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Reducing the gap of the Jensen's inequality by using the Wirtinger inequality

A. Seuret^{a,b} F. Gouaisbaut^{a,c}

^a*CNRS, LAAS, 7 avenue du Colonel Roche, 31077 Toulouse, France.
(e-mail: aseuret,fgouaisb@laas.fr).*

^b*Univ de Toulouse, LAAS, F-31400 Toulouse, France*

^c*Univ de Toulouse, UPS, LAAS, F-31400, Toulouse, France.*

Abstract

In the last decade, the Jensen's inequality has been intensively used in the context of time-delay or sampled-data systems since it is an appropriate tool to obtain tractable stability conditions expressed in terms linear matrix inequalities (LMI). However, it is also well-known that this inequality unavoidably introduces an undesirable conservatism in the stability conditions and looking at the literature, reducing this gap is a relevant issue and an open problem. In this paper, we propose an alternative inequality based on the Fourier Theory, more precisely on the Wirtinger inequalities. It is shown that this resulting inequality encompasses the Jensen's one and also leads to tractable LMI conditions. In order to illustrate the potential gain of employing this new inequality with respect to the Jensen's one, two applications on time-delay and sampled-data stability analysis are provided.

Key words: Jensen's Inequality, stability analysis, time-delay systems, sampled-data systems

1 Introduction

The last decade has shown an increasing research activity on time-delay and/or sampled-data systems analysis and control due to both emerging adapted theoretical tools and also practical issues in the engineering field and information technology (see [31], [22]) and references therein). In the case of linear system, many techniques allow to derive efficient criteria proving the stability of such systems. Among them, two frameworks, different in their spirits have been recognised as a powerful methodologies. The first one relies on Robust Analysis. In this framework, the time delay/sampled data system is transformed into a closed loop between a stable nominal system and a perturbation element depending either on the delay or the sampling process (which is also modelled by a time varying delay). The perturbation element is then embedded into some norm-bounded uncertainties and the use of scaled small gain theorem [24, 10], IQCs [18], or separation approach [3] allows then to derive efficient stability criteria. The challenge is then to reduce the conservatism either by constructing elaborated interconnections which generally include state augmentation [1] or by using finer L_2 induced norm upperbound evaluation [24], often based on Cauchy-Schwartz inequality [3]. Another popular approach is the use of a Lyapunov function to prove stability. For sampled data systems, two approaches have been successfully proposed. The first one relies on impulsive systems and some piecewise linear Lyapunov functions [25] but it often requires a constant sampling frequency. This approach have been then extended by considering discontinuous Lyapunov functions which allow to consider aperiodic sampling [21, 22]. In the second approach, the sampled state is modeled by a time varying delayed state. In that case, the original system is recasted into a time varying delay system where Lyapunov-Krasovskii functional [6] or Lyapunov-Razumikhin can be used directly. Hence, for sampled data systems and time delay systems as well, the last decade has seen a tremendous emergence of research devoted to the construction of Lyapunov-Krasovskii functional which aims at reducing the inherent conservatism of this approach. Several attempts has been done concerning the choice of the structure of V by choosing extended state based Lyapunov-Krasovskii functional ([1, 19]), discretized Lyapunov functional ([13])

or discontinuous Lyapunov functions [28]. Apart the choice of V , an important source of conservatism relies also on the way to bound some cross terms arisen when manipulating the derivative of V . According to the literature on this subject (see [27, 15, 30] for some recent papers), a common feature of all these techniques is the use of slack variables [16] and more or less refined Jensen inequality [30, 32, 22]. At this point, it is clear that for both the two frameworks - Robust Analysis and Lyapunov functional-, a part of the conservatism comes from the use of Jensen inequality or Cauchy Schwartz inequalities, usually used to get tractable criteria. But, as it has been shown in [4], Jensen's inequality can be viewed as a special case of Cauchy Schwartz inequality. Based on this observation, the objective of the present paper is then to propose to show how to use another class of inequalities called Wirtinger inequalities, which are well known in Fourier Analysis. Notice that this class of inequalities has been recently used to exhibit a new Lyapunov function to prove the stability of sampled-data system [22]. In this paper, its use combined with some special properties of sampled data systems has led to some interesting criteria expressed in terms of LMIs, which are less conservative at least on examples. In the present paper, contrary to the work of [22], we do not construct some new Lyapunov functional. We aim rather at developing new inequalities to be used to reduce the conservatism when computing the derivative of V . Wirtinger inequalities allow to consider a more accurate integral inequalities which can include the Jensen's one as a special case. The resulting inequalities depend not only on the state $x(t)$ and the delayed or sampled state but also on the integral of the state over a delay or sampling interval. This new signal is then directly integrated into a suitable classical Lyapunov function, highlighting so the features of Wirtinger inequality. Hence, it results some new stability criteria for time delay systems and sampled data systems directly expressed in terms of LMIs.

Notations: Throughout the paper \mathbb{R}^n denotes the n -dimensional Euclidean space with vector norm $|\cdot|$, $\mathbb{R}^{n \times m}$ is the set of all $n \times m$ real matrices, and the notation $P > 0$, for $P \in \mathbb{R}^{n \times n}$, means that P is symmetric and positive definite. The symmetric matrix $\begin{bmatrix} A & B \\ * & C \end{bmatrix}$ stands for $\begin{bmatrix} A & B \\ B^T & C \end{bmatrix}$. Moreover, for any square matrix $A \in \mathbb{R}^{n \times n}$, we define $\text{He}(A) = A + A^T$. The space of functions $\phi : [a, b] \rightarrow \mathbb{R}^n$, which are absolutely continuous on $[a, b]$, have a finite $\lim_{\theta \rightarrow b^-} \phi(\theta)$ and have square integrable first order derivatives is denoted by $W[a, b]$.

2 Preliminaries

In the sequel, the following notations will be considered. For any real numbers $a < b$, we consider a differentiable function $\omega : [a, b] \rightarrow \mathbb{R}^n$ and the vector $\Omega(a, b)$ given by

$$\Omega(a, b) = \left[\omega^T(b), \omega^T(a), \frac{1}{b-a} \int_a^b \omega^T(u) du \right]^T.$$

2.1 Jensen's inequality

Let recall the well-known Jensen's inequality.

Lemma 1 For given symmetric positive definite matrices $R > 0$ and for any differentiable signal ω in $[a, b] \rightarrow \mathbb{R}^n$, the following inequality holds:

$$\int_a^b \dot{\omega}(u) R \dot{\omega}(u) du \geq \frac{1}{b-a} \left(\int_a^b \dot{\omega}(u) du \right)^T R \left(\int_a^b \dot{\omega}(u) du \right) = \frac{1}{b-a} \Omega^T(a, b) W_1^T R W_1 \Omega(a, b), \quad (1)$$

where $W_1 = \begin{bmatrix} I & -I & 0 \end{bmatrix}$.

