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Continuous belief functions: focal intervals properties

Jean-Marc Vannobel

Abstract The set of focal elements resulting from a conjunctive or disjunctive combination of consonant belief functions is regrettably not consonant and is thus very difficult to represent.

In this paper, we propose a graphical representation of the cross product of two focal sets originating from univariate Gaussian pdfs. This representation allows to represent initial focal intervals as well as focal intervals resulting from a combination operation. We show in case of conjunctive or disjunctive combination operations, that the whole domain can be separated in four subsets of intervals having same properties. At last, we focus on identical length focal intervals resulting from a combination. We show that such intervals are organized in connected line segments on our graphical representation.

1 Introduction

1.1 Sources of information

Consider a source of information \mathcal{S}_i with knowledge modeled by a univariate convex (unimodal and consonant) probability density function $betf_i$ of a continuous random variable X . The support of $betf_i$ is called $\Omega_i = [\Omega_i^-, \Omega_i^+]$ with $\Omega_i^-, \Omega_i^+ \in \mathcal{R}$ [4]. The mode and the variance of $betf_i$ are respectively noted μ_i and σ_i^2 . Suppose now $\mathcal{E}_i = (\mathcal{F}_i, m_i)$, the *piece of evidence* deduced from $betf_i$. \mathcal{E}_i is totally described by the pair composed of m_i , the Least Committed isopignistic *basic belief density (bbd)* deduced from $betf_i$ [1] [4] and $\mathcal{F}_i = \{I \subseteq \Omega_i | m_i(I) > 0\}$, the focal set of intervals with elements in Ω_i .

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1.2 Focal intervals

An interval $A = [A^-, A^+]$ with $A^-, A^+ \in \mathcal{R}$ such as $m_i(A) \neq 0$ is called *focal interval* of \mathcal{E}_i thus $A \in \mathcal{F}_i$. All elements of \mathcal{F}_i are nested intervals in case of a consonant pdf $betf_i$ and correspond to horizontal cuts of $betf_i$ as shown in figure 1(a). It is convenient to label the elements of \mathcal{F}_i according to their inclusion order by a continuous index. This can be done for instance wrt the pdf value at focal interval bounds [2] or wrt the half-length of the focal interval [3]. This last option allows in general to define a single bbd's expression for a whole family of pdfs [6]. In case of symmetrical pdfs like Gaussian ones as well as Laplace ones, focal intervals can be labeled by an index z such as $A^z = [A^{z-}, A^{z+}]$ with:

$$z = \frac{|x - \mu|}{\sigma}, \quad z \in \mathcal{R}^+, \quad (1)$$

$$A^{z-} = \mu - \sigma z, \quad A^{z-} \in [\Omega^-, \mu], \quad (2)$$

$$A^{z+} = \mu + \sigma z, \quad A^{z+} \in [\mu, \Omega^+]. \quad (3)$$

2 Focal sets graphical representations

2.1 General considerations

We consider in what follows two pieces of evidence $\mathcal{E}_i = (\mathcal{F}_i, m_i)$ and $\mathcal{E}_j = (\mathcal{F}_j, m_j)$ deduced respectively from the Gaussian pdfs $betf_i(x; \mu_i, \sigma_i^2)$ and $betf_j(x; \mu_j, \sigma_j^2)$ with $\mu_i \leq \mu_j$. Focal intervals are denoted by $A_i^{z_k}$ with z_k the value taken by z_i the label obtained using relation (1). We assume the use of Gaussian pdfs since many sensors model uncertainty by such pdfs but any other symmetrical bell shaped pdfs like Laplace or Cauchy ones would also serve the purpose. $\mathcal{F}_{i,j}$ is the focal set resulting from a conjunctive (resp. disjunctive) combination of \mathcal{E}_i and \mathcal{E}_j . Elements of $\mathcal{F}_{i,j}$ correspond to the non empty intersection (resp. union) of pairs in $\mathcal{F}_i \times \mathcal{F}_j$. The content of $\mathcal{F}_{i,j}$ and the length dependencies between its elements depend of course on the chosen combination rule.

2.2 Bell shaped probability density functions

The graphical representation proposed in figure 1(a) shows the focal set \mathcal{F}_i obtained from a Gaussian pdf $Betf_i$. Elements of \mathcal{F}_i are ordered wrt label z , differing in that point from the graphical representation proposed by Strat [5].

