On the determination of the principal coefficient from boundary measurements in a KdV equation
Lucie Baudouin, Eduardo Cerpa, Emmanuelle Crépeau, Alberto Mercado

To cite this version:
Lucie Baudouin, Eduardo Cerpa, Emmanuelle Crépeau, Alberto Mercado. On the determination of the principal coefficient from boundary measurements in a KdV equation. Journal of Inverse and Ill-posed Problems, 2014, 22 (6), pp.819-845. hal-00765919v2

HAL Id: hal-00765919
https://hal.science/hal-00765919v2
Submitted on 8 Feb 2013

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L’archive ouverte pluridisciplinaire HAL, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d’enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.
On the determination of the principal coefficient from boundary measurements in a KdV equation

Lucie Baudouin*, Eduardo Cerpa†, Emmanuelle Crépeau‡ and Alberto Mercado†

February 8, 2013

Abstract

This paper concerns the inverse problem of retrieving the principal coefficient in a Korteweg-de Vries (KdV) equation from boundary measurements of a single solution. The Lipschitz stability of this inverse problem is obtained using a new global Carleman estimate for the linearized KdV equation. The proof is based on the Bukhgeım-Klibanov method.

Keywords: Inverse problem, Korteweg-de Vries equation, Carleman estimate

AMS subject classifications: 35R30, 35Q53.

1 Introduction

The Korteweg-de Vries (KdV) equation

\[ y_t(t, x) + y_{xxx}(t, x) + y_x(t, x) + y(t, x)y_x(t, x) = 0, \]

is a nonlinear dispersive equation that serves as a mathematical model to study the propagation of long water waves in channels of relatively shallow depth and flat bottom [33]. In this model, the function \( y = y(t, x) \) represents the surface elevation of the water wave at time \( t \) and at position \( x \). Since the works by Johnson [32] and Grimshaw [24] (see also the recent survey [25]), the study of water waves moving over variable

---

*CNRS, LAAS, 7 avenue du colonel Roche, F-31400 Toulouse, France Univ de Toulouse, LAAS ; F-31400 Toulouse, France. E-mail: lucie.baudouin@laas.fr
†Departamento de Matemática, Universidad Técnica Federico Santa María, Casilla 110-V, Valparaíso, Chile. E-mail: eduardo.cerpa@usm.cl, alberto.mercado@usm.cl
‡Laboratoire de Mathématiques, Université de Versailles Saint-Quentin en Yvelines, 78035 Versailles, France. E-mail: emmanuelle.crepeau@math.uvsq.fr
topography has been considered. If we denote \( h = h(x) \) the function describing the variations in depth of the channel, then the proposed model becomes (after scaling)

\[
y_t(t, x) + h^2(x) y_{xxx}(t, x) + (\sqrt{h(x)} y(t, x))_x + \frac{1}{\sqrt{h(x)}} y(t, x) y_x(t, x) = 0.
\]

(1)

In a recent paper, Israwi [31] proposes a correction to this model by saying that the main coefficient is not \( h^2(x) y_{xxx}(t, x) \) but \( h^{\frac{5}{2}}(x) y_{xxx}(t, x) \). Thus, we are led to consider variable coefficients KdV equations to model the water wave propagation in non-flat channels.

The original inverse problem which motivated this paper can be stated in an informal way as

"to recover information on the topography of a channel (denoted by \( h \)) by means of boundary measurements of a water wave (given by \( y \))."

This kind of coefficient inverse problem is by now classical and referred to as the determination of coefficients in evolution partial differential equations (pde’s) by means of the partial knowledge of a single solution.

Many other inverse problems on KdV equations could be studied with possible applications. For example the pulsatile blood flow in an arterial compartment can be modelized by KdV equation, see [20] for details. It leads to several open inverse problems, as the determination of the flow at the input of the arteria with only one measure at the output, or the determination of physical coefficients like stiffness of arterias. Those open inverse problems are very relevant for cardiovascular medicine.

Since the pioneer works by Bukhgeim, Klibanov and Malinsky [13, 35], there have been a number of papers devoted to the proof of uniqueness and stability for inverse problems concerning the determination of source or coefficients in pde’s by means of the localized knowledge of a single solution.

The problem of recovering sources has been first addressed for hyperbolic equations in [39, 40] and for parabolic equations in [27]. The determination of coefficients has also drawn the attention of many authors. Among the most classical results for the wave equation, we can mention the determination of a potential from boundary [45] or internal [28] measurements and the determination of the main coefficient from boundary [7, 36] or internal [29] observation. Regarding the problem of recovering coefficients in parabolic equations, we send the interested reader to the survey [44] by Yamamoto and the references therein.

Concerning dispersive equations, a stability result was obtained in [5] for the problem of recovering the potential in a Schrödinger equation. That paper motivated several further works as [14, 37], where appropriate Carleman estimates were proven in different contexts. To the best of our knowledge, there are no previous works concerning coefficient inverse problems for KdV equations.

Most of the references given above are inspired by what is often called the Bukhgeim-Klibanov method (see the recent book [6] or the survey [34]), which was meant to prove
global uniqueness results for single-measurement coefficients inverse problems. Roughly speaking, it consists in two steps. The first step comprises the time differentiation of the governing pde, getting an auxiliary system, in which the unknown coefficient appears now in the initial condition. The second step relies on the application of a Carleman estimate whose parameters and weight are crucial to drive to the conclusion. An idea of Imanuvilov and Yamamoto, presented for the first time in [27] (for parabolic equation) and then often used (for other pde’s), allows to use the appropriate global Carleman inequality to estimate the unknown coefficient in terms of internal or boundary measurements of the solution. The main drawback of this method, that is not lifted yet is the hypothesis that at least one initial condition, (or some of its derivatives) never vanishes in the domain.

This article is a first step in order to address the problem of recovering the topography of the channel by means of boundary measurements of the water wave. Indeed, we will deal with a KdV equation posed on a bounded interval \([0,L]\) with an unknown third-order coefficient:

\[
\begin{cases}
y_t + a(x)y_{xxx} + y_x + yy_x = 0, & \forall (x,t) \in (0,L) \times (0,T), \\
y(0,t) = y(L,t) = y_x(L,t) = 0, & \forall t \in (0,T), \\
y(x,0) = y_0(x), & \forall x \in (0,L),
\end{cases}
\]  

(2)

where the initial data \(y_0\) is known. The unknown coefficient \(a = a(x)\) is assumed to be time independent. The case where we have several unknown coefficients (as in the equation (1)) is very hard to deal with. There are few results in this direction and they ask many conditions on the coefficients. See for instance [15] where a Schrödinger equation is considered on an unbounded strip.

Up to our knowledge, this paper solves the first coefficient inverse problem for the KdV equation. As mentioned before, the Bukhgejm-Klibanov method requires a Carleman estimate for the equation. This type of inequalities has already been proven for the KdV equation on a bounded interval in the framework of control problems. The control of this equation from the left Dirichlet boundary condition have been studied in [43] and [23], where the authors get the null-controllability of this equation by proving one-parameter Carleman estimates. Unlike those papers, here we will prove a two-parameter estimate in order to be able to deal with a space dependent main coefficient.

The inverse problem we are interested in can be stated as follows.

**Inverse problem:** Retrieve the principal coefficient \(a = a(x)\) of equation (2) from the measurement of \(y_x(0,t)\), \(y_{xx}(0,t)\) and \(y_{xxx}(L,t)\) on \((0,T)\), where \(y\) is the solution of equation (2).

A partial and local answer for this nonlinear inverse problem will be proved in this paper. To be more specific, let us denote by \(y\) and \(\tilde{y}\) the solutions of Equation (2) corresponding to coefficients \(a\) and \(\tilde{a}\) respectively (we shall also use the notation \(y[a], y[\tilde{a}]\)).
Stability: Estimate $\|\tilde{a} - a\|_{L^2(0,L)}$ by suitable norms of the measurements $\|\tilde{y}_x(0,t) - y_x(0,t)\|$, $\|\tilde{y}_{xx}(0,t) - y_{xx}(0,t)\|$ and $\|\tilde{y}_{xx}(L,t) - y_{xx}(L,t)\|$.

