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ON L^2 SOLVABILITY OF BVPS FOR ELLIPTIC SYSTEMS

PASCAL AUSCHER, ALAN MCINTOSH, AND MIHALIS MOURGOGLOU

Abstract. In this article we prove solvability results for L^2 boundary value problems of some elliptic systems $Lu = 0$ on the upper half-space $\mathbb{R}^{n+1}_+, n \geq 1$, with transversally independent coefficients. We use the first order formalism introduced by Auscher-Axelsson-McIntosh and further developed with a better understanding of the classes of solutions in the subsequent work of Auscher-Axelsson. The interesting fact is that we prove only half of the Rellich boundary inequality without knowing the other half.

1. Introduction

The goal of this paper is to study Rellich type estimates for some elliptic systems by using the first order formalism introduced in [3].

We begin by recalling the classical Lax-Milgram existence theorems of variational (or energy) solutions for divergence form second order partial differential equations and make clear how we understand them in unbounded domains. It is well-known that both Neumann and regularity problems have unique solutions in the energy class. We reformulate this using the DB formalism in order to represent these solutions via a semi-group. We then obtain factorisations of the Dirichlet to Neumann map and of its inverse in $\dot{H}^{-1/2}$ topologies which are related to (abstract) boundary layer potentials. This part is completely general and true for any such elliptic system.

In the last part, we assume that the coefficients are block triangular (see below for definition). Although such an assumption is rather restrictive, it brings up a phenomenon which we think interesting on its own. Using these factorisations, we prove that, in the triangular situation which corresponds to making the conormal derivative proportional to the transversal derivative, the Neumann to Dirichlet map is bounded in the L^2 topology, that is, solutions (we shall make clear the meaning of solutions) satisfy half of the boundary Rellich estimate

$$\|\nabla_{tan} u\|_2 \leq C\|\partial_A u\|_2$$

which implies that the Neumann problem with L^2 data is solvable. For the adjoint situation, it is the Dirichlet to Neumann map that is bounded on L^2, hence the opposite Rellich inequality for solutions

$$\|\partial_A u\|_2 \leq C\|\nabla_{tan} u\|_2$$

holds so that the regularity problem with L^2 data is solvable. In each case, we do not know about boundedness of the inverse, that is, we do not know the other half of the Rellich estimate (and we do not expect it by any means) which is the alluded to phenomenon: usually Rellich estimates are proved both ways. We also show that

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the Dirichlet problem with L^2 data is solvable in the same block triangular situation as for the Neumann problem.

We remark that the conjunction of the two situations is the block-diagonal case for which

$$\|\nabla_{tan} u\|_2 \approx \|\partial_A u\|_2$$

is known to be equivalent to the Kato square root estimate [4].

As the reader might notice, a feature of our proof is the use of one equivalent formulation of the Kato square root estimate (for an auxiliary operator) which shows a tight connection between Rellich estimates and Tb technology.

In a subsequent paper of two of us (the first and third named author) with S. Hofmann, Neumann solvability results for Hardy space data are obtained in the triangular situations of this paper in the case of systems of such pde's satisfying the De Giorgi regularity condition.

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2. NOTATION

The system of equations is

$$(Lu)^\alpha(t,x) = \sum_{i,j=0}^{n}\sum_{\beta=1}^{m} \partial_i\Big(A_{i,j}^{\alpha,\beta}(x)\partial_j u^\beta(t,x)\Big) = 0, \qquad \alpha = 1,\dots,m \tag{1}$$

in $\mathbb{R}^{1+n}_+ = (0,\infty)\times\mathbb{R}^n$, where $\partial_0 = \frac{\partial}{\partial t}$ and $\partial_i = \frac{\partial}{\partial x_i}$, if $i = 1,\dots,n$. We assume

$$A = (A_{i,j}^{\alpha,\beta}(x))_{i,j=0,\dots,n}^{\alpha,\beta=1,\dots,m} \in L_\infty(\mathbb{R}^n;\mathcal{L}(\mathbb{C}^{2m})), \tag{2}$$

and that A is strictly accretive on the subspace $\mathcal{H}^0$ of $L_2(\mathbb{R}^n;\mathbb{C}^{(1+n)m})$ defined by $(f_j^\alpha)_{j=1,\dots,n}$ is curl free in $\mathbb{R}^n$ for all α, that is there exists $\lambda > 0$ such that for all $f \in \mathcal{H}^0$

$$\sum_{i,j=0}^{n}\sum_{\alpha,\beta=1}^{m}\int_{\mathbb{R}^n} \mathrm{Re}(A_{i,j}^{\alpha,\beta}(x) f_j^\beta(x)\overline{f_i^\alpha(x)})dx \geq \lambda \sum_{i=0}^{n}\sum_{\alpha=1}^{m}\int_{\mathbb{R}^n} |f_i^\alpha(x)|^2 dx. \tag{3}$$

The system (1) is always considered in the sense of distributions with weak solutions, that is, H^1_{loc} solutions in the particular domain under consideration.

We stress that our results are valid for any m but set for notational simplicity $m = 1$ from now on. In this case, the accretivity condition above is equivalent to the usual pointwise accretivity condition

$$\mathrm{Re}\sum_{i,j=0}^{n} A_{i,j}(x)\xi_j\overline{\xi_i} \geq \lambda \sum_{i=0}^{n}|\xi_i|^2.$$

It is convenient to write A in the block form

$$A(x) = \begin{bmatrix} a(x) & b(x) \\ c(x) & d(x)\end{bmatrix}$$

where a is scalar-valued, b, c vector-valued and d $n \times n$ matrix-valued. Call $\mathcal{A}$ the set of 2×2 block matrices A with these properties.

All our estimates in this article depend only on ellipticity constants $\Lambda = \|A\|_\infty$ and the largest λ in the accretivity inequality (3). For non negative quantities a, b, the notation $a \lesssim b$ means $a \leq Cb$ where C is a constant depending on the parameters at hand. The notation $a \approx b$ means both $a \lesssim b$ and $b \lesssim a$.

3. Energy solutions

In this section we recall the usual construction of variational or energy solutions. Although this is fairly classical and mostly follows [3], we need to stress a few points and set up some notation. This uses the homogeneous Sobolev space $\dot{H}^1(\mathbb{R}^{1+n}_+)$ of $u \in L^2_{loc}(\mathbb{R}^{1+n}_+)$ such that $\nabla_{t,x} u \in L^2(\mathbb{R}^{1+n}_+)$, equipped with the semi-norm $\|u\|^2_{\dot{H}^1} := \iint_{\mathbb{R}^{1+n}_+} |\nabla_{t,x} u|^2 \, dtdx$ (note that we could have supposed $u \in \mathcal{D}'(\mathbb{R}^{1+n}_+)$ as it is not hard to check that u can be identified with a $L^2_{loc}(\mathbb{R}^{1+n}_+)$ function when $\nabla u \in L^2$). This is a Banach space modulo constants.

Lemma 3.1. *$C^\infty_0(\overline{\mathbb{R}^{1+n}_+})$ is dense in $\dot{H}^1(\mathbb{R}^{1+n}_+)$. The trace on $\mathbb{R}^n$ is bounded from $\dot{H}^1(\mathbb{R}^{1+n}_+)$ onto the homogoneous Sobolev space $\dot{H}^{1/2}(\mathbb{R}^n)$, which we define as the closure of $C^\infty_0(\mathbb{R}^n)$ for the semi-norm $\left(\int_{\mathbb{R}^n} |\xi||\hat{f}(\xi)|^2 \, d\xi\right)^{1/2}$ (with $\hat{f}$ designating the Fourier transform).*

Proof. Once the density is shown, the trace result is classical. Let $u \in \dot{H}^1(\mathbb{R}^{1+n}_+)$. As for any compact subset K of $\mathbb{R}^n$ and $0 < t_0 < t_1 < \infty$,

$$\int_K |u(t_1, x) - u(t_0, x)|^2 \, dx \leq (t_1 - t_0) \iint_{\mathbb{R}^{1+n}_+} |\partial_t u(t, x)|^2 \, dtdx,$$

u extends to an element $u \in L^2_{loc}(\overline{\mathbb{R}^{1+n}_+})$. By the reflection principle, we can extend u to an element in $\dot{H}^1(\mathbb{R}^{1+n})$ which we still call u. We claim we can find a sequence u_k of $L^2(\mathbb{R}^{1+n})$ functions with compact support that converges to u in $\dot{H}^1(\mathbb{R}^{1+n}_+)$. Indeed, by Mazur's lemma, it suffices to have a sequence with weak convergence, that is such that $\nabla u_k \to \nabla u$ weakly in L^2. It is easy to see using Poincaré inequality, and this is where we need to know *a priori* that $u \in L^2_{loc}$, that $u_k = (u - c_k)\varphi_k$ has this property when c_k is the mean of u on the ball $B(0, 2^{k+1})$ and $\varphi_k(x) = \varphi(2^{-k}x)$ with φ is a smooth function that vanishes outside $B(0, 2)$ and that is 1 on $B(0, 1)$. Finally, it suffices to mollify this sequence to conclude. □