When labeling focal intervals wrt their length as defined in (1), the focal set \mathcal{F}_i is encompassed by two symmetrical half-lines defining an isosceles triangle. Equations of these half-lines are deduced from relations (2) and (3) and correspond to:

$$\begin{cases} z = \frac{|A_i^{z^-} - \mu_i|}{\sigma_i}, \\ z = \frac{|A_i^{z^+} - \mu_i|}{\sigma_i}. \end{cases} \quad (4)$$

As shown in figure 1(b), this is a convenient way to graphically compare focal intervals coming from different focal sets. One can see in this figure the result of the intersection or union of two intervals $A_i^{z_k} \in \mathcal{F}_i$ and $B_j^{z_l} \in \mathcal{F}_j$ which are indexed resp. by z_k and z_l . For instance, it also allows to show the domain of intervals $B_j \in \mathcal{F}_j$ that do not intersect with $A_i^{z_k}$ (if any). It is obvious from relation (1) that the label value of these B_j intervals is in $[0, \frac{|A_i^{z_k^+} - \mu_j|}{\sigma_j}]$.

2.3 Bounds relations of $\mathcal{F}_i \times \mathcal{F}_j$ elements

Figure 2(a) shows the pairs of intervals $(A_i^{z_i^x}, B_j^{z_j^x}) \in \mathcal{F}_i \times \mathcal{F}_j$ having a common bound $x \in \Omega$. The index pairs $(z_i^x, z_j^x) \in \mathcal{R}^{+2}$ corresponding to $(A_i^{z_i^x}, B_j^{z_j^x})$ draw the lines ①, ② and ③ as shown in Figure 2(a). These lines are defined by¹:

$$\begin{cases} \textcircled{1} : z_j^x = \frac{|\mu_i - \mu_j|}{\sigma_j} + \frac{\sigma_i}{\sigma_j} z_i^x, & x \in [-\infty, \mu_i], \\ \textcircled{2} : z_j^x = \frac{|\mu_i - \mu_j|}{\sigma_j} - \frac{\sigma_i}{\sigma_j} z_i^x, & x \in [\mu_i, \mu_j], \\ \textcircled{3} : z_j^x = \frac{-|\mu_i - \mu_j|}{\sigma_j} + \frac{\sigma_i}{\sigma_j} z_i^x, & x \in [\mu_j, +\infty]. \end{cases} \quad (5)$$

Pairs (z_i^x, z_j^x) on the half line called ① lead to $x \leq \mu_i$ such as:

$$\begin{cases} A_i^{z_i^x-} = B_j^{z_j^x-} = x, \\ A_i^{z_i^x+} \leq B_j^{z_j^x+}. \end{cases} \quad (6)$$

Pairs (z_i^x, z_j^x) on line segment ② correspond to $x \in [\mu_i, \mu_j]$ such as:

$$A_i^{z_i^x+} = B_j^{z_j^x-} = x. \quad (7)$$

At last, pairs (z_i^x, z_j^x) on the half line ③ correspond to $x \geq \mu_j$ such as:

$$\begin{cases} A_i^{z_i^x+} = B_j^{z_j^x+} = x, \\ A_i^{z_i^x-} \leq B_j^{z_j^x-}. \end{cases} \quad (8)$$

¹ proof is not given here due to lack of space

To outline existing partial conflict $k_{i,j}$ between the agents \mathcal{E}_i and \mathcal{E}_j [6], the z label values of the modes are denoted by K_i and K_j :

$$\begin{cases} K_i = \frac{|\mu_j - \mu_i|}{\sigma_i}, \\ K_j = \frac{|\mu_j - \mu_i|}{\sigma_j}. \end{cases} \quad (9)$$

Relations (5) show that the absolute value of line directions of ①, ② and ③ are equal to $|\arctg(\frac{\sigma_i}{\sigma_j})|$ and correspond to angles α_1, α_2 and α_3 in figure 2(a).

2.4 Focal intervals intersection or union overview

As shown in figure 2(b), focal intervals $A_i^{z_k} \in \mathcal{F}_i$ and $B_j^{z_l} \in \mathcal{F}_j$ can directly be drawn on the chart by respectively vertical and horizontal line segments since ①, ② and ③ correspond to the focal intervals bounds. The path defined by half-lines ①, ② and ③ covers Ω . It becomes thus easy to analyze pairs in $\mathcal{F}_i \times \mathcal{F}_j$ to deduce their intersection or union. For instance, intervals shown in figure 2(b) are such as $A_i^{z_k} \cap B_j^{z_l} = [B_j^{z_l-}, A_i^{z_k+}]$ and $A_i^{z_k} \cup B_j^{z_l} = [A_i^{z_k-}, B_j^{z_l+}]$. When considering z_k and z_l respectively as horizontal and vertical cursors it is also possible to find the z limits of intervals intersecting or including one another or not.

