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# DECAY ESTIMATES OF GLOBAL SOLUTION TO 2D INCOMPRESSIBLE INHOMOGENEOUS NAVIER-STOKES EQUATIONS WITH VARIABLE VISCOSITY

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ABSTRACT. In this paper, we investigate the time decay behavior to Lions weak solution of 2D incompressible inhomogeneous Navier-Stokes equations.

*Keywords:* Inhomogeneous Navier-Stokes equations, Decay estimates.

*AMS Subject Classification (2000):* 35Q30, 76D03

## 1. INTRODUCTION

The main purpose of this paper is to investigate the decay estimates for the global solutions of the following two-dimensional incompressible inhomogeneous Navier-Stokes equations with viscous coefficient depending on the density

$$(1.1) \quad \begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0, & (t, x) \in \mathbb{R}^+ \times \mathbb{R}^2, \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) - \operatorname{div}(\mu(\rho)\mathcal{M}(u)) + \nabla \Pi = 0, \\ \operatorname{div} u = 0, \end{cases}$$

where  $\rho, u = (u_1, u_2)$  stand for the density and velocity of the fluid respectively,  $\mathcal{M}(u) = \nabla u + \nabla^T u$ ,  $\Pi$  is a scalar pressure function, and in general, the viscosity coefficient  $\mu(\rho)$  is a smooth positive function on  $[0, \infty)$ . Such system describes a fluid which is obtained by mixing two immiscible fluids that are incompressible and that have different densities. It may also describe a fluid containing a melted substance. One may check [9] for the detailed derivation.

When  $\mu(\rho)$  is independent of  $\rho$ , i.e.  $\mu$  is a positive constant, and the initial density has a positive lower bound, Ladyženskaja and Solonnikov [8] first addressed the question of unique solvability of (1.1). More precisely, they considered the system (1.1) in a bounded domain  $\Omega$  with homogeneous Dirichlet boundary condition for  $u$ . Under the assumption that  $u_0 \in W^{2-\frac{2}{p}, p}(\Omega)$  ( $p > d$ ) is divergence free and vanishes on  $\partial\Omega$  and that  $\rho_0 \in C^1(\Omega)$  is bounded away from zero, then they [8] proved

- Global well-posedness in dimension  $d = 2$ ;
- Local well-posedness in dimension  $d = 3$ . If in addition  $u_0$  is small in  $W^{2-\frac{2}{p}, p}(\Omega)$ , then global well-posedness holds true.

Danchin [3] proved similar well-posedness result of (1.1) in the whole space case and the initial data in the almost critical spaces. In particular, in two dimension, he proved the global well-posedness of (1.1) provided the initial data  $(\rho_0, u_0)$  satisfying  $\rho_0 - 1 \in H^{1+\alpha}(\mathbb{R}^2)$ ,  $\rho_0 \geq m > 0$ , and  $u_0 \in H^\beta(\mathbb{R}^2)$  for any  $\alpha \in (0, 1)$  and  $\beta \in (0, 1]$ .

In general, Lions [9] (see also the references therein) proved the global existence of weak solutions to (1.1) with finite energy. Yet the uniqueness and regularities of such weak solutions are big open questions even in two space dimensions. Except under the additional assumptions that

$$(1.2) \quad \|\mu(\rho_0) - 1\|_{L^\infty(\mathbb{T}^2)} \leq \varepsilon \quad \text{and} \quad u_0 \in H^1(\mathbb{T}^2),$$

Desjardins [4] proved the following theorem.

**Theorem 1.1.** *Let  $\rho_0 \in L^\infty(\mathbb{T}^2)$  and  $\operatorname{div} u_0 = 0$ . Then there exists  $\varepsilon > 0$  such that under the assumption (1.2), Lions weak solutions ([9]) to (1.1) satisfy the following regularity properties hold for all  $T > 0$ :*

- (1)  $u \in L^\infty((0, T); H^1(\mathbb{T}^2))$  and  $\sqrt{\rho} \partial_t u \in L^2((0, T) \times \mathbb{T}^2)$ ,
- (2)  $\rho$  and  $\mu(\rho) \in L^\infty((0, T) \times \mathbb{T}^2) \cap C([0, T]; L^p(\mathbb{T}^2))$  for all  $p \in [1, \infty)$ ,
- (3)  $\nabla(\Pi - \mathcal{R}_i \mathcal{R}_j(\mu \mathcal{M}(u)_{ij}))$  and  $\nabla(\mathbb{P} \otimes \mathbb{Q}(\mu \mathcal{M}(u)_{ij})) \in L^2((0, T) \times \mathbb{T}^2)$ ,
- (4)  $\Pi$  may be renormalized in such a way that for some universal constant  $C_0 > 0$ ,  $\Pi$  and  $\nabla u \in L^2((0, T); L^p(\mathbb{T}^2))$  for all  $p \in [4, p^*)$ , where  $\frac{1}{p^*} = 2C_0 \|\mu(\rho_0) - 1\|_{L^\infty}$ .

