



**HAL**  
open science

# Approximation numbers of composition operators on the Dirichlet space

Pascal Lefèvre, Daniel Li, Luis Rodriguez-Piazza, Hervé Queffélec

► **To cite this version:**

Pascal Lefèvre, Daniel Li, Luis Rodriguez-Piazza, Hervé Queffélec. Approximation numbers of composition operators on the Dirichlet space. 2012. hal-00766018

**HAL Id: hal-00766018**

**<https://hal.science/hal-00766018>**

Preprint submitted on 18 Dec 2012

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# Approximation numbers of composition operators on the Dirichlet space

*Pascal Lefèvre, Daniel Li, Hervé Queffélec,  
Luis Rodríguez-Piazza\**

December 18, 2012

**Abstract.** *We study the decay of approximation numbers of compact composition operators on the Dirichlet space. We give upper and lower bounds for these numbers. In particular, we improve on a result of O. El-Fallah, K. Kellay, M. Shabankhah and A. Youssfi, on the set of contact points with the unit circle of a compact symbolic composition operator acting on the Dirichlet space  $\mathcal{D}$ . We extend their results in two directions: first, the contact only takes place at the point 1. Moreover, the approximation numbers of the operator can be arbitrarily sub-exponentially small.*

**Mathematics Subject Classification.** Primary: 47B33 – Secondary: 28A12; 30C85; 31A15; 46E20; 46E22; 47B06

**Key-words.** approximation numbers – capacity – composition operator – cusp map – Dirichlet space – Schatten classes

## 1 Introduction

### 1.1 Organization of the paper.

The paper deals with composition operators. This area is widely studied nowadays, on various spaces of analytic functions (Hardy, Bergman, Dirichlet...spaces): one may read for instance the monographs [14] or [4] to get an overview on the subject until the nineties, and [5] or [8] for some recent results in the framework of the Dirichlet space. It seems natural to try to apply again some of the techniques used in the framework of Hardy or Bergman spaces. Nevertheless it is far from being that simple. Actually it often turns out that the Dirichlet space is one of the most difficult “classical” spaces to handle. For instance, a first difficulty appears at the very beginning of the theory: the composition operators are not necessarily bounded when we only require the symbol to belong to the Dirichlet space (whereas all the composition operators are bounded on the Hardy and the Bergman spaces).

---

\*Supported by a Spanish research project MTM 2009-08934.

The study of approximation numbers of composition operators acting on classical spaces of analytic functions (like the Hardy or the Bergman spaces) was initiated in [9] and [10] by the three last named authors. In the present paper, we get interested in the same kind of questions but for composition operators acting on the Dirichlet space. Some results already appear in [8] (among other things), but we focus exclusively on this topic in the sequel.

The notations and definitions are precised in the next subsection.

In section 2, we show that some similar phenomena (as in Hardy and Bergman spaces) hold in the framework of Dirichlet spaces. More precisely, the approximation numbers of composition operators on the Dirichlet space cannot decay more rapidly than exponentially, and this speed of convergence can only be attained for symbols verifying  $\|\varphi\|_\infty < 1$  (see Theorem 2.1 and Theorem 2.2). On the other hand, we investigate the extremal case and it turns out that  $C_\varphi$  may have almost geometric decay (in particular belong to all Schatten classes) and may touch the boundary of the  $\mathbb{D}$  (see Theorem 2.8).

In section 3, we focus on composition operators whose symbol is a cusp map. It plays the same role in the theory as the lens maps in the theory of Hardy spaces. The rate of decay of its approximation numbers is given in Theorem 3.1.

At last, in section 4., we precise Theorem 2.8 and prove in Theorem 4.1 that the symbol (which will be the composition of a cusp map and a peak function) may belong to both the disk algebra and the Dirichlet space, and moreover meet the boundary precisely at  $\{1\}$  with a level set which is any compact set with zero logarithmic capacity.

## 1.2 Notation and background.

We denote by  $\mathbb{D}$  the unit open disk of the complex plane and by  $A$  the normalized area measure  $dx dy/\pi$  of  $\mathbb{D}$ . The unit circle is denoted by  $\mathbb{T} = \partial\mathbb{D}$ .

A Schur function is an analytic self-map of  $\mathbb{D}$  and the associated composition operator is defined, formally, by  $C_\varphi(f) = f \circ \varphi$ . The function  $\varphi$  is called the symbol of  $C_\varphi$ .

The Dirichlet space  $\mathcal{D}$  is the space of analytic functions  $f: \mathbb{D} \rightarrow \mathbb{C}$  such that:

$$(1.1) \quad \|f\|_{\mathcal{D}}^2 := |f(0)|^2 + \int_{\mathbb{D}} |f'(z)|^2 dA(z) < +\infty.$$

If  $f(z) = \sum_{n=0}^{\infty} c_n z^n$ , one has:

$$(1.2) \quad \|f\|_{\mathcal{D}}^2 = |c_0|^2 + \sum_{n=1}^{\infty} n |c_n|^2.$$

Then  $\|\cdot\|_{\mathcal{D}}$  is a norm on  $\mathcal{D}$ , making  $\mathcal{D}$  a Hilbert space. We consider its subspace  $\mathcal{D}_*$ , consisting of functions  $f \in \mathcal{D}$  such that  $f(0) = 0$ . In this paper, we call  $\mathcal{D}_*$  the *Dirichlet space*. For further information on the Dirichlet space, the reader may see [1] or [13].

Recall that, whereas every Schur function  $\varphi$  generates a bounded composition operator  $C_\varphi$  on the Hardy or Bergman spaces, it is no longer the case for the Dirichlet space (see [11], Proposition 3.12, for instance).

The Bergman space  $\mathfrak{B}$  is the space of analytic functions  $f: \mathbb{D} \rightarrow \mathbb{C}$  such that:

$$\|f\|_{\mathfrak{B}}^2 := \int_{\mathbb{D}} |f(z)|^2 dA(z) < +\infty.$$

If  $f(z) = \sum_{n=0}^{\infty} c_n z^n$ , one has  $\|f\|_{\mathfrak{B}}^2 = \sum_{n=0}^{\infty} \frac{|c_n|^2}{n+1}$ .

We denote by  $S_n$  the truncation operator: if  $f(z) = \sum_{k=1}^{\infty} c_k z^k$  is in  $\mathcal{D}_*$ , then

$$(1.3) \quad (S_n f)(z) = \sum_{k=1}^n c_k z^k.$$

The Carleson window centered at  $\xi \in \mathbb{T}$  and of size  $h \in (0, 1)$  is defined as:

$$(1.4) \quad S(\xi, h) = \{z \in \mathbb{D}; |z - \xi| < h\}.$$

The notation  $A \lesssim B$  (equivalently  $B \gtrsim A$ ) means that  $A \leq CB$  for some constant  $C > 0$ , and  $A \approx B$  means that we have both  $A \lesssim B$  and  $B \lesssim A$ .

## 2 Approximation numbers in the general case

### 2.1 Geometric decay of the approximation numbers

We saw in [9], that the approximation numbers of composition operators on the Hardy space  $H^2$  as well as on the (weighted) Bergman spaces  $\mathfrak{B}_\alpha$ ,  $\alpha > -1$ , cannot decay more rapidly than exponentially, and that this speed of convergence can only be attained for symbols mapping the unit disk  $\mathbb{D}$  into a smaller disk  $r\mathbb{D}$ , with  $0 < r < 1$ . In this section, we see that the same phenomenon holds for the Dirichlet space. The proofs will be adapted from those of [9].

