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Many bosons in a narrow resonance: the R^* -phase issue

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The many-boson problem in presence of an asymptotically narrow Feshbach resonance is considered. The energy spectrum is shown to be bounded from below in the limit of zero range interaction. This implies the promising possibility of achieving a strongly interacting bosonic phase in a dilute regime where the details of the actual interatomic forces are irrelevant. The integral relation between the energy and the one-body momentum distribution is derived.

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The magnetic Feshbach Resonance (FR) technique allows the study of highly correlated and dilute many-body systems in which the two-body scattering cross section is arbitrarily large. This has led to ways of studying the cross-over between fermionic (BCS) and bosonic (BEC) superfluidity in the two spin-component Fermi gas [1] and to obtain the first evidence of Efimov states [2–12] as reviewed in Ref. [13]. However, large atomic losses due to three-body recombinations prevent the observation of a thermalized resonant Bose gas. Nevertheless, the large scale separation between the scattering length (denoted by a) and the interaction range (denoted by b , of the order of few nanometers) suggests the existence of universal low energy properties in this system [14, 15]. The prediction [16–20] and the observation [21, 22] of universal four-body states are the first signatures of possible universal features for a many-boson system in presence of three-body Efimov states. The universal character of many-body properties at energies much smaller in absolute value than $E_b = \hbar^2/(Mb^2)$, should be revealed using a model where the interaction range is arbitrarily small. Unfortunately, structureless point-like interactions lead to the so-called Thomas collapse which makes the stability issue of the resonant Bose gas puzzling [23]. A well defined low energy model for the resonant Bose gas is thus needed. Eventually unlike the fermionic case, little is known about the resonant Bose gas which is now a fascinating issue at the cutting edge of the researches in ultracold atoms.

Recently, it has been shown that three bosons in an asymptotically narrow FR experience an Efimov effect without Thomas collapse in the limit of zero-range interaction [24]. In this system, the three-body parameter is only a function of the so-called width parameter R^* and is independent from the details of the actual interatomic forces [24, 25]. Moreover low energy states are predicted to be long lived [26]. Inspired by the three-body problem, asymptotically narrow FR are a relevant way to achieve a thermalized and dilute resonant many-boson system. In this letter it is shown that in such a regime the system has a well defined ground state, denoted as the R^* -phase, without Thomas collapse. The tail of the atomic momentum distribution $n_{\mathbf{k}}$ at high momentum \mathbf{k} is shown

to behave as

$$n_{\mathbf{k}} \underset{k \rightarrow \infty}{=} \frac{c_4}{k^4} + \frac{c_6}{k^6} + \dots \quad (1)$$

Furthermore, the energy E for bosons of mass M verifies

$$E - E^{\text{trap}} = \frac{\hbar^2}{2M} \int \frac{d^3k}{(2\pi)^3} \left(k^2 n_{\mathbf{k}} - \frac{a^2 c_4}{1 + k^2 a^2} \right) + \frac{\hbar^2 R^* c_6}{8M\pi} - \langle \hat{K}^{\text{mol}} \rangle, \quad (2)$$

where $\langle \hat{K}^{\text{mol}} \rangle$ is the mean translational kinetic energy of the molecular state and E^{trap} is the contribution coming from the interaction with a possible external potential. Equation (2) generalizes the Tan relation of Refs. [27] to the case of an asymptotically narrow FR.

The FR mechanism is based on the coherent coupling of atomic pairs (belonging to an open channel) with a molecular state (belonging to a closed channel). Using an external magnetic field \mathcal{B} permits the tuning of the molecular binding energy E_{mol} with $dE_{\text{mol}}/d\mathcal{B} = \delta\mu$. As a result, the scattering length in the vicinity of the FR located at $\mathcal{B} = \mathcal{B}_0$ can be arbitrarily fixed through the law $a = a_{\text{bg}}[1 - \Delta\mathcal{B}/(\mathcal{B} - \mathcal{B}_0)]$. Here, $\Delta\mathcal{B}$ is the width of the resonance and a_{bg} is the background scattering length. For a weak coherent coupling between the atomic pairs and the molecular state, the resonance is narrow ($|\delta\mu\Delta\mathcal{B}| \ll E_b$) and the width parameter $R^* = \hbar^2/(Ma_{\text{bg}}\delta\mu\Delta\mathcal{B})$ becomes essential in the description of low energy properties. Asymptotically narrow resonances where R^* is very much larger than b , are characterized by the scattering amplitude [24]

$$f_0(k) = -(1/a + R^*k^2 + ik)^{-1}, \quad (3)$$

where the actual range of the interaction b is negligible.

