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Abstract

Identification of real-world systems is often applied in closed loop due to stability, performance or safety constraints. However, in case of Linear Parameter-Varying (LPV) systems, closed-loop identification is not well-established despite the recent advances in prediction error approaches. Building on the available results, the paper proposes the closed-loop generalization of a recently introduced instrumental variable scheme for the identification of LPV-IO models with Box-Jenkins type of noise model structures. Estimation under closed-loop conditions with the proposed approach is analyzed from the stochastic point of view and the performance of the method is demonstrated through a representative simulation example.

Key words: Linear parameter-varying system; Identification; Refined instrumental variable; Box-Jenkins models.

1 Introduction

Identification of physical or chemical systems is often restricted to data gathered during closed-loop operation due to stability, performance/economical or safety constraints. As the control loop itself introduces correlation between the disturbances and the control signal, the statistically optimal (unbiased with minimal variance) estimation of the parameters of a chosen model structure w.r.t. the data-generating system is an essentially different problem than in the open-loop setting. Hence in the identification literature, especially in the *linear time-invariant* (LTI) case, many approaches have been proposed to provide well-applicable solutions in this problem setting. An overview of the recent developments in the LTI case can be found in [16] and [6].

Identification of *linear parameter-varying* (LPV) models for systems operated in open loop has recently seen significant improvements. The case of closed-loop LPV model identification has however remained sparsely studied (see [5, 14]). The main difficulty in this system class in terms of identification is that even if the signal relations between the inputs u and outputs y of the system are linear, these relations are allowed to change over time as a function of a so-called *scheduling variable* $p : \mathbb{Z} \to \mathbb{P}$ with $\mathbb{P} \subseteq \mathbb{R}^{n_{\mathrm{p}}}$. This allows to describe a large class of nonlinear/time-varying systems in an attractive structure on which the well-established LPV control-synthesis approaches, e.g. [11, 12], can be applied. On the other hand, this varying-relation prevents the use of crucial system theoretical relations, like transfer functions and commutativity of operators, on which most LTI closed-loop identification approaches are based. As a consequence, only preliminary closed-loop methods have been proposed in the literature without being able to exploit the existing tools and knowledge available in the LTI case. In [17], an approximation based LPV extension of a predictor subspace approach (PSBID) has been introduced which is also applicable in a closed-loop setting. In [3], also an approximation based LPV extension of the CLOE algorithm (see [8]) has been investigated w.r.t. LPV *output-error* (OE) type of models. In [1] and [4], a basic instrumental variable method has been introduced. It is a clear indicator of the immature state of this research direction that the stochastic properties of the estimation has only been analyzed in [4]. Unfortunately in [9], it has been shown recently that the formulation of the one-step IV approach, proposed in [4], does not allow to reach statistically optimal esti-

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mates due to the non-commutativity of certain filtering operations. This highlights that up to now, no method has been established which allows statistically optimal estimation under a closed-loop setting especially with general noise models like *Box-Jenkins* (BJ).

Based upon the available results of [1, 4] and using the theoretical advancements of the LPV prediction error-framework introduced in [14], we propose in this paper a closed-loop extension of a recently developed IV approach for LPV-BJ models [9]. In this setting, the noise v_o affecting the sampled output measurement $y(t_k)$ is assumed to have a rational spectral density which is allowed to be unrelated to the actual process dynamics of the data-generating system (general noise structure). As a first step towards the case of a p-dependent noise, it is also assumed that this rational spectral density does not depend on p. A stochastic analysis of the proposed closed-loop approach is provided, exploring the limitations and the benefits of this estimation strategy. The performance of the algorithm is also demonstrated on a representative simulation example.

2 Problem description

Consider the data-generating LPV system $\mathcal{S}_\mathrm{o}$ given in the closed-loop setting of Fig. 1, where $\mathcal{S}_\mathrm{o}$ is defined as:

$$A_\mathrm{o}(p_k, q^{-1})\chi_\mathrm{o}(t_k) = B_\mathrm{o}(p_k, q^{-1})u(t_{k-d}), \tag{1a}$$
$$y(t_k) = \chi_\mathrm{o}(t_k) + v_\mathrm{o}(t_k). \tag{1b}$$

Here $u(t_k)$ is the input of the plant, p_k is the value of the scheduling variable p at sample time t_k, χ_o is the noise-free output, v_o is the additive noise with bounded spectral density, y is the noisy output of the system, d is the delay, and q is the time-shift operator, i.e. $q^{-i}u(t_k) = u(t_{k-i})$. $A_\mathrm{o}(p_k, q^{-1})$ and $B_\mathrm{o}(p_k, q^{-1})$ are polynomials in q^{-1} of degree n_a and n_b respectively:

$$A_\mathrm{o}(p_k, q^{-1}) = 1 + \sum_{i=1}^{n_\mathrm{a}} a_i^\mathrm{o}(p_k)q^{-i}, \tag{2a}$$

$$B_\mathrm{o}(p_k, q^{-1}) = \sum_{j=0}^{n_\mathrm{b}} b_j^\mathrm{o}(p_k)q^{-j}, \tag{2b}$$

where the coefficients a_i^o and b_j^o are real meromorphic functions [2] with *static dependence* on p, i.e. dependence only on the instantaneous value of p at time t_k. It is assumed that each a_i^o and b_j^o is non-singular on $\mathbb{P}$, thus the solutions of $\mathcal{S}_\mathrm{o}$ are well-defined and the process part $\mathcal{G}_\mathrm{o}$, defined by (1a), is completely characterized by the coefficient functions $\{a_i^\mathrm{o}\}_{i=1}^{n_\mathrm{a}}$ and $\{b_j^\mathrm{o}\}_{j=0}^{n_\mathrm{b}}$. The noise v_o is assumed to be independent from p. Latter we will return to the more general case of p-dependent noise models, showing that in case of an IV-based estimation, violation of this assumption does not lead to a biased estimate.