2.5 Particular domains

Figure 2(b) shows also that the ①, ② and ③ lines separate $\mathcal{F}_i \times \mathcal{F}_j$ in four domains called $\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3$ and \mathcal{D}_4 ². For $\mu_i < \mu_j, z_i \geq 0, z_j \geq 0$, defining pairs $(A_i^{z_i}, B_j^{z_j}) \in \mathcal{F}_i \times \mathcal{F}_j$, we have:

$$\begin{cases} z_j > K_j + \frac{\sigma_i}{\sigma_j} z_i & \Rightarrow A_i^{z_i} \subset B_j^{z_j} & (\mathcal{D}_1), \\ z_j < -K_j + \frac{\sigma_i}{\sigma_j} z_i & \Rightarrow A_i^{z_i} \supset B_j^{z_j} & (\mathcal{D}_2), \\ z_j < K_j + \frac{\sigma_i}{\sigma_j} z_i, \quad z_j > K_j - \frac{\sigma_i}{\sigma_j} z_i, \quad z_j > -K_j + \frac{\sigma_i}{\sigma_j} z_i & \Rightarrow A_i^{z_i} \cap B_j^{z_j} \notin \{\emptyset, A_i^{z_i}, B_j^{z_j}\} & (\mathcal{D}_3), \\ z_j < K_j - \frac{\sigma_i}{\sigma_j} z_i, & \Rightarrow A_i^{z_i} \cap B_j^{z_j} = \emptyset & (\mathcal{D}_4). \end{cases} \quad (10)$$

When $\mu_1 = \mu_2$ only \mathcal{D}_1 and \mathcal{D}_2 exist and are separated by the half-line $z_2 = \frac{\sigma_1}{\sigma_2} z_1$.

² proof is not given here due to lack of space

2.6 Consonant subsets of $\mathcal{F}_{i,j}$

The focal set $\mathcal{F}_{i,j}$ obtained after a conjunctive or a disjunctive combination operation of \mathcal{E}_i and \mathcal{E}_j is composed of an infinite number of nested focal intervals subsets. These consonant subsets appear both in \mathcal{D}_1 , \mathcal{D}_2 and \mathcal{D}_3 shown in figure 2(b) and are partially represented in figure 1(b) with the dark gray area. These subsets are not disjoint and consequently if the same interval belongs to several of them, it is necessary to integrate to get its total weight. The domain \mathcal{D}_4 differs from the other ones as it is empty in case of a conjunctive combination operation and composed of nested non convex intervals in case of a disjunctive one.

3 Same length intervals resulting from intersection and union operations of focal ones

3.1 Intersection of focal intervals

The length l of the intersection of the pairs $(A_i^{z_i}, B_j^{z_j}) \in \mathcal{F}_i \times \mathcal{F}_j$ can be deduced from the characteristics of $\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3$ and \mathcal{D}_4 given by relations (10). l only depends on the pdfs $betf_i$ and $betf_j$ parameters and the focal intervals indexes z_i and z_j :

$$\begin{cases} \mathcal{D}_1 : A_i^{z_i} \cap B_j^{z_j} = A_i^{z_i}, & l(A_i^{z_i} \cap B_j^{z_j}) = 2\sigma_i z_i, \\ \mathcal{D}_2 : A_i^{z_i} \cap B_j^{z_j} = B_j^{z_j}, & l(A_i^{z_i} \cap B_j^{z_j}) = 2\sigma_j z_j, \\ \mathcal{D}_3 : A_i^{z_i} \cap B_j^{z_j} = [B_j^{z_j-}, A_i^{z_i+}], & l(A_i^{z_i} \cap B_j^{z_j}) = -|\mu_2 - \mu_1| + \sigma_i z_i + \sigma_j z_j, \\ \mathcal{D}_4 : A_i^{z_i} \cap B_j^{z_j} = \emptyset, & l(A_i^{z_i} \cap B_j^{z_j}) = 0. \end{cases} \quad (11)$$