In which, we denote  $\mathcal{R}$  as the Riesz transform:  $\mathcal{R} = \nabla \Delta^{-\frac{1}{2}}$ .  $\mathbb{Q} = \nabla \Delta^{-1} \operatorname{div}$  and  $\mathbb{P} = \mathbb{I} - \mathbb{Q}$  respectively denote the projection on the space of curl-free and divergence-free vector fields.

In order to investigate the global well-posedness of thus solutions, we first need to study the global-in-time type estimates. However, because of the difficulties of the continuity equation in (1.1) being of hyperbolic nature and the estimate of the diffusion term in the momentum equation, we shall first study the time decay of the solutions, which is very much motivated by [5, 10, 12].

**Theorem 1.2.** *For  $1 < p < 2$ , let  $u_0 \in L^p(\mathbb{R}^2) \cap H^1(\mathbb{R}^2)$ ,  $\rho_0 - 1 \in L^2(\mathbb{R}^2)$  and  $\rho_0 \in L^\infty(\mathbb{R}^2)$  with a positive lower bound. We assume that  $(\rho, u, \nabla p)$  is a given Lions weak solution of (1.1) with initial data  $(\rho_0, u_0)$ . Denote  $\mu(1) = \mu_0$ , then under the assumption*

$$(1.3) \quad \|\mu(\rho) - \mu_0\|_{L^\infty(\mathbb{R}^+; L^\infty(\mathbb{R}^2))} \leq \varepsilon_0,$$

for a small constant  $\varepsilon_0$ , there exists a constant  $C_1$  which depends on  $\|\rho_0 - 1\|_{L^2}$ ,  $\|u_0\|_{L^p}$  and  $\|u_0\|_{H^1}$  such that there hold

$$(1.4) \quad \|u(t)\|_{L^2}^2 \leq C_1(t+e)^{-2\beta(p)}, \quad \|\nabla u(t)\|_{L^2}^2 \leq C_1(t+e)^{-1-2\beta(p)+\varepsilon},$$

$$(1.5) \quad \int_0^\infty \|u_t\|_{L^2} + \|\mathbb{P} \operatorname{div}(\mu(\rho) \mathcal{M}(u))\|_{L^2} + \|\mathbb{Q} \operatorname{div}(\mu(\rho) \mathcal{M}(u)) - \nabla \Pi\|_{L^2} dt \leq C_1,$$

$$(1.6) \quad \int_0^\infty (t+e)^{1+2\beta(p)-\varepsilon} \left( \|u_t\|_{L^2} + \|\mathbb{P} \operatorname{div}(\mu(\rho) \mathcal{M}(u))\|_{L^2} + \|\mathbb{Q} \operatorname{div}(\mu(\rho) \mathcal{M}(u)) - \nabla \Pi\|_{L^2} \right)^2 dt \leq C_1,$$

with  $\beta(p) = \frac{1}{2}(\frac{2}{p} - 1)$  and any  $\varepsilon > 0$ .

**Remark 1.1.** *The first estimate of (1.4) coincides with the  $L^2$ -norm decay result in [10, 12] for the weak solutions of the two-dimensional classical Navier-Stokes system, and also coincides with the result in [5] for (1.1). When  $\mu(\rho)$  be a constant, we can get optimal decay of  $\|\nabla u\|_{L^2}^2$  with the order  $-1-2\beta(p)$ , see [6]. Notice the main ingredients of the proof in [6, 10, 12] are the usual energy estimates and the phase space analysis. In our case, due to the additional difficulties mentioned above, we not only need to apply phase space analysis, but also need more explicit energy estimates, see Proposition 3.1 below. We note also that the 3D case with constant viscosity was studied in [1]. Using energy estimates with weight in time and the Fourier splitting method of Schonbek [10] we can generalize this decay in time estimates to the 3D case with variable viscosity.*

Motivated by Proposition 3.1, we have a more general result. Indeed, using interpolation argument we obtain a similar decay rate of the solution, under a weaker assumption on the initial velocity.