[2] $f : \mathbb{R}^n \to \mathbb{R}$ is a real meromorphic function if $f = g/h$ with g, h analytic and $h \neq 0$.

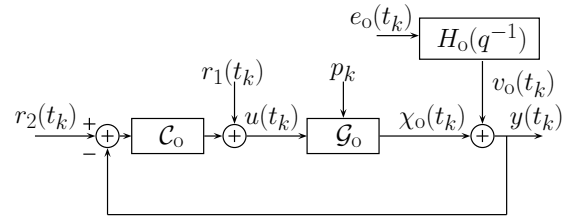

Fig. 1. Closed-loop LPV system.

If v_o has a rational spectral density, then the noise process can be represented by an *autoregressive moving average* (ARMA) model:

$$v_\mathrm{o}(t_k) = H_\mathrm{o}(q)e_\mathrm{o}(t_k) = \frac{C_\mathrm{o}(q^{-1})}{D_\mathrm{o}(q^{-1})}e_\mathrm{o}(t_k), \tag{3}$$

where $C_\mathrm{o}(q^{-1})$ and $D_\mathrm{o}(q^{-1}) \neq 0$ are monic polynomials with constant coefficients and with respective degree n_c and n_d. The corresponding proper transfer function $H_\mathrm{o}(q)$ is assumed to be stable and to have a stable inverse. In case $C_\mathrm{o}(q^{-1}) = D_\mathrm{o}(q^{-1}) = 1$, (3) defines an OE noise model, however with $C_\mathrm{o}(q^{-1}) \neq D_\mathrm{o}(q^{-1})$, (3) is general enough to represent BJ-type of noise models. In terms of the closed-loop setting of Fig. 1, $u(t_k) = r_1(t_k) + \mathcal{C}_\mathrm{o}(r_2(t_k) - y(t_k))$, where $\mathcal{C}_\mathrm{o}$ is the operator form of the controller and r_1, r_2 are reference signals. Note that, as in many digital control systems, u is known (the error introduced by the actuators can be modeled as an output additive (p-dependent) noise due to the linearity of the system). The controller can be any LTI, nonlinear or LPV controller, under the assumptions that

- $\mathcal{C}_\mathrm{o}$ is *a priori* known;
- $\mathcal{C}_\mathrm{o}$ ensures BIBO stability of the closed-loop system $\mathcal{S}_\mathrm{o}$ for any $p \in \mathbb{P}^\mathbb{Z}$.

Next we introduce a model structure and parametrization for the identification of $\mathcal{S}_\mathrm{o}$, where, according to (1a-b) and (3), the noise model and the process model are parameterized separately. The proposed LPV-BJ model, denoted in the sequel as $\mathcal{M}_\theta$, is defined as:

$$A(p_k, q^{-1}, \rho)\chi(t_k) = B(p_k, q^{-1}, \rho)u(t_{k-d}), \tag{4a}$$
$$D(q^{-1}, \eta)v(t_k) = C(q^{-1}, \eta)e(t_k), \tag{4b}$$
$$y(t_k) = \chi(t_k) + v(t_k), \tag{4c}$$

with parameters $\theta = [\,\rho^\top \;\; \eta^\top\,]$ and with $u(t_k) = r_1(t_k) + \mathcal{C}_\mathrm{o}(r_2(t_k) - y(t_k))$. The process model part of $\mathcal{M}_\theta$, denoted by $\mathcal{G}_\rho$, is defined in terms of the LPV-IO representation (4a) where A and B are polynomials with order n_a and n_b respectively and with p-dependent coefficients a_i and b_j parameterized as

$$a_i(p_k) = a_{i,0} + \sum_{l=1}^{n_\alpha} a_{i,l}f_l(p_k), \quad i = 1, \ldots, n_\mathrm{a}, \tag{5a}$$

$$b_j(p_k) = b_{j,0} + \sum_{l=1}^{n_\beta} b_{j,l}g_l(p_k), \quad j = 0, \ldots, n_\mathrm{b}. \tag{5b}$$

In this parametrization, $\{f_l\}_{l=1}^{n_\alpha}$ and $\{g_l\}_{l=1}^{n_\beta}$ are *a priori* chosen meromorphic functions of p bounded on $\mathbb{P}$, with static dependence, allowing the identifiability of the model (e.g. linearly independent functions on $\mathbb{P}$). The associated model parameters ρ are stacked column-wise: $\rho = [\ \mathtt{a}_1\ \dots\ \mathtt{a}_{n_\mathrm{a}}\ \mathtt{b}_0\ \dots\ \mathtt{b}_{n_\mathrm{b}}\ ]^\top \in \mathbb{R}^{n_\rho}$ where $\mathtt{a}_i = [\ a_{i,0}\ \ a_{i,1}\ \ \dots\ \ a_{i,n_\alpha}\ ]$, $\mathtt{b}_j = [\ b_{j,0}\ \ b_{j,1}\ \ \dots\ \ b_{j,n_\beta}\ ]$ and $n_\rho = n_\mathrm{a}(n_\alpha+1) + (n_\mathrm{b}+1)(n_\beta+1)$.