We can find the elements of $\mathcal{F}_i \times \mathcal{F}_j$ having an identical intersection length $L = l(A_i^{z_i} \cap B_j^{z_j})$ with the help of relations (11). For instance, the z indexes of these elements in \mathcal{D}_3 are:

$$\sigma_j z_j = L + |\mu_2 - \mu_1| - \sigma_i z_i \quad (12)$$

thus:

$$z_j = A - \frac{\sigma_i}{\sigma_j} z_i \text{ with } A = \frac{L}{\sigma_j} + K_j. \quad (13)$$

This is graphically represented in figure 2(c) by the line segment ⑥ bounded by points (F, C) and (D, E) and crossing the point $(z_i, z_j) = (0, A)$ with A as defined in relation (13). Pairs $(A_i^{z_i}, B_j^{z_j})$ in \mathcal{D}_1 having also an intersection length equal to L correspond to the points of the half-line ⑤ crossing (F, C) since $A_i^{z_i} \cap B_j^{z_j} = A_i^{z_i}$ in \mathcal{D}_1 . Pairs of intervals on the half line ⑦ crossing (D, E) in figure 2(c) have an intersection length equal to L too since $A_i^{z_i} \cap$

$B_j^{z_j} = B_j^{z_j}$ in \mathcal{D}_2 .

At last, values A to F are quite interdependent³ depending on K_i and K_j values. This is due to the symmetry properties of straight lines ① to ⑦ in figures 2(b) and 2(c). Many relations depending on these parameters lead to the value of L such as $L = 2\sigma_j(C - K_j) = 2\sigma_i(D - K_i)$ for instance.

3.2 Union of focal intervals

One can deduce from the relations (10) related to the domains $\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3, \mathcal{D}_4$ that for the union operation of pairs $(A_i^{z_i}, B_j^{z_j}) \in \mathcal{F}_i \times \mathcal{F}_j$, we have:

$$\begin{cases} \mathcal{D}_1 : A_i^{z_i} \cup B_j^{z_j} = B_j^{z_j}, & l(A_i^{z_i} \cup B_j^{z_j}) = 2\sigma_j z_j, \\ \mathcal{D}_2 : A_i^{z_i} \cup B_j^{z_j} = A_i^{z_i}, & l(A_i^{z_i} \cup B_j^{z_j}) = 2\sigma_i z_i, \\ \mathcal{D}_3 : A_i^{z_i} \cup B_j^{z_j} = [B_j^{z_j-}, A_i^{z_i+}], & l(A_i^{z_i} \cup B_j^{z_j}) = |\mu_2 - \mu_1| + \sigma_i z_i + \sigma_j z_j, \\ \mathcal{D}_4 : A_i^{z_i} \cap B_j^{z_j} = \emptyset, & l(A_i^{z_i} \cup B_j^{z_j}) = 2(\sigma_i z_i + \sigma_j z_j). \end{cases} \quad (14)$$

As expressed in (14), concerning \mathcal{D}_3 , one can write:

$$l(A_i^{z_i} \cup B_j^{z_j}) - |\mu_2 - \mu_1| = \sigma_i z_i + \sigma_j z_j. \quad (15)$$

From (15), pairs $(A_i^{z_i}, B_j^{z_j}) \in \mathcal{D}_3$ having a constant union length $L = l(A_i^{z_i} \cup B_j^{z_j})$ satisfy thus:

$$z_j = A - \frac{\sigma_i}{\sigma_j} z_i \text{ with } A = \frac{L}{\sigma_j} - K_j. \quad (16)$$

This relation corresponds in figure 2(d) to the line segment ⑥ bounded by points (F, C) and (D, E) and crossing the point $(z_i, z_j) = (0, A)$ with A as defined in relation (16). Pairs in \mathcal{D}_1 having a union length equal to L correspond to the points of the line segment ⑧. This is also the case in \mathcal{D}_2 for the points of the line segment ⑨. As in the case of intersection operation, many relations⁴ link the value of L to the parameters of pdfs $betf_i$ and $betf_j$. For instance we have $L = 2\sigma_j C = 2\sigma_i D$.