**Theorem 1.3.** *For  $1 < p < 2$  and  $0 < \alpha < 1$ , let  $u_0 \in L^p(\mathbb{R}^2) \cap H^\alpha(\mathbb{R}^2)$ ,  $\rho_0 - 1 \in L^2(\mathbb{R}^2)$  and  $\rho_0 \in L^\infty(\mathbb{R}^2)$  with a positive lower bound. We assume that  $(\rho, u, \nabla p)$  is a given Lions weak*

solution of (1.1) with initial data  $(\rho_0, u_0)$ . Then under the assumption (1.3), there exists a constant  $C_\alpha$  which depends on  $\|\rho_0 - 1\|_{L^2}$ ,  $\|u_0\|_{L^p}$  and  $\|u_0\|_{H^\alpha}$  such that there hold

$$(1.7) \quad \|u(t)\|_{L^2}^2 \leq C_\alpha(t+e)^{-2\beta(p)}, \quad \|\nabla u(t)\|_{L^2}^2 \leq C_\alpha(t+e)^{-1-2\beta(p)+\varepsilon},$$

$$(1.8) \quad \int_0^\infty \|u_t\|_{L^2} + \|\mathbb{P} \operatorname{div}(\mu(\rho)\mathcal{M}(u))\|_{L^2} + \|\mathbb{Q} \operatorname{div}(\mu(\rho)\mathcal{M}(u)) - \nabla \Pi\|_{L^2} dt \leq C_\alpha,$$

$$(1.9) \quad \int_0^\infty t^{1-r}(t+e)^{r+2\beta(p)-\varepsilon} \left( \|u_t\|_{L^2} + \|\mathbb{P} \operatorname{div}(\mu(\rho)\mathcal{M}(u))\|_{L^2} + \|\mathbb{Q} \operatorname{div}(\mu(\rho)\mathcal{M}(u)) - \nabla \Pi\|_{L^2} \right)^2 dt \leq C_\alpha,$$

with any  $\varepsilon > 0$  and  $0 < r < \alpha$ .

**Remark 1.2.** We note also that the 3D case with constant viscosity was studied in [1]. Using energy estimates with weight in time and the Fourier splitting method of Schonbek [10] we can generalize this decay in time estimates to the 3D case with variable viscosity.

In the second part of this paper, we investigate the regularity propagation of transport equation. We consider the transport equation:

$$(1.10) \quad \begin{cases} \partial_t \rho + u \nabla \rho = 0, & (t, x) \in \mathbb{R}^+ \times \mathbb{R}^2, \\ \operatorname{div} u = 0, \\ \rho|_{t=0} = \rho_0. \end{cases}$$

In the case of  $u \in L^1(\text{Lip})$ , for any small positive regularity, it is well known that

$$\|\rho(t)\|_{B_{p,r}^\varepsilon} \leq \|\rho_0\|_{B_{p,r}^\varepsilon} \exp(\|u\|_{L_t^1(\text{Lip})}).$$

And if the regularity index is 0, follows from [11], we have

$$\|\rho(t)\|_{B_{p,r}^0} \leq C \|\rho_0\|_{B_{p,r}^0} (1 + \|u\|_{L_t^1(\text{Lip})}).$$

We want to know how it changes from zero regularity to positive regularity. So we define a Besov space with logarithms regularity  $B_{\infty,1}^{\eta \ln}$ , which is just between zero regularity and positive regularity, see Definition 2.2 and Remark 2.1 below. So we gain a polynomial relation between the velocity and the density, which is the case between exponential and linear cases.

According these two results, we give an application about global existence to solutions of (1.1).

**Theorem 1.4.** For  $1 < p < \frac{4}{3}$ , let  $u_0 \in L^p(\mathbb{R}^2) \cap H^1(\mathbb{R}^2)$ . Let  $\rho_0 - 1 \in B_{2,1}^{1+\varepsilon}(\mathbb{R}^2)$  for any  $\varepsilon > 0$ , and  $\rho_0 \in L^\infty(\mathbb{R}^2)$  with positive lower bound. Then there exist positive constant  $\eta > 1$  and  $C_0, c_0$  such that if

$$(1.11) \quad \|\mu(\rho_0) - \mu_0\|_{B_{\infty,1}^{(\eta+1) \ln}} \left( \frac{(1 + \mu_0)G(\rho_0, u_0)}{\mu_0} \right)^{\eta+1} \exp\{(\eta+1) \exp(C_0 \|u_0\|_{L^2}^4)\} \leq c_0 \mu_0,$$