The noise-model part of $\mathcal{M}_\theta$, denoted by $\mathcal{H}_\eta$, is defined in terms of (4b) where C and D are monic polynomials with order n_c and n_d respectively and with constant coefficients c_i and d_j collected as $\eta = [\ c_1\ \dots\ c_{n_\mathrm{c}}\ d_1\ \dots\ d_{n_\mathrm{d}}\ ]^\top \in \mathbb{R}^{n_\eta}$, where $n_\eta = n_\mathrm{c} + n_\mathrm{d}$. As (4b) is LTI, it can be represented by a transfer function $H(q,\eta) = C(q^{-1},\eta)/D(q^{-1},\eta)$. It is further assumed that $\eta \in \Theta_\eta \subset \mathbb{R}^{n_\eta}$ such that $H(z,\eta)$ is stable (analytic in the exterior of the unit circle) and has a stable inverse.

Introduce also $\mathcal{G} = \{\mathcal{G}_\rho \mid \rho \in \mathbb{R}^{n_\rho}\}$ and $\mathcal{H} = \{\mathcal{H}_\eta \mid \eta \in \mathbb{R}^{n_\eta}\}$, as the collection of all process and noise models in the form of (4a) and (4b). Then the model set, denoted as $\mathcal{M}$, takes the form

$$\mathcal{M} = \left\{(\mathcal{G}_\rho, \mathcal{H}_\eta)\ \mid\ \mathrm{col}(\rho,\eta) = \theta \in \mathbb{R}^{n_\rho+n_\eta}\right\}, \tag{6}$$

corresponding to the set of candidate models in which we seek the model that explains data gathered from $\mathcal{S}_\mathrm{o}$ the best under a given identification criterion.

Denote $\mathcal{D}_N = \{y(t_k), u(t_k), p(t_k)\}_{k=1}^N$ a data sequence of $\mathcal{S}_\mathrm{o}$. In [9], it has been shown, that a one-step-ahead output predictor $\hat{y}_\theta(t_k|t_{k-1})$ can be formulated w.r.t. the considered model structure (4a-c) under the commonly used assumption that noise-free observation of the sequence $\{p_k, p_{k-1}, \dots\}$ is available. Recently it has been proven that using estimated moments, such a predictor can be formulated if p is observed up to an additive white noise independent from v_o [15]. However for the sake of simplicity, we will only consider the previous case. These results allow to formulate (in either case) the estimation of θ, as the minimization of the *one-step ahead prediction error*: $\varepsilon_\theta = y(t_k) - \hat{y}_\theta(t_k|t_{k-1})$. This minimization is formulated in terms of an identification criterion $W(\mathcal{D}_N,\theta)$, like the *least squares* criterion

$$W(\mathcal{D}_N,\theta) = \frac{1}{N}\sum_{k=1}^N \varepsilon_\theta^2(t_k) = \frac{1}{N}\|\varepsilon_\theta\|_{\ell_2}^2, \tag{7}$$

such that the parameter estimate is

$$\hat{\theta}_N = \arg\min_{\theta\in\mathbb{R}^{n_\rho+n_\eta}} W(\mathcal{D}_N,\theta). \tag{8}$$

Based on the previous considerations, the identification problem addressed in the sequel is defined as follows:

Problem 1 *Given a discrete-time LPV data-generating system $\mathcal{S}_\mathrm{o}$ in the closed-loop setting of Fig. 1 with a priori known stabilizing controller $\mathcal{C}_\mathrm{o}$. Based on the LPV-BJ model structure $\mathcal{M}_\theta$ defined by (4a-c) and a data set $\mathcal{D}_N$ collected from $\mathcal{S}_\mathrm{o}$, estimate the parameter vector θ as the minimization of* (7) *under the following assumptions:*

A1 $\mathcal{S}_\mathrm{o} \in \mathcal{M}$.
A2 *In the parametrization (5a-b) of A and B, $\{f_l\}_{l=1}^{n_\alpha}$ and $\{g_l\}_{l=1}^{n_\beta}$ are chosen such that $\mathcal{M}_\theta$ is identifiable.*
A3 *$\mathcal{D}_N$ is informative w.r.t. $\mathcal{M}_\theta$: if $\theta_1 \neq \theta_2$ then $\mathcal{M}_{\theta_1}$ and $\mathcal{M}_{\theta_2}$ lead to different prediction errors given $\mathcal{D}_N$.*
A4 *$\mathcal{S}_\mathrm{o}$ is BIBO stable, i.e. for any bounded $p \in \mathbb{P}^\mathbb{Z}$ and $u \in \mathbb{R}^\mathbb{Z}$, the output of $\mathcal{S}_\mathrm{o}$ is bounded [14].*
A5 *The closed-loop system with $\mathcal{C}_\mathrm{o}$ is BIBO stable.*

In [9], a *refined instrumental variable* (RIV) based approach has been introduced to solve the estimation problem of $\mathcal{M}_\theta$ in an open-loop setting. However, a major difference between the open-loop setting considered in [9] and the closed-loop setting of Prob. 1 is the correlation of u with e_o due to the feedback loop. Therefore in the following, we aim to solve the problem of determining a suitable instrument that allows to developing the LPV-RIV approach in the closed-loop case.