The proof is omitted and can be found in several reference books (see [13], [26]). In the context of time-delay systems and sampled-data systems, this inequality has been the core of several important contributions (see [13, 15] for time delay systems or [22] and references therein for sampled data systems) : it is usually used to bound some integral terms of the form $\int_a^b \dot{\omega}(u) R \dot{\omega}(u) du$ which arise when calculating the derivative of Lyapunov function. Naturally, it is likely to entail some inherent conservatism and several works have been devoted to the reduction of such a gap [4, 12]. In the present paper, we propose to use a different class of inequalities called Wirtinger inequalities in order to obtain new bounds for this integral and therefore to improve the results for the stability analysis of time-delay and/or sampled-data systems. In [29], a first result on the use of Wirtinger inequality was presented. The present paper proposes a more accurate analysis of this class of inequalities and its application to a larger class of problems.

2.2 Different Wirtinger inequalities

In the literature [17], Wirtinger inequalities are referred as inequalities which estimate the integral of the derivative function with the help of the integral of the function. Often proved using Fourier analysis, it exists several versions which depend on the characteristics or constraints we impose on the function. Let firstly recall the initial Wirtinger inequality adapted to our purpose.

Lemma 2 *Let $z \in W[a, b]$ such that $z(a) = z(b)$ and $\int_a^b z(u)du = 0$. Then, for any symmetric positive definite matrix $R = R^T \in \mathbb{R}^{n \times n}$, the following inequality holds*

$$\int_a^b \dot{z}^T(u)R\dot{z}(u)du \geq \frac{4\pi^2}{(b-a)^2} \int_a^b z^T(u)Rz(u)du, \quad (2)$$

and the equality holds when $z(u) = a_1 \sin\left(\frac{2\pi(u-a)}{b-a}\right) + b_1 \cos\left(\frac{2\pi(u-a)}{b-a}\right)$, for any a_1 and b_1 in \mathbb{R}^n .

Proof : The proof is based on the one-dimensional Wirtinger inequality provided in [17] and adapted to the case of vector function using the same method as in [22]. \diamond

Notice that in order to use this first inequality, we need to impose two constraints $z(a) = z(b)$ and $\int_a^b z(u)du = 0$ which may be difficult to fulfil. A classical exercise in Fourier analysis and which can be found for instance in [17] allows to derive another result described later on. The idea is to relax the constraint $\int_a^b z(u)du = 0$ at a price of an increasing gap between the two terms composing the inequality.

Lemma 3 *Let $z \in W[a, b]$ and $z(a) = z(b) = 0$. Then, for any symmetric positive definite matrix $R = R^T \in \mathbb{R}^{n \times n}$, the following inequality holds*

$$\int_a^b \dot{z}^T(u)R\dot{z}(u)du \geq \frac{\pi^2}{(b-a)^2} \int_a^b z^T(u)Rz(u)du, \quad (3)$$

and equality holds when $z(u) = a_1 \sin\left(\frac{\pi(u-a)}{b-a}\right)$ for any a_1 in \mathbb{R}^n .

Proof : Without loss of generality, consider that $a = 0$. Consider the function \tilde{z} defined over $[-b, b]$ such that $\tilde{z}(u) = z(u)$ for all $u \in [0, b]$ and $\tilde{z}(u) = -z(-u)$ for all $u \in [-b, 0)$. Since $\tilde{z}(0^+) = \tilde{z}(0^-) = 0$, the function \tilde{z} is continuous and piecewise differentiable over $[-b, b]$. Moreover, it is easy to see that this function \tilde{z} satisfies the conditions of Lemma 2 which allow to conclude the proof. \diamond

As it is explained in the next lemma, a last version of the Wirtinger inequality can be obtained by removing the constraint $z(b) = 0$.

Lemma 4 *Let $z \in W[a, b]$ and $z(a) = 0$. Then, for any symmetric positive definite matrix $R = R^T \in \mathbb{R}^{n \times n}$, the following inequality holds*

$$\int_a^b \dot{z}^T(u)R\dot{z}(u)du \geq \frac{\pi^2}{4(b-a)^2} \int_a^b z^T(u)Rz(u)du, \quad (4)$$

and equality holds when $z(u) = a_1 \sin\left(\frac{\pi(u-a)}{2(b-a)}\right)$ for any a_1 in \mathbb{R}^n .

Proof : Without loss of generality, consider this time that $b = 0$ and therefore $a < 0$. Consider the function \tilde{z} defined over $[a, -a]$ such that $\tilde{z}(u) = z(u)$ for all $u \in [a, 0]$ and $\tilde{z}(u) = z(-u)$ for all $u \in [0, -a)$. Since $\tilde{z}(0^+) = \tilde{z}(0^-)$, the function \tilde{z} is continuous and piecewise differentiable over $[a, -a]$. Moreover it is easy to see that this function \tilde{z} satisfies the conditions of Lemma 3 which allows to conclude the proof. \diamond

Remark 1 *The inequality (4) has been already employed in [22], leading to a new type of Lyapunov-Krasovskii functionals for sampled-data systems. Our approach differs significantly from [22] since we only use this inequality for estimating an upper-bound of the derivative of the Lyapunov functional. An interesting future feature should be the extension of our work by considering the techniques proposed by Fridman et al [22].*

Finally, we have proposed three different inequalities which are very closed to Jensen's inequality in its essence. Nevertheless, the function has to meet several constraints which are not generally satisfied if, for instance, the function z is related to the states of a dynamical system. The next section shows how to overcome such a problem and how to construct relevant new inequalities.

3 Application of the Wirtinger inequalities

The objective of this section is twofold. On the first hand, based on Lemma 2, 3 and 4, we aim at providing new tractable inequalities, which can be easily implemented into a convex optimisation scheme. On the other hand, we propose some inequalities which are proved to be less conservative than Jensen's one. Indeed, recall that the objectives of the present paper are to obtain new lower bounds of the integral $\int_a^b \dot{\omega}(u)R\dot{\omega}(u)du$, in order to be consistent with the Jensen's inequality. Thus a first step consists in defining appropriate function z such that this integral appears naturally in the developments. Thus a necessary condition is that the function z as the following form

$$z(u) = \gamma\omega(u) - y(u), \quad (5)$$

where $\omega \in W[a, b]$ is the vector function which was employed in the original Jensen's inequality in Lemma 1, γ is a constant and $y(u)$ is a function of u and are chosen so that the function z satisfies the different constraints imposed by Lemma 2, 3 or 4.