where

$$(1.12) \quad \begin{aligned} G(\rho_0, u_0) &= G_1 \exp(G_2), \\ G_1 &= \|\rho_0 - 1\|_{L^2} + \|\rho_0 - 1\|_{L^2}^7 + \|u_0\|_{L^p} + \|u_0\|_{L^p}^7 + \|u_0\|_{H^1} + \|u_0\|_{L^p}^7 \|u_0\|_{H^1}^7 \\ &\quad + (1 + \|\rho_0 - 1\|_{L^2}^7) \|u_0\|_{H^1}^{14}, \\ G_2 &= \|u_0\|_{L^p}^4 + \|u_0\|_{H^1}^4 + \|\rho_0 - 1\|_{L^2}^4 + \|u_0\|_{L^p}^4 \|u_0\|_{H^1}^4 + (1 + \|\rho_0 - 1\|_{L^2}^4) \|u_0\|_{H^1}^8, \end{aligned}$$

(1.1) has a global solution  $(\rho, u)$  such that  $\rho - 1 \in L^\infty((0, T); B_{2,1}^{1+\frac{\varepsilon}{2}}(\mathbb{R}^2))$  for any  $T > 0$ , and  $\nabla u \in L^1(\mathbb{R}^+; B_{\infty,2}^0)$ .

**Remark 1.3.** We don't mention the result of uniqueness, and this is well known in [3].

**The organization of the paper.** In the second section, we collect some basic facts on Littlewood-Paley theory and integral inequalities, which have been used throughout this paper. In Section 3, we shall present the proof of Theorem 1.2. In Section 4, we shall prove Theorem 1.3. In Section 5, we give an application of Theorem 1.2.

Let us complete this section by the notations we shall use in this context:

**Notation.** For  $a \lesssim b$ , we mean that there is a uniform constant  $C$ , which may be different on different lines, such that  $a \leq Cb$ .  $a \approx b$  means that there is two positive uniform constant  $c, C$  such that  $cb \leq a \leq Cb$ . We shall denote by  $(c_{j,r})_{j \in \mathbb{N}}$  to be a generic element of  $\ell^r(\mathbb{N})$  so that  $c_{j,r} \geq 0$  and  $\sum_{j \in \mathbb{N}} c_{j,r}^r = 1$ .

## 2. PRELIMINARIES

First, we are going to recall some facts on the Littlewood-Paley Theory, one may check [2] for details. Let  $B \stackrel{\text{def}}{=} \{\xi \in \mathbb{R}^2, |\xi| \leq \frac{4}{3}\}$  and  $\mathcal{C} \stackrel{\text{def}}{=} \{\xi \in \mathbb{R}^2, \frac{3}{4} \leq |\xi| \leq \frac{8}{3}\}$ . Let  $\chi \in C_c^\infty(B)$  and  $\varphi \in C_c^\infty(\mathcal{C})$  which satisfy

$$\chi(\xi) + \sum_{j \geq 0} \varphi(2^{-j}\xi) = 1, \quad \xi \in \mathbb{R}^2,$$

we denote  $h \stackrel{\text{def}}{=} \mathcal{F}^{-1}\varphi$  and  $\tilde{h} \stackrel{\text{def}}{=} \mathcal{F}^{-1}\chi$ . Then the Littlewood-Paley operators  $\Delta_j$  and  $S_j$  can be defined as follows

$$\begin{aligned} \Delta_j f &\stackrel{\text{def}}{=} \varphi(2^{-j}D)f = 2^{2j} \int_{\mathbb{R}^2} h(2^j y) f(x-y) dy, \quad \text{for } j \geq 0, \\ (2.1) \quad S_j f &\stackrel{\text{def}}{=} \chi(2^{-j}D)f = \sum_{-1 \leq k \leq j-1} \Delta_k f = 2^{2j} \int_{\mathbb{R}^2} \tilde{h}(2^j y) f(x-y) dy, \\ \Delta_{-1} f &\stackrel{\text{def}}{=} S_{-1} f, \quad S_{-2} f = 0. \end{aligned}$$

With the introduction of  $\Delta_j$  and  $S_j$ , we define two norm which will be used throughout of our work.