In what follows, the system is described by using a generic two-channel model which contains the FR mechanism [28]. In this model, the molecular state (of mass $2M$) is structureless: a reasonable approximation for the description of mechanisms at energies much smaller than E_b . The operator $a_{\mathbf{k}}^\dagger$ (or $b_{\mathbf{k}}^\dagger$) creates an atomic (or a molecular) plane wave $\langle \mathbf{r} | \mathbf{k} \rangle = \exp(i\mathbf{k} \cdot \mathbf{r})$ and obeys the usual commutation rules for bosons. The kinetic energy

of a particle of wave vector \mathbf{k} is denoted by $E_{\mathbf{k}} = \frac{\hbar^2 k^2}{2M}$, so that in absence of an external potential the free Hamiltonian for atoms and molecules is

$$H_0 = \int \frac{d^3k}{(2\pi)^3} \left[E_{\mathbf{k}} a_{\mathbf{k}}^\dagger a_{\mathbf{k}} + \left(\frac{E_{\mathbf{k}}}{2} + E_{\text{mol}} \right) b_{\mathbf{k}}^\dagger b_{\mathbf{k}} \right]. \quad (4)$$

The interchannel coupling converts a molecule into a pair of atoms *via* the operator V and *vice-versa* *via* the operator V^\dagger . It is modeled by a constant amplitude Λ (chosen real positive) and a generic cut-off function $\chi_\epsilon(k) = \exp\left(-\frac{k^2 \epsilon^2}{4}\right)$ (with $\epsilon \geq 0$);

$$V = \Lambda \int \frac{d^3K d^3k}{(2\pi)^6} \chi_\epsilon(k) b_{\mathbf{K}} a_{\mathbf{K}/2-\mathbf{k}}^\dagger a_{\mathbf{K}/2+\mathbf{k}}^\dagger. \quad (5)$$

The Hamiltonian is thus $H_0 + V + V^\dagger$ and the scattering amplitude denoted by $f_\epsilon(k)$ at energy $E = E_{\mathbf{k}}$ is given by

$$\frac{M[\chi_\epsilon(k)]^{-2}}{4\pi\hbar^2 f_\epsilon(k)} = \frac{E_{\text{mol}} - E}{2\Lambda^2} + \int \frac{d^3k'}{(2\pi)^3} \frac{[\chi_\epsilon(k')]^2}{z - 2E_{\mathbf{k}'}} , \quad (6)$$

where the variable z encapsulates the usual prescription $z = E + i0^+$ for $E > 0$ and $z = E$ otherwise. For vanishing values of ϵ , identification of $f_\epsilon(k_0)$ with Eq. (3) gives

$$E_{\text{mol}} = \frac{M\Lambda^2}{2\pi\hbar^2} \left(\frac{\sqrt{2}}{\epsilon\sqrt{\pi}} - \frac{1}{a} \right) \quad \text{and} \quad \Lambda = \frac{\hbar^2}{M} \sqrt{\frac{2\pi}{R^*}}. \quad (7)$$

In what follows, only states with asymptotically small values of ϵ are considered while a and R^* are kept fixed. In this zero range limit, Eq. (7) shows that E_{mol} has an arbitrarily large positive value. Thus the molecular state is occupied only through virtual processes. As a consequence of the interchannel coupling, an eigenstate for a system composed of N atoms is in general a coherent superposition of states containing 0 to $[N/2]$ molecules

$$|\Psi_\epsilon\rangle = \sum_{m=0}^{[N/2]} |\psi_\epsilon^m\rangle, \quad (8)$$

where $|\psi_\epsilon^m\rangle$ is a state with m molecules and $N - 2m$ atoms. Projection of the stationary Schrödinger's equation at energy E in the subspace with m molecules ($1 < m < [N/2]$) yields the recursive equation

$$(E - H_0)|\psi_\epsilon^m\rangle = V|\psi_\epsilon^{m+1}\rangle + V^\dagger|\psi_\epsilon^{m-1}\rangle. \quad (9)$$

