen that this result is true for spectrum of the discussed operator P_ε. However, we have not succeeded to show it in the case of asymptotic pseudo-lattice.
△

3.5.2. *Definition of monodromy of an asymptotic pseudo-lattice.* Let $(\Sigma(\varepsilon,h), U(\varepsilon))$ be an asymptotic pseudo-lattice as the 3.26. Assume that U is covered by a locally finite covering $\{B_\alpha)\}$ and that $(\Sigma(\varepsilon,h), U(\varepsilon))$ is covered by associated local pseudo-charts on $B_\alpha(\varepsilon) := \chi(B_\alpha)$:

$$\{(f(\cdot;\varepsilon,h), R(\chi_a,\varepsilon,h)) \quad a \in B_\alpha \quad \text{is a good value }\}.$$

We can see $\{(\widetilde{f}_{\alpha,0}(\varepsilon,h), B_\alpha)\}$ as the charts of U whose transition functions are in the linear group $GL(2,\mathbb{Z})$.
Indeed, on each nonempty intersection $B_\alpha \cap B_\beta \neq \emptyset$, the theorem 3.31 states that there exists a unique linear map $M_{\alpha\beta} \in GL(2,\mathbb{Z})$ satisfying

$$d(\widetilde{f}_{\alpha,0}) = M_{\alpha\beta}d(\widetilde{f}_{\beta,0}) \qquad \text{or} \qquad d\big(\widetilde{f}_{\alpha,0} \circ (\widetilde{f}_{\beta,0})^{-1}\big) = M_{\alpha\beta}. \tag{106}$$

The family $\{M_{\alpha\beta}\}$ defines 1-cocycle, denoted by $\mathcal{M}$ in the Čech cohomology with value in the linear group $GL(2,\mathbb{Z})$. We have therefore

Definition 3.34. *The class $[\mathcal{M}] \in \check{H}^1(U(\varepsilon), GL(2,\mathbb{Z}))$ is called (linear) monodromy of the asymptotic pseudo-lattice $(\Sigma(\varepsilon,h), U(\varepsilon))$.*

As with quantum monodromy, we can of course associate this cocycle class with an isomorphism class of bundles (bundle up to isomorphism) on $U(\varepsilon)$ with structure of the group $GL(2,\mathbb{Z})$ and fiber $\mathbb{Z}^2$. The transition functions between two adjacent trivializations of the bundle are $\{M_{\alpha\beta}\}$.

If $\mathcal{M}$ is some cocycle associated with trivialization of the bundle and let $\gamma(\varepsilon)$ be a closed loop, contained in $U(\varepsilon)$.
We define

$$\mu(\gamma(\varepsilon)) = M_{1,N} \circ M_{N,N-1} \circ \cdots M_{3,2} \circ M_{2,1},$$

where $M_{i,j}$ denote the corresponding transition function to each pair of nonempty intersection $(B_i(\varepsilon), B_j(\varepsilon))$, defined as in the theorem (3.31), here $(B_1(\varepsilon), \cdots, B_N(\varepsilon))$ is a numbered finite covering of $\gamma(\varepsilon)$ in $U(\varepsilon)$ with $B_i \cap B_{i+1} \neq \emptyset$.

The non triviality of $[\mathcal{M}]$ is equivalent to that of a group morphism, denoted also by μ, defined as follows

$$\begin{aligned} \mu : \pi_1(U(\varepsilon)) &\rightarrow GL(2,\mathbb{Z})/\{\sim\} \\ \gamma(\varepsilon) &\mapsto \mu(\gamma(\varepsilon)), \end{aligned} \tag{107}$$

where $\{\sim\}$ denote the modulo conjugation.
We call μ the representation of the monodromy $[\mathcal{M}]$.

3.6. **Linear Spectral Monodromy.** We reconsider the operator $P_\varepsilon = P + i\varepsilon Q$, the case $\{p,q\} = 0$ that we discussed in the section 3.4.

Let U a subset of regular values U_r of the map (p,q) with compact closure and denote $U(\varepsilon) = \chi(U)$.
We further assume that the spectrum of P_ε is discrete in $U(\varepsilon)$. Then it is clear that $(\sigma(P_\varepsilon), U(\varepsilon))$ is an asymptotic pseudo-lattice, adapted to the definition 3.26. We can therefore define the monodromy of P_ε as the monodromy of the asymptotic pseudo-lattice $(\sigma(P_\varepsilon), U(\varepsilon))$ according to the definition 3.34. We call it the (linear) spectral monodromy, denoted by $[\mathcal{M}_{sp}] \in \check{H}^1(U(\varepsilon), GL(2,\mathbb{Z}))$.