**Definition 2.1.** Let  $s \in \mathbb{R}$  and  $1 \leq p, r \leq \infty$ . The inhomogeneous Besov space  $B_{p,r}^s$  consists of all tempered distributions  $u$  such that

$$\|u\|_{B_{p,r}^s} \stackrel{\text{def}}{=} \|(2^{js} \|\Delta_j u\|_{L^p})_{j \geq -1}\|_{\ell^r} < \infty.$$

**Definition 2.2.** Let  $\eta > 0$ . The logarithms inhomogeneous Besov space  $B_{\infty,1}^{\eta \ln}$  consists of all tempered distributions  $u$  such that

$$\|u\|_{B_{\infty,1}^{\eta \ln}} \stackrel{\text{def}}{=} \sum_{j \geq -1} (2+j)^\eta \|\Delta_j u\|_{L^\infty} < \infty.$$

**Remark 2.1.** One may see that for any positive  $\varepsilon$  and  $\eta$ ,  $B_{\infty,1}^\varepsilon \subset B_{\infty,1}^{\eta \ln} \subset B_{\infty,1}^0$ .

Let us recall the following lemmas from [2].

**Lemma 2.1** (Bernstein's inequality). Let  $1 \leq p \leq q \leq \infty$ . Assume that  $f \in L^p$ , then there exists a positive constant  $C$  independent of  $f, j$  such that

$$\begin{aligned} \text{supp } \hat{f} \subset \{|\xi| \lesssim 2^j\} &\Rightarrow \|\partial^\alpha f\|_{L^q} \leq C 2^{j|\alpha|+2j(\frac{1}{p}-\frac{1}{q})} \|f\|_{L^p}, \\ \text{supp } \hat{f} \subset \{|\xi| \approx 2^j\} &\Rightarrow \|f\|_{L^p} \leq C 2^{-j|\alpha|} \|\partial^\alpha f\|_{L^p}. \end{aligned}$$

**Lemma 2.2.** *Let  $\phi$  be a smooth function supported in the annulus  $\{\xi \in \mathbb{R}^2 : |\xi| \approx 1\}$ . Then, there exist two positive constants  $c$  and  $C$  depending only on  $\phi$  such that for any  $1 \leq p \leq \infty$  and  $\lambda > 0$ , we have*

$$\|\phi(\lambda^{-1}D)e^{t\Delta}f\|_{L^p} \leq Ce^{-ct\lambda^2}\|\phi(\lambda^{-1}D)f\|_{L^p}.$$

In what follows, we will constantly use Bony's decomposition

$$(2.2) \quad uv = T_u v + T_v u + R(u, v)$$

where

$$T_u v = \sum_{j \geq -1} S_{j-1} u \Delta_j v \quad \text{and} \quad R(u, v) = \sum_{j \geq -1} \Delta_j u \tilde{\Delta}_j v,$$

where  $\tilde{\Delta}_j = \sum_{i=-1}^1 \Delta_{j+i}$ .

Also, we need some calculus inequalities which can be found in [12].

**Lemma 2.3.** *Let  $m \in \mathbb{R}^+$ ,  $0 \leq \alpha < 1$  and  $\beta > 0$ . Then*

- (1)  $\int_0^t (s+e)^{-1} \ln(s+e)^{-m} ds \leq \frac{1}{m-1}$  for  $m > 1$ ,
- (2) there is some  $\gamma_m > 0$  such that  $\int_0^t (s+e)^{-1-\beta} \ln(s+e)^m ds \leq \gamma_m \beta^{-(m+1)}$ ,
- (3) there is some  $\gamma_{m,\alpha} > 0$  such that, for all  $t > 0$ ,  
 $\int_0^t (s+e)^{-\alpha} \ln(s+e)^{-m} ds \leq \gamma_{m,\alpha} (t+e)^{1-\alpha} \ln(t+e)^{-m}$ .

Finally, we need the integral form of Gronwall's inequality, which is well known.

**Proposition 2.1.** *Let  $f, g, h$  be positive functions defined on  $\mathbb{R}^+$ , and  $h(t) \in L^1_{loc}$ . If*

$$f(t) \leq g(t) + \int_0^t h(s)f(s) ds,$$

then following estimate holds:

$$(2.3) \quad f(t) \leq g(t) + \int_0^t h(s)g(s) \exp\left(\int_s^t h(\tau) d\tau\right) ds.$$

### 3. THE PROOF OF THEOREM 1.2

In this section, we will prove Theorem 1.2. First, we have some energy estimates.