Based on the first Wirtinger inequality and choosing an appropriate function z , we propose a first corollary:

Corollary 5 *For a given symmetric positive definite matrix $R > 0$, any differentiable function ω in $[a, b] \rightarrow \mathbb{R}^n$, then the following inequality holds:*

$$\int_a^b \dot{\omega}(u)R\dot{\omega}(u)du \geq \frac{1}{(b-a)}\Omega^T(a, b)W_1^T R W_1 \Omega(a, b), \quad (6)$$

where $W_1 = \begin{bmatrix} I & -I & 0 \end{bmatrix}$.

Proof : The main contribution here is to propose an appropriate interesting signal z of the form given in (5) which satisfies the conditions of Lemma 2. Consider the following signal

$$z(u) = \omega(u) - \frac{1}{b-a} \int_a^b \omega(u)du - \left[\frac{u-a}{b-a} - \frac{1}{2} \right] (\omega(b) - \omega(a)),$$

which has been built in order to satisfy the condition of Lemma 2. Then, computing inequality (2) leads to

$$\int_a^b \dot{z}^T(u)R\dot{z}(u)du = \int_a^b \dot{\omega}^T(u)R\dot{\omega}(u)du - \frac{1}{b-a}(\omega(b) - \omega(a))^T R(\omega(b) - \omega(a)) \geq \frac{4\pi^2}{(b-a)^2} \int_a^b z^T(u)Rz(u)du.$$

Furthermore, applying the Jensen's inequality to the righthand side of (2) leads to

$$\int_a^b z^T(u)Rz(u)du \geq \frac{1}{b-a} \int_a^b z^T(u)du R \int_a^b z(u)du.$$

Noting that $\int_a^b z(u)du = 0$ allows to conclude the proof. ◇

Remark 2 *It is important to stress that the previous corollary is equivalent to the classical Jensen's inequality. Indeed the Jensen's inequality is included in the left-hand-side of the original Wirtinger inequality when using the proposed signal z . Consequently, the use of this lemma seems not relevant as it is presented now. However it can be noticed that the previous corollary provides another demonstration of the Jensen's inequality.*

The main problem comes from the constraint $\int_a^b z(u)du = 0$ which does not allow to give a lower bound of the left-hand side of (2). In the following corollary, we propose to use Lemma 3 in which this constraint has been removed.

Corollary 6 *For a given symmetric positive definite matrix $R > 0$, any differentiable function ω in $[a, b] \rightarrow \mathbb{R}^n$, then the following inequality holds:*

$$\int_a^b \dot{\omega}(u)R\dot{\omega}(u)du \geq \frac{1}{b-a}\Omega^T(a, b) [W_1^T R W_1 + \pi^2 W_2^T R W_2] \Omega(a, b), \quad (7)$$

where W_1 is given in Lemma 1, and $W_2 = \begin{bmatrix} I/2 & I/2 & -I \end{bmatrix}$.

Proof : For any function $\omega \in W[a, b]$, consider a signal z given by

$$z(u) = \omega(u) - \frac{u-a}{b-a}\omega(b) - \frac{b-u}{b-a}\omega(a), \quad \forall u \in [a, b].$$

By construction, the function $z(u)$ satisfies the conditions of Lemma 3, i.e. $z(a) = z(b) = 0$. The computation of the left-hand-side of the inequality stated in Lemma 3 leads to:

$$\begin{aligned} \int_a^b \dot{z}^T(u)R\dot{z}(u)du &= \int_a^b \dot{\omega}^T(u)R\dot{\omega}(u)du - \frac{1}{b-a}(\omega(b) - \omega(a))^T R(\omega(b) - \omega(a)) \\ &= \int_a^b \dot{\omega}^T(u)R\dot{\omega}(u)du - \frac{1}{b-a}\Omega^T(a, b)W_1^T R W_1 \Omega(a, b). \end{aligned} \quad (8)$$

Consider now the right-hand side of the inequality (3). Applying the Jensen's inequality, it yields

$$\frac{\pi^2}{(b-a)^2} \int_a^b z^T(u)Rz(u)du \geq \frac{\pi^2}{(b-a)^3} \left(\int_a^b z(u)du \right)^T R \left(\int_a^b z(u)du \right). \quad (9)$$

The last step of the proof consists in the computation of the integral $\int_a^b z(u)du$, which is obtained as follows

$$\begin{aligned} \int_a^b z(u)du &= - \left(\int_a^b \frac{(u-a)}{(b-a)} du \right) \omega(b) - \left(\int_a^b \frac{(b-u)}{(b-a)} du \right) \omega(a) + \int_a^b \omega(u)du \\ &= -(b-a) \left[\frac{1}{2}(\omega(b) + \omega(a)) - \frac{1}{b-a} \int_a^b \omega(u)du \right] \\ &= -(b-a)W_2\Omega(a, b). \end{aligned} \quad (10)$$

Then applying Lemma 3, we obtain

$$\int_a^b \dot{\omega}^T(u)R\dot{\omega}(u)du - \frac{1}{b-a}\Omega^T(a, b)W_1^T R W_1 \Omega(a, b) \geq \frac{\pi^2}{b-a}\Omega^T(a, b)W_2^T R W_2 \Omega(a, b),$$

which concludes the proof of Corollary 6. \diamond

Remark 3 *The previous corollary has been already presented in [29]. However its proof has been remarkably shorten.*

Remark 4 *The inequality proposed (7) encompasses the Jensen's inequality since the matrix $\pi^2 W_2^T R W_2$ is positive definite and the term $\frac{1}{b-a}\Omega^T(a, b)W_1^T R W_1 \Omega(a, b)$ is exactly the right-hand of the Jensen's inequality. It is also worth noting that this improvement is allowed by using an extra signal $\int_a^b \omega(u)du$ and not only the signals $\omega(b)$ and $\omega(a)$. Therefore, it suggests that in order to be useful, this inequality should be combined with a Lyapunov functional where the signal $\int_a^b \omega(u)du$ appears explicitly.*

The following corollary is based on Lemma 4 where only one constraint is imposed.

Corollary 7 For a given symmetric positive definite matrix $R > 0$, any differentiable function ω in $[a, b] \rightarrow \mathbb{R}^n$, then the following inequality holds:

$$\int_a^b \dot{\omega}^T(u) R \dot{\omega}(u) du \geq \frac{\pi^2}{4(b-a)} \Omega^T(a, b) W_3^T R W_3 \Omega(a, b) \quad (11)$$

where $W_3 = \begin{bmatrix} 0 & -I & I \end{bmatrix}$.