Our first result is on the geometric decay.

**Theorem 2.1** *Let  $\varphi: \mathbb{D} \rightarrow \mathbb{D}$  be an analytic self-map inducing a bounded composition operator on  $\mathcal{D}_*$ . Then, there exist positive constants  $c', c > 0$  and  $0 < r < 1$  such that the approximation numbers of the composition operator  $C_\varphi: \mathcal{D}_* \rightarrow \mathcal{D}_*$  satisfy:*

$$(2.1) \quad c' \sqrt{n} \|\varphi\|_{\infty}^n \geq a_n(C_\varphi) \geq cr^n, \quad n = 1, 2, \dots$$

**Proof.** We introduce two notations. First, we set, for any operator  $T$  on some Hilbert space  $H$ :

$$(2.2) \quad \beta(T) = \liminf_{n \rightarrow \infty} [a_n(T)]^{1/n}.$$

Next, let

$$\varphi^\#(z) = \frac{\varphi'(z)(1-|z|^2)}{1-|\varphi(z)|^2}$$

be the pseudo-hyperbolic derivative of  $\varphi$ ; we set:

$$(2.3) \quad [\varphi] = \sup_{z \in \mathbb{D}} |\varphi^\#(z)| = \|\varphi^\#\|_\infty.$$

Note that  $[\varphi] \leq 1$ , by the Schwarz-Pick inequality.

The upper bound is easy. We may assume that  $\|\varphi\|_\infty < 1$ . Note that, since  $C_\varphi$  is bounded on  $\mathcal{D}_*$ , we have  $\varphi^k = C_\varphi(z^k) \in \mathcal{D}_*$ . Then:

$$\begin{aligned} a_n(C_\varphi)^2 &\leq \|C_\varphi - C_\varphi S_{n-1}\|^2 \leq \|C_\varphi - C_\varphi S_{n-1}\|_{HS}^2 = \sum_{k=n}^{\infty} \frac{\|\varphi^k\|_{\mathcal{D}}^2}{k} \\ &= \sum_{k=n}^{\infty} \int_{\mathbb{D}} k |\varphi^{k-1}(z)|^2 |\varphi'(z)|^2 dA(z) \\ &= \int_{\mathbb{D}} \sum_{k=n}^{\infty} k |\varphi^{k-1}(z)|^2 |\varphi'(z)|^2 dA(z) \\ &\leq \int_{\mathbb{D}} \sum_{k=n}^{\infty} k \|\varphi\|_\infty^{2k-2} |\varphi'(z)|^2 dA(z) \\ &\leq K(\varphi) n \|\varphi\|_\infty^{2n} \|\varphi\|_{\mathcal{D}}^2, \end{aligned}$$

(we used that  $\sum_{k=n}^{\infty} k \rho^{k-1} = (\sum_{k=n}^{\infty} \rho^k)' = \frac{\rho^{n-1}[n-(n-1)\rho]}{(1-\rho)^2} \leq \frac{1}{(1-\rho)^2} n \rho^{n-1}$ , with  $\rho = \|\varphi\|_\infty^2$ ), implying

$$(2.4) \quad a_n(C_\varphi) \lesssim \sqrt{n} \|\varphi\|_{\mathcal{D}} \|\varphi\|_\infty^n \approx \sqrt{n} \|\varphi\|_\infty^n$$

and  $\beta(C_\varphi) \leq \|\varphi\|_\infty$ .

For the lower bound in (2.1), we shall prove that:

$$(2.5) \quad [\varphi]^2 \leq \beta(C_\varphi) \leq \|\varphi\|_\infty,$$

which will give the result, since for each  $\kappa < [\varphi]^2$ , there will be some constant  $c_\kappa > 0$  such that  $a_n(C_\varphi) \geq c_\kappa \kappa^n$ ,  $n \geq 1$ .

The inequality (2.5) is obtained as in the Hardy and Bergman cases in [9]. We may assume that  $C_\varphi$  is compact on  $\mathcal{D}_*$  (since otherwise  $\beta(C_\varphi) = 1$  and the result is trivial). Now, set  $\phi_u(z) = \frac{u-z}{1-\bar{u}z}$ ,  $u \in \mathbb{D}$ . Then, if  $\varphi$  is a symbol with  $C_\varphi$  compact on  $\mathcal{D}_*$  and  $a \in \mathbb{D}$ , let  $\psi = \phi_{\varphi(a)} \circ \varphi \circ \phi_a$ . Note that the compactness of  $C_\varphi$  on  $\mathcal{D}_*$  implies its compactness on  $\mathcal{D}$ . Hence we can write  $C_\psi = C_{\phi_a} \circ C_\varphi \circ C_{\phi_{\varphi(a)}}$ . Now, the relations  $\psi(0) = 0$ ,  $\psi'(0) = \frac{\varphi'(a)(1-|a|^2)}{1-|\varphi(a)|^2} = \varphi^\#(a)$  and the diagrams:

$$\mathcal{D}_* \xrightarrow{C_{\phi_a}} \mathcal{D} \xrightarrow{C_\varphi} \mathcal{D} \xrightarrow{C_{\phi_{\varphi(a)}}} \mathcal{D}_*,$$

with  $0 \xrightarrow{\phi_a} a \xrightarrow{\varphi} \varphi(a) \xrightarrow{\phi_{\varphi(a)}} 0$ , show that  $\psi \in \mathcal{D}_*$  and that  $C_\psi$  is also compact on  $\mathcal{D}_*$ . Now we notice that, for any compact composition operator  $C_\tau$  on  $\mathcal{D}_*$ , the solution  $\sigma$  of the König equation

$$\sigma \circ \tau = \tau'(0) \sigma, \quad \sigma(0) = 0, \quad \sigma'(0) = 1$$

has to belong to  $\mathcal{D}_*$  as this would be the case for any Hilbert space of analytic functions on  $\mathbb{D}$ . Hence, if  $\psi'(0) = \varphi^\#(a) \neq 0$ , the sequence of eigenvalues of  $C_\psi$  is  $([\psi'(0)]^n)_{n \geq 0}$ . It follows from [9], Lemma 3.2 (which is an easy consequence of Weyl's inequality) that  $\beta(C_\varphi) = \beta(C_\psi) \geq |\varphi^\#(a)|$ . Since this remains trivially true when  $\varphi^\#(a) = 0$ , Theorem 2.1 is proved.  $\square$

Now, we shall see that the geometric decay can take place only for symbols  $\varphi$  such that  $\|\varphi\|_\infty < 1$ .

**Theorem 2.2** *Let  $C_\varphi$  a bounded composition operator on  $\mathcal{D}_*$ . Then for each  $r \in (0, 1)$ , there exists  $s = s(r) \in (0, 1)$ , with  $s(r) \rightarrow 1$  as  $r \rightarrow 1$ , and such that:*

$$(2.6) \quad \|\varphi\|_\infty > r \quad \implies \quad a_n(C_\varphi) \gtrsim \frac{s^n}{\sqrt{n}}.$$

We shall see in the proof that we can take  $s = e^{-\varepsilon\pi}$ , with  $\varepsilon = 2\pi / \log \frac{1+r}{1-r}$  (see (2.9), where  $s$  is changed into  $s^2$ ).

Note that, in particular, with the notation (2.2), one has:

$$\|\varphi\|_\infty = 1 \quad \implies \quad \beta(C_\varphi) = 1.$$

The converse implication is true by (2.5).

