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December 18, 2012

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# Overlapping tile automata

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## Abstract

Premorphisms are monotonic mappings between partially ordered monoids where the morphism condition  $\varphi(xy) = \varphi(x)\varphi(y)$  is relaxed into the condition  $\varphi(xy) \leq \varphi(x)\varphi(y)$ . Their use in place of morphisms has recently been advocated in situations where classical algebraic recognizability collapses.

With languages of overlapping tiles, such an extension of classical recognizability by morphisms, called quasi-recognizability, has already proved both its effectiveness and its power; it is shown to essentially capture definability in monadic second order logic.

In this paper, we complete the theory of languages of such tiles by providing a notion of (finite state) tile automaton that is proved to be both sound and complete with respect to quasi-recognizability, i.e. every quasi-recognizable language of tiles is definable by a finite state tile automaton and, conversely, every language of tiles definable by a finite state tile automaton is quasi-recognizable.

## 1 Introduction

### 1.1 Motivations and background

There are many ways to describe one-dimensional overlapping tiles : the objects which are studied in this paper. Arising in inverse semigroup theory, they can be

defined as (representations of) elements McAlister monoid [13], linear and unidirectional birooted trees, used in studies [9, 10] of the structure of tiling (in the usual sense with no overlaps) of the  $d$ -dimensional Euclidian space  $\mathbb{R}^d$ .

Overlapping tiles can also be seen as two way string objects extended by extra history recording capacities that prevent a new letter from being placed in a position where another distinct letter has already been positioned in the past.

For instance, starting from a given string object, say  $ab$  with two distinct letters  $a$  and  $b$ , one can remove letter  $b$  from the right of that object. The resulting object is denoted by  $abb^{-1}$ . If these objects are treated just as standard string objects,  $b^{-1}$  acts as the *group inverse* of  $b$ , and thus  $bb^{-1} = 1$  henceforth  $abb^{-1} = a$ . If these string objects are treated as strings extended with recording capacity as mentioned above, then  $abb^{-1} \neq a$ . In that case,  $b^{-1}$  acts as the *pseudo-inverse* of  $b$  and  $bb^{-1}$  is seen rather as sort of a footprint of letter  $b$  that is kept on the right side of letter  $a$ .

Now if one adds that same letter  $b$  to the right of  $abb^{-1}$  again, this leads to build the building of the new object  $abb^{-1}b$  which, in both cases, equals the string object  $ab$ . Indeed, adding letter  $b$  to the right of its footprint  $bb^{-1}$  merely amounts to rebuilding  $b$ , i.e.  $bb^{-1}b = b$ .

On the contrary, if one tries to add letter  $a$  - distinct from  $b$  - to the right of  $abb^{-1}$  then the resulting object  $abb^{-1}a$  equals 0: the undefined tile. Indeed, with standard string objects the resulting value would be  $aa$  but with overlapping tiles, no other letter than the original can be positioned on the right footprint  $bb^{-1}$  of  $b$  hence  $bb^{-1}a = 0$  and thus  $abb^{-1}a = 0$ .

Surprisingly, this extension of the string data type turns out to be a mathematically well-founded and structurally rich extension of that type.

Adding and removing letters to the right or to the left of an extended string induces an associative product: the underlying algebraic structure is the monoid of overlapping tiles. The history recording mechanism induces a partial order relation: the natural order defined on tiles.

Recent modeling experiments [1], conducted with variants of overlapping tiles in computational music, further illustrate how the structural richness of the monoid of tiles can be used to great benefit. Indeed both sequential or parallel composition operators can be derived from the monoid structure.

There is thus a need to develop a language theory for overlapping tiles. Such a study has been initiated in [8]. However, an immediate difficulty is that, as already observed for inverse monoids [14, 16], classical language theory tools are not directly applicable. Indeed, on birooted trees [16] - in the free inverse monoids - or even on positive overlapping tiles [8] - in a quasi-inverse monoid -, the notion

of recognizability defined by morphisms into finite monoids - equivalently by finite state automata - collapses in terms of expressive power.

As a remedy, the use of premorphisms instead of morphisms has been successfully proposed [7]. Indeed, this variant of algebraic recognizability, called quasi-recognizability, is shown to *essentially* captures the expressive power of Monadic Second Order Logic (MSO) over tiles [8].

The purpose of the present paper is to extend and strengthen such an emerging algebraic theory by providing it with an automata theoretical counterpart.

## 1.2 Outline

Overlapping tile automata are finite state word automata with a semantics that is now defined in terms of overlapping tiles. In this paper, we prove that they capture the notion of quasi-recognizable languages of tiles [7]: every quasi-recognizable languages of tiles is definable by a finite state tile automaton and, conversely, every language of tiles definable by a finite state tile automaton is quasi-recognizable

Previously, our definition of recognizability by premorphisms was only defined for languages of positive tiles. This new automata theoretical approach induces a refined definition that, equivalent to our former proposal on positive tiles, is now applicable to languages of arbitrary positive and negative tiles.