**Proposition 3.1.** *Let  $v \in L^\infty(\mathbb{R}^+; L^2) \cap L^2(\mathbb{R}^+; \dot{H}^1)$ ,  $\operatorname{div} v = 0$ . Assume that  $u_0 \in H^1(\mathbb{R}^2)$  and  $\rho_0 \in L^\infty(\mathbb{R}^2)$  with positive lower bound.  $f(t)$  be a positive second-order differentiable function satisfies  $f'(t) \geq 0$  and  $f''(t) \geq 0$ .  $(\rho, u)$  be the global weak solution of the linear system:*

$$(3.1) \quad \begin{cases} \partial_t \rho + v \nabla \rho = 0, \\ \rho \partial_t u + \rho v \nabla u - \operatorname{div}(\mu(\rho)\mathcal{M}(u)) + \nabla \Pi = 0, \\ \operatorname{div} u = 0, \\ (\rho, u)|_{t=0} = (\rho_0, u_0). \end{cases}$$

Then under the assumption (1.3), we have the following estimates:

$$(3.2) \quad \begin{aligned} & \sup_{0 < t < \infty} f(t) \int_{\mathbb{R}^2} \mu(\rho) |\nabla u|^2(t) dx \\ & + \int_0^\infty f(t) \int_{\mathbb{R}^2} |\sqrt{\rho} u_t|^2 + |\mathbb{P} \operatorname{div}(\mu(\rho)\mathcal{M}(u))|^2 + |\mathbb{Q} \operatorname{div}(\mu(\rho)\mathcal{M}(u)) - \nabla \Pi|^2 dx dt \\ & \leq C(f(0) \|\nabla u_0\|_{L^2}^2 + \int_0^\infty f'(t) \int_{\mathbb{R}^2} \mu(\rho) |\nabla u|^2 dx dt) \exp\{C(1 + \|v\|_{L^\infty(L^2)}^2) \|\nabla v\|_{L^2(L^2)}^2\}, \end{aligned}$$

$$(3.3) \quad \sup_{0 < t < \infty} f'(t) \int_{\mathbb{R}^2} \rho |u|^2(t) dx + \int_0^\infty f'(t) \int_{\mathbb{R}^2} \mu(\rho) |\nabla u|^2 dx dt \leq C(f'(0) \|u_0\|_{L^2}^2 + \int_0^\infty f''(t) \int_{\mathbb{R}^2} \rho |u|^2 dx dt).$$

*Proof.* First, we follow the line of the proof of Theorem 1.1, see [4]. By taking  $L^2$  inner product of the momentum equation of (3.1) with  $f(t)u_t$  and using integration by parts, we deduce that

$$f(t) \int_{\mathbb{R}^2} |\sqrt{\rho}u_t|^2 dx + f(t) \int_{\mathbb{R}^2} (\rho v \nabla u) \cdot u_t dx + f(t) \int_{\mathbb{R}^2} \mu(\rho) \nabla u : \nabla u_t dx = 0.$$

Note that

$$\begin{aligned} f(t) \int_{\mathbb{R}^2} \mu(\rho) \nabla u : \nabla u_t dx &= \frac{1}{2} \partial_t [f(t) \int_{\mathbb{R}^2} \mu(\rho) |\nabla u|^2 dx] - \frac{1}{2} f'(t) \int_{\mathbb{R}^2} \mu(\rho) |\nabla u|^2 dx \\ &\quad - \frac{1}{2} f(t) \int_{\mathbb{R}^2} \partial_t \mu(\rho) |\nabla u|^2 dx, \end{aligned}$$

and from the derivation of (29) in [4] that

$$\begin{aligned} - \int_{\mathbb{R}^2} \partial_t \mu(\rho) |\nabla u|^2 dx &= \int_{\mathbb{R}^2} \operatorname{div}(\mu(\rho)v) |\nabla u|^2 dx \\ &= \int_{\mathbb{R}^2} (v \nabla) u \cdot \operatorname{div}(\mu(\rho)\mathcal{M}(u)) dx + \int_{\mathbb{R}^2} \mu(\rho) \operatorname{tr}(\nabla v \nabla u \mathcal{M}(u)) dx \\ &= \int_{\mathbb{R}^2} (v \nabla) u \cdot (\rho u_t + \rho v \nabla u + \nabla \Pi) dx + \int_{\mathbb{R}^2} \mu(\rho) \operatorname{tr}(\nabla v \nabla u \mathcal{M}(u)) dx, \end{aligned}$$

we have

$$\begin{aligned} &\frac{d}{dt} [f(t) \int_{\mathbb{R}^2} \mu(\rho) |\nabla u|^2 dx] + f(t) \int_{\mathbb{R}^2} |\sqrt{\rho}u_t|^2 dx \\ &\lesssim f'(t) \int_{\mathbb{R}^2} \mu(\rho) |\nabla u|^2 dx + f(t) \int_{\mathbb{R}^2} |\sqrt{\rho}v \nabla u|^2 dx \\ &\quad + f(t) \int_{\mathbb{R}^2} \mu(\rho) |\nabla v| |\nabla u|^2 dx + f(t) \left| \int_{\mathbb{R}^2} \Pi \partial_i v_j \partial_j u_i dx \right|. \end{aligned}$$