The proof follows the same pattern as in [9], with the following additional argument.

**Lemma 2.3** *Let  $\nu$  be a probability measure, compactly carried by  $\varphi(\mathbb{D})$ , and let  $R_\nu: \mathfrak{B} \rightarrow L^2(\nu)$  be the canonical inclusion. Then, we have:*

$$a_n(C_\varphi) \gtrsim a_n(R_\nu).$$

To prove this lemma, we need another one. For  $f \in \mathcal{H}(\mathbb{D})$  and  $0 < r < 1$ , we set as usual:

$$M(r, f) = \sup_{|z|=r} |f(z)|.$$

We then have:

**Sublemma 2.4** *Let  $g \in \mathcal{H}(\mathbb{D})$ , not identically zero, and  $0 < r < 1$ . Then, there exists  $C > 0$ , depending only on  $g$  and  $r$ , such that:*

$$(2.7) \quad M(r, f) \leq C \|fg\|_{\mathfrak{B}}, \quad \forall f \in \mathcal{H}(\mathbb{D}).$$

Therefore, for each compact subset  $L \subseteq \mathbb{D}$ , there exists a constant  $C = C(L, g)$  such that, for any  $f \in \mathcal{H}(\mathbb{D})$ , one has:

$$(2.8) \quad \|f\|_{C(L)} \leq C \|fg\|_{\mathfrak{B}}.$$

**Proof.** Since the zeros of  $g$  are at most countable, we can find  $r \leq \rho < 1$  such that  $g$  does not vanish on the circle of radius  $\rho$ . Hence there is some  $\mu_r > 0$  such that:

$$|g(a)| \geq \mu_r > 0 \quad \text{for } |a| = \rho.$$

Let  $\delta = 1 - \rho$ ,  $f \in \mathcal{H}(\mathbb{D})$  and  $|a| = \rho$ . By subharmonicity of  $|fg|^2$ , we have:

$$\mu_r^2 |f(a)|^2 \leq |f(a)g(a)|^2 \leq \frac{1}{\delta^2} \int_{D(a,\delta)} |fg|^2 dA \leq \frac{1}{\delta^2} \int_{\mathbb{D}} |fg|^2 dA,$$

whence  $M(\rho, f) \leq C \|fg\|_{\mathfrak{B}}$  with  $C = 1/(\delta\mu_r)$ . But  $M(r, f) \leq M(\rho, f)$  by the maximum modulus principle, and we get (2.7). That ends the proof of Sublemma 2.4, since if  $L \subseteq \overline{D(0, r)}$  and  $f \in \mathcal{H}(\mathbb{D})$ , then  $\|f\|_{\mathcal{C}(L)} \leq M(r, f) \leq C \|fg\|_{\mathfrak{B}}$  by the maximum modulus principle again.  $\square$

**Proof of Lemma 2.3.** Let  $D: \mathcal{D}_* \rightarrow \mathfrak{B}$  be the differentiation operator (which is a unitary operator by definition of the norms of these spaces). We can write (see [9], proof of Lemma 3.5):  $\nu = \varphi(\sigma)$ , for some probability measure  $\sigma$  carried by a compact subset  $L$  of  $\mathbb{D}$ . We then have, for any  $f \in \mathcal{D}_*$ , with help of Sublemma 2.4 applied to the non-zero function  $g = \varphi'$ :

$$\begin{aligned} \|R_\nu D f\|_{L^2(\nu)}^2 &= \int |f'|^2 d\nu = \int_L |f' \circ \varphi|^2 d\sigma \leq \|f' \circ \varphi\|_{\mathcal{C}(L)}^2 \\ &\leq C^2 \int_{\mathbb{D}} |f' \circ \varphi|^2 |\varphi'|^2 dA = C^2 \|C_\varphi f\|_{\mathcal{D}_*}^2. \end{aligned}$$

This implies  $a_n(R_\nu D) \leq C a_n(C_\varphi)$ , or, equivalently,  $a_n(R_\nu) \leq C a_n(C_\varphi)$ , since  $D$  is unitary.  $\square$

Recall now the following lemmas, borrowed from [9] (the first one will be used again latter, in Lemma 3.6).

**Lemma 2.5** ([9], Lemma 3.6) *For every  $r \in (0, 1)$  there exist  $s = s(r) < 1$  and  $f = f_r \in H^\infty$  with the following properties:*

- 1)  $\lim_{r \rightarrow 1^-} s(r) = 1$ ;
- 2)  $\|f\|_\infty \leq 1$ ;
- 3)  $f((0, r]) = s \partial\mathbb{D}$  in a one-to-one way.

Explicitly, one has:

$$(2.9) \quad s = e^{-\varepsilon\pi/2} \quad \text{with} \quad \varepsilon = \frac{2\pi}{\log \frac{1+r}{1-r}}.$$

Note that in [9], we defined  $\varepsilon$  with the help of a parameter  $\rho$ , but  $\frac{1+\rho}{1-\rho} = \sqrt{\frac{1+r}{1-r}}$ .

**Lemma 2.6** (see [9], Lemma 3.7 and its proof) *Let  $0 < r < 1$  and  $s$  be as in (2.9). Then, there exists a probability measure  $\mu$  carried by  $[0, r]$  such that, if  $R_\mu: \mathfrak{B} \rightarrow L^2(\mu)$  is the canonical inclusion, one has, for every  $n \geq 1$ :*

$$a_n(R_\mu) \gtrsim \frac{s^n}{\sqrt{n}}.$$

**Lemma 2.7** (see [9], Lemma 3.8 and its sequel) *Let  $\varphi: \mathbb{D} \rightarrow \mathbb{D}$  be a Schur function and suppose that 0 and  $r$  belong to  $\varphi(\mathbb{D})$ . Then, for any probability measure  $\mu$  carried by  $[0, r]$ , there exists a probability measure  $\nu$  compactly carried by  $\varphi(\mathbb{D})$  such that*

$$a_{2n}(R_\mu) \leq 2 a_n(R_\nu).$$

**Proof of Theorem 2.2 .** The three lemmas put together give the result. Indeed, assume that  $\|\varphi\|_\infty > r$ . By making a rotation, we may assume that  $r \in \varphi(\mathbb{D})$ . Let then  $\mu$  be as in Lemma 2.6 and  $\nu$  be as in Lemma 2.7 (that we may use since  $0 = \varphi(0) \in \varphi(\mathbb{D})$ ). Using Lemma 2.3, we obtain:

$$a_n(C_\varphi) \gtrsim a_n(R_\nu) \gtrsim a_{2n}(R_\mu) \gtrsim \frac{s^{2n}}{\sqrt{2n}},$$

and, changing  $s$  into  $s^2$ , this ends the proof of Theorem 2.2.  $\square$

## 2.2 Extremal behavior

In this section, we see that we may have very compact composition operators on  $\mathcal{D}_*$  whose image touches the boundary of  $\mathbb{D}$ .

**Theorem 2.8** *For every vanishing sequence  $(\varepsilon_n)_n$  of positive numbers, there exists a symbol  $\varphi$  with  $\|\varphi\|_\infty = 1$  and such that  $C_\varphi$  is compact on  $\mathcal{D}_*$ , but:*

$$(2.10) \quad a_n(C_\varphi) \lesssim e^{-n\varepsilon_n}.$$

*In particular,  $C_\varphi$  may be in all Schatten classes  $S_p(\mathcal{D}_*)$ ,  $p > 0$ , of the Dirichlet space.*

The result will follow from the forthcoming theorem, which is the analogue of Theorem 5.1 in [9].