Proof: As proposed in [22], consider the signal $z(u) = \omega(u) - \omega(a)$. This function satisfies the condition of Lemma 4. Then, we have

$$\int_a^b \dot{\omega}^T(u) R \dot{\omega}(u) du \geq \frac{\pi^2}{4(b-a)^2} \int_a^b (\omega(u) - \omega(a))^T R (\omega(u) - \omega(a)) du.$$

Applying the Jensen's inequality yields the results. \diamond

Remark 5 It is relevant to try a comparison between Jensen's inequality and the previous one. But, since the matrix $\frac{\pi^2}{4} W_3^T R W_3 - W_1^T R W_1$ is not definite positive, the quantity

$$\Omega^T(a, b) \left(\frac{\pi^2}{4} W_3^T R W_3 - W_1^T R W_1 \right) \Omega(a, b),$$

may be positive or negative, depending on the components of $\Omega(a, b)$. In that case, we cannot state that we improve Jensen's inequality.

4 Appropriate inequalities for robust stability analysis

In all the three inequalities (6),(7),(11), the resulting lower bound is rational with respect to $(b-a)$, which is ill-posed when this quantity tends to zero. At a price of an increasing computational burden, an equivalent formulation depending linearly on $b-a$ can be drawn as follows. Noting that, for all matrices Y_i , $i \in \{1, 2, 3\}$ in $\mathbb{R}^{n \times 3n}$, the matrix $\frac{1}{b-a} (R W_i - (b-a) Y_i)^T R^{-1} (R W_i - (b-a) Y_i)$ is positive for all $i \in \{1, 2, 3\}$, it yields

$$-\frac{1}{(b-a)} W_i^T R W_i \leq -Y_i^T W_i - W_i^T Y_i + (b-a) Y_i^T R^{-1} Y_i, \quad i \in \{1, 2, 3\}.$$

This inequality turns out to be relevant in the context of time-delay systems or sampled-data systems as it will be explained in the next sections 5 and 6. Applying the same inequality to the second term of the inequalities (6),(7), (11), one modifies the previous corollaries as follows.

Corollary 8 For a given symmetric positive definite matrix $R > 0$, any differentiable signal ω in $[a, b] \rightarrow \mathbb{R}^n$ and for any matrices Y_1 in $\mathbb{R}^{n \times 3n}$, the following inequality holds:

$$\int_a^b \dot{\omega}(u) R \dot{\omega}(u) du \geq \Omega^T(a, b) [-\text{He}\{Y_1^T W_1\} + (b-a) (Y_1^T R^{-1} Y_1)] \Omega(a, b). \quad (12)$$

Corollary 9 For a given symmetric positive definite matrix $R > 0$, any differentiable signal ω in $[a, b] \rightarrow \mathbb{R}^n$ and for any matrices Y_1 and Y_2 in $\mathbb{R}^{n \times 3n}$, the following inequality holds:

$$\int_a^b \dot{\omega}(u) R \dot{\omega}(u) du \geq \Omega^T(a, b) [-\text{He}\{Y_1^T W_1 - \pi^2 Y_2^T W_{20}\} + (b-a) (Y_1^T R^{-1} Y_1 + \pi^2 Y_2^T R^{-1} Y_2)] \Omega(a, b). \quad (13)$$

Corollary 10 For a given symmetric positive definite matrix $R > 0$, any differentiable signal ω in $[a, b] \rightarrow \mathbb{R}^n$ and for any matrices Y_3 in $\mathbb{R}^{n \times 3n}$, the following inequality holds:

$$\int_a^b \dot{\omega}(u) R \dot{\omega}(u) du \geq \frac{\pi^2}{4} \Omega^T(a, b) [-\text{He}\{Y_3^T W_3\} + (b-a) (Y_3^T R^{-1} Y_3)] \Omega(a, b). \quad (14)$$

Remark 6 *In the literature, several by-products of the Jensen's inequality have been proposed and employed (see for example [27, 19] and references therein). Obviously, the same by-products could be derived from the Corollaries proposed in Section 3 and therefore will not be presented in the present article.*

In the following, we will show how these inequalities can be applied to the stability analysis of time-delays systems and sampled-data systems. As expected, we will show that the use of these new inequalities reduces the conservatism of the stability conditions. It has to be noticed that, in these new inequalities, the functions to be considered are $\omega(b)$, $\omega(a)$ and $\frac{1}{b-a} \int_a^b \omega(u)du$. The two first signals appear naturally in the context of discrete time-delay or sampled-data systems but not the last one. It only appears in the context of distributed time-delay systems. At first sight, an expected consequence is that these new inequalities only help in the context of distributed time-delay systems. However we will present two solutions dealing with the context of discrete-time delay or sampled-data systems, respectively. The objective will be to show how this third signal is in relation with these classes of systems.

5 Application to the stability analysis of time-delay systems

Before entering into the details of this section, it is important to stress that the present paper does not focus on the development of new Lyapunov-Krasovskii functionals. The present section on the stability analysis of time-delay systems is provided to emphasize the potential gains of using the inequalities provided in the previous section. In this section, we will show how the previous inequalities can be straightforwardly applied to the following basic problems:

- Stability analysis of systems with discrete and distributed constant delays,
- Robust stability analysis of systems with unknown delays,
- Stability analysis of systems with time-varying delays.

5.1 Systems with constant and known delay

We present in this sub-section a first stability result for time delay systems, which is based on the use of the Wirtinger inequalities developed in section 3. This approach is based on a slightly modified Lyapunov-Krasovskii functional and allows us to establish the main theorem for the robust delay range stability analysis. Consider a linear time-delay system of the form:

$$\begin{cases} \dot{x}(t) = Ax(t) + A_d x(t-h) + A_D \int_{t-h}^t x(s)ds, & \forall t \geq 0, \\ x(t) = \phi(t), & \forall t \in [-h, 0], \end{cases} \quad (15)$$

where $x(t) \in \mathbb{R}^n$ is the state vector, ϕ is the initial condition and $A, A_d, A_D \in \mathbb{R}^{n \times n}$ are constant matrices. The delay is assumed to be known and constant. Based on the previous inequality and classical results on Lyapunov-Krasovskii functionals, two stability theorems are provided.