The recursion begins for $m = 0$ by the equation

$$|\psi_\epsilon^0\rangle = |\Phi_{\text{inc}}\rangle + G_0(z)V|\psi_\epsilon^1\rangle, \quad (10)$$

where $G_0(z) = (z - H_0)^{-1}$ is the resolvent and $|\Phi_{\text{inc}}\rangle$ is the possible incoming wave for a scattering process when $E \geq 0$ whereas $|\Phi_{\text{inc}}\rangle \equiv 0$ when $E < 0$. The recursion for $m = 1$ plays a central role in the rest of this letter. From Eqs. (9) and (10) it takes the form

$$[E - H_0 - V^\dagger G_0(z)V]|\psi_\epsilon^1\rangle = V^\dagger|\Phi_{\text{inc}}\rangle + V|\psi_\epsilon^2\rangle. \quad (11)$$

In the subsequent lines, the momentum representation is used and the hyper wavevector $(k)_p = (\mathbf{k}_1, \mathbf{k}_2, \dots, \mathbf{k}_p)$ [$(K)_m = (\mathbf{K}_1, \mathbf{K}_2, \dots, \mathbf{K}_m)$] denotes the wavevectors of p particles (or of m molecules). The m -molecule state can then be written as

$$|\psi_\epsilon^m\rangle = \int \frac{d(k)_p d(K)_m}{(2\pi)^{3p+3m}} \frac{\langle(k)_p; (K)_m | \psi_\epsilon^m\rangle}{\sqrt{p!} \sqrt{m!}} a_{\mathbf{k}_1}^\dagger \dots a_{\mathbf{k}_p}^\dagger b_{\mathbf{K}_1}^\dagger \dots b_{\mathbf{K}_m}^\dagger |0\rangle. \quad (12)$$

In the following, a superscript \hat{i} in $(k)_p^{\hat{i}}$ means that the wavevector \mathbf{k}_i is withdrawn from $(k)_p$. For instance $(k)_N^{\hat{1}\hat{2}} = (\mathbf{k}_3, \mathbf{k}_4, \dots, \mathbf{k}_N)$. The ket $|(k)_p; (K)_m\rangle$ describes the state where the atom i ($1 \leq i \leq p$) is in the plane wave of wave vector \mathbf{k}_i and the molecule j ($1 \leq j \leq m$) is in the plane wave of momentum \mathbf{K}_j . Furthermore $\mathbf{k}_{ij} = \frac{\mathbf{k}_i - \mathbf{k}_j}{2}$ (or $\mathbf{K}_{ij} = \mathbf{k}_i + \mathbf{k}_j$) denotes the relative (or the total) wave vector for the pair (i, j) .

As in the two-body states, for an arbitrarily large relative wavevector of the pair (ij) , it is expected that in the presence of $m + 1$ molecules (*i.e.* $|\psi_\epsilon^{m+1}\rangle \neq 0$) $\langle(k)_p; (K)_m | \psi_\epsilon^m\rangle$ has the asymptotic behavior $\chi_\epsilon(k_{ij})/k_{ij}^2$ [29]. This behavior is obtained by considering Eq. (9) divided by $E_{\text{mol}} \equiv O(1/\epsilon)$ in the zero range limit; the only non vanishing contribution in the right hand side of the resulting equation is $\langle(k)_p; (K)_m | V^\dagger | \psi_\epsilon^{m-1}\rangle$. This enables the m -molecule wave function to be expressed in terms of the $(m - 1)$ -molecule wave function:

$$\lim_{\epsilon \rightarrow 0} \langle(k)_p; (K)_m | \psi_\epsilon^m\rangle = -\sqrt{\frac{(p+2)!R^*}{(2m)^3\pi p!}} \sum_{i=1}^m \lim_{k \rightarrow \infty} \left[k^2 \times \lim_{\epsilon \rightarrow 0} \langle(k)_p, \frac{\mathbf{K}_i}{2} - \mathbf{k}, \frac{\mathbf{K}_i}{2} + \mathbf{k}; (K)_m^{\hat{i}} | \psi_\epsilon^{m-1}\rangle \right]. \quad (13)$$