The next theorem is the main result of this article and gives the stability result for the inverse problem under consideration.

**Theorem 1** Let $r_0$, $a_0$, $\alpha$ and $K$ be given positive constants. Assume that the initial data $y_0 \in \{w \in H^7(0,L); w(0) = w(L) = w'(L) = 0\}$ satisfies for every $x \in [0,L]$

$$y''_0(x) = y''_0(L-x)$$

and

$$\|y''_0(x)\| > r_0 > 0.$$  

Define the set

$$\Sigma(a_0, \alpha) = \left\{ a \in W^{6,\infty}(0,L) \mid \forall x \in [0,L], a(x) \geq a_0 > 0, a(x) = a(L-x), \right.$$  

$$\text{and } \|a\|_{W^{6,\infty}(0,L)} \leq \alpha \right\}. \quad (5)$$

Then, there exists a constant $C = C(L,T,r_0,a_0,\alpha,K) > 0$ such that

$$C\|a - \tilde{a}\|_{L^2(0,L)} \leq \|y_x(0,t) - \tilde{y}_x(0,t)\|_{H^1(0,T)} + \|y_{xx}(0,t) - \tilde{y}_{xx}(0,t)\|_{H^1(0,T)}$$

$$\quad + \|y_{xx}(L,t) - \tilde{y}_{xx}(L,t)\|_{H^1(0,T)} \quad (6)$$

for every $a, \tilde{a}$ in $\Sigma(a_0, \alpha)$ such that the corresponding solutions of Equation (2) satisfy

$$\max \{\|y\|_{W^{1,\infty}(0,T;W^{3,\infty}(0,L))}, \|\tilde{y}\|_{W^{1,\infty}(0,T;W^{3,\infty}(0,L))}\} \leq K.$$

**Remark 1** In the Appendix, we will sketch the proof of the fact that the required hypotheses on $a \in W^{6,\infty}(0,L)$ and $y_0 \in \{w \in H^7(0,L); w(0) = w(L) = w'(L) = 0\}$ are enough to guarantee that the solution of (2) lies in $W^{1,\infty}(0,T;W^{3,\infty}(0,L))$ (see Proposition 3). Thus, the set of trajectories where Theorem 1 is valid is not empty. We mention that hypothesis $y_0 \in H^7(0,L)$ is not sharp, see Remark 6 in the Appendix.

**Remark 2** The symmetry hypothesis on the initial data and the main coefficient are technical points coming from our proof. More precisely, they allow us to extend the solution of KdV equation to negative times, in order to apply the Bukhgeim-Klibanov method.

**Remark 3** Concerning the question of the minimal number of boundary measurement required in Theorem 1, one could expect to get estimate (6) with either one measurement at $x = L$ or two measurements at $x = 0$, but not with just one measurement at $x = 0$. This is based on the fact that Carleman estimates imply some observability inequalities, which do not hold for the linear KdV equation with only one observation at $x = 0$. This lack of observability holds for some critical lengths $L$, as one can read in [42, 19, 16, 17]. On the other hand, we need in this paper three measurements because of the symmetry assumptions and the extension to negative times used.
Other related papers on this type of inverse problem for pde’s can be listed: [11, 20] (one dimensional fourth order parabolic equation), [10, 9, 3] (discontinuous coefficients), [2] (network of one dimensional waves) [9, 8] (logarithmic stability estimates), [21] (parabolic system), [11] (unknown coefficient in the nonlinearity), [46, 15, 21] (two unknown coefficients). We also mention some important books which can be the starting point to study inverse problems [30], Carleman estimates [22], and control theory for partial differential equations [18].

Outline: In section 2, a two-parameters global Carleman estimate for the linearized KdV equation with non-constant main coefficient is obtained. It is then used in section 3, following the Bukhgeim-Klibanov method, to prove the desired stability result. As mentioned before, the Appendix A is concerned with the Cauchy problem.

2 Carleman estimate

In this section, a global Carleman inequality will be proven for the linearized KdV equation on the domain denoted here by \( Q := (0, L) \times (-T, T) \). As mentioned in the introduction, there are in the literature some Carleman estimates for the KdV equation with constant main coefficient posed on a bounded interval [43, 23]. In this work we deal with non-constant coefficients \( a = a(x) \). We prove a two-parameter \((s, \lambda)\) Carleman estimate in order to avoid extra hypothesis on the coefficients when solving the inverse problem.

Let us consider \( d \in L^\infty((-T, T) \times (0, L)), b \in L^\infty(-T, T; W^{1, \infty}(0, L)) \) and \( a \in W^{3, \infty}(0, L) \), with \( a(x) \geq a_0 \) for every \( x \in [0, L] \), and \( \|a\|_{W^{3, \infty}(0, L)} \leq \alpha \), where \( a_0, \alpha > 0 \). We define the operator

\[
P = \partial_t + a(x)\partial_{xxx} + b(x, t)\partial_x + d(x, t)
\]

and the space

\[
\mathcal{V} = \left\{ v \in L^2(-T, T; H^3 \cap H^1_0(0, L)) / Pv \in L^2((0, L) \times (-T, T)) \right\}.
\]

Consider \( \beta \in C^3([0, L]) \) such that for some \( r > 0 \) we have

\[
0 < r \leq \beta(x) \quad \text{and} \quad 0 < r \leq \beta'(x), \quad \forall x \in (0, L).
\]

We define, for each \( \lambda > 0 \), the functions

\[
\phi(x, t) = \frac{e^{2\lambda ||\beta||_\infty} - e^{\lambda \beta(x)}}{(T + t)(T - t)}, \quad \theta(x, t) = \frac{e^{\lambda \beta(x)}}{(T + t)(T - t)}
\]

for \((x, t) \in [0, L] \times (-T, T)\). Let us notice that there exists a constant \( C = C(T) \) such that

\[
0 < \frac{1}{C} \leq \theta(x, t) \quad \text{and} \quad |\theta_t(x, t)| \leq C\theta^2(x, t), \quad \forall (x, t) \in [0, L] \times (-T, T).
\]

We will prove the following Carleman estimate.
Therefore, \( \phi \). When looking for a one-parameter Carleman estimate (as in \([23, 43]\) where for all such that for every \( \theta \), we define the operator \( \phi \) as the linearized KdV operator defined in (7). There exist \( s_0 > 0 \), \( \lambda_0 > 0 \) and a constant \( C(L, T, s_0, \lambda_0, \alpha, a_0, r) > 0 \) such that for every \( s \geq s_0 \) and \( \lambda \geq \lambda_0 \),

\[
\int_{-T}^{T} \int_{0}^{L} e^{-2s\phi} \left( s^5 \lambda^6 \phi^5 |u|^2 + s^3 \lambda^4 \phi^3 |u_x|^2 + s \lambda^2 \theta |u_{xx}|^2 \right) \, dx \, dt \\
\leq C \int_{-T}^{T} \int_{0}^{L} e^{-2s\phi} |Pw|^2 \, dx \, dt + C s^3 \lambda^3 \int_{-T}^{T} e^{-2s\phi(L,t)} \phi^3(L,t) |u_x(L,t)|^2 \, dt \\
+ C s \lambda \int_{-T}^{T} e^{-2s\phi(L,t)} \phi(L,t) |u_{xx}(L,t)|^2 \, dt \quad (12)
\]

for all \( u \in \mathcal{V} \) defined by (8).

**Remark 4** When looking for a one-parameter Carleman estimate (as in \([23, 43]\) where \( \phi = \frac{\beta(x)}{(t+T)(t-T)} \), some hypothesis on the second derivative of the function \( \beta \) have to be imposed. The use of the second parameter \( \lambda \) allows to omit them.