**Theorem 2.9** *Let  $\varphi$  be a Schur function inducing a bounded composition operator on  $\mathcal{D}_*$ . Set:*

$$(2.11) \quad m(t) = \frac{1}{t^2} \int_{|w| \geq 1-t} n_\varphi dA \quad \text{and} \quad M(t) = \sum_{k=0}^{\infty} m(2^{-k}t).$$

*Then:*

$$(2.12) \quad a_n(C_\varphi) \lesssim \inf_{0 < t < 1} \left[ n(1-t)^n + \sqrt{M(t)} \right].$$

**Proof.** We shall need the following simple inequalities.

1) For  $f \in \mathcal{D}_*$  and  $a \in \mathbb{D}$ , one has:

$$(2.13) \quad |f'(a)| \leq \frac{\|f\|_{\mathcal{D}}}{1 - |a|^2}.$$

This is clear by using the Cauchy-Schwarz inequality or by using the formula  $f'(a) = \langle f, \frac{\partial K}{\partial \bar{a}}(a) \rangle$  (where  $K$  is the reproducing kernel of  $\mathcal{D}_*$ ).

2) If  $g \in \mathcal{D}_*$  and  $g = G'$ , then:

$$(2.14) \quad \|g\|_{\mathcal{D}} \leq \|zg\|_{\mathcal{D}} \quad \text{and} \quad \|G\|_{\mathcal{D}} \leq \|g\|_{\mathcal{D}}.$$

This is obvious by inspection of coefficients.

Let now  $R = C_\varphi S_{n-1}: \mathcal{D}_* \rightarrow \mathcal{D}_*$  be the operator of rank  $< n$  defined by:

$$R(f) = \sum_{k=1}^{n-1} \widehat{f}(k) \varphi^k,$$

so that  $(C_\varphi - R)(f) = C_\varphi(u)$  with:

$$u(z) = \sum_{k=n}^{\infty} \widehat{f}(k) z^k := z^n v(z)$$

and  $v \in \mathcal{D}$ .

Assume once and for all in the proof that  $\|f\|_{\mathcal{D}} \leq 1$ . Then  $\|v\|_{\mathcal{D}} \leq \|u\|_{\mathcal{D}} \leq \|f\|_{\mathcal{D}} \leq 1$ .

Fix  $0 < h < 1$ . We have, writing  $v = w'$ ,  $u'(z) = nz^{n-1}w'(z) + z^n v'(z)$ , and using (2.14):

$$\begin{aligned} \|(C_\varphi - R)(f)\|_{\mathcal{D}}^2 &= \|C_\varphi(u)\|_{\mathcal{D}}^2 = \int_{\mathbb{D}} |u'(z)|^2 n_\varphi(z) dA(z) \\ &= \int_{|z| \leq 1-h} |u'(z)|^2 n_\varphi(z) dA(z) \\ &\quad + \int_{1-h \leq |z| < 1} |u'(z)|^2 n_\varphi(z) dA(z) \\ &\lesssim n^2(1-h)^{2n} \int_{|z| \leq 1-h} |w'(z)|^2 n_\varphi(z) dA(z) \\ &\quad + (1-h)^{2n} \int_{|z| \leq 1-h} |v'(z)|^2 n_\varphi(z) dA(z) \\ &\quad + \int_{1-h \leq |z| < 1} |u'(z)|^2 n_\varphi(z) dA(z) \\ &:= I_1 + I_2 + I_3. \end{aligned}$$

Clearly, using that  $\|w\|_{\mathcal{D}} \leq \|v\|_{\mathcal{D}} \leq 1$ , we have:

$$I_1 \leq n^2(1-h)^{2n} \|C_\varphi(w)\|_{\mathcal{D}}^2 \lesssim n^2(1-h)^{2n}.$$

Similarly, we have:

$$I_2 \leq (1-h)^{2n} \|C_\varphi(v)\|_{\mathcal{D}}^2 \lesssim (1-h)^{2n}.$$

We estimate  $I_3$  by splitting into dyadic annuli:  $I_3 = \sum_{k=0}^{\infty} J_k$  with:

$$J_k = \int_{1-2^{-k}h \leq |z| < 1-2^{-k-1}h} |u'(z)|^2 n_{\varphi}(z) dA(z).$$

We now use the pointwise estimate (2.13) to get:

$$|u'(z)|^2 \leq \frac{\|u\|_{\mathcal{D}}^2}{(1-|z|)^2} \leq \frac{1}{(1-|z|)^2}.$$

In view of (2.9), this gives an estimate of the form:

$$J_k \leq 4^k h^{-2} \int_{|z| \geq 1-2^{-k}h} n_{\varphi}(z) dA(z) = m(2^{-k}h).$$

Summing up, we get  $I_3 \leq M(h)$ . It follows that

$$\|(C_{\varphi} - R)(f)\|^2 \lesssim [n^2(1-h)^{2n} + M(h)].$$

Taking the supremum on  $f$ , and then square roots, we then get:

$$a_n(C_{\varphi}) \leq \|C_{\varphi} - R\| \lesssim [n(1-h)^n + \sqrt{M(h)}].$$

Finally, taking the infimum on  $h$ , we end the proof of Theorem 2.9.  $\square$

**Remark.** In [9] (Theorem 4.1), we proved in the opposite direction, following [3], that a composition operator on the weighted Bergman space  $\mathfrak{B}_{\alpha}$  may be compact, but no little more. It is likely that the same occurs in  $\mathcal{D}_*$ , namely that for every vanishing sequence  $(\varepsilon_n)_n$  of positive numbers, there exists a symbol  $\varphi$  such that  $C_{\varphi}$  is compact on  $\mathcal{D}_*$  and for which  $\liminf_{n \rightarrow \infty} \frac{a_n(C_{\varphi})}{\varepsilon_n} > 0$  (in particular, if it happens to be true, we might have  $C_{\varphi}$  compact and in no Schatten class  $S_p(\mathcal{D}_*)$ ,  $p < \infty$ , of the Dirichlet space). But we do not succeed in proving that.

### 3 Approximation numbers of the cusp map

In [10], it is shown that there is a composition operator  $C_{\chi}: H^2 \rightarrow H^2$ , whose symbol is called the cusp map, defined on the Hardy space, such that, for some constants  $c_1 > c_2 > 0$ , one has:

$$(3.1) \quad e^{-c_1 n / \log n} \lesssim a_n(C_{\chi}: H^2 \rightarrow H^2) \lesssim e^{-c_2 n / \log n}, \quad n = 2, 3, \dots$$

In [8], we proved that every composition operator which is compact on the Dirichlet space is in all Schatten classes  $S_p(H^2)$ ,  $p > 0$ , on the Hardy space. Therefore the approximation numbers of  $C_{\varphi}: H^2 \rightarrow H^2$  must be (much) smaller than those of  $C_{\varphi}: \mathcal{D}_* \rightarrow \mathcal{D}_*$ . The next theorem gives, for the cusp map, this order of smallness.