In the center of mass frame, $|\psi_\epsilon^1\rangle$ can be factorized by a ket of $N - 2$ particles, which is denoted by $|D\rangle$ in the zero range limit:

$$\lim_{\epsilon \rightarrow 0^+} \frac{\langle(k)_N^{\hat{1}\hat{2}}; \mathbf{K}_{12} | \psi_\epsilon^1\rangle}{(2\pi)^3 \sqrt{R^* N(N-1)}} = \delta\left(\sum_{i=3}^N \mathbf{k}_i + \mathbf{K}_{12}\right) \langle(k)_N^{\hat{1}\hat{2}} | D\rangle. \quad (14)$$

Similarly, the possible incoming state is factorized as $\langle(k)_N | \Phi_{\text{inc}}\rangle = (2\pi)^3 \delta\left(\sum_{i=1}^N \mathbf{k}_i\right) \langle(k)_N | \phi_{\text{inc}}\rangle$. The equation for $|D\rangle$ is deduced from Eqs. (5), (11) and (13) and the identity $\lim_{k_{ij} \rightarrow \infty} k_{ij}^2 \langle(k)_N^{\hat{1}\hat{2}} | D\rangle = \lim_{k_{12} \rightarrow \infty} k_{12}^2 \langle(k)_N^{\hat{1}\hat{2}} | D\rangle$ which comes from the exchange symmetry of the two molecules described by $|\psi_\epsilon^2\rangle$ for $\{i, j\} \notin \{1, 2\}$ [30]:

$$\int \frac{d^3k_2}{(2\pi)^3} \frac{\sum_{3 \leq i < j \leq N} \langle(k)_N^{\hat{ij}} | D\rangle + 2 \sum_{i=3}^N \langle(k)_N^{\hat{i}} | D\rangle}{\sum_{i=2}^N k_i^2 + \sum_{2 \leq i < j \leq N} \mathbf{k}_i \cdot \mathbf{k}_j - Mz/\hbar^2} + \frac{R^*}{4\pi} \sum_{3 \leq i < j \leq N} \lim_{k_{12} \rightarrow \infty} \left[k_{12}^2 \langle(k)_N^{\hat{ij}} | D\rangle \right] - \int \frac{d^3k_{12}}{(2\pi)^3} \langle(k)_N | \phi_{\text{inc}}\rangle = \frac{\langle(k)_N^{\hat{1}\hat{2}} | D\rangle}{4\pi f_0(k^{\text{col}})}, \quad (15)$$

where $k^{\text{col}} = \sqrt{ME^{\text{col}}}/\hbar$ is the collisional wave number for the pair (1,2) defined from the energy $E^{\text{col}} = E - \sum_{n=3}^N E_{\mathbf{k}_n} - E_{\mathbf{K}_{12}}/2$. Surprisingly Eq. (15) supports solutions where $\lim_{\epsilon \rightarrow 0} |\psi_\epsilon^m\rangle = 0$ for $m \geq 2$ and thus $\lim_{k_{ij} \rightarrow \infty} k_{ij}^2 \langle (k)_N^{\hat{1}2} | D \rangle = 0$ for $\{i, j\} \notin \{1, 2\}$ [32].

Without loss of generality, the energy spectrum of Eq. (15) is shown to be bounded from below in the strict resonant limit $|a| = \infty$. To show this, we use the following inequality verified by negative energy solutions of Eq. (15):

$$qR^* |\langle (k)_N^{\hat{1}2} | D \rangle| < \frac{R^*}{4\pi q} \sum_{3 \leq i < j \leq N} \lim_{k_{12} \rightarrow \infty} |k_{12}^2 \langle (k)_N^{\hat{i}j} | D \rangle| + \sum_{i < j; (i,j) \neq (1,2)} \int \frac{d^3 k_{12}}{2\pi^2 q^3} \frac{|\langle (k)_N^{\hat{i}j} | D \rangle|}{k_{12}^2/q^2 + 1}, \quad (16)$$

where $q = \sqrt{-ME}/\hbar$ is the binding wavenumber. Then, we consider a hypothetical solution of Eq. (15) having an arbitrarily large and negative energy ($q \rightarrow \infty$). Assuming that $\langle (k)_N^{\hat{i}j} | D \rangle$ is a bounded function, integration in the domain where k_{12} is smaller or of the order of q in the right hand side of Eq. (16) gives a finite result in the limit where q is arbitrarily large. In the domain where k_{12} is much larger than q , the high momentum behavior of $\langle (k)_N^{\hat{i}j} | D \rangle$ [32] gives rise to a vanishing contribution after integration in Eq. (16). As for any fixed value of $(k)_N^{\hat{1}2}$ the left hand side of Eq. (16) is bounded from above by a finite quantity, qR^* cannot be arbitrarily large. The desired result is thus obtained by contradiction.