It must also be mentioned that, although we restrict our presentation to linear shaped tiles: avoiding the combinatorial complexity that would be induced by tree shaped tiles, further studies provide evidences that our proposed construction generalizes to birooted trees [15] and to (bottom or top-down) finite state tree automata.

This paper is organized as follows. After a series of preliminary remarks that essentially serve fix notations for classical concepts, the monoid of overlapping tiles is briefly presented in the following section. Special classes of premorphisms and ordered monoids, referred to as adequate, are defined and studied in the third section: they provide the appropriate concepts for defining an effective notion of quasi-recognizability. Tile automata are defined in the fourth section where we also prove that they capture quasi-recognizable languages. Several research perspectives are outlined as a conclusion.

## 1.3 Preliminaries

Given a finite alphabet  $A$ ,  $A^*$  denotes the free monoid generated by  $A$ ,  $1$  denotes the neutral element. The concatenation of two words  $u$  and  $v$  is denoted by  $u \cdot v$  or

simply  $uv$ . More generally, given a monoid  $S$ , the neutral element of  $S$  is denoted by 1 and the product of two elements  $x$  and  $y \in S$  is also denoted by  $x \cdot y$  or simply  $xy$ .

Relation  $\leq_p$  (resp.  $\leq_s$ ) stands for the prefix (resp. suffix) order over  $A^*$  with  $\vee_p$  (resp.  $\vee_s$ ) denoting the joint operator for the prefix (resp. suffix) order. It follows that  $u \vee_p v$  (resp.  $u \vee_s v$ ) is the least word whose both  $u$  and  $v$  are prefixes (resp. suffixes) for every word  $u$  and  $v \in A^*$ .

The extended monoid  $A^* + \{0\}$  (with  $0 \cdot u = u \cdot 0 = 0$  for every word  $u$ ), ordered by  $\leq_p$  (extended with  $u \leq_p 0$  for every word  $u$ ), is a lattice; in particular,  $u \vee_p v = 0$  whenever neither  $u$  is a prefix of  $v$ , nor  $v$  is a prefix of  $u$ . Symmetric properties hold in the suffix lattice.

Given  $\bar{A}$  a disjoint copy of  $A$ ,  $u \mapsto \bar{u}$  denotes the mapping from  $(A + \bar{A})^*$  to itself inductively defined by  $\bar{1} = 1$ , for every letter  $a \in A$ ,  $\bar{a}$  is the copy of  $a$  in  $\bar{A}$  and  $\overline{\bar{a}} = a$ , and, for every word  $u \in (A + \bar{A})^*$ ,  $\overline{\bar{u}} = u \cdot \bar{a}$ . By definition, for every word  $u$  and  $v \in (A + \bar{A})^*$  one has  $\overline{\bar{u}} = u$  and  $\overline{\bar{u}\bar{v}} = \bar{v} \cdot \bar{u}$ .

The free group  $FG(A)$  generated by  $A$  is the quotient of  $(A + \bar{A})^*$  by the least congruence  $\simeq$  such that, for every letter  $a \in A$ ,  $a\bar{a} \simeq 1$  and  $\bar{a}a \simeq 1$ . As well known, every class  $[u] \in FG(A)$  contains a unique element  $red(u)$  (the *reduced form* of  $u$ ) that contains no factors of the form  $a\bar{a}$  or  $\bar{a}a$ .

## 2 The monoid of overlapping tiles

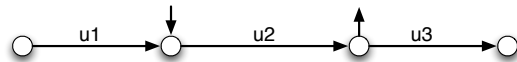
Here we briefly give a description of the monoid of overlapping tiles as presented and studied in [8].

A *tile* over the alphabet  $A$  is a triple of words  $x = (u_1, u_2, u_3) \in A^* \times (A^* + \bar{A}^*) \times A^*$  such that, if  $u_2 \in \bar{A}^*$ , its inverse  $\bar{u}_2$  is a suffix of  $u_1$  and a prefix of  $u_3$ .

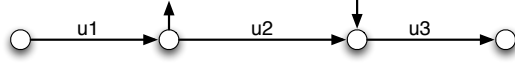
When  $u_2 \in A^*$  we say that  $x$  is a *positive tile*. When  $u_2 \in \bar{A}^*$  we say that  $x$  is a *negative tile*. When  $u_2 = 1$ , i.e. when  $x$  is both positive and negative, we say that  $x$  is a *context tile*. Sets  $T_A$ ,  $T_A^+$ ,  $T_A^-$  and  $C_A$  will respectively denote the set of tiles, the set of positive tiles, the set of negative tiles and the set of context tiles over  $A$ .

The *domain* of a tile  $x = (u_1, u_2, u_3)$  is the reduced form of  $u_1 u_2 u_3$  (always a word of  $A^*$ ); its *root* is the word  $u_2$ .