**Proof:** Let \( s > 0 \) and define \( \mathcal{W}_s = \{ e^{-s\phi} u / u \in \mathcal{V} \} \). For \( u \in \mathcal{V} \), we set \( w = e^{-s\phi} u \) and we define the operator \( P_\phi \) from \( \mathcal{W}_s \) to \( L^2((0,L) \times (-T,T)) \) by

\[
P_\phi w = e^{-s\phi} P(e^{s\phi} w).
\]

After straightforward computations, we write

\[
P_\phi w = P_1 w + P_2 w + Rw
\]

where, for a constant \( m > 0 \) to be chosen later,

\[
P_1 w = w_t + 3a^2 s \phi^2 w_x + aw_{xxx} + 3am s^2 \phi_x \phi_{xx} w, \quad (13)
\]

\[
P_2 w = a s^3 \phi^3 w_x + 3as \phi_x w_{xx} + 3sw_x(\alpha \phi_x)_x, \quad (14)
\]

and

\[
Rw = bs \phi_x w + bw_x + as \phi_{xxx} w + 3as^2 \phi_x \phi_{xx} w + dw \\
+ s \phi_t w - 3a s \phi_x w_x - 3am s^2 \phi_x \phi_{xx} w. \quad (15)
\]

Therefore,

\[
\|P_\phi w - Rw\|_{L^2(Q)}^2 = \|P_1 w\|_{L^2(Q)}^2 + 2 \langle P_1 w, P_2 w \rangle + \|P_2 w\|_{L^2(Q)}^2 \quad (16)
\]

where \( \langle \cdot, \cdot \rangle \) is the \( L^2(Q) \) scalar product.

**Step 1. Explicit calculations**

Notice that \( w(0,t) = w(L,t) = 0 \) for all \( t \in (-T, T) \) and \( w(x, \pm T) = 0 \) for all \( x \in (0,L) \). This is due to the fact that \( w = e^{-s\phi} u \), with \( u \in L^2(-T, T; H^2_0(0,L)) \) and \( \lim_{t \to \pm T} \phi(x,t) = +\infty \).
By developing \( \langle P_1 w, P_2 w \rangle \), we get
\[
\langle P_1 w, P_2 w \rangle = \sum_{i=1, \ldots, 4} \sum_{j=1, \ldots, 3} I_{i,j}
\]
where \( I_{i,j} \) is the \( L^2(Q) \) scalar product of the \( i \)-th term in (13) with the \( j \)-th term in (14). By integrations by parts, we obtain:

- \( I_{1,1} = -\frac{3s^3}{2} \int_Q \int a \phi_x^2 \phi_{xx} |w|^2 \, dx \, dt = \frac{3}{2} s^3 \lambda^3 \int_Q a \phi_x^2 \theta_x^2 |w|^2 \, dx \, dt \);
- \( I_{1,2} = -3s \int_Q (a \phi_x)_x w_x w_t \, dx \, dt + \frac{3s}{2} \int_Q a \phi_x |w_x|^2 \, dx \, dt \
  = -3s \int_Q (a \phi_x)_x w_x w_t \, dx \, dt - \frac{3}{2} s \lambda \int_Q a \phi_x \theta |w_x|^2 \, dx \, dt \);
- \( I_{1,3} = 3s \int_Q (a \phi_x)_x w_x w_t \, dx \, dt \);
- \( I_{2,1} = -\frac{3s^5}{2} \int_Q \int (a^2 \phi_x^5)_x |w|^2 \, dx \, dt \
  = \frac{3}{2} s^5 \lambda^5 \int_Q (a^2 \phi_x^5)_x \theta^5 |w|^2 \, dx \, dt + \frac{15}{2} s^5 \lambda^6 \int_Q a^2 \phi_x \theta^5 |w|^2 \, dx \, dt \);
- \( I_{2,2} = -\frac{9s^3}{2} \int_Q (a^2 \phi_x^3)_x w_x |w|^2 \, dx \, dt + \frac{9s^3}{2} \int_0^T a^2 \phi_x^3 |w_x|^2 \bigg|_{x=0}^L \, dt \
  = \frac{9}{2} s^3 \lambda^3 \int_Q (a^2 \phi_x^3)_x \theta^3 |w_x|^2 \, dx \, dt + \frac{27}{2} s^3 \lambda^4 \int_Q a^2 \phi_x \theta^3 |w_x|^2 \, dx \, dt \
  - \frac{9}{2} s^3 \lambda^3 \int_0^T a^2 \phi_x \theta^3 |w_x|^2 \bigg|_{x=0}^L \, dt \);
- \( I_{2,3} = 9s^3 \int_Q a \phi_x^2 (a \phi_x)_x |w_x|^2 \, dx \, dt \
  = -9s^3 \lambda^3 \int_Q a \phi_x^2 (a \phi_x)_x \theta^3 |w_x|^2 \, dx \, dt - 9s^3 \lambda^4 \int_Q a^2 \phi_x \theta^3 |w_x|^2 \, dx \, dt \);
- \( I_{3,1} = \frac{3s^3}{2} \int_Q \int (a^2 \phi_x^3)_x |w|^2 \, dx \, dt - \frac{s^3}{2} \int_Q (a^2 \phi_x^3)_x w_x |w|^2 \, dx \, dt - \frac{s^3}{2} \int_0^T a^2 \phi_x^3 |w_x|^2 \bigg|_{x=0}^L \, dt \
  = -\frac{3}{2} s^3 \lambda^3 \int_Q (a^2 \phi_x^3)_x \theta^3 |w_x|^2 \, dx \, dt - \frac{9}{2} s^3 \lambda^4 \int_Q a^2 \phi_x \theta^3 |w_x|^2 \, dx \, dt \
  + \frac{s^3 \lambda^3}{2} \int_Q (a^2 \phi_x^3)_x \theta^3 |w|^2 \, dx \, dt + \frac{s^3 \lambda^3}{2} \int_0^T a^2 \phi_x \theta^3 |w_x|^2 \bigg|_{x=0}^L \, dt \);
- \( I_{3,2} = -\frac{3s}{2} \int_Q (a^2 \phi_x)_x w_x |w|^2 \, dx \, dt + \frac{3s}{2} \int_0^T a^2 \phi_x |w_x|^2 \bigg|_{x=0}^L \, dt \
  = \frac{s}{2} s \lambda \int_Q (a^2 \phi_x)_x \theta |w_x|^2 \, dx \, dt + \frac{3}{2} s \lambda^2 \int_Q a^2 \phi_x \theta |w_x|^2 \, dx \, dt \
  - \frac{3}{2} s \lambda \int_0^T a^2 \phi_x |w_x|^2 \bigg|_{x=0}^L \, dt \);
\[ I_{3,3} = -3s \int_T \int_Q a(\alpha_x) |w_{xx}|^2 dx \, dt + \frac{3s}{2} \int_T \int_Q |a(\alpha_x) |x=0| w_x |^2 dx \, dt \\
- \frac{3s}{2} \int_0^T (a(\alpha_x) ) |x=0| w_x |^2 dx \\
= 3s \int_T \int_Q a(\alpha_x) \theta |w_{xx}|^2 dx \, dt + 3s \lambda \int_T \int_Q a^2 \beta_x^2 \theta |w_{xx}|^2 dx \, dt \\
+ \frac{3}{2} s \lambda \int_0^T (a(\alpha_x) ) |x=0| w_x |^2 dx \\
- \frac{3}{2} s \lambda \int_T \int_Q a(\alpha_x) \theta |w_x |^2 dx \, dt; \]

\[ I_{4,1} = 3ms^5 \int_T \int_Q a^2 \phi_x^4 |w_x|^2 dx \, dt \\
= -3ms^5 \lambda^5 \int_T \int_Q a^2 \beta_x^4 \theta^5 |w|^2 dx \, dt - 3ms^5 \lambda^6 \int_T \int_Q a^2 \beta_x^6 \theta^5 |w|^2 dx \, dt; \]