Theorem 11 *For a given constant delay h , assume that there exist $n \times n$ matrices $P = P^T > 0$, $S = S^T > 0$, $R = R^T > 0$, Q and $Z = Z^T$ such that the following LMIs are satisfied*

$$\Pi_1(h) = \begin{bmatrix} P & Q \\ * & Z + S/h \end{bmatrix} > 0, \quad (16)$$

$$\Pi_2^1(h) = \Pi_2^0(h) - \frac{1}{h} [W_1^T R W_1 + \pi^2 W_2^T R W_2] < 0, \quad (17)$$

where

$$\Pi_2^0(h) = \begin{bmatrix} \Delta_2^0 & P A_d - Q & h(P A_D + A^T Q + Z) \\ * & -S & h(A_d^T Q - Z) \\ * & * & h^2(A_D Q + Q^T A_D^T) \end{bmatrix} + h \begin{bmatrix} A^T \\ A_d^T \\ h A_D \end{bmatrix} R \begin{bmatrix} A^T \\ A_d^T \\ h A_D^T \end{bmatrix}^T,$$

and $\Delta_2^0 = PA + A^T P + S + Q + Q^T$. Then the system (22) is asymptotically stable for the constant delay h .

Theorem 12 For a given constant delay h , assume that there exist $n \times n$ matrices $P = P^T > 0$, $S = S^T > 0$, $R = R^T > 0$, Q and $Z = Z^T$ such that the following LMIs are satisfied

$$\Pi_1(h) > 0, \quad \Pi_2^2(h) = \Pi_2^0(h) - \frac{\pi^2}{4h} W_3^T R W_3 < 0 \quad (18)$$

where $\Pi_1(h)$ and $\Pi_2^0(h)$ are given in Theorem (11), and where the matrix W_2 is given in Corollary 6. Then the system (15) is asymptotically stable for the constant delay h .

Proof : The proof of Theorems 11 and 12 are presented here in a compact form. Consider a Lyapunov-Krasovskii functional of the form

$$V(x_t, \dot{x}_t) = \begin{bmatrix} x(t) \\ \int_{t-h}^t x(s) ds \end{bmatrix}^T \begin{bmatrix} P & Q \\ * & Z \end{bmatrix} \begin{bmatrix} x(t) \\ \int_{t-h}^t x(s) ds \end{bmatrix} + \int_{t-h}^t x^T(s) S x(s) ds + \int_{t-h}^t (h-t+s) \dot{x}^T(s) R \dot{x}(s) ds \quad (19)$$

The previous functional refers to a classical type of functionals to derive delay-dependent stability conditions (see for instance [13]). It is interesting to note that this class of functionals employed a signal of the form $\int_{t-h}^t x(s) ds$ which is related to the third signal introduced in the previous section. If the matrices Q and Z are set to zero, one recovers the usual functional employed in the literature. First of all, following [13] and using Jensen inequality, a lower-bound for V can be easily found:

$$V(x_t, \dot{x}_t) \geq \begin{bmatrix} x(t) \\ \int_{t-h}^t x(s) ds \end{bmatrix}^T \Pi_1(h) \begin{bmatrix} x(t) \\ \int_{t-h}^t x(s) ds \end{bmatrix} + \int_{t-h}^t (h-t+s) \dot{x}^T(s) R \dot{x}(s) ds, \quad (20)$$

and it is clear that the positive definiteness of the matrices P , S , R and $\Pi_1(h)$ implies the positive definiteness of the functional V . Classical computations show that the derivative of the functional along the trajectories of the system (15) satisfies

$$\dot{V}(x_t, \dot{x}_t) = \xi^T(t) \Pi_2^0(h) \xi(t) - \int_{-h}^0 \dot{x}^T(t+s) R \dot{x}(t+s) ds.$$

where Π_2^0 is defined in Theorem 11 and $\xi(t) = \text{col}\{x(t), x(t-h), \frac{1}{h} \int_{t-h}^t x(s) ds\}$. Applying Corollary 6 or Corollary 7, respectively, the following upper-bounds of the derivative of the functional is then obtained:

$$\dot{V}(x_t, \dot{x}_t) \leq \xi^T(t) \Pi_2^1(h) \xi(t), \quad \text{or} \quad \dot{V}(x_t, \dot{x}_t) \leq \xi^T(t) \Pi_2^2(h) \xi(t), \quad (21)$$

where $\Pi_2^1(h)$ and $\Pi_2^2(h)$ are defined in (17) and in (18), respectively. Then if the stability conditions from Theorem 11 or 12 are satisfied, the system (15) is asymptotically stable. \diamond

As it will be shown in the example section, the first Theorem 11 proposes less conservative conditions than Theorem 12. Thus in the sequel, only Corollary 6 and 9 will be used.

5.2 Systems with constant and unknown delay: Delay Range stability

Consider the case of a system with a single discrete delay (i.e. $A_D = 0$). Then we have

$$\begin{cases} \dot{x}(t) = Ax(t) + A_d x(t-h) \quad \forall t \geq 0, \\ x(t) = \phi(t) \quad \forall t \in [-h_{\max}, 0], \end{cases} \quad (22)$$

The delay h is a positive constant scalar which satisfies, from now on, the constraint $h \in [h_{\min}, h_{\max}]$ where h_{\min} , h_{\max} are given positive constants. In the following, we aim at assessing stability of system (22) with the delay constraints described above via the an appropriate Lyapunov-Krasovskii functional. The following result holds.

Theorem 13 For an uncertain constant delay $h \in [h_{min}, h_{max}]$, assume that there exist $n \times n$ -matrices $P = P^T > 0$, $S = S^T > 0$, $R = R^T > 0$, Q and $Z = Z^T$ and two $3n \times n$ -matrices Y_1 and Y_2 , such that $\Pi_1(h_{max}) > 0$ and

$$\Pi_3(h) = \begin{bmatrix} \Pi_3^0(h) - He\{Y_1 W_1 + \pi^2 Y_2 W_{20}\} & hY_1 & \pi^2 hY_2 \\ * & -hR & 0 \\ * & * & -\pi^2 hR \end{bmatrix} < 0, \quad (23)$$

for all $h \in [h_{min}, h_{max}]$ where Π_1 is given in (16) where $\Pi_3^0 = \Pi_2^0$ with $A_D = 0$. Then, the system (22) is asymptotically stable for all constant delay $h \in [h_{min}, h_{max}]$.