A positive tile  $x = (u_1, u_2, u_3)$  is conveniently drawn as a (linear, unidirectional and left to right) Munn's birooted word tree [15]:

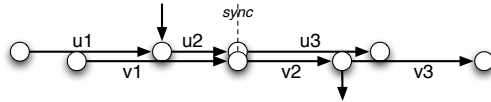


where the dangling input arrow (marking the beginning of the root) appears on the left of the dangling output arrow (marking the end of the root). The corresponding negative tile  $x^{-1} = (u_1u_2, \bar{u}_2, u_2u_3) \in A^* \times \bar{A}^* \times A^*$  is also drawn as a birooted word tree

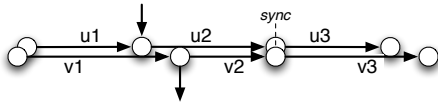


where the dangling input arrow appears on the right of the dangling output arrow. The mapping that maps every tile  $x = (u_1, u_2, u_3)$  to  $x^{-1} = (u_1u_2, \bar{u}_2, u_2u_3)$  is involutive. Tile  $x^{-1}$  is called the inverse of tile  $x$ .

The sequential product of two tiles  $x = (u_1, u_2, u_3)$  and  $y = (v_1, v_2, v_3)$  is defined by their superposition in such a way that the end of the root of the first tile coincides with the beginning of the root of the second tile. When both tiles are positive, this can be drawn as follows:



The product can be extended to arbitrary tiles, as illustrated by the following figure (with positive  $x$  and negative  $y$ ):



Formally, we extend the set  $T_A$  with a zero tile to obtain  $T_A^0 = T_A + \{0\}$ . The sequential product of two non-zero tiles  $x = (u_1, u_2, u_3)$  and  $y = (v_1, v_2, v_3)$  is defined as

$$x \cdot y = ((u_1u_2 \vee_s v_1)\bar{u}_2, u_2v_2, \bar{v}_2(u_3 \vee_p v_2v_3))$$

when both pattern matching conditions  $u_1u_2 \vee_s v_1 \neq 0$  and  $u_3 \vee_p v_2v_3 \neq 0$  are satisfied, and  $x \cdot y = 0$  otherwise. Product is extended to zero by taking  $x \cdot 0 = 0 \cdot x = 0$  for every  $x \in T_A^0$ .

Set  $T_A^0$  equipped with the above product is a *monoid* [8] isomorphic to McAslister monoid [13]. Extending the inverse mapping to 0 by taking  $0^{-1} = 0$ , for every tile  $x \in T_A^0$ , tile  $x^{-1}$  is the unique tile such that both  $xx^{-1}x = x$  and  $x^{-1}xx^{-1} = x$ . Monoid  $T_A^0$  is thus an *inverse monoid* [12].

As such, idempotents are elements of the form  $xx^{-1}$  (equivalently  $x^{-1}x$ ) and the *natural order* associated with the inverse monoid  $T_A^0$  is defined by  $x \leq y$  when  $x = xx^{-1}y$  (or, equivalently  $x = yx^{-1}x$ ).



One can easily check that for every non zero tile  $x = (u_1, u_2, u_3)$  and  $y = (v_1, v_2, v_3)$ , we have  $x \leq y$  if and only if  $u_1 \geq_s v_1$ ,  $u_2 = v_2$  and  $u_3 \geq_p v_3$ . In particular, for every non zero tile  $x = (u_1, u_2, u_3) \in T_A$ , one has  $x_L = x^{-1}x = (u_1u_2, 1, u_3)$  and  $x_R = xx^{-1} = (u_1, 1, u_2u_3)$  that are idempotents.

An immediate property worth being mentioned: the mapping  $u \mapsto (1, u, 1)$  from  $A^*$  to  $T_A$  is a one-to-one morphism. In the remainder of the text we may use the same notation for words of  $A^*$  and their images in  $T_A^0$ .

### 3 Quasi-recognizable languages of tiles

We define in this section a notion of quasi-recognizability extending the one proposed in [7, 6]. The major difference is that in [7, 6] our proposal was made for languages of positive tiles only while here, its extension holds for languages of arbitrary tiles.

#### 3.1 Adequately ordered monoids

Let  $S$  be a monoid partially ordered by a relation  $\leq_S$  (or just  $\leq$  when there is no ambiguity). We always assume that the order relation  $\leq$  is stable under product, i.e. if  $x \leq y$  then  $xz \leq yz$  and  $zx \leq zy$  for every  $x, y$  and  $z \in S$ . The set  $U(S)$  of *subunits* of the partially ordered monoid  $S$  is defined by  $U(S) = \{y \in S : y \leq 1\}$ .

A partially ordered monoid  $S$  is an *adequately ordered monoid* when all subunits of  $S$  are idempotents, and, for every  $x \in S$ , both the minimum of right local units

$$x_L = \min\{y \in U(S) : xy = x\}$$

and the minimum of left local units

$$x_R = \min\{y \in U(S) : yx = x\}$$

exist and belong to  $U(S)$ .

For every  $x \in S$ , the subunits  $x_L$  and  $x_R$  are also called the *left projection* and the *right projection* of  $x$ . Since subunits are assumed to be idempotents, one can check that they commute and thus, ordered by the monoid order, form a meet semilattice with the product as the meet operators. It follows that when  $x$  is itself a subunit, we have  $x = x_L = x_R$ . In other words, in an adequately ordered monoid, both left and right projection mappings are indeed projections from  $S$  onto  $U(S)$ .