3 RIV approach for closed-loop LPV systems

In [9] it has been shown that reformulation of (4a-c) as a *multiple-input single-output* (MISO) LTI model allows an elegant solution of the filtering problem associated with LPV-IV approaches. This reformulation is necessary to avoid the problem of non-commutativity of parameter-varying filters and hence required to solve Prob. 1. Consequently, (4a) is rewritten as

$$\underbrace{\chi(t_k) + \sum_{i=1}^{n_\mathrm{a}} a_{i,0}\chi(t_{k-i})}_{F(q^{-1})\chi(t_k)} + \sum_{i=1}^{n_\mathrm{a}}\sum_{l=1}^{n_\alpha} a_{i,l}\underbrace{f_l(p_k)\chi(t_{k-i})}_{\chi_{i,l}(t_k)} = \sum_{j=0}^{n_\mathrm{b}}\sum_{l=0}^{n_\beta} b_{j,l}\underbrace{g_l(p_k)u(t_{k-d-j})}_{u_{j,l}(t_k)}, \tag{9}$$

where $F(q^{-1}) = 1 + \sum_{i=1}^{n_\mathrm{a}} a_{i,0}q^{-i}$ and $g_0(\boldsymbol{.}) \equiv 1$. Note that in this way, the LPV-BJ model is rewritten as a MISO system with $(n_\mathrm{b}+1)(n_\beta+1) + n_\mathrm{a}n_\alpha$ inputs $\{\chi_{i,l}\}_{i=1,l=1}^{n_\mathrm{a},n_\alpha}$ and $\{u_{j,l}\}_{j=0,l=0}^{n_\mathrm{b},n_\beta}$. $F(q^{-1})$ does not depend on p_k, thus (9) and (4b-c) have the following LTI form:

$$y(t_k) = -\sum_{i=1}^{n_\mathrm{a}}\sum_{l=1}^{n_\alpha}\frac{a_{i,l}}{F(q^{-1})}\chi_{i,l}(t_k) + \sum_{j=0}^{n_\mathrm{b}}\sum_{l=0}^{n_\beta}\frac{b_{j,l}}{F(q^{-1})}u_{k,j}(t_k) + H(q)e(t_k). \tag{10}$$

Note that (10) is equivalent with (4a-c), but it is not a representation of the associated LPV system as it includes the lumped output variables in $\{\chi_{i,l}\}_{i=1,l=1}^{n_\mathrm{a},n_\alpha}$. Using (10), the estimation problem of the parameters can

be formulated as a linear regression which allows optimal solution of (7) w.r.t. LPV-BJ models. To achieve this solution, one possible way is to develop an extension of the closed-loop *refined instrumental variable* (RIV) approach of the LTI framework. Next we derive this extension, which, as we will see, provides an easily implementable iterative estimation scheme.

Rewrite (10) to the linear regression form:

$$y(t_k) = \varphi^\top(t_k)\rho + \tilde{v}(t_k), \tag{11}$$

where $\tilde{v}(t_k) = F(q^{-1},\rho)v(t_k)$ and

$$\begin{aligned}\varphi(t_k) = [\, &-y(t_{k-1}) \ \dots \ -y(t_{k-n_\mathrm{a}}) \ -\chi_{1,1}(t_k) \ \dots \\ &-\chi_{n_\mathrm{a},n_\alpha}(t_k) \ u_{0,0}(t_k) \ \dots \ u_{n_\mathrm{b},n_\beta}(t_k)\,]^\top, \\ \rho = [\, &a_{1,0} \ \dots \ a_{n_\mathrm{a},0} \ a_{1,1} \ \dots \ a_{n_\mathrm{a},n_\alpha} \ b_{0,0} \ \dots \ b_{n_\mathrm{b},n_\beta}\,]^\top.\end{aligned}$$

Two difficulties still remain to obtain the minimum of (7) based on (11): the regressor $\varphi(t_k)$ contains the unknown terms $\{\chi_{i,l}(t_k)\}_{i=1,l=0}^{n_\mathrm{a},n_\alpha}$ and all of its elements are corrupted with the colored noise $v(t_k)$. To resolve this problem, an appropriate instrument $\zeta(t_k)$ can be introduced such that the estimate of ρ can be given as [13]:

$$\begin{aligned}\hat{\rho}_{\mathtt{XIV}}(N) = \arg\min_{\rho\in\mathbb{R}^{n_\rho}} \Bigg\| &\left[\frac{1}{N}\sum_{k=1}^{N} L(q)\zeta(t_k)L(q)\varphi^\top(t_k)\right]\rho \\ &- \left[\frac{1}{N}\sum_{k=1}^{N} L(q)\zeta(t_k)L(q)y(t_k)\right]\Bigg\|_W^2\end{aligned} \tag{12}$$

where $\|x\|_W^2 = x^T W x$, with W a positive definite weighting matrix and $L(q)$ is a stable pre-filter. If $G_\mathrm{o} \in \mathcal{G}$, the estimate (12) is consistent under the following well-known conditions [3]:

C1 $\bar{\mathbb{E}}\{L(q)\zeta(t_k)L(q)\varphi^\top(t_k)\}$ is full column rank.
C2 $\bar{\mathbb{E}}\{L(q)\zeta(t_k)L(q)\tilde{v}(t_k)\} = 0$.