Recall that

$$-\mu_0 \Delta u = \operatorname{div}((\mu(\rho) - \mu_0)\mathcal{M}(u)) - \operatorname{div}(\mu(\rho)\mathcal{M}(u)),$$

so that we have

$$\mu_0 \partial_i u_j = \mathcal{R}_i \mathbb{P}_j \mathcal{R}((\mu(\rho) - \mu_0)\mathcal{M}(u)) - \mathcal{R}_i \mathbb{P}_j \mathcal{R}(\mu(\rho)\mathcal{M}(u)).$$

Estimating it in the  $L^4(\mathbb{R}^2)$  and using the Gagliardo-Nirenberg inequality, we can write

$$\begin{aligned} \|\nabla u\|_{L^4} &\lesssim \|\mathbb{P} \otimes \mathbb{Q}((\mu(\rho) - \mu_0)\mathcal{M}(u))\|_{L^4} + \|\mathbb{P} \otimes \mathbb{Q}(\mu(\rho)\mathcal{M}(u))\|_{L^4} \\ &\lesssim \|\mu(\rho) - \mu_0\|_{L^\infty(\mathbb{R}^+; L^\infty)} \|\nabla u\|_{L^4} + \|\mathbb{P} \otimes \mathbb{Q}(\mu(\rho)\mathcal{M}(u))\|_{L^2}^{\frac{1}{2}} \|\nabla(\mathbb{P} \otimes \mathbb{Q}(\mu(\rho)\mathcal{M}(u)))\|_{L^2}^{\frac{1}{2}} \end{aligned}$$

Finally, using (1.3) and the conservation of the momentum, we obtain that

$$(3.4) \quad \|\nabla u\|_{L^4} \lesssim \|\nabla u\|_{L^2}^{\frac{1}{2}} \|\mathbb{P}(\rho u_t + \rho v \nabla u)\|_{L^2}^{\frac{1}{2}},$$

Now letting  $(-\Delta)^{-\frac{1}{2}} \mathcal{R}$  operate on the equation of momentum, we get that

$$\Pi = \mathcal{R}_i \mathcal{R}_j (\mu(\rho) (\partial_i u_j + \partial_j u_i)) + (-\Delta)^{-\frac{1}{2}} \mathcal{R}(\rho u_t + \rho v \nabla u).$$

It follows that

$$\|\Pi - \mathcal{R}_i \mathcal{R}_j (\mu(\rho) \mathcal{M}(u))\|_{BMO} \lesssim \|\nabla(\Pi - \mathcal{R}_i \mathcal{R}_j (\mu(\rho) \mathcal{M}(u)))\|_{L^2} \lesssim \|\rho u_t + \rho v \nabla u\|_{L^2}.$$

We obtain that

$$\begin{aligned} \left| \int_{\mathbb{R}^2} \Pi \partial_i v_j \partial_j u_i dx \right| &\leq \|\nabla v\|_{L^2} \|\nabla u\|_{L^4}^2 + \|\Pi - \mathcal{R}_i \mathcal{R}_j (\mu(\rho) \mathcal{M}(u))\|_{BMO} \|\partial_i v_j \partial_j u_i\|_{\mathcal{H}^1} \\ &\leq \|\nabla v\|_{L^2} \|\nabla u\|_{L^2} \|\rho u_t + \rho v \nabla u\|_{L^2}, \end{aligned}$$

so that

$$f(t) \left| \int_{\mathbb{R}^2} \Pi \partial_i u_j \partial_j u_i dx \right| \leq C_\varepsilon f(t) \|\nabla v\|_{L^2}^2 \|\nabla u\|_{L^2}^2 + \varepsilon f(t) (\|\sqrt{\rho} u_t\|_{L^2}^2 + \|v \nabla u\|_{L^2}^2).$$

$$\begin{aligned} f(t) \int_{\mathbb{R}^2} \mu(\rho) |\nabla v| |\nabla u|^2 dx &\leq C f(t) \|\nabla v\|_{L^2} \|\nabla u\|_{L^4}^2 \\ &\leq C_\varepsilon f(t) \|\nabla v\|_{L^2}^2 \|\nabla u\|_{L^2}^2 + \varepsilon f(t) (\|\sqrt{\rho} u_t\|_{L^2}^2 + \|v \nabla u\|_{L^2}^2). \end{aligned}$$