*Examples.* Every monoid  $S$  extended with the trivial order  $x \leq y$  when  $x = y$  is a adequately ordered monoid with  $x_L = x_R = 1$  for every  $x \in S$ . These adequately ordered monoids are called trivial.

Every inverse monoid  $S$  ordered by the natural order relation defined by  $x \leq y$  when  $x = xx^{-1}y$  (or equivalently  $y = yx^{-1}x$ ) for every  $x$  and  $y \in S$  is a adequately ordered monoid with  $x_L = x^{-1}x$  and  $x_R = xx^{-1}$ .

As shown in the next section, the relation monoid  $\mathcal{P}(Q \times Q)$  ordered by inclusion is also an adequately ordered monoid.

*Remark.* For the reader familiar with the work initiated by Fountain [5], an ordered monoid  $S$  is adequately ordered exactly when it is  $U(S)$ -semiadequate [11].

This means that, conversely, a  $U$ -semiadequate is an adequate monoid when its natural order defined by  $x \preceq y$  when  $x = x_R y x_L$  can be extended into a partial order  $\leq$  that is stable under product with  $U = U(S)$ . In that case, both order  $\preceq$  and  $\leq$  coincide on subunits.

In [7], only  $U$ -semiadequate monoids with stable natural order were considered and shown to suffice for languages of positive tiles. Of course, every such a monoid is also an adequately ordered monoid.

Our more relaxed definition comes from the fact that the natural order on relation monoid  $\mathcal{P}(Q \times Q)$  is not itself stable under product.

**Lemma 1** *Let  $S$  be an adequately monoid. For every  $x$  and  $y \in S$ , if  $x$  and  $y$  are  $\mathcal{R}$ -equivalent (i.e. if  $x \cdot S = y \cdot S$ ) then  $x_R = y_R$  and, symmetrically, if  $x$  and  $y$  are  $\mathcal{L}$ -equivalent (i.e. if  $S \cdot x = S \cdot y$ ) then  $x_L = y_L$ .*

In other words, left and right canonical local identities of a given element can be seen as some approximation of its left and right Green's classes.

*Remark.* It is worth being mentioned that [7]: every monoid  $S$  can be embedded into an adequately ordered monoid  $\mathcal{Q}(S)$ , the *quasi-inverse expansion* of  $S$ , so that every (images of) two elements  $x$  and  $y \in S$ , one has  $x_L = y_L$  (resp.  $x_R = y_R$ ) in  $\mathcal{Q}(S)$  if and only if  $x$  and  $y$  are  $\mathcal{L}$ -equivalent (resp.  $\mathcal{R}$ -equivalent). We will use a similar idea in the proof of Theorem 5 below.

## 3.2 Adequate premorphisms

A mapping  $\varphi : S \rightarrow T$  between two adequately ordered monoids is a *premorphisms* when  $\varphi(1) = 1$  and, for every  $x$  and  $y \in S$ , if  $x \leq_S y$  then  $\varphi(x) \leq_T \varphi(y)$  and  $\varphi(xy) \leq_T \varphi(x)\varphi(y)$ .

A premorphism  $\varphi$  is an *adequate premorphism* when  $\varphi(x_L) = (\varphi(x))_L$  and  $\varphi(x_R) = (\varphi(x))_R$  for every  $x \in S$ , and if  $x_L \vee y_R = 1$  then  $\varphi(xy) = \varphi(x)\varphi(y)$  for every  $x$  and  $y \in S$ . In that latter case we say that the product  $xy$  is disjoint.

**Lemma 2** *Let  $\varphi : T_A^0 \rightarrow S$  be an adequate premorphism. The restriction of  $\varphi$  to  $A^*$  is a morphism and, for every positive tile  $(u, v, w) \in T_A^+$  one has*

$$\varphi((u, v, w)) = (\varphi(u))_L \cdot \varphi(v) \cdot (\varphi(w))_R$$

*Symmetrically, the restriction of  $\varphi$  to the inverses of  $A^*$  is also a morphism and, for every negative tile  $(uv, \bar{v}, vw)$  one has*

$$\varphi((uv, \bar{v}, vw)) = (\varphi(w))_R \cdot \varphi(v^{-1}) \cdot (\varphi(u))_L$$

*In particular, when  $S$  is finite, for every tile  $x \in T_A^0$ , the image  $\varphi(x)$  of  $x$  by  $\varphi$  is effectively computable from the images of  $\varphi(A)$ ,  $\varphi(\bar{A})$  and the left and the right canonical projections.*

*Proof.* Every word of  $A^*$  (seen as a tile) is the disjoint product of its letters (also seen as tiles) hence, restricted to on  $A^*$ , mapping  $\varphi$  behave like a morphism. By symmetry, the same holds for inverses of words of  $A^*$ . Indeed, for every tile  $x$  and  $y$ , if  $xy$  is a disjoint product then so is  $y^{-1}x^{-1}$ .