From now the Effective Range Model (ERM) is used to derive Eqs. (1) and (2). To avoid any confusion with the two-channel model, the eigenstates in the ERM are underlined (as $|\underline{\Psi}\rangle$ for instance). The ERM is a one-channel and zero-range approach which supports the scattering amplitude in Eq. (3) whereas the molecular field is hidden. The main idea of the ERM is to filter asymptotically (i.e at interparticle distances much larger than b) correct wavefunctions among the set of singular wavefunctions satisfying a Schrödinger's equation for point-like interactions. This filtering process can be performed by using a pseudo-potential. Following Refs. [33, 34], the pseudo-potential of the ERM is defined by introducing a Gaussian shape $\langle \mathbf{r} | \delta_\epsilon \rangle$ tending to the Dirac distribution in the limit where the short range parameter ϵ goes to zero. In the momentum representation $\langle \mathbf{k} | \delta_\epsilon \rangle = \chi_\epsilon(k)$ i.e. it coincides with the interchannel coupling function of the two-channel model. For a many-body system, a short range parameter ϵ_{ij} and a pseudo-potential $V_{ij}^{\epsilon_{ij}}$ is introduced for each interacting pair of particles (i, j) :

$$V_{ij}^{\epsilon_{ij}} \cdot = \frac{4\pi\hbar^2 a}{M} |\delta_{\epsilon_{ij}}\rangle \lim_{\epsilon_{ij} \rightarrow 0^+} \left(\partial_{\epsilon_{ij}} - \sqrt{\frac{\pi}{2}} \frac{R^*}{2} \partial_{\epsilon_{ij}}^2 \right) \epsilon_{ij} \langle \delta_{\epsilon_{ij}} | \cdot \quad (17)$$

where the ket $|\delta_{\epsilon_{ij}}\rangle$ is associated with the function $\delta_{\epsilon_{ij}}$ for the relative coordinates of the pair. Instead of introducing the pseudo-potential, the filtering process of the

ERM can be defined through the notion of contact condition. To see this, one can notice that the action of the pseudo-potential $V_{ij}^{\epsilon_{ij}}$ on an eigenstate $|\underline{\Psi}\rangle$ gives rise to a δ -source term for each interacting pair in the stationary Schrödinger's equation ;

$$\left(E - \sum_{i=1}^N \frac{\hbar^2 \hat{\mathbf{k}}_i^2}{2M} \right) |\underline{\Psi}\rangle = \frac{4\pi\hbar^2}{M} \sum_{i < j} |A_{\underline{\Psi}}^{(ij)}\rangle, \delta_{\epsilon_{ij}}. \quad (18)$$

Here, $|A_{\underline{\Psi}}^{(ij)}\rangle$ depends on the set of short range parameters $\{\epsilon_{pq}\}$ such that $(p, q) \neq (i, j)$ and verifies $|A_{\underline{\Psi}}^{ij}\rangle = -\sqrt{\pi/2} \lim_{\epsilon_{ij} \rightarrow 0^+} (\epsilon_{ij} \langle \delta_{\epsilon_{ij}} | \underline{\Psi} \rangle)$ [35]. The ERM can then be defined from Eq. (18) together with a boundary condition at the contact of each interacting pair (i, j) [35]

$$\lim_{\epsilon_{ij} \rightarrow 0^+} B_{\epsilon_{ij}} \langle \delta_{\epsilon_{ij}} | \underline{\Psi} \rangle = 0, \quad (19)$$

where B_ϵ is the boundary operator defined by

$$B_\epsilon \cdot = \left[\partial_\epsilon - \sqrt{\frac{\pi}{2}} \frac{R^*}{2} \partial_\epsilon^2 + \sqrt{\frac{\pi}{2}} \frac{1}{a} \right] \epsilon \cdot \quad (20)$$