**Theorem 3.1** *Let  $\chi$  be the cusp map. There exist two constants  $0 < c' < c$  such that the approximation numbers  $a_n(C_\chi)$  of the associated composition operator  $C_\chi: \mathcal{D}_* \rightarrow \mathcal{D}_*$  satisfy:*

$$(3.2) \quad e^{-c\sqrt{n}} \lesssim a_n(C_\chi: \mathcal{D}_* \rightarrow \mathcal{D}_*) \lesssim e^{-c'\sqrt{n}} \quad n = 1, 2, \dots$$

Recall the definition of the cusp map  $\chi$ , introduced in [6], and later used, with a slightly different definition in [10]. Actually, as in [8], we have to modify it slightly again in order to have  $\chi(0) = 0$ . We first define:

$$\chi_0(z) = \frac{\left(\frac{z-i}{iz-1}\right)^{1/2} - i}{-i\left(\frac{z-i}{iz-1}\right)^{1/2} + 1};$$

we note that  $\chi_0(1) = 0$ ,  $\chi_0(-1) = 1$ ,  $\chi_0(i) = -i$ ,  $\chi_0(-i) = i$ , and  $\chi_0(0) = \sqrt{2} - 1$ . Then we set:

$$\chi_1(z) = \log \chi_0(z), \quad \chi_2(z) = -\frac{2}{\pi} \chi_1(z) + 1, \quad \chi_3(z) = \frac{a}{\chi_2(z)},$$

and finally:

$$\chi(z) = 1 - \chi_3(z),$$

where:

$$(3.3) \quad a = 1 - \frac{2}{\pi} \log(\sqrt{2} - 1) \in (1, 2)$$

is chosen in order that  $\chi(0) = 0$ . The image  $\Omega$  of the (univalent) cusp map is formed by the intersection of the inside of the disk  $D(1 - \frac{a}{2}, \frac{a}{2})$  and the outside of the two disks  $D(1 + \frac{ia}{2}, \frac{a}{2})$  and  $D(1 - \frac{ia}{2}, \frac{a}{2})$ .

### 3.1 Proof of the upper bound of Theorem 3.1

We need some lemmas.

**Lemma 3.2** *We have:*

$$(3.4) \quad \|\chi^n\|_{\mathcal{D}} \leq C n^{-\delta},$$

for every  $n \geq 1$ , where  $C$  and  $\delta$  are positive numerical constants.

**Proof.** Since  $\chi$  is univalent, we have, for every  $0 < h < 1$ :

$$\|\chi^n\|_{\mathcal{D}}^2 = \int_{\mathbb{D}} n^2 |w|^{2n-2} n_\chi(w) dA(w) \leq n^2 (1-h)^{2n-2} + n^2 \int_{|w| \geq 1-h} n_\chi(w) dA(w)$$

But  $\int_{|w| \geq 1-h} n_\chi(w) dA(w)$  is the area of  $\chi(\mathbb{D}) \cap \{|w| \geq 1-h\}$ ; since  $\chi(\mathbb{D})$  is delimited at the cuspidal point 1 by two circular arcs, this area is  $\approx h^3$ . We get hence:

$$\|\chi^n\|_{\mathcal{D}} \lesssim n[e^{-nh} + h^{3/2}].$$

The choice  $h = 2(\log n/n)$  gives  $\|\chi^n\|_{\mathcal{D}} \lesssim n^{-1/2}(\log n)^{3/2}$  and hence the lemma, with any  $\delta < 1/2$ .  $\square$

An immediate corollary, in which  $S_N$  denotes the operator of  $N$ th-partial sum, as defined in (1.3), is the following.

**Corollary 3.3** *We have:*

$$\|C_\chi - C_\chi S_N\| \lesssim N^{-\delta}.$$

**Proof.** Using the Hilbert-Schmidt norm, we get:

$$\begin{aligned} \|C_\chi - C_\chi S_N\|^2 &\leq \|C_\chi - C_\chi S_N\|_{HS}^2 = \sum_{n>N} \frac{\|\chi^n\|^2}{n} \\ &\lesssim \sum_{n>N} n^{-1-2\delta} \lesssim N^{-2\delta}. \end{aligned} \quad \square$$

Now, the idea for majorizing  $a_n(C_\chi)$  is to write, for every operator  $R$  with rank  $< n$ :

$$\|C_\chi - R\| \leq \|C_\chi - C_\chi S_N\| + \|C_\chi S_N - R\|,$$

which gives, taking the infimum over all such  $R$ :

$$(3.5) \quad a_n(C_\chi) \leq \|C_\chi - C_\chi S_N\| + a_n(C_\chi S_N).$$

Using the corollary, we get:

$$(3.6) \quad a_n(C_\chi) \lesssim N^{-\delta} + a_n(C_\chi S_N)$$

and our goal is to give a good upper bound of  $a_n(C_\chi S_N)$ .

**Lemma 3.4** *For some numerical constant  $\varepsilon > 0$ , we have:*

$$(3.7) \quad a_n(C_\chi S_N) \lesssim \sqrt{N} e^{-\varepsilon\sqrt{n}}.$$

With this estimation, we get:

$$a_n(C_\chi) \lesssim [N^{-\delta} + \sqrt{N} e^{-\varepsilon\sqrt{n}}]$$

and, by adjusting  $N = \lceil e^{\varepsilon\sqrt{n}} \rceil$ , we obtain the upper bound in (3.2).

**Proof of Lemma 3.4.** To prove (3.7), we shall replace  $C_\chi S_N$  by a ‘‘dominating’’ operator.

We begin with observing that, if  $f(z) = \sum_{j=1}^{\infty} c_j z^j \in \mathcal{D}_*$ , we have by the change of variable formula, setting  $d\mu = n_\chi dA = \mathbb{1}_{\chi(\mathbb{D})} dA$ :

$$(3.8) \quad \begin{aligned} \|C_\chi S_N f\|_{\mathcal{D}}^2 &= \int_{\mathbb{D}} \left| \sum_{j=1}^N j c_j w^{j-1} \right|^2 n_\chi(w) dA(w) \\ &= \int_{\mathbb{D}} \left| \sum_{j=1}^N j c_j w^{j-1} \right|^2 d\mu(w). \end{aligned}$$

Now, denote by  $\Delta_N: \mathcal{D}_* \rightarrow H^2$  the map defined by:

$$\Delta_N f(w) = \sum_{j=1}^N j c_j w^{j-1}.$$

Observe that:

$$\|\Delta_N f\|_{H^2}^2 = \sum_{j=1}^N j^2 |c_j|^2 \leq N \sum_{j=1}^N j |c_j|^2 \leq N \|f\|_{\mathcal{D}}^2,$$

so that  $\|\Delta_N\| \leq \sqrt{N}$ .

Let also  $J$  be the canonical inclusion  $J: H^2 \rightarrow L^2(\mu)$ . The equality (3.8) reads  $\|C_\chi S_N f\|_{\mathcal{D}}^2 = \|J \Delta_N f\|_{L^2(\mu)}^2$ ; therefore there is a contraction  $C_N: \mathcal{D}_* \rightarrow \mathcal{D}_*$  such that

$$(3.9) \quad C_\chi S_N = C_N J \Delta_N.$$

The ideal property of approximation numbers now implies:

$$a_n(C_\chi S_N) = a_n(C_N J \Delta_N) \leq \|C_N\| \cdot a_n(J) \cdot \|\Delta_N\| \leq \sqrt{N} a_n(J),$$

and we are left with the task of majorizing  $a_n(J)$ . To that effect, we use the Gelfand numbers  $c_n$  ([12] or [2]). Recall that if  $T: X \rightarrow X$  is an operator on some Banach space  $X$ , then  $c_n(T) = \inf\{\|T|_Z\|; Z \subseteq X, \text{codim } Z < n\}$ , and if  $X = H$  is a Hilbert space, then  $c_n(T) = a_n(T)$ .