Proof : The proof uses the same Lyapunov-Krasovskii functional as in Theorem 11. Similar calculations lead to

$$\dot{V}(x_t, \dot{x}_t) = \xi^T(t) \Pi_3^0(h) \xi(t) - \int_{-h}^0 \dot{x}^T(t+s) R \dot{x}(t+s) ds.$$

Applying Corollary 9 and the Schur complement ensure that the derivative of the Lyapunov-Krasovskii functional along the trajectories of (22) is negative definite if $\Pi_3(h)$ is negative definite for this h . Since the matrix Π_3 is affine with respect to h . The conditions $\Pi_3(h_{min}) < 0$ and $\Pi_3(h_{max}) < 0$ ensures that $\Pi_3(h) < 0$ for all $h \in [h_{min}, h_{max}]$. \diamond

5.3 Systems with a time-varying delay

In this section we consider the situation where the delay is time varying. The assumptions on the delay function are classical and are given by:

$$\forall t \geq 0, \quad h(t) \in [h_{min}, h_{max}], \quad d_{min} \leq \dot{h}(t) \leq d_{max}. \quad (24)$$

In this situation, the following theorem holds.

Theorem 14 For a given time varying delay $h(t)$, assume that there exist $n \times n$ matrices $P = P^T > 0$, $S = S^T > 0$, $R = R^T > 0$, Q, Y_1, Y_2 of appropriate dimensions and $Z = Z^T$ such that $\Pi_1(h_{max}) > 0$ and

$$\Pi_4^1(h, \dot{h}) = \begin{bmatrix} \Pi_4^0(h, \dot{h}) - He\{Y_1 W_1 + \pi^2 Y_2 W_{20}\} & hY_1 & \pi^2 hY_2 \\ * & -hR & 0 \\ * & * & -\pi^2 hR \end{bmatrix} < 0, \quad (25)$$

holds for $h \in [h_{min}, h_{max}]$ and $\dot{h} \in [d_{min}, d_{max}]$, where

$$\Pi_4^0(h, \dot{h}) = \begin{bmatrix} \Delta_2^0 P A_d - (1 - \dot{h})Q & h(A^T Q + Z) \\ * & -(1 - \dot{h})S & h(A_d^T Q - (1 - \dot{h})Z) \\ * & * & 0 \end{bmatrix} + h \begin{bmatrix} A^T \\ A_d^T \\ 0 \end{bmatrix} R \begin{bmatrix} A^T \\ A_d^T \\ 0 \end{bmatrix}^T,$$

and Δ_2^0 is given in Theorem 11. Then, the system (22) is asymptotically stable for any time-varying delay $h(t)$ which satisfies (24).

Proof : Consider a Lyapunov-Krasovskii functional of the form

$$V(x_t, \dot{x}_t) = \begin{bmatrix} x(t) \\ \int_{t-h(t)}^t x(s) ds \end{bmatrix}^T \begin{bmatrix} P & Q \\ * & Z \end{bmatrix} \begin{bmatrix} x(t) \\ \int_{t-h(t)}^t x(s) ds \end{bmatrix} + \int_{t-h(t)}^t x^T(s) S x(s) ds + \int_{t-h_{max}}^t (h_{max} - t + s) \dot{x}^T(s) R \dot{x}(s) ds. \quad (26)$$

Following the proof of Theorem 11, the condition $\Pi_1(h_{max}) > 0$ ensures the positive definiteness of the functional. The derivative of the functional along the trajectories of (22) leads to

$$\dot{V}(x_t) = \zeta^T(t)\Pi_4^0(h, \dot{h})\zeta(t) - \int_{t-h_{max}}^t \dot{x}^T(s)R\dot{x}(s)ds \leq \zeta^T(t)\Pi_4^0(h, \dot{h})\zeta(t) - \int_{t-h(t)}^t \dot{x}^T(s)R\dot{x}(s)ds.$$

The conditions (25) are then obtained by applying Corollary 9 and the Schur Complement and by noting that the final condition is affine in $h(t)$ (for a fixed $\dot{h}(t)$) and similiary affine in $\dot{h}(t)$ (for a fixed $h(t)$). \diamond

It is important to stress that the previous calculations are very classical for the stability analysis of time-delay systems. The idea is to prove that the use of the inequalities given in Corollary 6 or 9 fits perfectly with this stability analysis. Of course, there are many possibilities to reduce the conservatism of the previous theorems. One can consider, for instance, more involved Lyapunov-Krasovskii functionals, a discretized version of the functional (see for instance [4],[13]), among many others. They will not be presented in the present paper and are left for future works.

5.4 Examples

The purpose of the following section is to show how the inequalities given in Section 3 leads to a relevant reduction of conservatism in the stability condition. In it is important to stress that, our goal is not to find the best result on several examples. Our goal is to show the gap between existing results based on the Jensen's inequality and the ones proposed in the article.

5.4.1 Constant delay case

In this section, we will consider the two following examples. On the first hand, the linear time-delay systems (22) with the matrices with the matrices

$$A = \begin{bmatrix} -2 & 0 \\ 0 & -0.9 \end{bmatrix}, \quad A_d = \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix}, \quad A_D = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}. \quad (27)$$

is under consideration. This system is a well-known delay dependent stable system, that is the delay free system is stable and the maximum allowable delay $h_{max} = 6.1721$ can be easily computed by delay sweeping techniques. To demonstrate the effectiveness of our approach, results are compared to the literature and are reported in Table 1. All papers except [18] use Lyapunov theory in order to derive stability criteria. Many recent papers give the same result since they are intrinsically based on the same Lyapunov functional and use the same bounding cross terms technique i.e. Jensen inequality. Some papers [3],[32], which use an augmented Lyapunov can go further but with a numerically increasing burden, compared to our proposal. The robust approach [18] gives a very good upper-bound with a similar computational complexity than our result. The discretized Lyapunov functional proposed by [13] gives a delay upperbound very closed to the maximum allowable delay with an increasing numerical complexity.

Theorem 13 addresses also the stability of systems with *interval delays*, which may be unstable for small delays (or without delays) as it is illustrated with the second example.

$$A = \begin{bmatrix} 0 & 1 \\ -2 & 0.1 \end{bmatrix}, \quad A_d = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad A_D = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}. \quad (28)$$

As $Re(eig(A + A_d)) = 0.05 > 0$, the delay free system is unstable and in this case, the results to assess stability of this system are much more scarce. They are often related to robust analysis [3] or discretized Lyapunov-Krasovskii functionals [13]. The results are reported in Table 2. In this example, our result gives better result than [13] and [3] with a fewer numbers of variables to be optimized. Notice that with the discretization technique from [13], increasing N yields to a better result approaching the analytical bound.