Now, one can observe that for every tile  $x = (u, v, w)$  we have  $x = u_L \cdot v \cdot w_R$  with only disjoint products. By symmetry, this also shows that  $x^{-1} = w_R \cdot v^{-1} \cdot u_L$  since subunits are self inverses. Adequacy assumption enables us to conclude the proof since  $\varphi(u_L) = (\varphi(u))_L$  and  $\varphi(w_R) = (\varphi(w))_R$ .  $\square$

*Remark.* As opposed to our former definition [7], the actual definition of adequate premorphism, built via disjoint products, is applicable to arbitrary tiles and, even beyond the scope of that paper, to arbitrary birooted trees.

### 3.3 Quasi-recognizable languages

A language  $L \subseteq T_A$  is a *quasi-recognizable language of tiles* when there exists an adequate premorphism  $\varphi : T_A^0 \rightarrow S$  in a finite adequately ordered monoid  $S$  such that  $L = \varphi^{-1}(\varphi(L))$ .

Lemma 2 ensures that quasi-recognizable languages are MSO definable (see [8] for a definition of MSO logic over tiles). The following Lemma (proved in [7]) tells that quasi-recognizable languages of tiles are presumably not all MSO definable languages of tiles since they satisfy some coherent context closure property.

**Lemma 3** *For every language  $L \subseteq T_A$  recognized by premorphism  $\varphi : T_A \rightarrow S$ , for every  $(u, v, w) \in L$ , for every  $u' \in A^*$  (resp.  $w' \in A^*$ ), if  $S \cdot \varphi(u') = S \cdot \varphi(u)$  (resp.  $\varphi(w') \cdot S = \varphi(w) \cdot S$ ) then  $(u', v, w) \in L$  (resp.  $(u, v, w') \in L$ ).*

However, quasi-recognizability *essentially* capture MSO in the following sense. Given a new letter  $\# \notin A$ , given  $\#(L) \subseteq T_{A+\#}$  defined for every language  $L \subseteq T_A$  by  $\#(L) = \{(\#u, v, w\#) : (u, v, w) \in L\}$ , we have: language  $\#(L)$  is quasi-recognizable if and only if  $L$  is definable in MSO.

Indeed, this follows from Theorem 4 presented below and the fact that, in presence of  $\#$  on both sides of tile's domain, the left and right equivalences induced by any adequate premorphism recognizing  $\#(L)$  are trivial.

## 4 Tile automata and quasi-recognizability

In this section, we define finite state tile automata and proved that they capture quasi-recognizable languages of tiles.

### 4.1 Tile automata

A finite tile automaton is a triple  $\mathcal{A} = \langle Q, \delta, K \rangle$  with a finite set of states  $Q$ , transition function  $\delta : A \rightarrow \mathcal{P}(Q \times Q)$  and an accepting set  $K \subseteq \mathcal{P}(Q \times Q)$ .

Given a word  $u \in A^*$ , we say that automaton  $\mathcal{A}$  reads  $u$  from state  $p$  to state  $q$ , which is denoted by  $p \xrightarrow{u} q$ , when  $u = a_1 \cdots a_n$ , and there is  $n + 1$  states  $q_0 = p, q_1, q_2, \dots, q_n = q \in Q$  such that for every  $1 \leq i \leq n$ ,  $(q_{i-1}, q_i) \in \delta(a_i)$ .

A run of automaton  $\mathcal{A}$  on a positive tile  $x = (u, v, w) \in T_A^+$  is a pair of states  $(p, q) \in Q \times Q$ , the input state  $p$  and the output state  $q$ , such that there exist two other states: a start state  $s$  and an end state  $e$ , such that  $s \xrightarrow{u} p, p \xrightarrow{v} q$  and  $q \xrightarrow{w} e$ .

The run image of the positive tile  $x = (u, v, w) \in T_A^+$  is then defined as the set of pairs of states

$$\varphi_{\mathcal{A}}((u, v, w)) = \{(p, q) \in Q \times Q : \exists s, e \in Q, s \xrightarrow{u} p, p \xrightarrow{v} q, q \xrightarrow{w} e\}$$

i.e. the set of all runs of the tile automaton  $\mathcal{A}$  over  $u$ . Then we say that  $x$  is accepted by  $\mathcal{A}$  when  $\varphi_{\mathcal{A}}(x) \in K$ . The set  $L^+(\mathcal{A}) \subset T_A^+$  of positive tiles recognized by automaton  $\mathcal{A}$  is then defined as  $L^+(\mathcal{A}) = \{x \in T_A^+ : \varphi_{\mathcal{A}}(x) \in K\}$ .

This definition is extended to negative tiles by saying that for every negative tile  $x = (uv, \bar{v}, vw)$ :

$$\varphi_{\mathcal{A}}((uv, \bar{v}, vw)) = \{(q, p) \in Q \times Q : \exists s, e \in Q, s \xrightarrow{u} p, p \xrightarrow{v} q, q \xrightarrow{w} e\}$$

and we put  $L^-(\mathcal{A}) = \{x \in T_A^- : \varphi_{\mathcal{A}}(x) \in K\}$ .

The language  $L(\mathcal{A}) \subseteq T_A$  of tiles recognized by  $\mathcal{A}$  is then defined by  $L(\mathcal{A}) = L^+(\mathcal{A}) + L^-(\mathcal{A})$ .