Moreover, the minimum variance estimator can be achieved if [7, 13, 18]:

C3 $W = I$ and $\zeta(t_k)$ is chosen as the noise-free version of (11) and is therefore defined in the LPV case as:

$$\begin{aligned}\zeta(t_k) = \Big[&-\mathring{\chi}(t_{k-1}) \ \dots \ -\mathring{\chi}(t_{k-n_\mathrm{a}}) \ -\mathring{\chi}_{1,1}(t_k) \ \dots \\ &-\mathring{\chi}_{n_\mathrm{a},n_\alpha}(t_k) \ \mathring{u}_{0,0}(t_k) \ \dots \ \mathring{u}_{n_\mathrm{b},n_\beta}(t_k)\Big]^\top\end{aligned}$$

where $\mathring{u}$ and $\mathring{\chi}$ are the signals from the auxiliary model as presented in Fig. 2.

C4 $\mathcal{G}_\mathrm{o} \in \mathcal{G}$ and n_ρ is equal to the minimal number of parameters required to represent $\mathcal{G}_\mathrm{o}$ with the considered model structure.

[3] The notation $\bar{\mathbb{E}}\{.\} = \lim_{N\to\infty}\frac{1}{N}\sum_{t=1}^{N}\mathbb{E}\{.\}$ is adopted from the prediction error framework of [10].

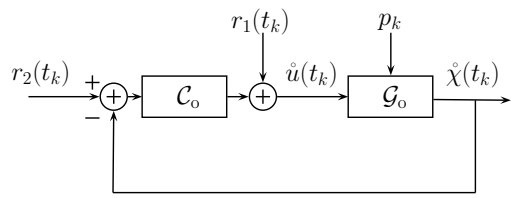

Fig. 2. Auxiliary model

C5 $L(q)$ is chosen as [9]:

$$L(q) = \frac{D(q^{-1})}{F(q^{-1})C(q^{-1})}. \tag{13}$$

Full column rank of $\bar{\mathbb{E}}\{L(q)\zeta(t_k)L(q)\varphi^\top(t_k)\}$ is ensured by the informativity of the data set (A3). Note that conditions about informativity w.r.t. LPV-BJ model structures is in the focus of current research (see [2, 4, 14] for results in the ARX case). To fulfill C1 under A3, the discussion can be found in [13]. It is also important to note that in a practical situation none of $F(q^{-1},\rho)$, $C(q^{-1},\eta)$, $D(q^{-1},\eta)$ or $\{a_{i,l}(\rho)\}_{i=1,l=0}^{n_\mathrm{a},n_\alpha}$, $\{b_{j,l}(\rho)\}_{j=0,l=0}^{n_\mathrm{b},n_\beta}$ is known *a priori*. Therefore, the RIV estimation normally involves an iterative (or relaxation) algorithm in which, at each iteration, an 'auxiliary model' is used to generate the instrumental variables (which guarantees C2), as well as the associated pre-filters. Convergence of such an iterative RIV algorithm has not been proven so far and is only empirically assumed [19].

Based on the previous considerations, the iterative scheme of the RIV algorithm can be extended to the closed-loop LPV case as described by Algorithm 1.

Based on a similar concept, the so-called *simplified* LPV-CLRIV (LPV-CLSRIV) method, can also be developed for the estimation of LPV-OE models. This method is based on a model structure (4a-c) with $C(q^{-1},\eta) = D(q^{-1},\eta) = 1$ and consequently, Step 7 of Algorithm 1 can be skipped.

Remark 1 *During the formulation of the approach it has been assumed that v_o is independent from p (see (3)). However, to have a general noise model in the LPV setting it is necessary to consider p-dependence of the noise as has been pointed out in [14]. However the RIV setting, even using an OE type of noise model (like in the SRIV case), leads to unbiased estimates also in the case when the assumed noise model is incorrect (or correlated with p). The role of a more accurate noise model in this setting is to lower the variance of the resulting estimates. This explains why using a general BJ structure is important to increase the accuracy of the RIV method. However in the LPV case, allowing the noise model to be also p-dependent not only makes the estimation problem more difficult from the filtering point of view, but also increases significantly the number of parameters to be estimated. This results in a trade-off problem, where the simple estimation scheme proposed in this paper provides an attractive and practically applicable approach. Finding an*

Algorithm 1 LPV-CLRIV

1: set $\tau = 0$ and let $\hat{\theta}^{(0)} = [\,(\hat{\rho}^{(0)})^\top \;\; (\hat{\eta}^{(0)})^\top\,]^\top$ be an initial parameter estimate given by the *least square* (LS) approach (ARX estimate of $\mathcal{M}_\theta$).

2: **repeat**

3: compute an estimate of $\chi(t_k)$ via

$$A(p_k, q^{-1}, \hat{\rho}^{(\tau)})\hat{\chi}(t_k) = B(p_k, q^{-1}, \hat{\rho}^{(\tau)})u(t_{k-d}),$$

where $\mathcal{G}_{\hat{\rho}^{(\tau)}}$ is assumed to be stable (A5) and compute $\{\hat{\chi}_{i,l}(t_k)\}_{i=1,l=0}^{n_{\mathrm{a}},n_\alpha}$ according to (9).

4: compute the estimated filter:

$$\hat{L}(q^{-1}, \hat{\theta}^{(\tau)}) = \frac{D(q^{-1}, \hat{\eta}^{(\tau)})}{C(q^{-1}, \hat{\eta}^{(\tau)})F(q^{-1}, \hat{\rho}^{(\tau)})}$$

and the filtered signals $\{u^{\mathrm{f}}_{j,l}(t_k)\}_{j=0,l=0}^{n_{\mathrm{b}},n_\beta}$, $y_{\mathrm{f}}(t_k)$ and $\{\hat{\chi}^{\mathrm{f}}_{i,l}(t_k)\}_{i=1,l=0}^{n_{\mathrm{a}},n_\alpha}$.