Theorem 3.2. *Given $\ell \in \dot{H}^{-1/2}(\mathbb{R}^n)$, there exists $u \in \dot{H}^1(\mathbb{R}^{1+n}_+)$, unique up to a constant, such that $\iint_{\mathbb{R}^{1+n}_+} A\nabla u \cdot \overline{\nabla \phi} \, dtdx = \langle \ell, \varphi \rangle$, for any $\phi \in \dot{H}^1(\mathbb{R}^{1+n}_+)$ with trace φ. This function u is a weak solution of $Lu = 0$ in $\mathbb{R}^{1+n}_+$. We define the conormal derivative of u at the boundary to be $\partial_{\nu_A} u_{|t=0} = -\ell$ (we use the inward unit normal convention).*

We say that u is the *energy solution of the Neumann problem for $Lu = 0$ with Neumann data $-\ell$*.

Proof. This is just the Lax-Milgram theorem in the Hilbert space $\dot{H}^1(\mathbb{R}^{1+n}_+)/\mathbb{C}$ using that $\dot{H}^{-1/2}(\mathbb{R}^n)$ is the dual space of $\dot{H}^{1/2}(\mathbb{R}^n)$. □

Theorem 3.3. *Given $f \in \dot{H}^{1/2}(\mathbb{R}^n)$, there exists $v \in \dot{H}^1(\mathbb{R}^{1+n}_+)$, unique up to a constant, such that $Lv = 0$ in $\mathbb{R}^{1+n}_+$ and $v_{|t=0} = f$ with equality in $\dot{H}^{1/2}(\mathbb{R}^n)$. Furthermore, there exists $\ell \in \dot{H}^{-1/2}(\mathbb{R}^n)$ such that $\iint_{\mathbb{R}^{1+n}_+} A\nabla u \cdot \overline{\nabla \phi}\, dtdx = \langle \ell, \varphi \rangle$ for any $\varphi \in \dot{H}^{1/2}(\mathbb{R}^n)$ and any extension $\phi \in \dot{H}^1(\mathbb{R}^{1+n}_+)$ of φ.*

We say that v is the *energy solution for the regularity problem $Lv = 0$ with data* $\nabla_x f$ and we have $\partial_{\nu_A} v_{|t=0} = -\ell$.

Proof. Pick an extension $w \in \dot{H}^1(\mathbb{R}^{1+n}_+)$ of f. Let $\dot{H}^1_0(\mathbb{R}^{1+n}_+)$ be the subspace of $\dot{H}^1(\mathbb{R}^{1+n}_+)$ consisting of all u with constant trace on $\mathbb{R}^n$ (alternately this is the closure of $C^\infty_0(\mathbb{R}^{1+n}_+)$ in $\dot{H}^1(\mathbb{R}^{1+n}_+)$). By the Lax-Milgram theorem, there exists a unique $u \in \dot{H}^1_0(\mathbb{R}^{1+n}_+)$ solving

$$\iint_{\mathbb{R}^{1+n}_+} A\nabla u \cdot \overline{\nabla \phi}\, dtdx = -\iint_{\mathbb{R}^{1+n}_+} A\nabla w \cdot \overline{\nabla \phi}\, dtdx$$

for all $\phi \in \dot{H}^1_0(\mathbb{R}^{1+n}_+)$. Then $v = u + w$ is the solution.

Next, the integral $\iint_{\mathbb{R}^{1+n}_+} A\nabla u \cdot \overline{\nabla \phi}\, dtdx$ depends only on the trace modulo constants of $\phi \in \dot{H}^1(\mathbb{R}^{1+n}_+)$. Thus, the map $\varphi \mapsto \iint_{\mathbb{R}^{1+n}_+} A\nabla u \cdot \overline{\nabla \phi}\, dtdx$ is bounded from $\dot{H}^{1/2}(\mathbb{R}^n)$ to $\mathbb{C}$ and this defines ℓ. □

Observe that ∇_x is injective with closed range from $\dot{H}^{1/2}(\mathbb{R}^n)$ into $\dot{H}^{-1/2}(\mathbb{R}^n; \mathbb{C}^n)$. We set $\dot{\mathcal{H}}^{-1/2}_\parallel$ the range of this map and for a reason that will become clear later also set $\dot{\mathcal{H}}^{-1/2}_\perp = \dot{H}^{-1/2}(\mathbb{R}^n)$. By a Fourier transform argument, one sees that $\dot{\mathcal{H}}^{-1/2}_\parallel = \mathcal{R}(\dot{\mathcal{H}}^{-1/2}_\perp)$, where $\mathcal{R} = \nabla(-\Delta)^{-1/2}$ is the array of Riesz transforms on $\mathbb{R}^n$ (the Hilbert transform if $n = 1$) and Δ is the ordinary self-adjoint Laplace operator on $L^2(\mathbb{R}^n)$. With this notation, one defines the Neumann to Dirichlet map

$$\Gamma^A_{ND}\ell = \nabla_x u_{|t=0}, \quad \ell \in \dot{\mathcal{H}}^{-1/2}_\perp \tag{4}$$

where u is the energy solution of the Neumann problem for $Lu = 0$ with Neumann data $-\ell$ and the Dirichlet to Neumann map

$$\Gamma^A_{DN} g = \partial_{\nu_A} v_{|t=0}, \quad g \in \dot{\mathcal{H}}^{-1/2}_\parallel, \tag{5}$$

where v is the energy solution of $Lv = 0$ of the regularity problem with data g.

Theorem 3.4. *Γ^A_{ND} is a bounded and invertible map from $\dot{\mathcal{H}}^{-1/2}_\perp$ onto $\dot{\mathcal{H}}^{-1/2}_\parallel$ with inverse Γ^A_{DN}.*

The proof is an obvious consequence of the above results with our definitions. We have $\Gamma^A_{ND}(\partial_{\nu_A} u_{|t=0}) = \nabla_x u_{|t=0}$ and $\Gamma^A_{DN}(\nabla_x u_{|t=0}) = \partial_{\nu_A} u_{|t=0}$ for any energy solution u in the upper half-space of $Lu = 0$.

We finish this section with a standard result.

Lemma 3.5. *Let $u \in \dot{H}^1(\mathbb{R}^{1+n})$ be a solution of $Lu = 0$ in $\mathbb{R}^{1+n}$. Then $u = 0$ (modulo constants).*

Proof. By definition, we have

$$\iint_{\mathbb{R}^{1+n}} A\nabla u \cdot \overline{\nabla \phi}\, dtdx = 0$$

for all $\phi \in C_0^\infty(\mathbb{R}^{1+n})$, hence for all $\phi \in \dot{H}^1(\mathbb{R}^{1+n})$ by density as in Lemma 3.1. We conclude taking $\phi = u$ and using the accretivity of A. □

4. The first order formalism

Following [3] and [1], we can characterize weak solutions u to the divergence form equation (1), by replacing u by its conormal gradient $\nabla_A u$ as the unknown function. More precisely (1) for u is replaced by (7) for

$$F(t,x) = \nabla_A u(t,x) = \begin{bmatrix} \partial_{\nu_A} u(t,x) \\ \nabla_x u(t,x) \end{bmatrix},$$

where $\partial_{\nu_A} u := (A\nabla_{t,x} u)_\perp$, that is the first component of $A\nabla_{t,x}u$. This is the inward conormal derivative of u for the upper half-space and the outward conormal derivative for the lower half-space. Here we use the notation $v = \begin{bmatrix} v_\perp \\ v_\parallel \end{bmatrix}$ for vectors in $\mathbb{C}^{1+n}$, where $v_\perp \in \mathbb{C}$ is called the scalar part and $v_\parallel \in \mathbb{C}^n$ the tangential part of v. For example, $\partial_t u = (\nabla_{t,x} u)_\perp$ and $\nabla_x u = (\nabla_{t,x} u)_\parallel$.