Let  $B$  be a Blaschke product of length  $< n$ , let  $E = BH^2$  which is a subspace of  $H^2$  of codimension  $< n$ . We have:

$$a_n(J) = c_n(J) \leq \|J|_E\|.$$

The majorization is then made using the Carleson embedding theorem. Let  $r$  be the greatest integer  $< \sqrt{n}$ , and  $B_0$  a Blaschke product with  $r$  zeros well distributed on the interval  $(0, 1)$ . More precisely,  $B_0$  has its zeros at the points

$$z_j = 1 - 2^{-j}, \quad 1 \leq j \leq r.$$

Set  $\Omega = \chi(\mathbb{D})$  and observe that:

$$(3.10) \quad z \in \Omega \text{ and } \Re z \geq 1 - h \implies |\Im z| \lesssim h^2$$

$$(3.11) \quad A[S(\xi, h) \cap \Omega] \lesssim h^3 \quad \text{for every } \xi \in \mathbb{T}.$$

Let now  $B = B_0^r$ . This is a Blaschke product of length  $r^2 < n$ . Using the Carleson embedding theorem (for the measure  $d\mu = n_\chi dA$ ), as in [9] and [10], and the univalence of  $\chi$ , we get:

$$(3.12) \quad \|J|_E\|^2 \lesssim \sup_{0 < h < 1, |\xi|=1} \frac{1}{h} \int_{S(\xi, h) \cap \Omega} |B|^2 dA.$$

To estimate the supremum in the right-hand side of (3.12), we may assume that  $h = 2^{-l}$  and we separate two cases.

- $l \geq r$ . Then, using (3.11) and the fact that  $|B| \leq 1$ , we have:

$$(3.13) \quad \frac{1}{h} \int_{S(\xi, h) \cap \Omega} |B|^2 dA \lesssim \frac{1}{h} h^3 = h^2 = 2^{-2l} \leq 2^{-2r}.$$

- $l < r$ . Then, we have:

$$\begin{aligned} \frac{1}{h} \int_{S(\xi, h) \cap \Omega} |B|^2 dA &\leq \frac{1}{h} \int_{\{|z-1| \leq 2^{-r}\} \cap \Omega} |B(z)|^2 dA(z) \\ &\quad + \sum_{j=l+1}^r \frac{1}{h} \int_{C_j \cap \Omega} |B|^2 dA, \end{aligned}$$

where  $C_j$  is the annulus

$$C_j = \{z \in \mathbb{D}; 2^{-j} \leq |z-1| \leq 2^{-j+1}\}.$$

The first term is handled as before. Now, since  $\Omega$  is contained in some sector  $1 - |z| \geq \delta |1 - z|$ , we have, for  $z \in C_j \cap \Omega$ :

$$1 - |z| \geq \delta 2^{-j} \quad \text{and} \quad 1 - |z_j| = 2^{-j},$$

whereas

$$|z - z_j| = |z - 1 + 2^{-j}| \leq |z - 1| + 2^{-j} \leq 3 \cdot 2^{-j}.$$

This implies that, for some absolute constant  $M > 0$ :

$$|z - z_j| \leq M \min(1 - |z|, 1 - |z_j|)$$

and, by [7], Lemma 2.3, the  $j$ -th factor of  $B_0$  is, in modulus, less than  $\kappa = \frac{M}{\sqrt{M^2+1}} < 1$ . Therefore  $|B| = |B_0^r| \leq \kappa^r$  on all sets  $C_j \cap \Omega$ , so that

$$\begin{aligned} \sum_{j=l+1}^r \frac{1}{h} \int_{C_j \cap \Omega} |B|^2 dA &\leq \sum_{j=l+1}^r \frac{1}{h} \int_{C_j \cap \Omega} \kappa^{2r} dA \\ &\lesssim 2^l \kappa^{2r} A[S(\xi, 2^{-l})] \lesssim \kappa^{2r} 2^l 2^{-2l} \lesssim \kappa^{2r}. \end{aligned}$$

This finally shows, thanks to (3.12) and (3.13), that  $\|J_E\| \lesssim \kappa^r$ , or, in setting  $\kappa = e^{-\varepsilon}$ , that (recall that  $r$  is the greatest integer  $< \sqrt{n}$ , and hence  $r \approx \sqrt{n}$ ):

$$\|J_E\| \lesssim e^{-\varepsilon \sqrt{n}}.$$

This proves (3.7) and ends the proof of the upper bound in Theorem 3.1.  $\square$

### 3.2 Proof of the lower bound of Theorem 3.1

Recall that  $\mu$  is the measure  $d\mu = n_\chi dA$  and that  $\Omega = \chi(\mathbb{D})$ . Consider the diagram

$$H^2 \xrightarrow{P} \mathcal{D}_* \xrightarrow{C_\chi} \mathcal{D}_* \xrightarrow{D} L^2(\mu),$$

in which

$$P \left( \sum_{n=0}^{\infty} c_n z^n \right) = \sum_{n=0}^{\infty} c_n \frac{z^{n+1}}{n+1}$$

is the ‘‘primitivation’’ operator and  $D$  is the differentiation operator. We have:

$$D C_\chi P f = (f \circ \chi) \chi'.$$

We note that, by definition of the norms,  $\|P\| \leq 1$ . For  $0 < h < 1$  fixed, let also:

$$R: H^2 \rightarrow L^\infty([0, 1-h])$$

be the canonical injection.

The rest of the proof consists of two steps, the first of which consists of showing that  $a_n(C_\chi)$  is not much smaller than  $a_n(R)$ .

**Lemma 3.5** *We have:*

$$a_n(C_\chi) \geq \frac{h^2}{4} a_n(R).$$

**Proof.** We first notice that, if  $f \in H^2$ , and  $0 \leq x \leq 1-h$ , we have:

$$(3.14) \quad \|R(f)\|_{L^\infty([0, 1-h])} \leq \frac{4}{h^2} \|f\|_{L^2(\mu)}.$$

To that effect, we observe that (recall that  $a \in (1, 2)$  in defined in (3.3)):

$$(3.15) \quad 0 < h \leq a-1 \quad \text{and} \quad 0 \leq x \leq 1-h \quad \implies \quad D(x, h^2/4a) \subseteq \Omega.$$

Indeed, if  $z = u + iv \in D(x, h^2/4a)$  and  $0 \leq x \leq 1-h$ , we have  $1-u \geq h - (h^2/4a) \geq h/2$ , as well as  $|v| < h^2/4a$ , and:

$$\left| z - \left( 1 + \frac{ia}{2} \right) \right|^2 = (1-u)^2 + \left( v - \frac{a}{2} \right)^2 \geq \frac{h^2}{4} + v^2 - a|v| + \frac{a^2}{4} > \frac{a^2}{4}.$$

Similarly  $|z - (1 - \frac{ia}{2})| > \frac{a}{2}$ . Moreover, since  $1 - \frac{a}{2} \leq \frac{a}{2} - h$ , we have  $|z - (1 - \frac{a}{2})| \leq |z - x| + |x - (1 - \frac{a}{2})| \leq \frac{h^2}{4a} + \frac{a}{2} - h < \frac{a}{2}$ . Hence  $z \in \Omega$ .

Therefore, by subharmonicity of the function  $|f|^2$ :

$$\begin{aligned} |f(x)|^2 &\leq \frac{16 a^2}{h^4} \int_{D(x, h^2/4a)} |f|^2 dA \\ &\leq \frac{16 a^2}{h^4} \int_{\Omega} |f|^2 dA = \frac{16 a^2}{h^4} \int_{\mathbb{D}} |f|^2 n_\chi dA \\ &= \frac{16 a^2}{h^4} \int_{\mathbb{D}} |f|^2 d\mu, \end{aligned}$$

which proves (3.14).