5: build the estimated filtered regressor as:

$$\hat{\varphi}_{\mathrm{f}}(t_k) = [\,-y_{\mathrm{f}}(t_{k-1}) \;\ldots\; -y_{\mathrm{f}}(t_{k-n_{\mathrm{a}}}) \;\; -\hat{\chi}^{\mathrm{f}}_{1,1}(t_k) \;\ldots -\hat{\chi}^{\mathrm{f}}_{n_{\mathrm{a}},n_\alpha}(t_k) \;\; u^{\mathrm{f}}_{0,0}(t_k) \;\ldots\; u^{\mathrm{f}}_{n_{\mathrm{b}},n_\beta}(t_k)\,]^\top$$

and compute the filtered instrument $\hat{\zeta}_{\mathrm{f}}(t_k)$ by simulating $\mathcal{G}_{\hat{\rho}^{(\tau)}}$ according to Fig. 2:

$$\hat{\zeta}_{\mathrm{f}}(t_k) = [\,-\hat{\hat{\chi}}_{\mathrm{f}}(t_{k-1}) \;\ldots\; -\hat{\hat{\chi}}_{\mathrm{f}}(t_{k-n_{\mathrm{a}}}) \;\; -\hat{\hat{\chi}}^{\mathrm{f}}_{1,1}(t_k) \;\ldots -\hat{\hat{\chi}}^{\mathrm{f}}_{n_{\mathrm{a}},n_\alpha}(t_k) \;\; \hat{\hat{u}}^{\mathrm{f}}_{0,0}(t_k) \;\ldots\; \hat{\hat{u}}^{\mathrm{f}}_{n_{\mathrm{b}},n_\beta}(t_k)\,]^\top$$

6: With $W = I$ and $\hat{L}(q^{-1}, \hat{\theta}^{(\tau)})$ (according to C3-C5), compute the solution of (12) via

$$\hat{\rho}^{(\tau+1)} = \left[\frac{1}{N}\sum_{k=1}^{N}\hat{\zeta}_{\mathrm{f}}(t_k)\hat{\varphi}_{\mathrm{f}}^\top(t_k)\right]^{-1}\sum_{k=1}^{N}\frac{1}{N}\hat{\zeta}_{\mathrm{f}}(t_k)y_{\mathrm{f}}(t_k).$$

7: compute the estimate of the noise signal v as

$$\hat{v}(t_k) = y(t_k) - \hat{\chi}(t_k, \hat{\rho}^{(\tau)}).$$

Based on $\hat{v}$, estimation of $\hat{\eta}^{(\tau+1)}$ follows using e.g. the `ARMA` estimation algorithm of the `MATLAB` identification toolbox (an IV approach can also be used for this purpose, see [19]).

8: increase τ by 1

9: **until** $\theta^{(\tau)}$ has converged

IV-based estimation scheme for p-dependent BJ models, which preserves the simplicity of the proposed approach, could lead to further decrease of the variance of the estimates and hence it is in the focus of future research.

4 Simulation Example

In the following example we aim to demonstrating the performance of proposed closed-loop RIV scheme on a relevant simulation example. In this demonstration, we will focus on estimation under different noise conditions/models, hence the model structure of the process part is assumed to be known.

Consider the example of a mass connected to a spring and a varying damper depicted in Fig. 3. This problem is one of the typical phenomena occurring in the motion control of many mechatronic systems like in active suspension. Denote x the position (in [m]) of the mass m (in [kg]), $\mathrm{k_s} > 0$ the stiffness of the spring and $\mathrm{c_d} > 0$ the varying damping. Introduce F as the force (in [N]) acting on the mass m. Then in *continuous time* (CT), the behavior of the system is defined by:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(m\frac{\mathrm{d}}{\mathrm{d}t}x(t)\right) = F(t) - \mathrm{k_s}x(t) - \mathrm{c_d}(t)\frac{\mathrm{d}}{\mathrm{d}t}x(t). \tag{14}$$

By considering $F(t)$ as the input $u(t)$, $x(t)$ as the output $y(t)$ and $p(t) : \mathbb{R} \to [0, 1]$ as the scheduling variable such that $\mathrm{c_d}(t) = \mathrm{c_d}^{(0)} + \mathrm{c_d}^{(1)}p(t)$, (14) can be rewritten as a CT-LPV system. By using a simple *backward Euler* type of discretization in a *zero-order-hold* setting with sampling period $T_{\mathrm{d}} > 0$, this LPV system can be formulated as a *discrete-time* (DT) LPV system:

$$y(t_k) = \tfrac{T_{\mathrm{d}}\mathrm{c_d}^{(1)}p_k + T_{\mathrm{d}}\mathrm{c_d}^{(0)} + 2m}{L(p_k)}y(t_{k-1}) - \tfrac{m}{L(p_k)}y(t_{k-2}) + \tfrac{T_{\mathrm{d}}^2}{L(p_k)}u(t_k) \tag{15}$$

where $u(t_k)$, $y(t_k)$, p_k denote the sampled signals of $F(t)$, $x(t)$, and $p(t)$ respectively and $L(p_k) = T_{\mathrm{d}}\mathrm{c_d}^{(1)}p_k + T_{\mathrm{d}}\mathrm{c_d}^{(0)} + m + \mathrm{k_s}T_{\mathrm{d}}^2$. To simplify the problem, approximate $\frac{1}{L}$ by its 1st-order Taylor approximation at the mid-point of $\mathbb{P}$, i.e. at $p_k = 0.5$:

$$\left.\frac{1}{L(p)}\right|_{p=0.5} \approx \frac{1}{\frac{\tau_1}{2} + \tau_0} - \frac{\tau_1}{(\frac{\tau_1}{2} + \tau_0)^2}(p - 0.5), \tag{16}$$

where $\tau_0 = T_{\mathrm{d}}\mathrm{c_d}^{(0)} + m + \mathrm{k_s}T_{\mathrm{d}}^2$ and $\tau_1 = T_{\mathrm{d}}\mathrm{c_d}^{(1)}$. Then the resulting DT-LPV representation reads as

$$y(t_k) = -\big(a_{10} + a_{11}p_k + a_{12}p_k^2\big)y(t_{k-1}) - \big(a_{20} + a_{21}p_k\big)y(t_{k-2}) + \big(b_{00} + b_{01}p_k\big)u(t_k), \tag{17}$$

where $a_{11} = -\tau' T_{\mathrm{d}}\mathrm{c_d}^{(1)} + \tau''(T_{\mathrm{d}}\mathrm{c_d}^{(0)} + 2m)$, $a_{10} = -\tau'(T_{\mathrm{d}}\mathrm{c_d}^{(0)} + 2m)$, $a_{12} = \tau'' T_{\mathrm{d}}\mathrm{c_d}^{(1)}$, $a_{20} = \tau' m$, $a_{21} = -\tau'' m$, $b_{00} = \tau' T_{\mathrm{d}}^2$, $b_{01} = -\tau'' T_{\mathrm{d}}^2$ with $\tau' = \frac{2}{\tau_1 + 2\tau_0} + \frac{2\tau_1}{(\tau_1 + 2\tau_0)^2}$ and $\tau'' = \frac{4\tau_1}{(\tau_1 + 2\tau_0)^2}$.

Consider (15) as $\mathcal{G}_{\mathrm{o}}$ in the closed loop setting of Fig. 1 with parameters $T_{\mathrm{d}} = 0.05\mathrm{s}$, $m = 0.01$, $\mathrm{k_s} = 0.85$, $\mathrm{c_d}^{(1)} = 0.5$, $\mathrm{c_d}^{(0)} = 0.5$. To control the motion of this system an LPV-PI controller $\mathcal{C}_{\mathrm{o}}$ has been designed:

$$u(t_k) = u(t_{k-1}) + K_{\mathrm{C}}(p_k)\big(w(t_k) - w(t_{k-1})\big) + \frac{K_{\mathrm{C}}(p_k)T_{\mathrm{d}}}{T_{\mathrm{I}}(p_k)}w(t_k), \tag{18}$$

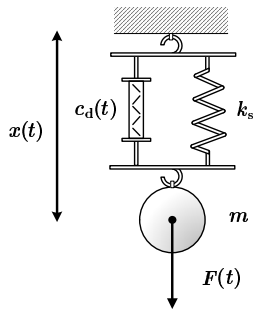

Fig. 3. Mass-spring-damper system with varying damping.

where $w(t_k) = r(t_k) - y(t_k)$, $K_C(p_k) = 3.33 + 0.99p_k + 2.06p_k^2$ and $T_I(p_k) = 0.225 + 0.215p_k - 0.09p_k^2$. Additionally, a BJ type of noise v is added to the output. Consequently the data generating system can be expressed:

$$\mathcal{S}_o \begin{cases} A_o(q, p_k) = 1 + a_1^o(p_k)q^{-1} + a_2^o(p_k)q^{-2} \\ B_o(q, p_k) = b_0^o(p_k) \\ H_o(q) = \dfrac{1}{1 - 0.5q^{-1} - 0.2q^{-2}} \end{cases} \tag{19}$$

where $v(t_k) = H_o(q)e_o(t_k)$, $e_o(t_k) \in \mathcal{N}(0, \sigma_e^2)$ and

$$a_1^o(p_k) = -1.135 - 0.1738p_k + 0.2538p_k^2, \tag{20a}$$
$$a_2^o(p_k) = 0.2523 - 0.1015p_k, \tag{20b}$$
$$b_0^o(p_k) = 0.0631 - 0.0254p_k. \tag{20c}$$

In the sequel, both the scheduling signal p and the input $r(t_k)$ are considered as white noises with uniform distributions $\mathcal{U}(0.05, 0.95)$ and $\mathcal{U}(-0.5, 0.5)$ respectively. Realizations of these signals with length $N = 2000$ are used to generate data sets $\mathcal{D}_N$ of $\mathcal{S}_o$. To demonstrate the efficiency of the presented method, the proposed approach is compared to the conventional *least-squares* (LPV-LS) method (see [2]) and also to the direct nonlinear minimization of the least-squares criteria (7) (NL-PEM) w.r.t. the LPV-BJ model. The latter approach is computed via the `LSQNONLIN` algorithm of the optimization toolbox in `MATLAB`. Note that the NL-PEM approach provides statistically optimal results in case the model structure is optimally chosen and the nonlinear optimization succeeds to find the global minimum of (7) (depending on the initialization). Therefore to use the NL-PEM as a base-line result for assessing the performance of the IV approach, it is initialized with the true parameters and hence the provided estimates can be considered optimal.