Proposition 4.1. *The pointwise transformation*

$$A \mapsto \hat{A} := \begin{bmatrix} 1 & 0 \\ c & d \end{bmatrix} \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} a^{-1} & -a^{-1}b \\ ca^{-1} & d - ca^{-1}b \end{bmatrix} \tag{6}$$

is a self-inverse bijective transformation of the set of matrices in $\mathcal{A}$.

For a pair of coefficient matrices $A = \hat{B}$ and $B = \hat{A}$, the pointwise map $\nabla_{t,x} u \mapsto F = \nabla_A u$ gives a one-one correspondence, with inverse $F \mapsto \nabla_{t,x} u = \begin{bmatrix} (BF)_\perp \\ F_\parallel \end{bmatrix}$, between gradients of weak solutions $u \in H^1_{\mathrm{loc}}(\mathbb{R}^{1+n}_+)$ to (1) and solutions $F \in L^2_{\mathrm{loc}}(\mathbb{R}^{1+n}_+; \mathbb{C}^{1+n})$ of the generalized Cauchy–Riemann equations

$$\partial_t F + \begin{bmatrix} 0 & \mathrm{div}_x \\ -\nabla_x & 0 \end{bmatrix} BF = 0, \quad \mathrm{curl}_x F_\parallel = 0, \tag{7}$$

where the derivatives are taken in the $\mathbb{R}^{1+n}_+$ distributional sense.

This transformation was introduced in [3] and the proposition is proved in this generality in [1]. We shortly review the L^2 theory in [1]. Denote by $\mathbf{D}$ the self-adjoint operator on $\mathcal{H} = L^2(\mathbb{R}^n; \mathbb{C}^{1+n})$ defined by

$$\mathbf{D} := \begin{bmatrix} 0 & \mathrm{div}_x \\ -\nabla_x & 0 \end{bmatrix}, \qquad \mathsf{D}(\mathbf{D}) = \begin{bmatrix} \mathsf{D}(\nabla) \\ \mathsf{D}(\mathrm{div}) \end{bmatrix}.$$

The closure of the range of $\mathbf{D}$ is the set of $F \in \mathcal{H}$ such that $\mathrm{curl}_x F_\parallel = 0$, that is $\overline{\mathsf{R}(\mathbf{D})} = \mathcal{H}^0$. It is shown in [7] that the operators $\mathbf{D}B$ and $B\mathbf{D}$ with respective domains $B^{-1}\mathsf{D}(\mathbf{D})$ and $\mathsf{D}(\mathbf{D})$ and ranges $\mathsf{R}(\mathbf{D})$ and $B\mathsf{R}(\mathbf{D})$ are bi-sectorial operators with bounded holomorphic functional calculi on the closure of their range $\mathcal{H}^0$ and $B^{-1}\mathcal{H}^0$. Observe the similarity relation

$$B(\mathbf{D}B) = (B\mathbf{D})B \quad \text{on } \mathsf{D}(\mathbf{D}B) \tag{8}$$

that allows to transfer functional properties between $\mathbf{D}B$ and $B\mathbf{D}$. In particular, if $\mathrm{sgn}(z) = 1$ for $\mathrm{Re}\, z > 0$ and -1 for $\mathrm{Re}\, z < 0$, the operators $\mathrm{sgn}(\mathbf{D}B)$ and $\mathrm{sgn}(B\mathbf{D})$ are well-defined bounded involutions on $\mathcal{H}^0$ and $B^{-1}\mathcal{H}^0$ respectively. One defines the spectral spaces $\mathcal{H}^{0,\pm}_{\mathbf{D}B} = \mathsf{N}(\mathrm{sgn}(\mathbf{D}B) \mp I)$ and $\mathcal{H}^{0,\pm}_{B\mathbf{D}} = \mathsf{N}(\mathrm{sgn}(B\mathbf{D}) \mp I)$. They topologically split $\mathcal{H}^0$ and $B^{-1}\mathcal{H}^0$ respectively. The restriction of $\mathbf{D}B$ to the invariant

space $\mathcal{H}^{0,+}_{\mathbf{D}B}$ is sectorial of type less than $\pi/2$, hence it generates an analytic semi-group $e^{-t\mathbf{D}B}$, $t \geq 0$, on it. Similarly, the restriction of $B\mathbf{D}$ to the invariant space $\mathcal{H}^{0,+}_{B\mathbf{D}}$ is sectorial of type less than $\pi/2$, hence it generates an analytic semi-group $e^{-tB\mathbf{D}}$, $t \geq 0$, on $\mathcal{H}^{0,+}_{B\mathbf{D}}$.

Theorem 4.2. *Let $u \in H^1_{loc}(\mathbb{R}^{1+n}_+)$. Then,*

(1) *u is a weak solution of $Lu = 0$ with $\|\widetilde{N}(\nabla u)\|_2 < \infty$ if and only if there exists $F_0 \in \mathcal{H}^{0,+}_{\mathbf{D}B}$ such that $\nabla_A u = e^{-t\mathbf{D}B}F_0$. Moreover, F_0 is unique and $\|F_0\|_2 \approx \|\widetilde{N}(\nabla u)\|_2$.*

(2) *u is a weak solution of $Lu = 0$ with $\iint_{\mathbb{R}^{1+n}_+} t|\nabla_{t,x}u|^2 dtdx < \infty$ if and only if there exists $\widetilde{F}_0 \in \mathcal{H}^{0,+}_{B\mathbf{D}}$ such that $\nabla_A u = \mathbf{D}e^{-tB\mathbf{D}}\widetilde{F}_0$. Moreover, $\widetilde{F}_0$ is unique, $\|\widetilde{F}_0\|_2 \approx (\iint_{\mathbb{R}^{1+n}_+} t|\nabla_{t,x}u|^2 dtdx)^{1/2}$ and u is given by $u = -(e^{-tB\mathbf{D}}\widetilde{F}_0)_\perp + c$ for some constant $c \in \mathbb{C}$.*

The if part was obtained in [3] and the only if part in [1, Theorems 8.2 and 9.2]. Here $\widetilde{N}(g)$ is the Kenig-Pipher modified non-tangential function $\|\widetilde{N}(\nabla u)\|_2$, where

$$\widetilde{N}(g)(x) := \sup_{t>0} t^{-(1+n)/2}\|f\|_{L_2(W(t,x))}, \qquad x \in \mathbb{R}^n,$$

with $W(t,x) := (c_0^{-1}t, c_0 t) \times B(x; c_1 t)$, for some fixed constants $c_0 > 1$, $c_1 > 0$.

Remark 4.3. Although we do not need that, the same proof shows when coefficients are t-independent that for the equivalence of (1) to hold one could replace $\|\widetilde{N}(\nabla u)\|_2$ by the weaker condition $\sup_{t>0}(\frac{1}{t}\int_t^{2t}\|\nabla_{s,x}u\|_2^2\,ds)^{1/2}$ or by $\sup_{t>0}\|\nabla_{t,x}u\|_2$ or even by the square function $(\iint_{\mathbb{R}^{1+n}_+} t|\partial_t\nabla_{t,x}u|^2 dtdx)^{1/2}$, so that in the end all these quantities are *a priori* equivalent for weak solutions.

In (2), the function $\widetilde{F}_0$ is formally built as $\mathbf{D}\widetilde{F}_0 = \nabla_A u_{|t=0}$ (which belongs to an adapted Sobolev space of order -1: we shall make this more precise).

The spectral spaces with negative signs correspond to estimates for solutions to $Lu = 0$ in the lower-half space, and the similar statement holds using the semigroups $e^{t\mathbf{D}B}$ and $e^{tB\mathbf{D}}$.

Our aim is to extend this formalism to Sobolev spaces. However, a difficulty is that $\mathsf{R}(B\mathbf{D})$ is really an L^2 object as it depends on B. We shall modify the setup to prepare this extension. Recall that $\mathcal{H}^0 = \overline{\mathsf{R}(\mathbf{D})}$ is a closed subspace of $\mathcal{H} = L^2(\mathbb{R}^n; \mathbb{C}^{1+n})$. Let $S = \mathbf{D}_{|\mathcal{H}^0}$ with domain $\mathsf{D}(\mathbf{D}) \cap \mathcal{H}^0$. Then S is a one-one, self-adjoint operator. Define Π as the orthogonal projection from $\mathcal{H}$ onto $\mathcal{H}^0$. It is the identity if $n = 1$ but not otherwise. Let $\mathcal{B}$ be the operator on $\mathcal{H}^0$ defined by $\mathcal{B}u = \Pi Bu = \Pi B\Pi u, u \in \mathcal{H}^0$. As B is a strictly accretive operator on $\mathcal{H}^0$ (for equations this is true on $\mathcal{H}$ but only on $\mathcal{H}^0$ for systems, that is, when $m > 1$), the restriction of Π on $B\mathcal{H}^0$ is an isomorphism onto $\mathcal{H}^0$ and $\mathcal{B}$ is a strictly accretive operator on $\mathcal{H}^0$.