\[ I_{4,2} = -9ms^3 \int_T \int_Q a^2 \phi_x^2 |w_x|^2 dx \, dt + \frac{9ms^3}{2} \int_T \int_Q (a^2 \phi_x^2 \phi_{xx}) |w|^2 dx \, dt \\
= 9ms^3 \lambda^3 \int_T \int_Q a^2 \beta_x^2 \theta^3 |w|^2 dx \, dt + 9ms^3 \lambda^4 \int_T \int_Q a^2 \beta_x^4 \theta^3 |w|^2 dx \, dt \\
- \frac{9m}{2} s^3 \lambda^3 \int_T \int_Q (a^2 \beta_x^2 \theta^3 ) |w| dx \, dt - \frac{9m}{2} s^3 \lambda^4 \int_T \int_Q (a^2 \beta_x^4 \theta^3 ) |w| dx \, dt; \]

\[ I_{4,3} = -\frac{9ms^3}{2} \int_T \int_Q (a\phi_{xx}(a\phi_x) |w|^2 dx \, dt \\
= \frac{9m}{2} s^3 \lambda^3 \int_T \int_Q (a\beta_x \beta_{xx} \theta^2 (a\beta_x \theta) ) |w|^2 dx \, dt + \frac{9m}{2} s^3 \lambda^4 \int_T \int_Q (a\beta_x \theta^2 (a\beta_x \theta) ) |w|^2 dx \, dt. \]

Therefore,
\[ \langle P_1 w, P_2 w \rangle = \left( \frac{15}{2} - 3m \right) s^5 \lambda^6 \int_T \int_Q a^2 \beta_x^6 \theta^5 |w|^2 dx \, dt \]
\[ + 9ms^3 \lambda^4 \int_T \int_Q a^2 \beta_x^4 \theta |w_x|^2 dx \, dt + \frac{9m}{2} s^3 \lambda^2 \int_T \int_Q a^2 \beta_x^2 \theta |w_{xx}|^2 dx \, dt + B + X_1 \]

where \( X_1 \) and \( B \) satisfy the following. The term \( X_1 \) gathers the non-dominating terms and satisfies that there exists a positive constant \( C = C(L, T, \alpha, r) \) independent of \( s \) and \( \lambda \) (which may change from line to line) such that, using (11), we get
\[ |X_1| \leq C(s^3 \lambda^6 + s^5 \lambda^5) \int_T \int_Q \theta |w|^2 dx \, dt + C(s \lambda^4 + s^3 \lambda^3) \int_T \int_Q \theta^2 |w_x|^2 dx \, dt. \]

The term \( B \) gathers the boundary terms. We split them as follows
\[ B := B^+ + B^- + B^* \]

with the notations
\[ B^+ = 4s^3 \lambda^3 \int_T \int_T (a^2 \beta_x^3 \theta^3 |w_x| dx \, dt + \frac{3}{2} s \lambda \int_T (a^2 \beta_x \theta) |w_x| dx \, dt, \]

\[ B^- = \frac{9m}{2} s^3 \lambda^3 \int_T \int_Q (a\beta_x \beta_{xx} \theta^2 (a\beta_x \theta) ) |w|^2 dx \, dt + \frac{9m}{2} s^3 \lambda^4 \int_T \int_Q (a\beta_x \theta^2 (a\beta_x \theta) ) |w|^2 dx \, dt. \]

\[ 8 \]
From (18) we obtain that there exists a constant $B^*$ only on $B^\ast$

we obtain that

By choosing, and fixing, $m$ such that

we obtain that there exists a constant $C = C(L, T, a_0, r) > 0$ such that:

\[
\left( \frac{15}{2} - 3m \right) s^5 \lambda^6 \int_Q a^2 \beta_x^{3} \theta^{3} |w|^2 dxdt \geq Cs^5 \lambda^6 \int_Q \theta^5 |w|^2 dxdt;
\]

\[
9ms^3 \lambda^4 \int_Q a^2 \beta_x^{4} \theta^{3} |w_x|^2 dxdt \geq Cs^3 \lambda^4 \int_Q \theta^3 |w_x|^2 dxdt;
\]

\[
\frac{9}{2} s^2 \lambda^2 \int_Q a^2 \beta_x^{2} \theta |w_{xx}|^2 dxdt \geq Cs^2 \lambda^2 \int_Q \theta |w_{xx}|^2 dxdt.
\]

We introduce the notation given by

\[
\|w\|^2_{s, \lambda, \theta} = s^5 \lambda^6 \int_Q \theta^5 |w|^2 dxdt + s^3 \lambda^4 \int_Q \theta^3 |w_x|^2 dxdt + s^2 \lambda^2 \int_Q \theta |w_{xx}|^2 dxdt.
\]

From (18) we obtain that

\[
|X_1| \leq C \left( \frac{1}{s^2} + \frac{1}{\lambda} \right) \|w\|^2_{s, \lambda, \theta}.
\]

Therefore, choosing $s$ and $\lambda$ large enough, from (17) we get some $C > 0$ depending only on $L, T, \alpha, a_0, r, s_0$ and $\lambda_0$ and is such that, for all $s > s_0, \lambda > \lambda_0$,

\[
C \|w\|^2_{s, \lambda, \theta} \leq \langle P_1 w, P_2 w \rangle - B.
\]

**Step 2. Dominating terms**

By choosing, and fixing, $m$ such that

we obtain that there exists a constant $C = C(L, T, a_0, r) > 0$ such that:

we obtain that there exists a constant $C = C(L, T, a_0, r) > 0$ such that:

\[
\left( \frac{15}{2} - 3m \right) s^5 \lambda^6 \int_Q a^2 \beta_x^{3} \theta^{3} |w|^2 dxdt \geq Cs^5 \lambda^6 \int_Q \theta^5 |w|^2 dxdt;
\]

\[
9ms^3 \lambda^4 \int_Q a^2 \beta_x^{4} \theta^{3} |w_x|^2 dxdt \geq Cs^3 \lambda^4 \int_Q \theta^3 |w_x|^2 dxdt;
\]

\[
\frac{9}{2} s^2 \lambda^2 \int_Q a^2 \beta_x^{2} \theta |w_{xx}|^2 dxdt \geq Cs^2 \lambda^2 \int_Q \theta |w_{xx}|^2 dxdt.
\]

We introduce the notation given by

\[
\|w\|^2_{s, \lambda, \theta} = s^5 \lambda^6 \int_Q \theta^5 |w|^2 dxdt + s^3 \lambda^4 \int_Q \theta^3 |w_x|^2 dxdt + s^2 \lambda^2 \int_Q \theta |w_{xx}|^2 dxdt.
\]

From (18) we obtain that

\[
|X_1| \leq C \left( \frac{1}{s^2} + \frac{1}{\lambda} \right) \|w\|^2_{s, \lambda, \theta}.
\]

Therefore, choosing $s$ and $\lambda$ large enough, from (17) we get some $C > 0$ depending only on $L, T, \alpha, a_0, r, s_0$ and $\lambda_0$ and is such that, for all $s > s_0, \lambda > \lambda_0$,

\[
C \|w\|^2_{s, \lambda, \theta} \leq \langle P_1 w, P_2 w \rangle - B.
\]

**Step 3. Boundary terms**

From (19) and (21), considering that $B^+$ and $-B^-$ are non-negative, we write

\[
C \|w\|^2_{s, \lambda, \theta} + B^+ \leq \langle P_1 w, P_2 w \rangle - B^- - B^*.
\]