The LPV-LS method assumes an LPV-ARX type of structure defined in this case as:

$$\mathcal{M}_\theta^{\texttt{LPV-ARX}} \begin{cases} A(p_k, q^{-1}, \rho) = 1 + a_1(p_k)q^{-1} + a_2(p_k)q^{-2} \\ B(p_k, q^{-1}, \rho) = b_0(p_k) \\ H(p_k, q, \rho) = A^\dagger(p_k, q^{-1}, \rho) \end{cases}$$

where $A^\dagger$ is the adjoint of A and

$$a_1(p_k) = a_{1,0} + a_{1,1}p_k + a_{1,2}p_k^2, \tag{21a}$$
$$a_2(p_k) = a_{2,0} + a_{2,1}p_k, \tag{21b}$$
$$b_0(p_k) = b_{0,0} + b_{0,1}p_k. \tag{21c}$$

The proposed closed-loop LPV *Refined Instrumental Variable* method (LPV-CLRIV) and the NL-PEM method are based on the LPV-BJ model structure:

$$\mathcal{M}_\theta^{\texttt{LPV-BJ}} \begin{cases} A(p_k, q^{-1}, \rho) = 1 + a_1(p_k)q^{-1} + a_2(p_k)q^{-2} \\ B(p_k, q^{-1}, \rho) = b_0(p_k) \\ H(p_k, q, \eta) = \dfrac{1}{1 + d_1q^{-1} + d_2q^{-2}} \end{cases}$$

with $a_1(p_k)$, $a_2(p_k)$, $b_0(p_k)$, $b_1(p_k)$ as given in (21a-c), and hence represent the situation $\mathcal{S}_o \in \mathcal{M}$.

The robustness of the proposed and existing algorithms are investigated with respect to different *signal-to-noise ratios*: $\text{SNR} = 10 \log \frac{P_{\chi_o}}{P_{v_o}}$, where P_{χ_o} and P_{v_o} are the average power of the signals χ_o and v_o respectively. To provide representative results, a Monte Carlo simulation of $N_{\text{MC}} = 200$ runs with new noise realizations is accomplished at SNR levels: 18dB, 13dB and 8dB.

Table 1 shows the norm of the bias (BN) $||\rho_o - \bar{\mathbb{E}}(\hat{\rho})||_{\ell_2}$ and variance norm (VN) $||\bar{\mathbb{E}}(\hat{\rho} - \bar{\mathbb{E}}(\hat{\rho}))||_{\ell_2}$ of the estimated parameter vector w.r.t. the process part, where $\bar{\mathbb{E}}$ is the mean operator over the Monte Carlo simulation. The table also displays the mean number of iterations (Nit) the algorithms needed to converge to the estimated parameter vector. Table 1 demonstrates that the `LPV-CLRIV` method is unbiased according to the theoretical results, while the LPV-LS method exhibits a considerable bias. For SNR down to 8dB, the LPV-CLRIV achieves a lower variance for $\hat{\rho}$ than the LPV-LS method. Moreover, the achieved variance is close to result of the statistically optimal NL-PEM estimates.

Table 1
Bias and variance results using the full model structure $\mathcal{M}_\theta^{\texttt{LPV-BJ}}$.

Method		18dB	13dB	8dB
LPV-LS	BN	0.0849	0.2158	0.4134
	VN	0.0752	0.1232	0.1876
NL-PEM	BN	40e-04	71e-04	0.0138
	VN	0.0464	0.0819	0.1393
LPV CLRIV	BN	39e-04	62e-04	0.0103
	VN	0.0505	0.0922	0.1638
	Nit	9	12	12

To show the advantage of the proposed IV method w.r.t. NL-PEM approach, consider a more realistic scenario when the model structure of the noise is unknown. Thus, identification of $\mathcal{S}_o$ is considered with the following LPV-OE model structure (corresponding to $\mathcal{G}_o \in \mathcal{G}$, $\mathcal{H}_o \notin \mathcal{H}$):

$$\mathcal{M}_\theta^{\texttt{LPV-OE}} \begin{cases} A(p_k, q^{-1}, \rho) = 1 + a_1(p_k)q^{-1} + a_2(p_k)q^{-2} \\ B(p_k, q^{-1}, \rho) = b_0(p_k)q^{-1} + b_1(p_k)q^{-2} \\ H(p_k, q, \eta) = 1 \end{cases}$$

This model is identified using the proposed simplified closed-loop RIV approach (LPV-CLSRIV) and the NL-PEM method. The results are presented in Table 2. It appears that the NL-PEM is affected by the incorrect structure of the noise model both in terms of bias and variance, while the proposed LPV-CLSRIV method only suffers from a mild variance increase (remaining close to the LPV-LS variance) but remains unbiased. Consequently, this representative example suggests that the proposed method is robust to the modeling error on the noise and displays a better bias/variance trade-off.

Table 2
Bias and variance results using the inaccurate model structure $\mathcal{M}_\theta^{\mathtt{LPV-OE}}$.

Method		18dB	13dB	8dB
LPV-LS	BN	0.0849	0.2158	0.4134
	VN	0.0752	0.1232	0.1876
NL-PEM	BN	0.010	0.027	0.072
	VN	0.0621	0.1069	0.1742
LPV CLSRIV	BN	34e-04	59e-04	0.0134
	VN	0.0612	0.1085	0.2124
	Nit	13	15	18

5 Conclusion

In this paper, a closed-loop LPV-RIV approach is introduced to provide an efficient solution for the closed-loop identification of LPV systems with Box-Jenkins type of noise models. The approach is formulated with the assumption of prior knowledge of the controller but without any restriction on its structure. It is shown that under given conditions the proposed method provides consistent estimates. The performance of the approach is demonstrated on a representative example pointing out that its particular advantage is being robust to noise and modeling errors.

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