Define

$$T : \mathcal{H}^0 \to \mathcal{H}^0, \quad T = \mathcal{B}S = \Pi B\mathbf{D}_{|\mathcal{H}^0} \text{ with } \mathsf{D}(T) = \mathsf{D}(S)$$

and

$$\underline{T} : \mathcal{H}^0 \to \mathcal{H}^0, \quad \underline{T} = S\mathcal{B} = \mathbf{D}\Pi B_{|\mathcal{H}^0} = \mathbf{D}B_{|\mathcal{H}^0} \text{ with } \mathsf{D}(\underline{T}) = \mathcal{B}^{-1}\mathsf{D}(S).$$

Theorem 4.4. (1) *$\underline{T}$ is a one-one bi-sectorial operator with bounded holomorphic functional calculus on $\mathcal{H}^0$. A function $u \in H^1_{loc}(\mathbb{R}^{1+n}_+)$ is a weak solution of $Lu = 0$ with $\|\widetilde{N}(\nabla u)\|_2 < \infty$ if and only if there exists $H_0 \in \mathcal{H}^{0,+}_{\underline{T}}$ such that $\nabla_A u = e^{-t\underline{T}} H_0$. Moreover, H_0 is unique and $\|H_0\|_2 \approx \|\widetilde{N}(\nabla u)\|_2$.*

(2) *T is a one-one bi-sectorial operator with bounded holomorphic functional calculus on $\mathcal{H}^0$. A function $u \in H^1_{loc}(\mathbb{R}^{1+n}_+)$ is a weak solution of $Lu = 0$ with $\iint_{\mathbb{R}^{1+n}_+} t|\nabla_{t,x} u|^2 dtdx < \infty$ if and only if there exists $\widetilde{H}_0 \in \mathcal{H}^{0,+}_T$ such that $\nabla_A u = Se^{-tT}\widetilde{H}_0$. Moreover, $\widetilde{H}_0$ is unique, $\|\widetilde{H}_0\|_2 \sim (\iint_{\mathbb{R}^{1+n}_+} t|\nabla_{t,x} u|^2 dtdx)^{1/2}$ and $u = -(e^{-tT}\widetilde{H}_0)_\perp + c$ for some constant c.*

Proof. We begin with the first point. As $\underline{T}$ is the restriction of $\mathbf{D}B$ to $\mathcal{H}^0$, it is a one-one bi-sectorial operator with bounded holomorphic functional calculus. In particular, the spectral spaces $\mathcal{H}^{0,\pm}_{\underline{T}}$ coincide with $\mathcal{H}^{0,\pm}_{\mathbf{D}B}$. The function H_0 is nothing but F_0 in Theorem 4.2, (1). This proves the first point.

Let us turn to the second point. That T is one-one, bi-sectorial with bounded holomorphic functional calculus on $\mathcal{H}^0$ follows from the similarity relation $\mathcal{B}\underline{T} = T\mathcal{B}$. To obtain the new equations for u is not as direct. Denote by $U : B\mathcal{H}^0 \to \mathcal{H}^0$ the restriction of Π on $B\mathcal{H}^0$. For $G \in \mathcal{H}^0$, we have $\Pi(U^{-1}G - G) = 0$ so $U^{-1}G \in \mathsf{D}(\mathbf{D})$ if and only if $G \in \mathsf{D}(\mathbf{D})$. A calculation shows that $TG = \Pi B\mathbf{D}G = UB\mathbf{D}U^{-1}G$. So T is also similar to the restriction of $B\mathbf{D}$ on $B\mathcal{H}^0$. Next, the equation $\widetilde{F} = e^{-tB\mathbf{D}}\widetilde{F}_0$ with $\widetilde{F}_0 \in \mathcal{H}^{0,+}_{B\mathbf{D}}$ is equivalent to $U\widetilde{F} = e^{-tT}U\widetilde{F}_0$ with $U\widetilde{F}_0 \in \mathcal{H}^{0,+}_T$. Hence,

$$Se^{-tT}U\widetilde{F}_0 = \mathbf{D}Ue^{-tB\mathbf{D}}\widetilde{F}_0 = \mathbf{D}e^{-tB\mathbf{D}}\widetilde{F}_0$$

and

$$(e^{-tT}U\widetilde{F}_0)_\perp = (Ue^{-tB\mathbf{D}}\widetilde{F}_0)_\perp = (e^{-tB\mathbf{D}}\widetilde{F}_0)_\perp$$

as U leaves the scalar component invariant. This concludes the proof with $\widetilde{H}_0 = U\widetilde{F}_0 = \Pi\widetilde{F}_0$, where $\widetilde{F}_0$ is the function specified in Theorem 4.2, (2). □

The point is that both operators T and $\underline{T}$ act on the same space $\mathcal{H}^0$ which is independent of the action of B. Thus, we use the relation between quadratic estimates and interpolation developed in [5, Section 8] for bi-sectorial operators and we strongly suggest the reader to have this reference handy from now on.

For $s \in \mathbb{R}$ define $\dot{\mathcal{H}}^s$ as the completion of $\mathcal{H}^0$ for the quadratic norm

$$\|F\|_{S,s} = \left\{ \int_0^\infty t^{-2s} \|\psi_t(S)F\|_2^2 \frac{dt}{t} \right\}^{1/2}$$

where ψ is a suitable holomorphic function on bi-sectors, for example $\psi(z) = z^k e^{-z\,\mathrm{sgn}(z)}$, $\mathrm{Re}\, z \neq 0$ and $\mathbb{N} \ni k > \max(s, 0)$. Remark that from the spectral theorem $\dot{\mathcal{H}}^0 = \mathcal{H}^0$ and it can be checked that $\|F\|_{S,s} = c_{\psi,s}\||S|^s F\|_2$ where $|S| = (S^2)^{1/2}$. Note that S extends to an isomorphism between $\dot{\mathcal{H}}^s$ and $\dot{\mathcal{H}}^{s-1}$. Classically, the intersection of $\dot{\mathcal{H}}^s$ for s in a bounded interval is dense in each of them.

We define similarly for $\dot{\mathcal{H}}^s_T$ and $\dot{\mathcal{H}}^s_{\underline{T}}$ replacing S by T and $\underline{T}$. The quadratic norms are equivalent under changes of suitable ψ that are non degenerate on both components of the bi-sectors.

Note that $|S|$ preserves the normal and tangential components so we can write $\dot{\mathcal{H}}^s = \dot{\mathcal{H}}^s_\perp \oplus \dot{\mathcal{H}}^s_\parallel$ which agrees with our earlier notation when $s = 0$ and $s = -1/2$.

If $n = 1$, we have $S = \mathbf{D} = \begin{bmatrix} 0 & d_x \\ -d_x & 0 \end{bmatrix}$ so the quadratic norm defines the usual homogeneous Sobolev space $\dot{H}^s(\mathbb{R};\mathbb{C}^2)$ which is also the completion of $\mathsf{D}(|S|^s)$ for the homogeneous norm $\||S|^s F\|_2$. It follows that $\dot{\mathcal{H}}^s_\perp = \dot{\mathcal{H}}^s_\parallel = \dot{H}^s(\mathbb{R})$.

For $n \geq 2$, recall that $\mathcal{R} = \nabla(-\Delta)^{-1/2}$ denotes the array of Riesz transforms and let $\mathcal{R}^* = -(-\Delta)^{-1/2}\mathrm{div}$ be its adjoint. The operator

$$V = \begin{bmatrix} I & 0 \\ 0 & -\mathcal{R} \end{bmatrix} : L^2(\mathbb{R}^n;\mathbb{C}^2) \to \mathcal{H}^0$$

is an isometry with inverse

$$V^{-1} = V^* = \begin{bmatrix} I & 0 \\ 0 & -\mathcal{R}^* \end{bmatrix} : \mathcal{H}^0 \to L^2(\mathbb{R}^n;\mathbb{C}^2)$$

and $VV^* = \Pi$. A calculation using $\nabla = \mathcal{R}(-\Delta)^{1/2}$ shows that

$$V^{-1}SV = \begin{bmatrix} 0 & (-\Delta)^{1/2} \\ (-\Delta)^{1/2} & 0 \end{bmatrix}.$$

Thus, $\dot{\mathcal{H}}^s_\perp = \dot{H}^s(\mathbb{R}^n)$ and $\dot{\mathcal{H}}^s_\parallel = \mathcal{R}\dot{H}^s(\mathbb{R}^n)$ where $\dot{H}^s(\mathbb{R}^n)$ is the usual homogeneous Sobolev space with semi-norm $\|(-\Delta)^{s/2}f\|_2$.