Now, we shall deal with the boundary terms in $B^*$, that has no sign. We will bound $|B^*|$ by above. For the four integrals one by one, using (11) and Cauchy-Schwarz inequality, this gives:

\[
\left| \frac{3}{2} s \lambda \int_{-T}^{T} (a(a\beta_x \theta) \phi) \left| w_x(L) \right|^2 dt \right| \leq C s \lambda^3 \int_{-T}^{T} a^2(L) \beta_x^2(L) \theta(L) \left| w_x(L) \right|^2 dt,
\]

\[
\left| \frac{3}{2} s \lambda \int_{-T}^{T} (a(a\beta_x \theta) \phi) \left| w_x(L) \right|^2 dt \right| \leq C s \lambda^3 \int_{-T}^{T} a^2(L) \beta_x^2(L) \theta(L) \left| w_x(L) \right|^2 dt,
\]
\[
\left| \frac{3}{2} s \lambda \int_{-T}^{T} (a(a(\beta_x)_{xx})_x)_{xx} \right|_{x=0} |w_x(0)|^2 dt \leq C s \lambda^3 \int_{-T}^{T} a^2(0) \beta^3_x(0) \theta(0) |w_x(0)|^2 dt,
\]
\[
3 s \lambda \int_{-T}^{T} (a(a(\beta_x)_{xx})_x)_{x=L} w_x(L) w_{xx}(L) dt \leq C s^2 \lambda^3 \int_{-T}^{T} a^2(L) \beta^3_x(L) \theta(L) |w_x(L)|^2 dt + C \lambda \int_{-T}^{T} a^2(L) \beta_x(L) \theta(L) |w_{xx}(L)|^2 dt,
\]
and
\[
3 s \lambda \int_{-T}^{T} (a(a(\beta_x)_{xx})_x)_{x=0} w_x(0) w_{xx}(0) dt \leq C s^2 \lambda^3 \int_{-T}^{T} a^2(0) \beta^3_x(0) \theta(0) |w_x(0)|^2 dt + C \lambda \int_{-T}^{T} a^2(0) \beta_x(0) \theta(0) |w_{xx}(0)|^2 dt.
\]

From these last four inequalities we get that
\[
|B^*| \leq \frac{C}{s} B^+ + \frac{C}{s} |B^-|.
\]

Therefore, taking \( s_0 \) large enough, inequality (22) becomes, for all \( s > s_0 \),
\[
\|w\|_{s,\lambda,\theta}^2 + B^+ \leq C \langle P_1 w, P_2 w \rangle + C |B^-|.
\]

**Step 4. Carleman estimate for \( w \).**

Since \( B^+ \geq 0 \), inequality (23) yields
\[
\|w\|_{s,\lambda,\theta}^2 \leq C \langle P_1 w, P_2 w \rangle + C s \lambda^3 \int_{-T}^{T} a^2(L) \beta^3_x(L) \theta^3(L, t) |w_x(L, t)|^2 dt + C s \lambda \int_{-T}^{T} a^2(L) \beta_x(L) \theta(L, t) |w_{xx}(L, t)|^2 dt.
\]

Moreover, from (15) we have
\[
\|P_\phi w - Rw\|_{L^2(Q)}^2 \leq 2 \|P_\phi w\|_{L^2(Q)}^2 + 2 \|R w\|_{L^2(Q)}^2 \leq 2 \|P_\phi w\|_{L^2(Q)}^2 + C s^4 \lambda^6 \int_{Q} \theta^3 |w|^2 dxdt + C s^2 \lambda^2 \int_{Q} |w_x|^2 dxdt.
\]

Therefore, choosing again \( s_0 \) and \( \lambda_0 \) large enough, we have proved the following.

**Proposition 1** There exist \( s_0, \lambda_0 > 0 \) and a constant \( C = C(L, T, s_0, \lambda_0, \alpha, a_0, r) > 0 \) such that for all \( s \geq s_0 \), for all \( \lambda \geq \lambda_0 \),
\[
s^5 \lambda^6 \int_{-T}^{T} \int_{0}^{L} \theta^5 |w|^2 dxdtdt + s^3 \lambda^4 \int_{-T}^{T} \int_{0}^{L} \theta^3 |w_x|^2 dxdtdt + s \lambda^2 \int_{-T}^{T} \int_{0}^{L} \theta |w_{xx}|^2 dxdtdt
\]
\[
+ \int_{-T}^{T} \int_{0}^{L} |P_1 w|^2 dxdtdt + \int_{-T}^{T} \int_{0}^{L} |P_2 w|^2 dxdtdt \leq C \int_{-T}^{T} \int_{0}^{L} |P_\phi w|^2 dxdtdt
\]
\[
+ C s^3 \lambda^3 \int_{-T}^{T} \theta^3(L, t) |w_x(L, t)|^2 dt + C s \lambda \int_{-T}^{T} \theta(L, t) |w_{xx}(L, t)|^2 dt
\]
for all \( w \in \mathcal{W}_s := \{ e^{-s\phi} v : v \in \mathcal{V} \} \).
In order to estimate Step 2. Extension to negative time
seek to estimate, appearing not only in the source term of the equation, but also into where $f$

We consider now Step 1. Local solution of the inverse problem

rem 1 about the inverse problem of retrieving the main coefficient $a$ in Equation (2) from boundary measurements of the solution. We recall here that we will use the Bukhgeim-Klibanov method. In the sake of clarity, we divide the proof in several steps.

Step 1. Local solution of the inverse problem
We consider two coefficients $a, \tilde{a}$ and the corresponding solutions of (2), $y$ and $\tilde{y}$, and we define

$$ u(x, t) := y(x, t) - \tilde{y}(x, t), \quad \text{and} \quad \sigma(x) := \tilde{a}(x) - a(x). $$

The function $u$ satisfies

$$
\begin{cases}
  u_t + a(x)u_{xxx} + (1 + \tilde{y})u_x + y_xu = \sigma y_{xxx}, & \forall (x, t) \in (0, L) \times (0, T), \\
  u(0, t) = 0, & u(L, t) = 0, \quad u_x(L, t) = 0 \quad \forall t \in (0, T), \\
  u(x, 0) = 0, & \forall x \in (0, L).
\end{cases}
$$

(26)

We consider now $z(x, t) := u_t(x, t)$, which satisfies the system

$$
\begin{cases}
  z_t + a(x)z_{xxx} + (1 + \tilde{y})z_x + y_xz = f, & \forall (x, t) \in (0, L) \times (0, T), \\
  z(0, t) = 0, & z(L, t) = 0, \quad z_x(L, t) = 0 \quad \forall t \in (0, T), \\
  z(x, 0) = \sigma(x)\tilde{y}_{xxx}(x), & \forall x \in (0, L),
\end{cases}
$$

(27)

where $f = \sigma(x)\tilde{y}_{xxx} - y_xu - \tilde{y}_tu_x$. One should notice that we have now $\sigma$, that we seek to estimate, appearing not only in the source term of the equation, but also into the initial condition.

Step 2. Extension to negative time
In order to estimate $\sigma$ through $z(x, 0)$ using the Carleman estimate of the previous
section, we shall extend the partial differential equations \((26)\) in \(u\) and \((27)\) in \(z\) to negative times. In order to extend these linearized KdV equations to the interval \((-T, T)\), we define the *symmetric extension* of any function \(g\) defined on \([0, L] \times [0, T]\) by

\[
\hat{g}(x,t) = \begin{cases} 
  g(x,t) & \text{if } x \in [0, L], \ t \in [0, T], \\
  g(L-x,-t) & \text{if } x \in [0, L], \ t \in [-T, 0).
\end{cases}
\]  

(28)

One should notice that this extension satisfies \(\hat{f} \hat{g} = \hat{f} \hat{g}\). We also define the following *anti-symmetric extension* to \([0, L] \times [-T, T]\) of any function \(g\) defined on \([0, L] \times [0, T]\) by

\[
\check{g}(x,t) = \begin{cases} 
  g(x,t) & \text{if } x \in [0, L], \ t \in [0, T], \\
  -g(L-x,-t) & \text{if } x \in [0, L], \ t \in [-T, 0).
\end{cases}
\]  

(29)