*Remark.* With such a definition, for all tile  $x \in T_A$ , for every pair  $(p, q) \in \varphi_{\mathcal{A}}(x)$ , there is a run of  $\mathcal{A}$  on the domain of  $x$  such that state  $p$  is labeling the beginning of the root of  $x$  and state  $q$  is labeling the end of the root of  $x$  regardless  $x$  is positive or negative. This means that the inversion of roots, from positive to negative tiles, is detectable by tile automata.

Indeed, the simplest automaton with two states  $Q = \{1, 2\}$  and transitions  $\delta(a) = \{(1, 1), (1, 2), (2, 2)\}$  for every  $a \in A$  will distinguish positive tiles with the pair of states  $(1, 2)$  from negative tiles with the pair of states  $(2, 1)$ .

## 4.2 From tile automata to quasi-recognizability

**Theorem 4** *Every language of tiles definable by a finite state tile automaton is quasi-recognizable.*

*Proof.* Let  $\mathcal{A} = \langle Q, \delta, K \subseteq Q \times Q \rangle$  be a tile automaton and let  $\varphi_{\mathcal{A}} : T_A \rightarrow \mathcal{P}(Q \times Q)$  be the run mapping induced by  $\mathcal{A}$ . We essentially have to prove that  $\mathcal{P}(Q \times Q)$  equipped with the product  $X \cdot Y = \{(p, q) \in Q \times Q : \exists r \in Q, (p, r) \in X, (r, q) \in Y\}$  and ordered by inclusion is an adequately ordered monoid and that  $\varphi_{\mathcal{A}}$  extended to 0 by taking  $\varphi_{\mathcal{A}}(0) = \emptyset$  is an adequate premorphism.

The fact  $\mathcal{P}(Q \times Q)$  equipped with the above product is a monoid with neutral element  $I_Q = \{(q, q) \in Q \times Q : q \in Q\}$  is a classical result. The fact inclusion is stable w.r.t. to product is also well known. It suffices thus to prove that canonical left and right identities exist.

Let  $X \in \mathcal{P}(Q \times Q)$  and let  $X_R = \{(p, p) \in Q \times Q : \exists q \in Q, (p, q) \in X\}$  and let  $X_L = \{(q, q) \in Q \times Q : \exists p \in Q, (p, q) \in X\}$ . One can easily check that  $X = X_R \cdot X = X \cdot X_L$  for every  $X \subseteq Q \times Q$ . Let then  $Y \subseteq I_Q$  such that  $Y \cdot X = X$  (resp.  $X \cdot Y = X$ ). It is an immediate observation that this implies  $X_R \subseteq Y$  (resp.  $X_L \subseteq Y$ ). In other words,  $X_L$  (resp.  $X_R$ ) is indeed the least right (left) local unit for  $X$ .

The fact  $\varphi_{\mathcal{A}}$  extended to zero as defined above is an premorphism raises no real difficulty. By definition, we have  $\varphi_{\mathcal{A}}(1) = I_Q$  and it is rather immediate that  $\varphi_{\mathcal{A}}(u) \subseteq \varphi_{\mathcal{A}}(v)$  whenever  $u \leq v$  in  $T_A$ . The fact that we also have  $\varphi_{\mathcal{A}}(uv) \subseteq \varphi_{\mathcal{A}}(u)\varphi_{\mathcal{A}}(v)$  for every  $u$  and  $v \in T_A^0$  is a little more complex to check but with no special difficulty.

The fact  $\varphi_{\mathcal{A}}$  is also adequate is somehow simpler and essentially follows from the definition. In particular, the disjoint product case just mimics the classical case where  $\varphi_{\mathcal{A}}$  is defined over words only.  $\square$

*Remark.* Observe that with our automata semantics via  $\varphi_{\mathcal{A}}$ , writing

$$X^{-1} = \{(q, p) \in Q \times Q : (p, q) \in X\}$$

for every relation  $X \subseteq Q \times Q$ , we have

$$\varphi_{\mathcal{A}}(u^{-1}) = (\varphi_{\mathcal{A}}(u))^{-1}$$

and thus we also have

$$L^-(\mathcal{A}) = \{u \in T_{\mathcal{A}}^- : (\varphi_{\mathcal{A}}(u^{-1}))^{-1} \in K\}$$

In general, this property is not satisfied by an arbitrary adequate premorphism. However, it is an immediate consequence of our two main results that any adequate premorphism can still be translated into an equivalent automaton and thus into an equivalent premorphism of the form  $\varphi_{\mathcal{A}}$ .

### 4.3 From quasi-recognizability to tile automata

**Theorem 5** *Every quasi-recognizable language of tiles is definable by a finite state tile automaton.*

*Proof.* Let  $\psi : T_{\mathcal{A}}^0 \rightarrow S$  be an adequate premorphism into a finite adequately ordered monoid  $S$  let  $K_{\psi} \subseteq S$  and let  $L = \psi^{-1}(K_{\psi})$ . We want to build a finite state automaton  $\mathcal{A} = \langle Q, \delta, K \rangle$  such that  $\varphi_{\mathcal{A}}^{-1}(K) = \psi^{-1}(K_{\psi})$ .