Proposition 4.5. (1) *For $s \in \mathbb{R}$, for all bounded holomorphic functions b in appropriate bi-sectors, $b(T)$ extends to a bounded operator on $\dot{\mathcal{H}}^s_T$. In particular, this holds for $\mathrm{sgn}(T)$ which is a bounded self-inverse operator on $\dot{\mathcal{H}}^s_T$, and T and $|T| = \mathrm{sgn}(T)T$ extend to isomorphisms between $\dot{\mathcal{H}}^s_T$ and $\dot{\mathcal{H}}^{s-1}_T$. The operator $|T|$ extends to a sectorial operator on $\dot{\mathcal{H}}^s_T$.*

(2) *$\dot{\mathcal{H}}^s$ topologically splits as the sum of the two spectral closed subspaces $\dot{\mathcal{H}}^{s,+}_T = \mathsf{N}(\mathrm{sgn}(T) - I)$ and $\dot{\mathcal{H}}^{s,-}_T = \mathsf{N}(\mathrm{sgn}(T) + I)$.*

(3) *The same two items hold with T replaced by $\underline{T}$.*

(4) *For $0 \leq s \leq 1$, $\dot{\mathcal{H}}^s_T = \dot{\mathcal{H}}^s$ and for $-1 \leq s \leq 0$, $\dot{\mathcal{H}}^s_{\underline{T}} = \dot{\mathcal{H}}^s$ with equivalence of norms.*

(5) *Furthermore, for $-1 \leq s < 0$, we have for $\|F\|_{S,s} \approx \left\{\int_0^\infty t^{-2s}\|e^{-t|\underline{T}|}F\|_2^2 \frac{dt}{t}\right\}^{1/2}$.*

Proof. The first four items are contained in [5, Theorem 8.3] and the previous sections therein with the exception of the cases $s = 1$ and $s = -1$ of (4).

For $s = 1$, observe that $\|F\|_{T,1} = c\||T|F\|$. Thus, the bounded holomorphic functional calculus of T on $\mathcal{H}^0$ implies that $\||T|F\| \approx \|TF\|$ [5, Theorem 8.2]. Finally $\|TF\| \approx \|SF\| \approx \|F\|_{S,1}$.

For $s = -1$, we observe the intertwining relation $\underline{T}S = ST$ (as unbounded operators) so the functional calculi also intertwine. Hence, $\psi_t(\underline{T})SF = S\psi_t(T)F = t^{-1}\mathcal{B}^{-1}\tilde{\psi}_t(T)F$ for $F \in \mathsf{D}(T) = \mathsf{D}(S)$ with $\tilde{\psi}(z) = z\psi(z)$. This implies that $\|SF\|_{\underline{T},-1} \approx \|F\|_{T,0} \approx \|F\|_{S,0} \approx \|SF\|_{S,-1}$ and the result follows from the fact that the functions SF, $F \in \mathsf{D}(T) = \mathsf{D}(S)$, form a dense subset of $\dot{\mathcal{H}}^{-1}$.

To prove (5), we proceed as above and a calculation with $\psi(z) = ze^{-z\,\mathrm{sgn}(z)}$ shows that $e^{-t|\underline{T}|}SF = Se^{-t|T|}F = t^{-1}\mathcal{B}^{-1}\psi_t(T)F$ for $F \in \mathsf{D}(T)$. It follows easily for appropriate F and using $0 \leq s + 1 < 1$ that

$$\left\{\int_0^\infty t^{-2s}\|e^{-t|\underline{T}|}F\|_2^2 \frac{dt}{t}\right\}^{1/2} \approx \|S^{-1}F\|_{T,s+1} \approx \|S^{-1}F\|_{S,s+1} \approx \|F\|_{S,s}.$$

□

We can give another useful characterization of solutions with square function estimates in terms of the negative Sobolev space.

Corollary 4.6. *Let $u \in H^1_{loc}(\mathbb{R}^{1+n}_+)$. Then u is a weak solution of $Lu = 0$ with $\iint_{\mathbb{R}^{1+n}_+} t|\nabla_{t,x}u|^2 dtdx < \infty$ if and only if there exists $H_0 \in \dot{\mathcal{H}}^{-1,+}_{\underline{T}}$ such that $\nabla_A u = e^{-t\underline{T}}H_0$. Moreover, H_0 is unique and $\|H_0\|_{\underline{T},-1} \approx (\iint_{\mathbb{R}^{1+n}_+} t|\nabla_{t,x}u|^2 dtdx)^{1/2}$.*

Proof. This is a reformulation of the previous results. By Theorem 4.4, (2), given u, we have $\nabla_A u = Se^{-tT}\widetilde{H}_0 = e^{-t\underline{T}}S\widetilde{H}_0$, where $e^{-t\underline{T}}$ is now the semi-group extended to $\dot{\mathcal{H}}^{-1,+}_{\underline{T}}$. Setting $H_0 = S\widetilde{H}_0$, we have $H_0 \in \dot{\mathcal{H}}^{-1,+}_{\underline{T}}$ as $\widetilde{H}_0 \in \mathcal{H}^{0,+}_T$ and $\|H_0\|_{\underline{T},-1} \approx \|H_0\|_{S,-1} \approx \|\widetilde{H}_0\|_2$ by Proposition 4.5, (4).

Conversely, let $H_0 \in \dot{\mathcal{H}}^{-1,+}_{\underline{T}}$ and set $\widetilde{H}_0 = S^{-1}H_0 \in \dot{\mathcal{H}}^{0,+}_T$. Then $e^{-t\underline{T}}H_0 = e^{-t\underline{T}}S\widetilde{H}_0 = Se^{-tT}\widetilde{H}_0$ is the conormal gradient of a solution u by Theorem 4.4, (2). □

Now, we prove a representation for energy solutions

Proposition 4.7. *Let $u \in H^1_{loc}(\mathbb{R}^{1+n}_+)$. Then u is a weak solution of $Lu = 0$ with $\iint_{\mathbb{R}^{1+n}_+} |\nabla_{t,x}u|^2 dtdx < \infty$ (i.e., u is an energy solution) if and only if there exists $H_0 \in \dot{\mathcal{H}}^{-1/2,+}_{\underline{T}}$ such that $\nabla_A u = e^{-t\underline{T}}H_0$. Moreover, H_0 is unique and $\|H_0\|_{\underline{T},-1/2} \approx (\iint_{\mathbb{R}^{1+n}_+} |\nabla_{t,x}u|^2 dtdx)^{1/2}$.*

Proof. Let us prove the converse first. If $H_0 \in \dot{\mathcal{H}}^{-1/2,+}_{\underline{T}}$, then $H_\varepsilon = e^{-\varepsilon\underline{T}}H_0 \in \dot{\mathcal{H}}^{-1/2,+}_{\underline{T}} \cap \dot{\mathcal{H}}^{1/2,+}_{\underline{T}} \subset \dot{\mathcal{H}}^{0,+}_{\underline{T}}$ for any $\varepsilon > 0$. By Theorem 4.4, $e^{-t\underline{T}}H_\varepsilon$ is the conormal gradient of some solution u_ε. As H_ε converges to H_0 in $\dot{\mathcal{H}}^{-1/2,+}_{\underline{T}}$, it is easy to conclude that u_ε converges to a solution u is the energy space and $(\iint_{\mathbb{R}^{1+n}_+} |\nabla_{t,x}u|^2 dtdx)^{1/2} \approx \|H_0\|_{\underline{T},-1/2}$.