Therefore, we first assume that the initial data \(z_0(x) = z(x,0)\) of equation \((27)\) satisfies

\[
z_0(x) = z_0(L-x) \quad \forall x \in [0, L],
\]

(30)

meaning that we have to assume this for \(a, \tilde{a}\) and \(g_0''\), as we did in assumptions \((3)\) and \((5)\). We define the extension of \(y\) (resp. \(\tilde{y}\)) solution of equation \((2)\) by \(\hat{y}\) (resp. \(\check{y}\)). Now we can define \(v := \hat{z}\) on \([0, L] \times [-T, T]\), which satisfies the equation given by

\[
\begin{align*}
  v_t + a(x)v_{xxx} + (1 + \hat{\gamma})v_x + \hat{\gamma}_x v &= \hat{f}, & \forall x \in (0, L), \ t \in (-T, T), \\
  v(0, t) &= 0, \quad v(L, t) = 0, & \forall t \in (-T, T), \\
  v_x(L, t) &= -z_x(0,-t), & \forall t \in (-T, 0), \\
  v(x, 0) &= \sigma(x)g_0''(x), & \forall x \in (0, L).
\end{align*}
\]

(31)

The partial differential operator of this equation is eligible for the Carleman estimate of Theorem \((2)\) according to \((7)\) with \(b = 1 + \check{\gamma}\) and \(d = \check{y}_x\), provided that \(b \in L^\infty(-T, T; W^{1, \infty}(0, L))\) and \(d \in L^\infty((-T, T) \times (0, L))\). This regularity is fulfilled thanks to the hypothesis on the regularity of \(y_0, a\) and \(\check{a}\), which ensures the corresponding solutions \(y\) and \(\check{y}\) to be regular enough. See Proposition \((3)\) in the Appendix.

**Step 3. First use of the Carleman estimate**

Following the definitions stated in the proof of Theorem \((1)\) we set \(w = e^{-s\phi}v\) and since we have \(w(0, t) = w(L, t) = 0\) and \(w(x, \pm T) = 0\) (because of the weight function \(\phi\)), we can either compute or estimate the following integral

\[
I := \int_{-T}^{0} \int_{0}^{L} w P_1 w \, dx \, dt
\]

(32)

\[
= \int_{-T}^{0} \int_{0}^{L} w(w_t + 3as^2\phi_x^2w_x + aw_{xxx} + 3as^2\phi_x\phi_{xxx}w) \, dx \, dt
\]

where \(P_1\) has been defined in \((13)\) and \(m = 1\) chosen according to \((20)\).

First of all, making integrations by parts, it is not difficult to see that

\[
I = \frac{1}{2} \int_{0}^{L} |w(x, 0)|^2 dx + R,
\]
where

\[
R = \int_{-T}^{T} \int_{0}^{L} (3a s^2 \phi_x^2 w_x w + aw_{xxx}w + 3a s^2 \phi_x \phi_{xx} |w|^2) \, dx \, dt
\]

\[
= -\frac{3}{2} s^2 \lambda^2 \int_{-T}^{T} \int_{0}^{L} (a \beta_x^2 \theta^2) |w|^2 \, dx \, dt - \int_{-T}^{T} \int_{0}^{L} aw_{xx}w_x \, dx \, dt + \int_{-T}^{T} \int_{0}^{L} a_x |w|^2 \, dx \, dt
\]

\[
- \frac{1}{2} \int_{-T}^{T} \int_{0}^{L} a_{xxx} |w|^2 \, dx \, dt + 3ms^2 \lambda^2 \int_{-T}^{T} \int_{0}^{L} a \beta_x (\beta_{xx} + \lambda \beta_x^2) \theta^2 |w|^2 \, dx \, dt
\]

Using \(a \in \Sigma(a_0, \alpha), \beta \in C^3([0, L])\), the property \(\ref{11}\) of \(\theta\) and Cauchy-Schwartz inequality, one obtains

\[
|R| \leq C \int_{-T}^{T} \int_{0}^{L} (s^2 \lambda^3 \theta^3 |w|^2 + |w_x|^2 + |w_{xx}|^2) \, dx \, dt
\]

for the same generic constant \(C > 0\) as in the previous section. Hence, estimating the quantity \(s^2 I\) by Carleman inequality \(\ref{25}\), and choosing \(s_0\) large enough to absorb the terms of \(R\) by the dominant ones of the left hand side of \(\ref{25}\), we prove

\[
s^2 \int_{0}^{L} |w(x, 0)|^2 \, dx \leq 2s^5 \int_{-T}^{T} \int_{0}^{L} |w|^2 \, dx \, dt + 2 \int_{-T}^{T} \int_{0}^{L} |P_1 w|^2 \, dx \, dt + 2s^5 |R|
\]

\[
\leq C \int_{-T}^{T} \int_{0}^{L} e^{-2s\phi} |\tilde{f}(x, t)|^2 \, dx \, dt + Cs^3 \lambda^3 \int_{-T}^{T} \theta^3(L, t) |w_x(L, t)|^2 \, dt
\]

\[
+ Cs \lambda \int_{-T}^{T} \theta(L, t) |w_{xx}(L, t)|^2 \, dt.
\]

On the one hand, since we assume that \(|y'''(x)| \geq r_0 > 0\), we have

\[
s^2 \int_{0}^{L} |w(x, 0)|^2 \, dx = s^2 \int_{0}^{L} e^{-2s\phi(0)} |\sigma(x) y_x'''(x)|^2 \, dx \geq s^2 r_0^2 \int_{0}^{L} e^{-2s\phi(0)} |\sigma(x)|^2 \, dx.
\]

Therefore, by these two last estimates, fixing \(\lambda > \lambda_0\), we get

\[
s^2 \int_{0}^{L} e^{-2s\phi(x,0)} |\sigma(x)|^2 \, dx \leq C \int_{-T}^{T} \int_{0}^{L} e^{-2s\phi} |\tilde{f}(x, t)|^2 \, dx \, dt
\]

\[
+ Cs^3 \int_{-T}^{T} \theta^3(L, t) |w_x(L, t)|^2 \, dt + Cs \int_{-T}^{T} \theta(L, t) |w_{xx}(L, t)|^2 \, dt
\]

and using the definition of \(\tilde{f}\) from \(\ref{29}\), and the fact that \(\theta\) and \(\phi\) are even in time, \(w = e^{-s\phi} v = e^{-s\phi} z\) with definition \(\ref{28}\), and the boundary properties of \(z\) in \(\ref{27}\), we obtain

\[
s^2 \int_{0}^{L} e^{-2s\phi(x,0)} |\sigma(x)|^2 \, dx \leq C \int_{0}^{L} \int_{0}^{L} \left(e^{-2s\phi(x,t)} + e^{-2s\phi(L-x,t)}\right) |f(x, t)|^2 \, dx \, dt
\]

\[
+ Cs^3 \int_{0}^{L} e^{-2s\phi(L,t)} \theta^3(L, t) |z_x(0, t)|^2 \, dt
\]

\[
+ Cs \int_{0}^{T} e^{-2s\phi(L,t)} \theta(L, t) |z_{xx}(0, t)|^2 \, dt + Cs \int_{0}^{T} e^{-2s\phi(L,t)} \theta(L, t) |z_{xx}(L, t)|^2 \, dt. \tag{33}
\]
On the other hand, since \( \phi(x, t) \geq \phi(x, 0) \) for all \((x, t) \in (0, L) \times (-T, T)\), and \( \tilde{y}_t, y_{xt} \) and \( \tilde{y}_{xxx} \) belong to \( L^\infty((0, L) \times (0, T)) \) (recall that \( y_0 \in \{ w \in H^2(0, L); w(0) = w(L) = w'(L) = 0 \} \) and \( a, \tilde{a} \in \Sigma(a_0, \alpha) \), which implies that \( y, \tilde{y} \in W^{1, \infty}(0, T; W^{3, \infty}(0, L)) \)), we can write