To achieve such a goal it is sufficient to define an automaton  $\mathcal{A}$  that, given any positive tile  $x = (u, v, w)$  (resp. negative tile  $x^{-1} = (uv, \bar{v}, vw)$ ) as input, computes, via  $\varphi_{\mathcal{A}}(x)$  (resp.  $\varphi_{\mathcal{A}}(x^{-1})$ ), the *left ideal*  $S.\psi(u)$  associated to  $\psi(u)$ , the *right ideal*  $\psi(w).S$  associated to  $\psi(w)$  and the *image*  $\psi(v)$  of the root  $v$  of  $x$  (resp. the *image*  $\psi(v^{-1})$  of the root  $\bar{v}$  of  $x^{-1}$ ).

Indeed, by Lemma 1, computing these left and right ideals is enough to compute the expected left and right canonical identities  $\psi(u_L)$  and  $\psi(w_R)$ . Then, by applying Lemma 2, together with the value of  $\psi(v)$  (resp.  $\psi(v^{-1})$ ), we can compute the value  $\psi(x)$  of  $x$  (resp.  $\psi(x^{-1})$  of  $x^{-1}$ ) by premorphism  $\psi$ .

Extended the idea described in automata section in order to distinguish positive from negative tiles, automaton  $\mathcal{A}$  is defined as follows.

The set of state  $Q$  is defined to be  $Q = S \times S \times S \times M$  with set of modes  $M = \{P, S, PR, NR, c, p_1, p_2, n_1, n_2\}$  where  $P, S, PR$  and  $NR$  respectively stand for some stable automaton mode prefix, suffix, positive root and negative root automaton

modes, and  $c, p_1, p_2, n_1$  and  $n_2$  stand for some frontier modes that occur at most once in between stable modes.

The set  $\delta(a)$  of transition labeled by  $a$  is defined, for every  $a \in A$ , to be the union of the following sets of transitions:

▷ “prefix” transitions: from mode  $P$  to mode  $n \in \{P, c, p_1, n_1\}$ ,

$$\{((x, y, z, P), (x \cdot \varphi(a), y, z, n)) : x, y, z \in Q\}$$

▷ “positive root” transitions: from mode  $m \in \{PR, p_1\}$  to mode  $n \in \{PR, p_2\}$ ,

$$\{((x, y, z, m), (x, y \cdot \varphi(a), z, n)) : x, y, z \in S\}$$

▷ “negative root” transition: from mode  $m \in \{PN, n_1\}$  to mode  $n \in \{PN, n_2\}$ ,

$$\{((x, y, z, m), (x, \varphi(a^{-1}) \cdot y, z, n)) : x, y, z \in S\}$$

▷ “suffix” transitions: from mode  $m \in \{S, c, p_2, n_2\}$  to mode  $S$ :

$$\{((x, y, \varphi(a) \cdot z, m), (x, y, z, S)) : x, y, z \in S\}$$

Of course, such an automaton will run freely on tiles regardless it is running on the prefix, the root or the suffix part of a tile. However, by watching the states in frontier modes occurring at the extremities of the root, we can collect all information we need.

More precisely, the next step is then to keep from the set of all runs of  $\mathcal{A}$  on any given input tile  $X$  only the relevant data. This is achieved by the following mapping. For every  $X \subseteq \mathcal{P}(Q^+ \times Q^+)$ , we define the relevant image  $f(X) \subseteq S \times S \times S$  of  $X$  “in”  $S$  by taking:

$$f(X) = \{(x, 1, z) \in S \times S \times S : ((x, 1, z, c), (x, 1, z, c)) \in X\} \quad (1)$$

$$\cup \{(x, y, z) \in S \times S \times S : ((x, 1, z, p_1), (x, y, z, p_2)) \in X\} \quad (2)$$

$$\cup \{(x, y, z) \in S \times S \times S : ((x, 1, z, n_2), (x, y, z, n_1)) \in X\} \quad (3)$$

where line (1) treats the case of context tiles, line (2) treats the case of (strictly) positive tiles and line (3) treats the case of (strictly) negative tiles.

With that construction, one can show that for every  $x = (u, v, w) \in T_A^+$  we have:

$$f(\varphi_{\mathcal{A}}(x)) = S \cdot \psi(u) \times \{\psi(v)\} \times \psi(w) \cdot S$$

and, for every  $x^{-1} = (uv, \bar{v}, vw) \in T_A^-$  we have:

$$f(\varphi_A(x^{-1})) = S \cdot \psi(u) \times \{\psi(v^{-1})\} \times \psi(w) \cdot S$$

In other words, for every  $x \in T_A$ , the finite value of  $f(\varphi_A(x))$  completely characterizes  $\psi(x)$  thus we conclude the proof by taking

$$\begin{aligned} K = & f^{-1}(\{S \cdot \psi(u) \times \{\psi(v)\} \times \psi(w) \cdot S : \psi(u_L) \cdot \psi(v) \cdot \psi(w_R) \in K_\psi\}) \\ & f^{-1}(\{S \cdot \psi(u) \times \{\psi(v^{-1})\} \times \psi(w) \cdot S : \psi(w_R) \cdot \psi(v^{-1}) \cdot \psi(u_L) \in K_\psi\}) \end{aligned}$$