Definition 3.35. *For $\varepsilon, h > 0$ small such that $h \ll \varepsilon \leqslant h^\delta, 0 < \delta < 1$, the class $[\mathcal{M}_{sp}] \in \check{H}^1(U(\varepsilon), GL(2,\mathbb{Z}))$ is called spectral monodromy of operator P_ε on the domain $U(\varepsilon)$.*

4. Relationship between the spectral monodromy and the classical monodromy

We will also make the link with classical monodromy that illuminate the existence of linear spectral monodromy. The first, we recall the classical monodromy that is given by J. Duistermaat in the article [18].

4.1. **Classical Monodromy.**
Let (M,ω) be a symplectic manifold of dimension $2n$.

Definition 4.1. *A completely integrable system on M is given n real-valued functions $f_1, \ldots, f_n$ in $C^\infty(M)$ in involution whose differentials are almost everywhere linearly independent.*
In this case, the map $F = (f_1, \ldots, f_n) : M \to \mathbb{R}^n$ is called ***momentum map****.*

A point $m \in M$ is regular for the momentum map F if its differential $dF(m)$ has maximal rank n. For $c \in \mathbb{R}^n$, we say that Λ_c is a sheet of F if it is a connected component of $F^{-1}(c)$. And more this sheet is regular if all its points are regular point for F.
A famous theorem called "action-angle theorem" of Liouville, Mineur and Arnold (see [32], [28], [3],[14]) says that if Λ_c is compact, then all sheets of F in a neighborhood of Λ_c are lagrangian tori of rank n (diffeomorphic to $\mathbb{T}^n := \mathbb{R}^n/\mathbb{Z}^n$, called by "Liouville torus") on which the joint flow of F becomes linear. We say that there exists local "action-angle coordinates". However, we have maybe no global existence of these action-angle coordinates. A obstruction of that global existence

is a geometrical invariant, called monodromy, given the first time in 1980 by J. Duistermaat in the article [18]. For more on this monodromy, we can also see [46], [48].

Theorem 4.2 (Action-Angle theorem)**.** *Let $F = (f_1, \dots, f_n)$ be completely integrable system and Λ_c be a compact regular sheet of F. Then there exists a neighborhood Ω of Λ_c in M, a small open disk D with center c in $\mathbb{R}^n$ and a symplectomorphism $\Psi : \mathbb{T}^n \times D \to \Omega$ such that:*

(1) $\Psi(\mathbb{T}^n \times \{c\}) = \Lambda_c$.
(2) *Ω is saturated, i.e all sheets that pass a point of Ω are tori, included in Ω.*
(3) *We have*

$$F \circ \Psi(x, \xi) = \varphi(\xi)$$

for all $x = (x_1, \dots, x_n) \in \mathbb{T}^n$, all $\xi = (\xi_1, \dots \xi_n) \in D$ and here $\varphi : D \to \varphi(D)$ in a (local) diffeomorphism with $\varphi(c) = c$.
In particular, the flow of all f_i in Ω is complete.
On Ω, $(x, \xi) = \Psi^{-1}$ and we say that $x \in \mathbb{T}^n$ are (local) angle variables and $\xi \in D \subset \mathbb{R}^n$ are (local) action variables.

Recall that $c \in \mathbb{R}^n$ is a regular value of F if all points of the fiber $F^{-1}(c)$ are regular points for F. Let $U_r \subseteq \mathbb{R}^n$ be the set of regular values of de F. We assume more that F is proper (so that the fibers are compact), B_r and fiber $F^{-1}(c)$ are connected. In this case, we can apply the action-angle theorem at each point of the set of regular sheets $F^{-1}(B_r) \subseteq M$. Moreover, there exists in general a integer structure on the space of regular sheets (see [18], [48]): let U_α,U_β are two any small enough opens in U_r with nonempty intersections such that there are action-angle coordinates associated on $F^{-1}(U_\alpha) \subset M$ and $F^{-1}(U_\beta) \subset M$. With notation as in previous theorem, then on $D_\alpha \cap D_\beta \neq \emptyset$, the function $\varphi_{\alpha\beta} := \varphi_\alpha^{-1} \circ \varphi_\beta$ is a affine map: for $\xi \in D_\alpha \cap D_\beta$,

$$\varphi_{\alpha\beta}(\xi) = A_{\alpha\beta} \cdot \xi = M_{\alpha\beta} \cdot \xi + C_{\alpha\beta}, \tag{108}$$

with $M_{\alpha\beta}$ is a integer constant matrix of $GL(n, \mathbb{Z})$ and a constant $C_{\alpha\beta} \in \mathbb{R}^n$.