\[
\int_0^T \int_0^L e^{-2s\phi(x, t)}|f(x, t)|^2 \, dx \, dt = \int_0^T \int_0^L e^{-2s\phi(x, t)}|\sigma(x)\tilde{y}_{xxx} - y_{xt}u - \tilde{y}_tu_x|^2 \, dx \, dt
\]

\[
\leq C \int_0^L e^{-2s\phi(0, 0)}|\sigma(x)|^2 \, dx + C \int_0^T \int_0^L e^{-2s\phi(x, t)}(|u|^2 + |u_x|^2) \, dx \, dt.
\]

and also,

\[
\int_0^T \int_0^L e^{-2s\phi(L-x, t)}|f(x, t)|^2 \, dx \, dt = \int_0^T \int_0^L e^{-2s\phi(L-x, t)}|\sigma(x)\tilde{y}_{xxx} - y_{xt}u - \tilde{y}_tu_x|^2 \, dx \, dt
\]

\[
\leq C \int_0^L e^{-2s\phi(L-x, 0)}|\sigma(x)|^2 \, dx + C \int_0^T \int_0^L e^{-2s\phi(L-x, t)}(|u|^2 + |u_x|^2) \, dx \, dt
\]

\[
\leq C \int_0^L e^{-2s\phi(0, 0)}|\sigma(x)|^2 \, dx + C \int_0^T \int_0^L e^{-2s\phi(x, t)}(|\tilde{u}|^2 + |\tilde{u}_x|^2) \, dx \, dt
\]

Thus, we get,

\[
\int_{-T}^T \int_0^L e^{-2s\phi}\hat{f}(x, t)^2 \, dx \, dt
\]

\[
\leq C \int_0^L e^{-2s\phi(0, 0)}|\sigma(x)|^2 \, dx + C \int_{-T}^T \int_0^L e^{-2s\phi(x, t)}(|\tilde{u}|^2 + |\tilde{u}_x|^2) \, dx \, dt
\]  \( (34) \)

Here, the second term in the right hand side has to be estimated, and this is done using again a Carleman inequality.

**Step 4. Second use of the Carleman estimate**

Applying now Carleman estimate \([12]\) to the equation satisfied by \( \tilde{u} \) that is the extension of \( \tilde{y} \) to negative times, and where the first and zero-th order potentials are \( b(x, t) = 1 + \tilde{y}(x, t) \), and \( d(x, t) = \tilde{y}_{xx}(x, t) \), we obtain

\[
\int_{-T}^T \int_0^L e^{-2s\phi}|\tilde{u}_t|^2 + |\tilde{u}_x|^2 \, dx \, dt
\]

\[
\leq C \int_{-T}^T \int_0^L e^{-2s\phi}\tilde{y}_{xxx}^2 \, dx \, dt + Cs^3 \int_{-T}^T e^{-2s\phi(L, t)}\theta^3(L, t)|\tilde{u}_x(L, t)|^2 \, dt
\]

\[
+ Cs \int_{-T}^T e^{-2s\phi(L, t)}\theta(L, t)|\tilde{u}_{xx}(L, t)|^2 \, dt
\]

\[
\leq C \int_0^L e^{-2s\phi(0, 0)}|\sigma(x)|^2 \, dx + Cs^3 \int_0^T e^{-2s\phi(L, t)}\theta(L, t)|u_x(0, t)|^2 \, dt
\]

\[
+ Cs \int_0^T e^{-2s\phi(L, t)}\theta(L, t)|u_{xx}(L, t)|^2 \, dt + Cs \int_0^T e^{-2s\phi(L, t)}\theta(L, t)|u_{xx}(L, t)|^2 \, dt.
\]
Thus, from (34), we can write,
\[
\int_0^T \int_0^L e^{-2s\phi} |\tilde{f}(x,t)|^2dxdt \\
\leq C \int_0^L e^{-2s\phi(x,0)} |\sigma(x)|^2 dx + C s^3 \int_0^T e^{-2s\phi(L,t)} \theta(L,t) |u_x(0,t)|^2 dt \\
+ Cs \int_0^T e^{-2s\phi(L,t)} \theta(L,t) |u_{xx}(0,t)|^2 dt + Cs \int_0^T e^{-2s\phi(L,t)} \theta(L,t) |u_{xx}(L,t)|^2 dt \\
(35)
\]
and from (33) and (35), choosing \(s_0\) large enough, we deduce that
\[
\frac{s^2}{2} \int_0^L e^{-2s\phi(0)} |\sigma(x)|^2 dx \leq C s^3 \int_0^T e^{-2s\phi(L,t)} \theta(L,t) (|u_x(0,t)|^2 + |z_x(0,t)|^2) dt \\
+ Cs \int_0^T e^{-2s\phi(L,t)} \theta(L,t) (|u_{xx}(0,t)|^2 + |z_{xx}(0,t)|^2) dt \\
+ Cs \int_0^T e^{-2s\phi(L,t)} (|u_{xx}(L,t)|^2 + |z_{xx}(L,t)|^2) dt \\
(36)
\]
Taking into account that \(z = u_t = \partial_t(y - \tilde{y})\) the result of Theorem 1 directly follows from (36).

A Cauchy problem

In this appendix, we state the well-posedness results for the KdV equation considered in the paper. We only give the main ideas in the proofs because the tools are pretty similar to those applied for the constant coefficient case, which is already standard in the literature (see [11], [12], [12], [19] and [18]). It is worthwhile to emphasize that these results allow to get, under smoothness hypothesis on the data, the existence of solutions for the KdV equation with the regularity required for our approach in order to prove the stability of the inverse problem.

We consider a main coefficient satisfying
\[
\exists a_0 \in \mathbb{R}, \forall x \in [0, L] \quad a(x) \geq a_0 > 0. \\
(37)
\]
Let us introduce, for any \(s \geq 0\) the space
\[
B^s = C([0,T],H^s(0,L)) \cap L^2(0,T;H^{s+1}(0,L))
\]
and consider only the principal part of the equation, i.e. we study the problem
\[
\begin{cases}
  y_t + a(x)y_{xxx} = f, & \forall (x,t) \in (0,L) \times (0,T), \\
  y(0,t) = 0, & y(L,t) = 0, \quad \forall t \in (0,T), \\
  y_x(L,t) = 0, & \forall t \in (0,T), \\
  y(0,x) = y_0(x), & \forall x \in (0,L).
\end{cases}
(38)
\]
In order to take into account the coefficient \( a = a(x) \) we work in the space \( X = L^2(0,L) \) endowed with the inner product

\[
\langle w_1, w_2 \rangle_X := \int_0^L \frac{1}{a(x)} w_1(x) w_2(x) dx.
\]

Notice that since \( a \in L^\infty(0,L) \) satisfies \( 37 \), the norm defined by the inner product \( \langle \cdot, \cdot \rangle_X \) is equivalent to the \( L^2 \)-norm.

In the domain \( D(A) = \{ w \in H^3(0,L) / w(0) = w(L) = w'(L) = 0 \} \), we define the operator \( A : D(A) \subset X \rightarrow X \) as \( A(w) = -aw''' \). It is easy to see that both \( A \) and its adjoint operator \( A^* \) are dissipative, and therefore, by standard semigroup theory (for instance a corollary of Lumer-Phillips Theorem, see Chapter 1 in \( 38 \)), we get that \( A \) generates a strongly continuous semigroup in \( L^2(0,L) \).

Thus, if \( y_0 \in L^2(0,L) \), \( a \in L^\infty(0,L) \) satisfies \( 37 \) and \( f \in L^1(0,T,L^2(0,L)) \) then the linear KdV equation \( 38 \) has a unique solution (called mild solution) in the space \( C([0,T],L^2(0,L)) \). Applying the multipliers technique, we get a Kato smoothing effect, which implies \( y \in L^2(0,T;H^1(0,L)) \) and thus \( y \in B^0 \). This regularity is indeed obtained by multiplying formally equation \( 38 \) by \( \frac{x}{a(x)} y(t,x) \) and integrating in \( (0,L) \times (0,T) \). This proves the following result.