Assume now that u is a weak solution of $Lu = 0$ with $\iint_{\mathbb{R}^{1+n}_+} |\nabla_{t,x}u|^2 dtdx < \infty$. Set $H_0 = \nabla_A u_{|t=0} \in \dot{\mathcal{H}}^{-1/2}$. Decomposing $H_0 = H_0^+ + H_0^-$ according to the spectral decomposition $\dot{\mathcal{H}}^{-1/2} = \dot{\mathcal{H}}^{-1/2,+}_{\underline{T}} \oplus \dot{\mathcal{H}}^{-1/2,-}_{\underline{T}}$ from Proposition 4.5 (2) and (4), and using the implication just proved, H_0^+ is the trace of the conormal gradient of an energy solution u^+ in the upper half-space and, by the same result in the lower half-space, H_0^- is the trace of the conormal gradient of an energy solution u^- in the lower half-space. It is then easy to see that the function v, defined by $v = u - u^+$ on the upper half-space and $v = u^-$ in the lower half-space, is an energy solution of $Lv = 0$ in $\mathbb{R}^{1+n}$. By Lemma 3.5, $v = 0$ (modulo constants) so that $u = u^+$ and $H_0 = H_0^+$. Thus, $\nabla_A u = \nabla_A u^+ = e^{-t\underline{T}}H_0^+$ from the first part of the proof. □

Corollary 4.8. *The elements of $\dot{\mathcal{H}}^{-1/2,+}_{\underline{T}}$ are $\begin{bmatrix} f \\ \Gamma^A_{ND}f \end{bmatrix}$, $f \in \dot{\mathcal{H}}^{-1/2}_\perp$. They can also be written $\begin{bmatrix} \Gamma^A_{DN}g \\ g \end{bmatrix}$, $g \in \dot{\mathcal{H}}^{-1/2}_\parallel$.*

Proof. It suffices to check the first representation by Theorem 3.4. By the previous result, $H_0 \in \dot{\mathcal{H}}^{-1/2,+}_{\underline{T}}$ if and only if $H_0 = \nabla_A u_{|t=0}$ for some solution u in the energy space. So if f is the conormal derivative $\partial_A u_{|t=0}$ in $\dot{\mathcal{H}}^{-1/2}_\perp$ then the tangential gradient at $t = 0$ is $\Gamma^A_{ND}f$ by (5).

Conversely, if $f \in \dot{\mathcal{H}}_{\perp}^{-1/2}$, then we let u be the energy solution to $Lu = 0$ with Neumann datum f. We know $\Gamma_{ND}^{A} f = \nabla_x u_{|t=0}$ and that $\nabla_A u = e^{-t\underline{T}} H_0$ for some $H_0 \in \dot{\mathcal{H}}_{\underline{T}}^{-1/2,+}$. It follows that $\begin{bmatrix} f \\ \Gamma_{ND}^{A} f \end{bmatrix} = H_0 \in \dot{\mathcal{H}}_{\underline{T}}^{-1/2,+}$. □

Remark 4.9. It is interesting to compare Theorem 4.4, Corollary 4.6 and Proposition 4.7. The use of $\underline{T}$ allows the same representation for conormal gradients of solutions and the only difference is the space to which the trace belongs. Solving Neumann, Dirichlet and/or regularity problems amount to proving boundedness of Neumann to Dirichlet map and/or the Dirichlet to Neumann map with different topologies on the boundary.

Remark 4.10. In [3, section 5], some *a posteriori* identification of the solutions coming from the DB formalism in L^2 is made with energy solutions, upon some further assumptions. But note that energy solutions require a different trace space. Here, the extension of this formalism to Sobolev spaces allows to prove *a priori* representation (Proposition 4.7) for energy solutions.

Remark 4.11. We note that Proposition 4.7 was stated without proof for systems on the ball in [2], even with some radially dependent coefficients. While this article was in its final stage, we learned that this kind of representations of solutions with data in Sobolev spaces was pursued in more generality by Andreas Rosén [12], and has some overlap with ours.

5. The operator theoretic lemma

The main ingredient in our proof is the next lemma which will provide a factorisation of the boundary maps with simpler operators to analyze. The following lemma is essentially taken from [6, Section 6].

Lemma 5.1. *Let X_1, X_2 be Banach spaces and let $\mathcal{Z} = X_1 \oplus X_2$ whose elements are written as $\begin{bmatrix} u_\perp \\ u_\parallel \end{bmatrix}$. Let $S = \begin{bmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{bmatrix} \in \mathcal{L}(\mathcal{Z})$ such that*

(1) $S^2 = I_{\mathcal{Z}}$.
(2) *There exists $c \in (0,1)$ such that for all $u \in N(S \pm I)$, one has*

$$c^{-1}\|u_\perp\|_{X_1} \leq \|u_\parallel\|_{X_2} \leq c\|u_\perp\|_{X_1}.$$

Then, $s_{12}, s_{21}, s_{11} \pm I, s_{22} \pm I$ are one-one with closed range in the respective $\mathcal{L}(X_i, X_j)$.

Proof. Let $P^\pm = \frac{1}{2}(I \pm S)$ be the pair of projectors on $\mathcal{Z}$ associated to S by (1) and $Q_\pm$ the pair of projectors on $X_1 \oplus \{0\}$ and $\{0\} \oplus X_2$. The assumption (2) implies

$$\|Q_+ P^\pm u\| \approx \|Q_- P^\pm u\| \approx \|P^\pm u\|, \quad u \in \mathcal{Z}.$$

We infer that

$$\|P^+ Q_\pm u\| \approx \|P^- Q_\pm u\| \approx \|Q_\pm u\|.$$

For example, using $Q_- Q_+ = 0$, then $\|Q_+ u\| \leq \|P^+ Q_+ u\| + \|P^- Q_+ u\| \lesssim \|P^+ Q_+ u\| + \|Q_- P^- Q_+ u\| \lesssim \|P^+ Q_+ u\| + \|Q_- P^+ Q_+ u\| \lesssim \|P^+ Q_+ u\| \lesssim \|Q_+ u\|$.

Going further, one easily sees that

$$\|Q_- P^+ Q_\pm u\| \approx \|Q_+ P^+ Q_\pm u\|| \approx \|Q_\pm u\| \approx \|Q_- P^- Q_\pm u\| \approx \|Q_+ P^- Q_\pm u\|.$$

Thus, let $\phi \in X_2$ and $u = \begin{bmatrix} 0 \\ \phi \end{bmatrix} = Q_- u$. One checks that

$$\begin{bmatrix} s_{12}\phi \\ 0 \end{bmatrix} = Q_+ S u = Q_+(2P^+ - I)u = 2Q_+ P^+ Q_- u$$

and

$$\begin{bmatrix} 0 \\ (I \pm s_{22})\phi \end{bmatrix} = 2Q_- P^\pm u = 2Q_- P^+ Q_- u,$$

hence

$$\|s_{12}\phi\|_{X_1} \approx \|(I \pm s_{22})\phi\|_{X_2} \approx \|\phi\|_{X_2}.$$

Similarly, with $\phi \in X_1$ and $u = \begin{bmatrix} \phi \\ 0 \end{bmatrix} = Q_+ u$, one obtains

$$\|s_{21}\phi\|_{X_2} \approx \|(I \pm s_{11})\phi\|_{X_1} \approx \|\phi\|_{X_1}.$$

□

Lemma 5.2. (1) *The operator* $\operatorname{sgn}(\underline{T})$ *satisfies the hypotheses of the above lemma on* $\dot{\mathcal{H}}^{-1/2}_{\underline{T}} = \dot{\mathcal{H}}^{-1/2} = \dot{\mathcal{H}}^{-1/2}_\perp \oplus \dot{\mathcal{H}}^{-1/2}_\parallel$. *Moreover, the operators* $s_{12}(\underline{T})$, $s_{21}(\underline{T})$, $s_{11}(\underline{T}) \pm I$, $s_{22}(\underline{T}) \pm I$ *are invertible.*

(2) *The same conclusion holds replacing* $\underline{T}$ *by* T *on* $\dot{\mathcal{H}}^{1/2}_T = \dot{\mathcal{H}}^{1/2} = \dot{\mathcal{H}}^{1/2}_\perp \oplus \dot{\mathcal{H}}^{1/2}_\parallel$.