Let now  $f \in H^2$  and  $g = P f \in \mathcal{D}_*$ , so that  $f = D g$ . As follows from (3.14) and from the change of variable formula, we have:

$$\begin{aligned} \|Rf\|_\infty^2 &\leq \frac{64}{h^4} \int_{\mathbb{D}} |f(w)|^2 n_\chi(w) dA(w) = \frac{64}{h^4} \int_{\mathbb{D}} |Dg(w)|^2 n_\chi(w) dA(w) \\ &= \frac{64}{h^4} \int_{\mathbb{D}} |g'(\chi(z))|^2 |\chi'(z)|^2 dA(z) = \frac{64}{h^4} \|D C_\chi g\|_{L^2(\mathbb{D})}^2 \\ &= \frac{64}{h^4} \|C_\chi P f\|_{\mathcal{D}}^2. \end{aligned}$$

Therefore, there exists  $C: \mathcal{D}_* \rightarrow L^\infty([0, 1-h])$  such that:

$$R = C C_\chi P \quad \text{and} \quad \|C\| \leq \frac{4}{h^2}.$$

All this implies, by the ideal property of approximation numbers:

$$a_n(R) \leq \|C\| a_n(C_\chi) \|P\| \leq \frac{4}{h^2} a_n(C_\chi),$$

which ends the proof of Lemma 3.5.  $\square$

The second step consists of a minoration of  $a_n(R)$ , which uses the comparison with Bernstein numbers and a good choice of an  $n$ -dimensional space  $E$ .

**Lemma 3.6** *Let  $0 < r < 1$  and  $s$  as in (2.9). We have:*

$$a_n(R) \geq \frac{s^n}{\sqrt{n}}.$$

Recall (see [12] for example) that, if  $X$  and  $Y$  are two Banach spaces, and  $T: X \rightarrow Y$  is a compact operator, the  $n$ -th Bernstein number of  $T$  is:

$$b_n(T) = \sup_{\dim E=n} \inf_{f \in S_E} \|Tf\|,$$

where  $S_E$  denotes the unit sphere of  $E$ , and we have:

$$(3.16) \quad a_n(T) \geq b_n(T).$$

**Proof of Lemma 3.6.** Let  $f = f_r$  be as in Lemma 2.5, and write  $r = 1 - h$ . Consider the  $n$ -dimensional space

$$E = [1, f, \dots, f^{n-1}],$$

and let  $g = \sum_{j=0}^{n-1} \alpha_j f^j \in E$  with  $\|g\|_\infty = 1$ . By Lemma 2.5 and the Cauchy-Schwarz inequality, we have:

$$1 \leq \sum_{j=0}^{n-1} |\alpha_j| \|f^j\|_\infty \leq \sum_{j=0}^{n-1} |\alpha_j| \leq \sqrt{n} \left( \sum_{j=0}^{n-1} |\alpha_j|^2 \right)^{1/2}.$$

On the other hand, Lemma 2.5 again gives us:

$$\begin{aligned} \|R(g)\|_\infty &\geq \left\| \sum_{j=0}^{n-1} \alpha_j s^j e^{ij\theta} \right\|_{L^\infty(\mathbb{T})} \geq \left\| \sum_{j=0}^{n-1} \alpha_j s^j e^{ij\theta} \right\|_{L^2(\mathbb{T})} \\ &= \left( \sum_{j=0}^{n-1} |\alpha_j|^2 s^{2j} \right)^{1/2} \geq s^n \left( \sum_{j=0}^{n-1} |\alpha_j|^2 \right)^{1/2} \\ &\geq \frac{s^n}{\sqrt{n}}. \end{aligned}$$

Therefore,  $b_n(R) \geq s^n/\sqrt{n}$ . Using (3.16), we get  $a_n(R) \geq s^n/\sqrt{n}$  as well.  $\square$

Let us now indicate how Lemma 3.5 and Lemma 3.6 allow to finish the proof. Write  $h = e^{-A}$  where  $A > 0$ . Then, with the notation (2.9), we have:

$$\varepsilon = \frac{2\pi}{\log \frac{1+r}{1-r}} \lesssim \frac{1}{\log \frac{1}{1-r}} = \frac{1}{\log(1/h)} = \frac{1}{A}$$

and

$$s \gtrsim e^{-c/A},$$

for some constant  $c > 0$ . Therefore Lemma 3.5 and Lemma 3.6 give:

$$a_n(C_\chi) \gtrsim h^2 a_n(R) \gtrsim h^2 \frac{s^n}{\sqrt{n}} \gtrsim e^{-c'(A+n/A)}.$$

The optimal choice  $A = \sqrt{n}$  gives the lower bound in Theorem 3.1.  $\square$

**Remark:** One sees that the approximation numbers of  $C_\chi$  behave quite differently on the Hardy space  $H^2$  (like  $e^{-cn/\log n}$ , see [10]) and on the Dirichlet space (like  $e^{-c\sqrt{n}}$ ). This seems to be due to the following. On the Hardy space, the important fact is the parametrization  $t \mapsto \chi(e^{it})$  where logarithms are involved. On the Dirichlet space, we only need to know the geometry of  $\chi(\mathbb{D})$ , a domain limited by three circles, where logarithms are no longer involved.

## 4 Capacity of the set of contact points

Here is now the improvement of a theorem in ([5]) in terms of approximation numbers (see also [8], Theorem 4.1). This improvement is definitely optimal in view of our previous Theorem 2.2, stating that, for every bounded composition operator  $C_\varphi$  on  $\mathcal{D}_*$  of symbol  $\varphi$ , one has:

$$\|\varphi\|_\infty = 1 \quad \implies \quad \beta(C_\varphi) := \liminf_{n \rightarrow \infty} [a_n(C_\varphi)]^{1/n} = 1.$$

Recall the following notation, where  $\varphi$  belongs to the disk algebra  $A(\mathbb{D})$ , i.e. the space of continuous functions  $f: \mathbb{D} \rightarrow \mathbb{C}$  which are analytic in  $\mathbb{D}$ :

$$E_\varphi = \{e^{it} \in \mathbb{T}; |\varphi(e^{it})| = 1\}.$$

**Theorem 4.1** *Let  $K$  be a compact set of the circle  $\mathbb{T}$  with logarithmic capacity  $\text{Cap } K = 0$ , and  $(\varepsilon_n)_n$  a sequence of positive numbers with limit 0. Then, there exists a Schur function  $\varphi$  generating a composition operator  $C_\varphi$  bounded on  $\mathcal{D}$  and with the following properties:*

- 1)  $\varphi \in A(\mathbb{D}) \cap \mathcal{D}$ , the “Dirichlet algebra”;
- 2)  $E_\varphi = K$  and  $E_\varphi = \{e^{it} \in \mathbb{T}; \varphi(e^{it}) = 1\}$ ;
- 3)  $a_n(C_\varphi) \lesssim e^{-n\varepsilon_n}$  for all  $n \geq 1$ .

Before proving this theorem, we need two results [8]. The first one is the existence of a peculiar peaking function. Recall that a function  $q \in A(\mathbb{D})$  is said to peak on a compact subset  $K \subseteq \partial\mathbb{D}$ , and is called a peaking function, if:  $q(z) = 1$  for  $z \in K$  and  $|q(z)| < 1$  for  $z \in \mathbb{D} \setminus K$ .