On the other hand, for all $c \in U_r$, on Λ_c, as the flow of each f_i is complete , then the joint flow of F, denoted by φ_F^t defines a locally free group action $(\mathbb{R}^n, +)$,

$$\varphi_F^t : \mathbb{R}^n \times \Lambda_c \to \Lambda_c.$$

We define the stabiliser of torus Λ_c, denoted by S_c, which is defined independently of the choice of m in Λ_c:

$$S_c = \{\tau \in \mathbb{R}^n : \varphi_F^\tau(m) = m\}. \tag{109}$$

It is know that S_c is a discrete subgroup of rank n of $\mathbb{R}^n$ (so isomorphic to $\mathbb{Z}^n$) and the set of all these stabilizers forms a bundle, called the period bundle over U_r, $\mathscr{F} : S_c \to c \in U_r$. It is locally trivial by the action-angle theorem, but can be globally nontrivial.
Moreover, with notation as in action-angle theorem, for all $b \in \varphi(D) \subseteq B_r$ near c, a basic of S_b, denoted by $(\tau)(b)$ is given by the formula:

$$(\tau)(b) = (\tau^{(1)}(b), \dots, \tau^{(n)}(b)) = [d(\varphi^{-1})(b)].$$

Then, from previous relation and by noting the equation (108), we obtain that the transition function between two trivialization of the bundle $\mathscr{F}$ are $\{{}^tM_{\alpha\beta}^{-1}\}$ in $GL(n, \mathbb{Z})$- the integer linear group.

On the other hand, by noting that a $n-$tuple $\tau = (\tau_1, \dots, \tau_n) \in \mathbb{R}^n$ such that the Hamiltonian $\tau_1 f_1 + \dots + \tau_n f_n$ admits on torus Λ_c a periodic flow of period 1 if and only if $\tau \in S_c$, then we get so an isomorphism between the stabiliser S_c and the homology group $H_1(\Lambda_c, \mathbb{Z})$:

$$S_c \simeq H_1(\Lambda_c, \mathbb{Z}).$$

The the bundle $\mathscr{F}$ can be identified with the homology bundle $H_1(\Lambda_c, \mathbb{Z}) \to c \in B_r$. The possible nontriviality of this bundle is called (linear classical) monodromy of completely integrable system

F. This monodromy is equivalent to the global existence of integral action on the space of regular sheets.

4.2. **Relationship.** We recall that the (linear) classical monodromy is given by J.Duistermaat [18] (presented in the last section) is defined as a bundle $H_1(\Lambda_c, \mathbb{Z}) \to c \in U$, associated with a cocycle, denoted by $[\mathcal{M}_{cl}]$ in $\check{H}^1(U, GL(2,\mathbb{Z}))$ of transition functions:

$$\{{}^t(M^{cl}_{\alpha\beta})^{-1} = {}^t(d(\varphi_\alpha^{-1} \circ \varphi_\beta))^{-1}\}.$$

The remark (3.32) gives us thus the following relationship between two monodromy types.

Theorem 4.3. *The linear spectral monodromy is the adjoint of the linear classical monodromy*

$$[\mathcal{M}_{sp}] = {}^t[\mathcal{M}_{cl}]^{-1}.$$

In other words,if the corresponding representations of monodromy of $[\mathcal{M}_{sp}]$ *and* $[\mathcal{M}_{cl}]$ *are*

$$\begin{aligned} \mu^{sp} : \pi_1(U(\varepsilon)) &\to GL(2,\mathbb{Z})/\{\sim\} \\ \mu^{cl} : \pi_1(U) &\to GL(2,\mathbb{Z})/\{\sim\}, \end{aligned} \tag{110}$$

then $\mu^{sp} = {}^t(\mu^{cl})^{-1}$.

Remark 4.4 A particular case when $[P,Q] = 0$, this implies $\{p,q\} = 0$. We can so have two spectral monodromy types for P_ε: the affine spectral monodromy, defined in section 2 and the linear spectral monodromy, defined in this section. It is obvious that the linear spectral monodromy is the linear part of the affine spectral monodromy.
Note also that in this case, by definition, the affine spectral monodromy is equal to the quantum monodromy and it is know from a result of S. Vu Ngoc [45] that the last monodromy is adjoint of classical monodromy. That asserts once again the result of previous theorem in the integrable quantum case. △

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