4 Conclusion and Acknowledgment

As we have seen, the focal set $\mathcal{F}_{i,j}$ obtained in case of a conjunctive (resp. disjunctive) combination of two pieces of evidence \mathcal{E}_i and \mathcal{E}_j with consonant

³ proof is not given here due to lack of space

⁴ proof is not given here due to lack of space

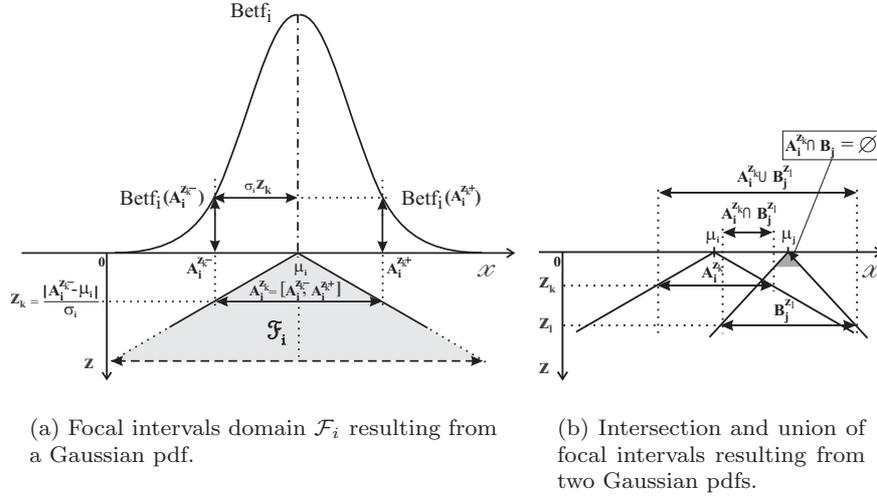


Fig. 1 Focal intervals graphical representation relatively to the z label value.

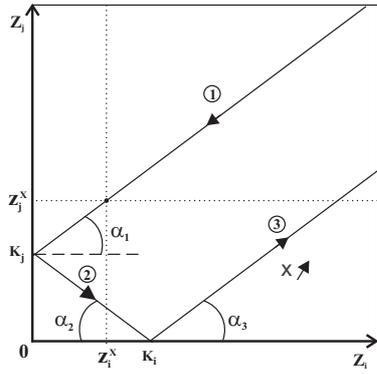
focal domains is not as heterogeneous as it seems to be. Intervals belonging to $\mathcal{F}_{i,j}$ are sorted into only four domains. In each of these domains, pairs $(A_i \in \mathcal{F}_i, B_j \in \mathcal{F}_j)$ of focal intervals share common properties regarding intersection $A_i \cap B_j$ and union $A_i \cup B_j$. These four domains can be graphically represented in a linear space where they are separated by straight lines when the focal sets \mathcal{F}_i and \mathcal{F}_j are composed of centered and consonant intervals. At last, elements of $\mathcal{F}_{i,j}$ having a same length are linked by linear relations. This can be useful in problems where interval lengths have to be taken into account.

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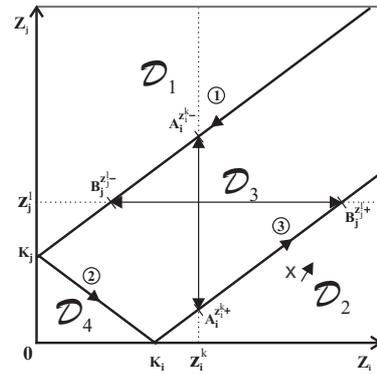
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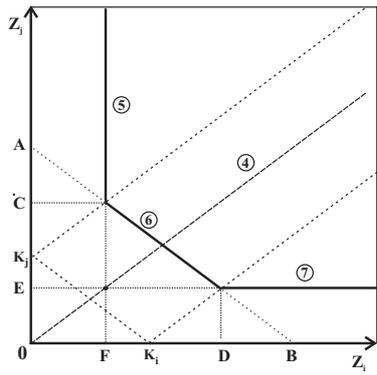
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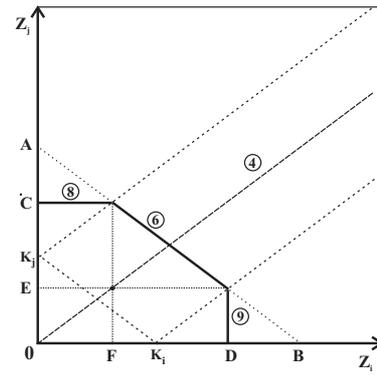
(a) Pairs of intervals having a common lower or upper bound



(b) Domains of identical properties



(c) Same length intervals resulting from a conjunctive combination



(d) Same length intervals resulting from a disjunctive combination

Fig. 2 Graphical representations of focal intervals properties.