By construction, for every tile  $x \in T_A$ , we indeed have  $x \in \varphi_A^{-1}(K)$  if and only if  $x \in \psi^{-1}(K_\psi)$ .  $\square$

*Remark.* If we consider a language  $L \subseteq T_A^+$  of *positive* tiles that is recognizable by a premorphism  $\psi$  from the monoid of positive tiles  $T_A^+$  into an adequately ordered monoid  $S$  then the above proof can easily be adapted so that  $\varphi_A : T_A^0 \rightarrow \mathcal{P}(Q \times Q)$  still recognizes  $L$ .

In other words, a quasi-recognizable language of positive tiles is also quasi-recognizable as a language of arbitrary tiles. This proves that the work presented here indeed generalizes the results formerly obtained in [7].

## 5 Conclusion

We have shown that the emerging notion of quasi-recognizability, defined in [7] as a remedy to the collapse of classical recognizability, can be equipped with a notion of finite state automata that captures its expressiveness for languages of tiles.

Compared to our former proposal [7], the notion of adequate premorphism has also been refined - especially via the notion of disjoint products - and can now be applied to more general settings. As already mentioned, further studies show that, in particular, extending the present work to birooted trees creates no particular difficulty even with labeled vertices.

Relying on the notion of  $U$ -semiadequate monoids [11], our present work also provides a rather unexpected application for the extension of inverse monoid theory initiated by Fountains [4, 5] in the 70's. Along the research lines proposed in [2, 3], our construction seems to raise new questions and possibilities.

In particular, we have seen that the adequately ordered monoids induced by automata are relational monoids  $\mathcal{P}(Q \times Q)$ . Although such a monoid is neither an inverse nor a regular monoid, an inverse operator is nevertheless defined over



relations. Then one can observe that, for every relation  $X$ , we have  $X_L \subseteq X^{-1} \cdot X$  and  $X_R \subseteq X \cdot X^{-1}$  hence, by stability,  $X = X_R \cdot X = X \cdot X_L \subseteq X \cdot X^{-1} \cdot X$ . In other words, the (relation) inverse operator induces a notion of *weak pseudo-inverse* about which, in the setting of partially ordered monoid with local units, further studies are probably worth being conducted.

## References

- [1] F. Berthaut, D. Janin, and B. Martin. Advanced synchronization of audio or symbolic musical patterns: an algebraic approach. In *Sixth IEEE International Conference on Semantic Computing (ICSC)*, pages 202–209. IEEE Society Press, 2012.
- [2] C. Cornock. *Restriction Semigroups: Structure, Varieties and Presentations*. PhD thesis, Departement of Mathematics University of York, 2011.
- [3] C. Cornock and V. Gould. Proper two-sided restriction semigroups and partial actions. *Journal of Pure and Applied Algebra*, 216:935–949, 2012.
- [4] J. Fountain. Right pp monoids with central idempotents. *Semigroup Forum*, 13:229–237, 1977.
- [5] J. Fountain. Adequate semigroups. *Proc. Edinburgh Math. Soc.*, 22(2):113–125, 1979.
- [6] D. Janin. Quasi-inverse monoids (and premorphisms). Technical Report RR-1459-12, LaBRI, Université de Bordeaux, 2012.
- [7] D. Janin. Quasi-recognizable vs MSO definable languages of one-dimensional overlapping tiles. In *Mathematical Foundations of computer Science (MFCS)*, volume 7464 of *LNCS*, pages 516–528, 2012.
- [8] D. Janin. On languages of one-dimensional overlapping tiles. In *International Conference on Current Trends in Theory and Practice of Computer Science (SOFSEM)*, LNCS, 2013, to appear.
- [9] J. Kellendonk. The local structure of tilings and their integer group of coinvariants. *Comm. Math. Phys.*, 187:115–157, 1997.
- [10] J. Kellendonk and M. V. Lawson. Universal groups for point-sets and tilings. *Journal of Algebra*, 276:462–492, 2004.

- [11] M. V. Lawson. Semigroups and ordered categories. i. the reduced case. *Journal of Algebra*, 141(2):422 – 462, 1991.
- [12] M. V. Lawson. *Inverse Semigroups : The theory of partial symmetries*. World Scientific, 1998.
- [13] M. V. Lawson. McAlister semigroups. *Journal of Algebra*, 202(1):276 – 294, 1998.
- [14] S. W. Margolis and J.-E. Pin. Languages and inverse semigroups. In *ICALP*, volume 172 of *Lecture Notes in Computer Science*, pages 337–346. Springer, 1984.
- [15] W. D. Munn. Free inverse semigroups. *Proceedings of the London Mathematical Society*, 29(3):385–404, 1974.
- [16] P. V. Silva. On free inverse monoid languages. *ITA*, 30(4):349–378, 1996.