**Proposition 2** Let \( y_0 \in L^2(0,L) \), \( a \in L^\infty(0,L) \) verifying \( 37 \) and \( f \in L^1(0,T,L^2(0,L)) \). Then the linear KdV equation \( 38 \) has a unique solution in the space \( B^0 \). Moreover, there exist \( C > 0 \) such that

\[
\| y \|_{B^0} \leq C \left( \| y_0 \|_{L^2(0,T)} + \| f \|_{L^1(0,T,L^2(0,L))} \right).
\]

Depending on the regularity of the data, we can prove the existence of more regular solutions. Let us consider \( y \) solution of \( 38 \). The function \( u := y_t \) satisfies

\[
\begin{align*}
  u_t + a(x) u_{xxx} &= f, & \forall (x,t) \in (0,L) \times (0,T), \\
  u(0,t) = 0, & u(L,t) = 0, & \forall t \in (0,T), \\
  u_x(L,t) = 0, & \forall t \in (0,T), \\
  u(0,x) = f(0,x) - a(x)y_{0xxx}, & \forall x \in (0,L).
\end{align*}
\]

(39)

Assuming that \( f \in L^1(0,T;L^2(0,L)) \) and \( y_0 \in D(A) \), we have that \( (f(0,x) - a(x)y_{0xxx}) \in L^2(0,L) \), and then Proposition 2 implies that \( u = y_t \in B^0 \).

If additionally we ask \( f \in B^0 \), then by equation \( 38 \) we have \( a(x)y_{xxx} = f - y_t \in B^0 \), and if \( a \in W^{1,\infty}(0,L) \) satisfies \( 37 \) we get that the original solution satisfies \( y \in B^3 \).

Analyzing in the same way each one of the equations fulfilled by \( y_{ttt} \) and \( y_{ttt} \), we get the following result.

**Proposition 3** Let \( y_0 \in H^9(0,L) \cap D(A) \), \( a \in W^{6,\infty}(0,L) \) verifying \( 37 \), and \( f \in B^6 \) such that \( f_t \in B^3, f_{tt} \in B^0, \) and \( f_{ttt} \in L^1(0,T,L^2(0,L)) \). Then the linear KdV equation \( 38 \) has a unique solution in the space \( B^3 \). This solution also satisfies \( y_t \in B^6, y_{tt} \in B^3 \) and \( y_{ttt} \in B^0 \).
Remark 5 The hypothesis \( a \in W^{6,\infty}(0,L) \) is used to ensure that the initial condition of the problem solved by \( y_{ttt} \) belongs to the space \( L^2(0,L) \).

In order to prove a similar result for the nonlinear system

\[
\begin{aligned}
&y_t + a(x)yy_{xxx} + y_x + yy_x = 0, \quad \forall (x,t) \in (0,L) \times (0,T), \\
y(0,t) = 0, \quad y(L,t) = 0, \quad \forall t \in (0,T), \\
y_x(L,t) = 0, \quad \forall t \in (0,T), \\
y(0,x) = y_0(x), \quad \forall x \in (0,L),
\end{aligned}
\]

we define the map

\[ \Pi : \check{y} \in \mathcal{B}^0 \rightarrow \Pi(\check{y}) = y \in \mathcal{B}^0 \]

where \( \Pi(\check{y}) = y \) is the solution of \( (38) \) with \( f \) replaced by \( (-\check{y}_x - \check{y}\check{y}_x) \). For \( \check{y} \in \mathcal{B}^0 \), we have \( \check{y} \in L^2(0,T; H^1(0,L)) \), and therefore \( \check{y}\check{y}_x \in L^1(0,T; L^2(0,L)) \). Thus we are able to use Proposition 2 with the right-hand side \( f = (-\check{y}_x - \check{y}\check{y}_x) \) to define \( \Pi(\check{y}) = y \in \mathcal{B}^0 \).

Applying a fixed point argument in small time and global a priori estimates to deal with the long-time case, we obtain the following

Proposition 4 Let \( y_0 \in L^2(0,L) \) and \( a \in L^\infty(0,L) \) verifying \( (37) \). Then, the nonlinear KdV equation \( (40) \) has a unique solution in the space \( \mathcal{B}^0 \).

We can also prove the nonlinear result in the more regular framework.

Proposition 5 Let \( y_0 \in H^3(0,L) \cap D(A) \) and \( a \in W^{6,\infty}(0,L) \) verifying \( (37) \). Then, the nonlinear KdV equation \( (40) \) has a unique solution in the space \( \mathcal{B}^9 \).

The proof of Proposition 5 can be done as before (with a fixed point argument and global a priori estimates) with the same map \( \Pi \) but now defined in the space

\[ \mathcal{Y} := \{ y \in \mathcal{B}^9 / y_t \in \mathcal{B}^6, \ y_{tt} \in \mathcal{B}^3 \text{ and } y_{ttt} \in \mathcal{B}^0 \} \]

instead of \( \mathcal{B}^0 \). Let us notice that the nonlinearity \( \check{y}\check{y}_x \) satisfies all the hypothesis required in Proposition 4 for the right hand side \( f \) provided that \( \check{y} \in \mathcal{Y} \).

Using some interpolation arguments, we can prove the following result, which gives us the existence of solutions as we require in order to solve our Inverse Problem in Theorem 1.

Theorem 3 Let \( y_0 \in H^7(0,L) \cap D(A) \) and \( a \in W^{6,\infty}(0,L) \) satisfying \( (37) \). Then the nonlinear KdV equation \( (40) \) has a unique solution in the space \( \mathcal{B}^7 \). Furthermore \( y_t \in \mathcal{B}^4 \) and therefore \( y \in W^{1,\infty}(0,T; W^{3,\infty}(0,L)) \).

Indeed, the proof of this result can be deduced from the Interpolation Theorem by Bona and Scott \( ([11], \text{see Theorem 4.3 in [12]}) \). This theorem is applied in the following way. Denoting by \( S \) the mapping which sends the initial condition \( y_0 \) to the solution \( y \) of the equation \( (40) \), by Propositions 4 and 5 we have that

\[ S : L^2(0,L) \longrightarrow \mathcal{B}^0 \quad \text{and} \quad S : H^3(0,L) \cap D(A) \longrightarrow \mathcal{B}^9 \]

are well defined. Additionally, two inequalities are needed (see (i) and (ii) from Theorem 4.3 in [12]):
• The first one is the estimation of \( \| S(y_{01}) - S(y_{02}) \|_{B^0} \) in terms of \( \| y_{01} - y_{02} \|_{L^2(0,L)} \), which can be proven exactly in the same way as in the proof of Proposition 4.2 of [12], but now using our Proposition 2.

• The second one consists in estimating \( \| S(y_0) \|_{B^9} \) by \( \| y_0 \|_{H^9(0,L)} \), which can be proven from Proposition 5 and following Proposition 15 in [19].

From the Interpolation Theorem we get that \( S : H^7(0,L) \cap D(A) \to B^7 \) is well defined, with gives us the proof of Theorem 3.

Remark 6 A sharper result can be obtained with \( y_0 \in H^s(0,L) \cap D(A) \) where \( s > 6 + \frac{1}{2} \). In that case, previous arguments give the existence of a solution \( y \in B^s \) with \( y_t \in B^{s-3} \), and then \( y_{txxx} \in C([0,T],H^q(0,L)) \) with \( q > 1/2 \). Thus, we obtain for system (40) a solution \( y \in W^{1,\infty}(0,T;W^{3,\infty}(0,L)) \).

Acknowledgements. This work has been partially supported by MathAmsud project CIP-PDE, by the ANR Project CISIFS number NT09-437023 (L. Baudouin, E. Crépeau), by Fondecyt grant 1120610 (A. Mercado), by Basal CMM U. de Chile, and by Conicyt grant ACT 1106.

References