Theorems	h_{max}	number of variables
[11]	4.472	$1.5n^2 + 1.5n$
[15]	4.472	$3n^2 + 3n$
[30]	4.472	$2.5n^2 + 1.5n$
[32]	4.472	$3n^2 + 3n$
[18]	6.1107	$1.5n^2 + 9n + 9$
[3]	5.120	$7n^2 + 4n$
[32]	5.02	$18n^2 + 18n$
[19]	4.97	$69n^2 + 5n$
[13] (N=1)	6.059	$5.5n^2 + 2.5n$
[1]	5.120	$6.5n^2 + 3.5n$
Th.11	5.901	$3n^2 + 2n$
Th.11 with $Q = Z = 0$	4.472	$1.5n^2 + 1.5n$
Th.12	4.484	$3n^2 + 2n$
Th.12 with $Q = Z = 0$	\emptyset	$1.5n^2 + 1.5n$

Table 1
Results for Example (27) for constant delay h .

Theorems	h_{min}	h_{max}	number of variables
[15]	\emptyset	\emptyset	$3n^2 + 3n$
[3]	0.102	1.424	$7n^2 + 4n$
[13] (N=1)	0.1006	1.4272	$5.5n^2 + 2.5n$
Th.12	0.1006	1.473	$3n^2 + 2n$
Th.12 with $Q = Z = 0$	\emptyset	\emptyset	$1.5n^2 + 1.5n$

Table 2
Results for Example (29) for constant delay h .

5.4.2 Constant distributed delay case

Consider the linear time-delay systems (15) with the matrices taken from [5]

$$A = \begin{bmatrix} 0.2 & 0 \\ 0.2 & 0.1 \end{bmatrix}, A_d = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, A_D = \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix} \quad (29)$$

For this example, the conditions from [34] and [20] are not able to characterize stability of this system. In [35], stability is guaranteed for delays over the interval [0.2090, 1.1942]. The stability conditions proposed in [5] ensures stability for any constant delay $h \in [0.2001, 1.6339]$. Using our new inequality, Theorem 11 ensures stability for all constant delay which belongs to the interval [0.200, 1.877] which is much larger than the interval found in the literature. Note that an eigenvalue analysis provides that the system remains stable for all constant delays in the interval [0.200, 2.04]. This shows the potential of the new inequality.

5.4.3 Time-varying delay case

Consider the linear time-delay systems (22) with the matrices given in equation (27). A comparison between existing results and the ones given in the present paper are shown in Table 3. Once more, one can see that with a few number of decision variables, Theorem 14 proposes a relevant alternative to the existing results from the literature.

As it was mentioned above, Theorem 14 only represents a first step. Our objective is to show the potential gain

d	0.1	0.2	0.5	0.8	1
[8]	3.604	3.033	2.008	1.364	0.999
[33]	3.604	3.033	2.008	1.364	-
[18]	3.604	3.033	2.008	1.364	0.999
[9]	1.632	1.632	1.632	1.632	1.632
[18]	4.714	3.807	2.280	1.608	1.360
[14]	3.605	3.039	2.043	1.492	1.345
[15]	3.605	3.039	2.043	1.492	1.345
[2]	4.794	3.995	2.682	1.957	1.602
[32]	3.611	3.047	2.072	1.590	1.529
[1]	4.081	3.448	2.528	2.152	1.991
Th.14	4.525	3.626	2.095	1.524	1.258

Table 3

The maximal allowable delays h_{max} for system described in Example (27).

of using the inequality proposed in the paper in hand. Further improvements of the stability conditions could be derived easily but will not be presented here.

6 Application to sampled-data systems

In this section, the aim is to show how the previous lemmas reduce the conservatism of the stability conditions for sampled-data systems. Let $\{t_k\}_{k \in \mathbb{N}}$ be an increasing sequence of positive scalars such that $\bigcup_{k \in \mathbb{N}} [t_k, t_{k+1}) = [0, +\infty)$, for which there exist two positive scalars $T_{min} \leq T_{max}$ such that

$$\forall k \in \mathbb{N}, \quad T_k = t_{k+1} - t_k \in [T_{min}, T_{max}]. \quad (30)$$

Consider the following sampled-data system

$$\forall t \in [t_k, t_{k+1}), \quad \dot{x}(t) = A_0 x(t) + B u(t_k), \quad (31)$$

where $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$ represent the state and the input vectors respectively. The sequence $\{t_k\}_{k \in \mathbb{N}}$ represents the sampling instants of the controller. The matrices A_0 and B are constant, known and of appropriate dimensions. The control law is a linear state feedback, $u = Kx$ with a given gain $K \in \mathbb{R}^{m \times n}$. The system is therefore governed by

$$\forall \tau \in [0, T_k), \quad \dot{x}(t_k + \tau) = A_0 x(t_k + \tau) + BKx(t_k). \quad (32)$$

An interesting difference between time-delay and sampled-data systems comes from the fact that the vector $x(t_k)$ is constant over $[t_k, t_{k+1})$. In order to take into account this additional information into the stability analysis, we integrate the dynamics of the system (32) over the interval $[0, \tau]$ for any $\tau \in [0, T_k]$. Since the vector $x(t_k)$ is constant over this interval, the following equation is derived

$$\int_0^\tau \dot{x}(t_k + s) ds = x(t_k + \tau) - x(t_k) = A_0 \int_0^\tau x(t_k + s) ds + \tau BKx(t_k) \quad (33)$$

This expression shows that there exists a relation between the vectors $x(t_k + \tau)$, $x(t_k)$ and $\frac{1}{\tau} \int_0^\tau \dot{x}(t_k + s) ds$. This linear relation is a particular characteristic of sampled-data systems. In the following we will show how this relation can be included in the stability analysis of sampled-data system and how it help reducing the conservatism of the resulting conditions.

Adopting the method proposed in [28], the following results is proposed

Theorem 15 *Let $0 < T_{min} \leq T_{max}$ be two positive scalars. Assume that there exist $n \times n$ -matrices $P = P^T > 0$,*

$R = R^T > 0$, $S = S^T$, $Q = Q^T$ and $X = X^T$ and $3n \times n$ -matrices Y_1 Y_2 and Y_3 that satisfy

$$\Psi_1(T) = \Pi_1 + T(\Pi_2 + \Pi_3) < 0, \quad \Psi_2(T) = \begin{bmatrix} \Pi_1 - T(\Pi_3 + \Pi_4) & TY_1 & \pi^2 TY_2 \\ * & -TR & 0 \\ * & * & -\pi^2 TR \end{bmatrix} < 0, \quad (34)$$

for $T \in \{T_{min}, T_{max}\}$ and where

$$\begin{aligned} \Pi_1 &= \Pi_1^0 - \text{He}\{Y_1 W_1 - \pi^2 Y_2 W_{20}\}, \\ \Pi_1^0 &= \text{He}\{M_1^T P M_0 - W_1^T Q M_2 - Y_3 W_1\} - W_1^T S W_1, \\ \Pi_2 &= M_0^T R M_0 + \text{He}\{M_0^T S W_1 + M_0^T Q M_2\}, \\ \Pi_3 &= M_2^T X M_2, \quad \Pi_4 = Y_3 (B K M_2 + A M_3), \end{aligned} \quad (35)$$

with $M_0 = [A_0 \ BK \ 0]$, $M_1 = [I \ 0 \ 0]$, $M_2 = [0 \ I \ 0]$, and $M_3 = [0 \ 0 \ I]$ and where W_1 and W_{20} are given in Corollary 9. Then the system (32) is asymptotically stable for any asynchronous sampling satisfying (30).