**Theorem 4.2 ([8] Theorem 4.2)** *For every compact set  $K \subseteq \partial\mathbb{D}$  of logarithmic capacity  $\text{Cap } K = 0$ , there exists a Schur function  $q \in A(\mathbb{D}) \cap \mathcal{D}_*$  which peaks on  $K$  and such that the composition operator  $C_q: \mathcal{D}_* \rightarrow \mathcal{D}_*$  is bounded (and even Hilbert-Schmidt).*

The other one is a lemma borrowed from the proof of Theorem 3.3 in [8].

**Lemma 4.3** *Let  $\delta: (0, 1) \rightarrow (0, \infty)$  be a positive function with  $\lim_{h \rightarrow 0} \delta(h) = 0$ . Then, there exists a univalent Schur function  $\gamma \in A(\mathbb{D})$  such that  $\gamma(1) = 1$  and that:*

$$(4.1) \quad \int_{|w| \geq 1-h} n_\gamma(w) dA(w) = A[\gamma(\mathbb{D}) \cap \{w; 1-h \leq |w| < 1\}] \leq \delta(h).$$

**Proof of Theorem 4.1.** It suffices to use Lemma 4.3 to construct a generalized cusp map  $\gamma$  in order to have  $a_n(C_\gamma) \lesssim e^{-n\varepsilon_n}$ . Indeed, we then compose this generalized cusp map  $\gamma$  with a symbol  $q$  peaking on  $K$ , as given by Theorem 4.2; namely consider  $\varphi = \gamma \circ q$ . Then, we know that  $E_\varphi = \{e^{it}; \varphi(e^{it}) = 1\} = K$ . Moreover,  $C_\varphi = C_q \circ C_\gamma$ , so that, using the fact that  $\|C_q\| < \infty$ :

$$a_n(C_\varphi) \leq \|C_q\| a_n(C_\gamma) \lesssim e^{-n\varepsilon_n}.$$

It remains to find such a generalized cusp map  $\gamma$ . Set:

$$\delta_n = \varepsilon_n + \frac{\log n}{n}.$$

Let  $\Phi$  be a positive, continuous, concave, and increasing function  $\Phi: [0, 1] \rightarrow [0, 1]$  such that  $\Phi(0) = 0$  and  $\Phi(1/n) \geq \delta_n$ . Let  $\Psi = \Phi^{-1}$ . By Lemma 4.3, we can adjust  $\gamma$  so as to have, using the notation (2.11):

$$m(h) \leq h \rho^2(h),$$

where

$$\rho(h) = \exp\left(-\frac{h}{\Psi(h)}\right).$$

Note that  $\rho$  is increasing. We then see that:

$$M(h) \leq \sum_{k=0}^{\infty} 2^{-k} h \rho^2(2^{-k}h) \lesssim h \rho^2(h) \leq \rho^2(h)$$

and, plugging that in (2.12), we get:

$$a_n(C_\gamma) \lesssim \inf_{0 < h < 1} [n(1-h)^n + \rho(h)] \leq \inf_{0 < h < 1} \left[ n \exp(-nh) + \exp\left(-\frac{h}{\Psi(h)}\right) \right].$$

In particular, if we choose  $h = \Phi(1/n)$ , we obtain:

$$a_n(C_\gamma) \lesssim n e^{-n\Phi(1/n)} \leq n e^{-n\delta_n} = e^{-n\varepsilon_n}.$$

In view of the initial observation, this ends the proof of Theorem 4.1.  $\square$

## References

- [1] N. Arcozzi, R. Rochberg, E. T. Sawyer and B. D. Wick, The Dirichlet space: a survey, *New York J. Math.* 17A (2011), 45–86.
- [2] B. Carl and I. Stephani, *Entropy, Compactness and the Approximation of Operators*, Cambridge Tracts in Mathematics, vol. 98 (1990).
- [3] T. Carroll., C. C. Cowen, Compact composition operators not in the Schatten classes, *J. Operator Theory* 26 (1991), 109–120.
- [4] C. C. Cowen and B. D. MacCluer, *Composition Operators on Spaces of Analytic Functions*, Studies in Advanced Mathematics, CRC Press, Boca Raton, FL (1995).
- [5] O. El-Fallah, K. Kellay, M. Shabankhah, and H. Youssfi, Level sets and composition operators on the Dirichlet space, *J. Funct. Anal.* 260 (2011), 1721–1733.
- [6] P. Lefèvre, D. Li, H. Queffélec and L. Rodríguez-Piazza, Compact composition operators on Bergman-Orlicz spaces, *Trans. Amer. Math. Soc.* (*to appear*).
- [7] P. Lefèvre, D. Li, H. Queffélec and L. Rodríguez-Piazza, Some new properties of composition operators associated to lens maps, *Israel J. Math.* (*to appear*).
- [8] P. Lefèvre, D. Li, H. Queffélec, and L. Rodríguez-Piazza, Compact composition operators on the Dirichlet space and capacity of sets of contact points, *To appear in J. Funct. Anal.*.
- [9] D. Li, H. Queffélec, and L. Rodríguez-Piazza, On approximation numbers of composition operators, *J. Approx. Theory* 164 (2012), no. 4, 431–459.

- [10] D. Li, H. Queffélec, and L. Rodríguez-Piazza, Estimates for approximation numbers of some classes of composition operators on the Hardy space, *To appear in Ann. Acad. Sci. Fenn. Math.*
- [11] B. McCluer and J. Shapiro, Angular derivatives and compact composition operators on the Hardy and Bergman spaces, *Canad. J. Math.* 38, no. 4 (1986), 878–906.
- [12] A. Pietsch,  $s$ -numbers of operators in Banach spaces, *Studia Math.* LI (1974), 201–223.
- [13] W. T. Ross, The classical Dirichlet space, Recent advances in operator-related function theory, 171–197, *Contemp. Math.* 393, Amer. Math. Soc., Providence, RI (2006).
- [14] J. H. Shapiro, *Composition Operators and Classical Function Theory*, Universitext, Tracts in Mathematics, Springer-Verlag, New York (1993).

Pascal Lefèvre, *Univ Lille-Nord-de-France UArtois,*  
*Laboratoire de Mathématiques de Lens EA 2462,*  
*Fédération CNRS Nord-Pas-de-Calais FR 2956,*  
*F-62 300 LENS, FRANCE*  
*pascal.lefevre@euler.univ-artois.fr*

Daniel Li, *Univ Lille-Nord-de-France UArtois,*  
*Laboratoire de Mathématiques de Lens EA 2462,*  
*Fédération CNRS Nord-Pas-de-Calais FR 2956,*  
*Faculté des Sciences Jean Perrin,*  
*Rue Jean Souvraz, S.P. 18,*  
*F-62 300 LENS, FRANCE*  
*daniel.li@euler.univ-artois.fr*

Hervé Queffélec, *Univ Lille-Nord-de-France USTL,*  
*Laboratoire Paul Painlevé U.M.R. CNRS 8524,*  
*F-59 655 VILLENEUVE D’ASCQ Cedex, FRANCE*  
*queff@math.univ-lille1.fr*

Luis Rodríguez-Piazza, *Universidad de Sevilla,*  
*Facultad de Matemáticas, Departamento de Análisis Matemático & IMUS,*  
*Apartado de Correos 1160,*  
*41 080 SEVILLA, SPAIN*  
*piazza@us.es*