From Eq. (18) one finds that two non degenerate eigenstates of the ERM, say $|\underline{\Psi}\rangle$ (of energy E_Ψ) and $|\underline{\Phi}\rangle$ (of energy $E_\Phi \neq E_\Psi$), are not mutually orthogonal:

$$\langle \underline{\Phi} | \underline{\Psi} \rangle = \frac{4\pi\hbar^2}{M} \sum_{i < j} \frac{\langle \underline{\Phi} | A_{\underline{\Psi}}^{(ij)} \rangle, \delta_{\epsilon_{ij}} - \langle A_{\underline{\Phi}}^{(ij)} \rangle, \delta_{\epsilon_{ij}} | \underline{\Psi} \rangle}{E_\Psi - E_\Phi}. \quad (21)$$

Interestingly, a modified scalar product denoted by $(\cdot | \cdot)_0$ which makes the ERM self-adjoint can be constructed by using Eqs. (19), (20) and (21):

$$(\underline{\Phi} | \underline{\Psi})_0 = \lim_{\{\epsilon\} \rightarrow 0} \left(\prod_{i < j} B_{\epsilon_{ij}} \right) \langle \underline{\Phi} | \underline{\Psi} \rangle, \quad (22)$$

where $\lim_{\{\epsilon\} \rightarrow 0}$ denotes the limit in which all the short range parameters tend to zero. Equation (22) generalizes the modified scalar product introduced in Ref. [36] for two interacting particles. The equivalence of this modified scalar product with the usual scalar product of the two-channel model has been already depicted at the two-body level in Ref. [28]. This mapping can be generalized in the N -body case as follows. In the zero-range limit, analogously to Eq. (13), the hidden m -molecule wave function is revealed in the state $|\underline{\Psi}\rangle$ by the existence of a k_{ij}^{-2} behavior in the large momentum limit for m distinct pairs (ij) . To be more explicit, one introduces the state $|A_{\underline{\Psi}}^{(ij), \dots, (ln), (pq)}\rangle$ [37] corresponding to the contact of the two particles in the distinct pairs $(ij), \dots, (ln), (pq)$. It is defined iteratively from the state $|A_{\underline{\Psi}}^{(ij)}\rangle$ by

$$\langle (k)_N^{\hat{i}j, \dots, \hat{ln}}; \mathbf{K}_{ij} \dots \mathbf{K}_{ln} | A_{\underline{\Psi}}^{(ij), \dots, (ln)} \rangle = \frac{\chi_{\epsilon_{pq}}(k_{pq})}{k_{pq} \rightarrow \infty} \frac{\chi_{\epsilon_{pq}}(k_{pq})}{k_{pq}^2} \times \langle (k)_N^{\hat{i}j, \dots, \hat{ln}, \hat{pq}}; \mathbf{K}_{ij} \dots \mathbf{K}_{ln} | A_{\underline{\Psi}}^{(ij), \dots, (ln), (pq)} \rangle. \quad (23)$$

In the scalar product $\langle \Phi | \Psi \rangle$, the contact for m distinct pairs $(ij), \dots (ln)$ manifests itself by the monom

$$(-1)^m \frac{16\pi^2}{(2\pi)^{3m/2}} \epsilon_{ij} \dots \epsilon_{ln} \langle A_{\Phi}^{(ij), \dots (ln)} | A_{\Psi}^{(ij), \dots (ln)} \rangle. \quad (24)$$

The number of monoms of the same type as in Eq. (24) is equal to $P_m^N = N!/[2^m m!(N-2m)!]$ *i.e.* the number of collections of m distinct pairs taken from the set of N particles. Applying all of the boundary operators to the right hand side of Eq. (22), and identifying the result with the scalar product in the two-channel model, one finds the following mapping in the zero-range limit:

$$\lim_{\{\epsilon\} \rightarrow 0} |A_{\Psi}^{(ij), \dots (ln)}\rangle = \left(\frac{4\pi}{R^*}\right)^{m/2} \frac{1}{4\pi\sqrt{P_m^N}} \lim_{\epsilon \rightarrow 0} |\psi_{\epsilon}^m\rangle \quad (25)$$