Proof: Consider a given integer $k \geq 0$ and the associated $T_k \in [T_{min} \ T_{max}]$. The stability analysis can be performed using the quadratic function $V(x) = x^T P x$ where P is a symmetric positive definite matrix and a functional \mathcal{V}_0 of the form

$$\begin{aligned} \mathcal{V}_0(\tau, x_{[t_k \ t_{k+1}]}) &= (T_k - \tau)(x(t_k + \tau) - x(t_k))^T (S(x(t_k + \tau) - x(t_k)) + 2Qx(t_k)) \\ &\quad + (T_k - \tau)\tau x^T(t_k) X x(t_k) + (T_k - \tau) \int_0^\tau \dot{x}^T(t_k + s) R \dot{x}(t_k + s) ds, \end{aligned}$$

where $x_{[t_k \ t_{k+1}]}$ represents the solution of the system over the interval $[t_k \ t_{k+1}]$, and where $R = R^T > 0$, $S = S^T$, $X = X^T$ and Q are $n \times n$ -matrices. This functional satisfies the boundary conditions as suggested in Theorem 1 in [28], i.e. $\mathcal{V}_0(0, x_{[t_k \ t_{k+1}]}) = \mathcal{V}_0(T_k, x_{[t_k \ t_{k+1}]}) = 0$. Define the functional $\mathcal{W}(\tau, x_{[t_k \ t_{k+1}]}) = V(x(t_k + \tau)) + \mathcal{V}_0(\tau, x_{[t_k \ t_{k+1}]})$. According to Theorem 1 from [28], if the derivative of this functional \mathcal{W} is strictly negative over the sampling interval $[t_k \ t_{k+1}]$, then the sampled-data system (32) is asymptotically stable. In the sequel, we will show how Corollary 9 improves the existing results. Introduce the vectors $\nu_k(\tau) = \frac{1}{\tau} \int_0^\tau x(t_k + s) ds$ and $\xi_k(\tau) = \begin{bmatrix} x^T(t_k + \tau) & x^T(t_k) & \nu_k^T(\tau) \end{bmatrix}$. According to equation (33), there exists a linking conditions between the components of $\xi_k(\tau)$ which can be stated as follows. For any matrix $Y_3 \in \mathbb{R}^{3n \times n}$ the following equality holds

$$2\xi_k^T(\tau) Y_3 [-W_1 + \tau(BK M_2 + A_0 M_3)] \xi_k(\tau) = 0 \quad (36)$$

Hence, following the proof of Theorem 2 in [28], the computation of the derivative of \mathcal{W} together with the linking relation (36)

$$\dot{\mathcal{W}}(\tau, x_{[t_k \ t_{k+1}]}) = \xi_k^T(\tau) [\Pi_1^0 + (T_k - \tau)\Pi_2 + (T_k - 2\tau)\Pi_3 + \tau\Pi_4] \xi_k(\tau) - \int_0^\tau \dot{x}^T(t_k + s) R \dot{x}(t_k + s) ds.$$

Applying Lemma 9 yields

$$\dot{\mathcal{W}}(\tau, x_{[t_k \ t_{k+1}]}) = \xi_k^T(\tau) [\Pi_1 + (T_k - \tau)\Pi_2 + (T_k - 2\tau)\Pi_3 + \tau\bar{\Pi}_4] \xi_k(\tau).$$

where $\bar{\Pi}_4 = \Pi_4 + Y_1 R^{-1} Y_1^T + \pi^2 Y_2 R^{-1} Y_2^T$. Since the previous inequality is affine with respect to $\tau \in [0, T_k]$, the right-hand side of the previous inequality is strictly if and only if it is negative when $\tau = \{0, T_k\}$. By use of the Schur complement, this is then equivalent to the conditions $\Psi_1(T_k) < 0$ and $\Psi_2(T_k) < 0$. Using the same argument on the variable $T_k \in [T_{min} \ T_{max}]$ yields the result. \diamond

6.1 Examples

Consider again the two linear systems (22) provided in the examples (27) and (29) given in the section on the stability analysis of time-delay systems with $A = A_0$ and $A_d = BK$. Additionally, we will also consider the following

Theorems	Ex. (27)	Ex. (29)	Ex. (37)
Theoretical bounds ¹	(0, 3.269]	[0.2007; 2.020]	(0, 1.729]
[7]	(0, 0.999]	\emptyset	(0, 0.869]
[25]	(0, 1.99]	\emptyset	(0, 1.113]
[6]	(0, 2.53]	\emptyset	(0, 1.695]
[23]	(0, 2.53]	\emptyset	(0, 1.695]
[28]	(0, 2.62]	[0.4, 1.25]	(0, 1.721]
Theorem 15	(0, 2.74]	[0.4, 1.33]	(0, 1.729]
Theorem 15 with $Y_3 = 0$	(0, 2.62]	[0.4, 1.25]	(0, 1.721]

Table 4
Interval of allowable asynchronous samplings.

example taken from [7],[25].

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ 0 & -0.1 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ -0.1 \end{bmatrix} \begin{bmatrix} 3.75 \\ 11.5 \end{bmatrix} x(t_k) \quad (37)$$

For this example, when the sampling period is chosen constant (i.e. $T_1 = T_2 = T_k$, for all $k \geq 0$), an eigenvalues analysis of the transition matrix ensures that the system is stable for all constant sampling period in $(0, 1.729]$. Applying Theorem 15, we prove that system (37) is asymptotically stable for all asynchronous sampling over the interval $[0, 1.724]$, encompassing many results of the literature as it can be seen in the table 4.

7 Conclusions

In this paper, we have provided new useful inequalities which encompass the Jensen's inequality. In combination with a simple Lyapunov-Krasovskii functional, this inequality leads to new stability criteria for linear time delay system and sampled-data systems. This new result has been expressed in terms of LMIs and has shown on numerical examples a large improvement of existing results using only a limited number of matrix variables. More generally, this new inequality could be copied to more elaborated Lyapunov functional.

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