Proof. We prove (1). Recall first that $\dot{\mathcal{H}}^{-1/2}_{\underline{T}} = \dot{\mathcal{H}}^{-1/2} = \dot{\mathcal{H}}^{-1/2}_\perp \oplus \dot{\mathcal{H}}^{-1/2}_\parallel$ follows from Proposition 4.5, (2) and (4). Next, equality $\operatorname{sgn}(\underline{T})^2 = I$ on $\dot{\mathcal{H}}^{-1/2}_{\underline{T}}$ holds by the bounded holomorphic functional calculus and $\operatorname{sgn}^2(z) = 1$ on $\mathbb{C} \setminus \mathbb{R}$. Recall that $\mathsf{N}(\operatorname{sgn}(\underline{T}) - I) = \mathcal{H}^{-1/2,+}_{\underline{T}}$ and by Corollary 4.8 and setting $\Gamma^+(\underline{T}) = \Gamma^A_{ND} : \dot{\mathcal{H}}^{-1/2}_\perp \to \dot{\mathcal{H}}^{-1/2}_\parallel$, one has for any $u \in \dot{\mathcal{H}}^{-1/2}_{\underline{T}}$

$$u_\parallel = \Gamma^+(\underline{T}) u_\perp \iff \begin{bmatrix} u_\perp \\ u_\parallel \end{bmatrix} \in \mathsf{N}(\operatorname{sgn}(\underline{T}) - I) = \mathcal{H}^{-1/2,+}_{\underline{T}}.$$

Similarly, the Neumann to Dirichlet map for the lower half-space yields the corresponding operator $\Gamma^-(\underline{T})$ characterized by for any $u \in \dot{\mathcal{H}}^{-1/2}_{\underline{T}}$

$$u_\parallel = \Gamma^-(\underline{T}) u_\perp \iff \begin{bmatrix} u_\perp \\ u_\parallel \end{bmatrix} \in \mathsf{N}(\operatorname{sgn}(\underline{T}) + I) = \mathcal{H}^{-1/2,-}_{\underline{T}}.$$

Details are left to the reader. Thus, the second assumption of Lemma 5.1 is granted and we have lower bounds for all six operators in the statement. Thus, when $A = Id$, $\underline{T} = S$ and so $\operatorname{sgn}(\underline{T}) = \operatorname{sgn}(S) = \begin{bmatrix} 0 & H \\ H^* & 0 \end{bmatrix}$ if $n = 1$ where H is the Hilbert transform and $\operatorname{sgn}(\underline{T}) = \operatorname{sgn}(S) = \begin{bmatrix} 0 & -\mathcal{R}^* \\ -\mathcal{R} & 0 \end{bmatrix}$ on $\mathcal{H}^0$ if $n \geq 2$. Thus, the six operators are invertible in this case and in particular onto. The ontoness for general A follows from the method of continuity. This proves (1).

To prove (2), we observe that the intertwining relation $S\operatorname{sgn}(T) = \operatorname{sgn}(\underline{T})S$ extends to $\dot{\mathcal{H}}^{1/2}_T = \dot{\mathcal{H}}^{1/2}$ using that S is an isomorphism from $\dot{\mathcal{H}}^{1/2}$ onto $\dot{\mathcal{H}}^{-1/2}$. Thus, the conclusion follows straightforwardly from (1). □

Lemma 5.3. *In* $\mathcal{L}(\dot{\mathcal{H}}^{-1/2}_\perp, \dot{\mathcal{H}}^{-1/2}_\parallel)$,

$$\Gamma^A_{ND} = s_{12}(\underline{T})^{-1}(I - s_{11}(\underline{T})) = (I - s_{22}(\underline{T}))^{-1} s_{21}(\underline{T}).$$

Proof. We have seen that the operators $\Gamma^{\pm}(\underline{T})$ are bounded and invertible. To obtain the formulas, one uses their operator defining relations

$$\begin{bmatrix} s_{11}(\underline{T}) & s_{12}(\underline{T}) \\ s_{21}(\underline{T}) & s_{22}(\underline{T}) \end{bmatrix} \begin{bmatrix} I \\ \Gamma^{\pm}(\underline{T}) \end{bmatrix} = \pm \begin{bmatrix} I \\ \Gamma^{\pm}(\underline{T}) \end{bmatrix},$$

and then solve for $\Gamma^{\pm}(\underline{T})$ using the invertibilities of the operators in Lemma 5.2. We conclude using $\Gamma^A_{ND} = \Gamma^+(\underline{T})$. (As mentioned, the operator $\Gamma^-(\underline{T})$ is the Neumann to Dirichlet operator for the lower half-space and has similar representations $\Gamma^-(\underline{T}) = -s_{12}(\underline{T})^{-1}(I + s_{11}(\underline{T})) = -(I + s_{22}(\underline{T}))^{-1}s_{21}(\underline{T})$.) □

6. BLOCK TRIANGULAR MATRICES

So far, everything holds for arbitrary (t independent) coefficients and can be used in full generality. Assume that A is block lower-triangular: $A(x) = \begin{bmatrix} a(x) & 0 \\ c(x) & d(x) \end{bmatrix}$. In the PDE language for L, this means that the conormal derivative is proportional to the t derivative: $\partial_{\nu_A} = a\partial_t$. If $B = \hat{A}$, as in the second section, then this is equivalent to B being block lower-triangular as well.

Let us write $B(x) = \begin{bmatrix} a'(x) & 0 \\ c'(x) & d'(x) \end{bmatrix}$, which we identify with the operator of multiplication by B. Then $\mathcal{B} = \Pi B \Pi$ also has the same structure $\begin{bmatrix} \alpha & 0 \\ \gamma & \delta \end{bmatrix}$, where α, γ, δ are the following operators: if $n = 1$, they are the operators of multiplication by a', c', d' respectively. If $n \geq 2$, then $\alpha = a'$, $\gamma = \mathcal{R}\mathcal{R}^* c'$ and $\delta = \mathcal{R}\mathcal{R}^* d' \mathcal{R}\mathcal{R}^*$.

Lemma 6.1. *If the coefficients are block lower-triangular then $s_{12}(\underline{T})$ is invertible from $\mathcal{H}^0_{\parallel}$ onto $\mathcal{H}^0_{\perp}$.*

Proof. We write the proof when $n \geq 2$. The proof when $n = 1$ is similar and simpler as we do not have to consider the Riesz transforms. By Lemma 5.2, $s_{12}(\underline{T})$ is invertible from $\dot{\mathcal{H}}^{-1/2}_{\parallel}$ onto $\dot{\mathcal{H}}^{-1/2}_{\perp}$. By the characterization of $\dot{\mathcal{H}}^{-1/2}_{\parallel}$ using $\mathcal{R}$, this is equivalent to $s_{12}(\underline{T})\mathcal{R}$ invertible from $\dot{H}^{-1/2}(\mathbb{R}^n)$ onto itself. Remark that the same holds for $s_{12}(T)\mathcal{R}$ from $\dot{H}^{1/2}(\mathbb{R}^n)$ onto itself. Thus, looking at the upper off-diagonal coefficient in the relation $\mathcal{B}\mathrm{sgn}(\underline{T})\Pi = \mathrm{sgn}(T)\mathcal{B}$ between bounded operators for the L^2 topology, we find

$$\alpha s_{12}(\underline{T})\mathcal{R}\mathcal{R}^* = s_{12}(T)\delta.$$

Hence we have the equality in $\mathcal{L}(L^2(\mathbb{R}^n))$

$$s_{12}(\underline{T})\mathcal{R} = a'^{-1}(s_{12}(T)\mathcal{R})(\mathcal{R}^* d' \mathcal{R})$$

and this implies that $s_{12}(\underline{T})\mathcal{R}$ extends to an invertible operator from $(\mathcal{R}^* d' \mathcal{R})^{-1}\dot{H}^{1/2}(\mathbb{R}^n)$ onto $a'^{-1}\dot{H}^{1/2}(\mathbb{R}^n)$. Using the complex interpolation equalities

$$[a'^{-1}\dot{H}^{1/2}(\mathbb{R}^n), \dot{H}^{-1/2}(\mathbb{R}^n)]_{1/2} = L^2(\mathbb{R}^n)$$

and

$$[(\mathcal{R}^* d' \mathcal{R})^{-1}\dot{H}^{1/2}(\mathbb{R}^n), \dot{H}^{-1/2}(\mathbb{R}^n)]_{1/2} = L^2(\mathbb{R}^n),$$

we obtain the desired conclusion.

It remains to explain the interpolation equalities. The first one was actually first observed (without proof) in [10] as a consequence of the solution of the 1-dimensional Kato square root problem [8] using that a' is an accretive function. See [11, Section

6] for some account. It can also be seen as a consequence of the Tb theorem. See [9, Théorème 9] for an explicit statement.