and $\lim_{\{\epsilon\} \rightarrow 0} |\Psi\rangle = \lim_{\epsilon \rightarrow 0^+} |\psi_{\epsilon}^0\rangle$. Equation (15) is recovered using this mapping and applying the contact condition in Eq. (19) on the state $|\Psi\rangle$. For an eigenstate normalized with Eq. (22) *i.e.* $\langle \Psi | \Psi \rangle_0 = 1$, the one-particle momentum distribution (taking into account the contributions of the m -molecule states) is $n_{\mathbf{k}} = \langle \Psi | a_{\mathbf{k}}^{\dagger} a_{\mathbf{k}} | \Psi \rangle_0$. This definition yields the correct normalization *i.e.*

$$\int \frac{d^3k}{(2\pi)^3} n_{\mathbf{k}} = N - 2N^{\text{mol}}; \quad N^{\text{mol}} = \lim_{\epsilon \rightarrow 0} \sum_{m=1}^{[N/2]} m \langle \psi_{\epsilon}^m | \psi_{\epsilon}^m \rangle \quad (26)$$

where N^{mol} is the mean number of molecules [38]. The ERM gives $c_4 = 8\pi N^{\text{mol}}/R^*$ in Eq. (1): a result which also can be deduced from the Hellmann-Feynman theorem [39, 40]. The expression of the coefficient c_6 in Eq. (1) depends on the possible interaction of the particles with a trapping potential $V^{\text{trap}}(\mathbf{r})$:

$$c_6 = \frac{32\pi^2 M}{\hbar^2} N(N-1) \langle A_{\Psi}^{(12)} | E - 2V^{\text{trap}}(\mathbf{R}_{12}) - \sum_{n=3}^N \left[\frac{\hbar^2 \hat{\mathbf{k}}_n^2}{2M} + V^{\text{trap}}(\mathbf{r}_n) \right] | A_{\Psi}^{(12)} \rangle_0. \quad (27)$$

Equation (2) is obtained using the identity $\langle \Psi | V_{ij}^{\epsilon_{ij}} | \Psi \rangle_0 = 0$, which is valid provided that the state $|\Psi\rangle$ verifies the contact condition in Eq. (19) for all

pairs (i, j) . Thus, applying $\lim_{\{\epsilon\} \rightarrow 0} \left(\prod_{i < j} B_{\epsilon_{ij}} \right)$ on the mean value of the Hamiltonian of the ERM evaluated for the normalized eigenstate $|\Psi\rangle$ yields

$$E - E^{\text{trap}} = \langle \Psi | \int \frac{d^3k}{(2\pi)^3} \frac{\hbar^2 k^2}{2M} a_{\mathbf{k}}^{\dagger} a_{\mathbf{k}} | \Psi \rangle_0, \quad (28)$$

where $E^{\text{trap}} = \langle \Psi | \sum_{n=1}^N V^{\text{trap}}(\mathbf{r}_n) | \Psi \rangle_0$ includes the contribution of the molecular state. The regularization of the kinetic energy in the right hand side of Eq. (28) through the modified scalar product permits to recover Eq. (2) with $\langle \hat{K}^{\text{mol}} \rangle = \langle \Psi | \int \frac{d^3k}{(2\pi)^3} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} E_{\mathbf{k}} | \Psi \rangle_0 / 2$.

To conclude, the main result of this letter is the stability of the resonant dilute many-boson system in an asymptotically narrow FR. The system is described in terms of a two-channel model of the FR. The mapping with the ERM permits to obtain the integral relation satisfied by the energy and the one-particle momentum distribution. In the ERM, the hidden molecular field is described through pairwise singularities in the N -particle wavefunction. The absence of Thomas collapse for three bosons in an asymptotically narrow FR is at the heart of the stability issue of this many-boson problem. The attractive effective force between one boson and one pair of bosons close together, is screened for interparticle distances smaller than the width parameter R^* [25]. Because of this feature one expects the system to be organized in pairs of bosons with a large population of the molecular state and a mean intermolecular distance of the order of R^* . The liquid or crystalline nature of the R^* -phase as a function of the particle density is an open fascinating issue. It is worth pointing out that Eq. (3) is valid in a small interval of magnetic field such that $|\mathcal{B} - \mathcal{B}_0| \ll |\Delta\mathcal{B}|$. Otherwise, off-resonant effects must be taken into account [41]. Ultra narrow FR (where $R^* \gg b$) have been already found in many atomic species [42] and new techniques allowing a fine tuning of the magnetic field are promising for the observation of the R^* -phase [43].

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