The second equality is a consequence of the solution of the n-dimensional Kato square root problem [4] and the results in [5] as follows. Remark that $\mathcal{R}^* d'\mathcal{R}$ is a strictly accretive and bounded operator on $L^2(\mathbb{R}^n)$. Combining [5, Theorems 5.3, 7.2, 7.3] using that $\dot{H}^s(\mathbb{R}^n)$ is defined by the semi-norms $\|(-\Delta)^{s/2} f\|_2$, we have

$$[(\mathcal{R}^* d'\mathcal{R})^{-1}\dot{H}^{1/2}(\mathbb{R}^n), \dot{H}^{-1/2}(\mathbb{R}^n)]_{1/2} = \dot{\mathcal{H}}^0_{\mathcal{T}},$$

where $\mathcal{T}$ is the sectorial operator $(-\Delta)^{1/2}(\mathcal{R}^* d'\mathcal{R})$. But $\dot{\mathcal{H}}^0_{\mathcal{T}} = L^2(\mathbb{R}^n)$ means that $\mathcal{T}$ has a bounded holomorphic functional calculus on $L^2(\mathbb{R}^n)$, which is the same as $(\mathcal{R}^* d'\mathcal{R})(-\Delta)^{1/2}$ has a bounded holomorphic functional calculus on $L^2(\mathbb{R}^n)$. By [5, Theorem 10.1], this is equivalent to the claim

$$\|L_\parallel^{1/2} f\|_2 \approx \|(-\Delta)^{1/2} f\|_2, \quad f \in \mathsf{D}(L_\parallel^{1/2}),$$

where $L_\parallel$ is the operator $(-\Delta)^{1/2}(\mathcal{R}^* d'\mathcal{R})(-\Delta)^{1/2}$. As $L_\parallel$ is nothing but the divergence operator $-\mathrm{div} d'\nabla$, the claim is proved in [4]. □

Theorem 6.2. *The Neumann problem for operators L with block lower-triangular, t-independent coefficients A is well-posed for L^2 data.*

Proof. From the relation $\Gamma^A_{ND} = s_{12}(\underline{T})^{-1}(I - s_{11}(\underline{T}))$ and the previous lemma, we have that Γ^A_{ND} is bounded from $\dot{\mathcal{H}}^0_\perp$ to $\dot{\mathcal{H}}^0_\parallel$. Thus, given a Neumann data $f \in L^2(\mathbb{R}^n)$, $\Gamma^A_{ND} f \in L^2(\mathbb{R}^n, \mathbb{C}^n)$, and from this it is easy to conclude that $H_0 = \begin{bmatrix} f \\ \Gamma^A_{ND} f \end{bmatrix} \in \mathcal{H}^{0,+}_{\underline{T}}$ with $\|H_0\|_2 \approx \|f\|_2$. By Theorem 4.4, (i), we can define a weak solution u of $Lu = 0$ with $\|\widetilde{N}(\nabla u)\|_2 \approx \|H_0\|_2$ by $\nabla_A u = e^{-t\underline{T}} H_0$. □

Remark 6.3. Under the same assumptions, by interpolation, Γ^A_{ND} is bounded from $\dot{\mathcal{H}}^{-s}_\perp$ to $\dot{\mathcal{H}}^{-s}_\parallel$ for $0 \le s \le 1/2$. When $0 < s \le 1/2$, given any $f \in \dot{H}^{-s}(\mathbb{R}^n)$ the above formalism furnishes a solution to $Lu = 0$ with conormal derivative f and $\iint_{\mathbb{R}^{1+n}_+} t^{-1+2s}|\nabla_{t,x} u|^2 dt dx \approx \|f\|^2_{\dot{H}^{-s}(\mathbb{R}^n)}$. By Sneiberg's perturbation result [13], this can be pushed to $s < 1/2 + \varepsilon$ for some $\varepsilon > 0$ because $s_{11}(\underline{T}) \in \mathcal{L}(\dot{\mathcal{H}}^{-s}_\perp)$ and $s_{12}(\underline{T}) \in \mathcal{L}(\dot{\mathcal{H}}^{-s}_\parallel, \dot{\mathcal{H}}^{-s}_\perp)$ for all $s \in [0,1]$, the spaces are complex interpolation scales and invertibility holds at $s = 1/2$.

The same arguments apply to block upper-triangular coefficients: this time, the lower off-diagonal block coefficient c' vanishes but not the upper off-diagonal block coefficient b'.

Theorem 6.4. *The regularity problem for operators L with block upper-triangular t-independent coefficients A is well-posed for L^2 data.*

Proof. It follows from Theorem 3.4 and Lemma 5.3 that $\Gamma^A_{DN} = s_{21}(\underline{T})^{-1}(I - s_{22}(\underline{T}))$ in $\mathcal{L}(\dot{\mathcal{H}}^{-1/2}_\parallel, \dot{\mathcal{H}}^{-1/2}_\perp)$ from the invertibility of $s_{21}(\underline{T})$ from $\dot{\mathcal{H}}^{-1/2}_\perp$ onto $\dot{\mathcal{H}}^{-1/2}_\parallel$. By Theorem 4.4, (i), it is enough to show the invertibility of $s_{21}(\underline{T})$ from $\dot{\mathcal{H}}^0_\perp$ onto $\dot{\mathcal{H}}^0_\parallel$, that is the invertibility of $\mathcal{R}^* s_{21}(\underline{T})$ on $L^2(\mathbb{R}^n)$. This is based on the interpolation argument and the formula

$$\mathcal{R}^* s_{21}(\underline{T}) a' = (\mathcal{R}^* d'\mathcal{R})(\mathcal{R}^* s_{21}(T))$$

in $\mathcal{L}(L^2(\mathbb{R}^n))$ that can be checked as above using that the coefficient c' vanishes. Details are left to the reader. □

Remark 6.5. Observations similar to the ones in Remark 6.3 apply for regularity problems with data (the tangential gradient at the boundary) in $\dot{\mathcal{H}}^{-s}_{\parallel}$ for block upper-triangular coefficients.

Theorem 6.6. *The Dirichlet problem for operators L with block lower-triangular t-independent operators A is well-posed for L^2 data.*

Proof. Here, well-posedness is within the class of solutions with $\iint_{\mathbb{R}^{1+n}_+} t|\nabla_{t,x}u|^2 dtdx < \infty$ and by [1], this implies then that $\|\widetilde{N}(u)\|_2 \approx \iint_{\mathbb{R}^{1+n}_+} t|\nabla_{t,x}u|^2 dtdx \approx \|u_{|t=0}\|_2$. Well-posedness follows from the duality principle [2, Proposition 17.6] that the regularity problem for L with data in L^2 is well-posed in the class with modified non-tangential estimate if and only if the Dirichlet problem for L^* with L^2 data is well-posed in the above class. □

Remark 6.7. We point out that the relation between L and A is a correspondence only when L and A are imposed to be self-adjoint. Otherwise, one L may be represented by many different strictly accretive matrices A. As a consequence, there are as many Neumann problems for L as choices of A since the conormal derivative depends on A. However, for the regularity and Dirichlet problems, the choice of A is irrelevant. Although the solution algorithms depend on the coefficients A, they produce in the end the same solutions (note that the tangential part of the conormal gradient is independent of the coefficients).

For example, one does not modify L by adding to A any matrix of the form $M_\gamma = \begin{bmatrix} 0 & \gamma(x)^t \\ -\gamma(x) & 0 \end{bmatrix}$ with γ a $\mathbb{R}^n$-valued, measurable and bounded function with $\mathrm{div}_x\gamma = 0$. First, the divergence free equation implies $-\mathrm{div}_{t,x}(A + M_\gamma)\nabla_{t,x} = -\mathrm{div}_{t,x}A\nabla_{t,x}$ in the sense of forms. Second, the accretivity constant of $A + M_\gamma$ is the lower bound of $\frac{1}{2}(A + M_\gamma + A^* + M_\gamma^*)$, but $M_\gamma + M_\gamma^* = 0$ when γ is $\mathbb{R}^n$-valued, so this is the same as the accretivity constant of A. This argument also shows that $A + M_\gamma$ remains strictly accretive if γ is $\mathbb{C}^n$-valued provided the norm of its imaginary part is not too large.

For systems of size m, this example has to be modified as follows. $\mathbb{R}^n$-valued is replaced by $\mathcal{S}^n$-valued where $\mathcal{S}$ is the space of real and symmetric $m \times m$ matrices. The divergence free equation is understood coefficientwise $\sum_{j=1}^n \partial_j \gamma_j^{\alpha,\beta} = 0$.

Remark 6.8. Recall that any of these well-posedness results is stable under complex perturbation in L^∞ within the class of t-independent coefficients A by [3, Theorem 2.2] and [1] (the latter reference extends the solution spaces for the results of the former to hold).

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