Also

$$\begin{aligned} \|v \nabla u\|_{L^2}^2 &\leq \|v\|_{L^4}^2 \|\nabla u\|_{L^4}^2 \leq \|v\|_{L^2} \|\nabla v\|_{L^2} \|\nabla u\|_{L^2} \|\rho u_t + \rho v \nabla u\|_{L^2}, \\ f(t) \int_{\mathbb{R}^2} |\sqrt{\rho} v \nabla u|^2 dx &\leq C_\varepsilon f(t) \|v\|_{L^2}^2 \|\nabla v\|_{L^2}^2 \|\nabla u\|_{L^2}^2 + \varepsilon f(t) \|\sqrt{\rho} u_t\|_{L^2}^2. \end{aligned}$$

Consequently,

$$\begin{aligned} (3.5) \quad &\frac{d}{dt} [f(t) \int_{\mathbb{R}^2} \mu(\rho) |\nabla u|^2 dx] + f(t) \int_{\mathbb{R}^2} |\sqrt{\rho} u_t|^2 dx \\ &\lesssim f'(t) \int_{\mathbb{R}^2} \mu(\rho) |\nabla u|^2 dx + f(t) \|\nabla u\|_{L^2}^2 \|\nabla v\|_{L^2}^2 (1 + \|v\|_{L^2}^2). \end{aligned}$$

Second, we act the Leray projector  $\mathbb{P}$  on the momentum equation of (3.1) to get that

$$\begin{aligned} \mathbb{P} \operatorname{div}(\mu(\rho) \mathcal{M}(u)) &= \mathbb{P}(\rho u_t + \rho v \nabla u), \\ \mathbb{Q} \operatorname{div}(\mu(\rho) \mathcal{M}(u)) - \nabla \Pi &= \mathbb{Q}(\rho u_t + \rho v \nabla u). \end{aligned}$$

Along with (3.5), we have

$$\begin{aligned} &\frac{d}{dt} [f(t) \int_{\mathbb{R}^2} \mu(\rho) |\nabla u|^2 dx] \\ &\quad + f(t) \int_{\mathbb{R}^2} |\sqrt{\rho} u_t|^2 + |\mathbb{P} \operatorname{div}(\mu(\rho) \mathcal{M}(u))|^2 + |\mathbb{Q} \operatorname{div}(\mu(\rho) \mathcal{M}(u)) - \nabla \Pi|^2 dx \\ &\leq C(f'(t) \int_{\mathbb{R}^2} \mu(\rho) |\nabla u|^2 dx + f(t) \|\nabla u\|_{L^2}^2 \|\nabla v\|_{L^2}^2 (1 + \|v\|_{L^2}^2)). \end{aligned}$$

Note that  $v \in L^\infty(L^2) \cap L^2(\dot{H}^1)$ , so that

$$\int_0^\infty (1 + \|v\|_{L^2}^2) \|\nabla v\|_{L^2}^2 dt \leq (1 + \|v\|_{L^\infty(L^2)}^2) \|\nabla v\|_{L^2(L^2)}^2,$$

and (3.2) holds.

The same strategy can be held for  $f'(t)u$ , we have

$$\frac{1}{2} \frac{d}{dt} [f'(t) \int_{\mathbb{R}^2} |\sqrt{\rho} u|^2 dx] + f'(t) \int_{\mathbb{R}^2} \mu(\rho) |\nabla u|^2 dx = \frac{1}{2} f''(t) \int_{\mathbb{R}^2} |\sqrt{\rho} u|^2 dx,$$

so that

$$\sup_{0 < t < \infty} f'(t) \int_{\mathbb{R}^2} \rho |u|^2(t) dx + \int_0^\infty f'(t) \int_{\mathbb{R}^2} \mu(\rho) |\nabla u|^2 dx dt \leq C(f'(0) \|u_0\|_{L^2}^2 + \int_0^\infty f''(t) \int_{\mathbb{R}^2} \rho |u|^2 dx dt).$$

□

According these two energy estimates, letting  $v = u$ , we can prove Theorem 1.2. More precisely, we have the following theorem.



**Theorem 3.1.** *Under the assumption of Theorem 1.2, (1.4), (1.5) and (1.6) hold. More precisely, we have*

$$(3.6) \quad \|u(t)\|_{L^2}^2 \lesssim (K + K^7) \exp(K^2)(t + e)^{-2\beta(p)}, \quad \|\nabla u(t)\|_{L^2}^2 \lesssim (K + K^7) \exp(K^2)(t + e)^{-1-2\beta(p)+\varepsilon},$$

$$(3.7) \quad \int_0^\infty \|u_t\|_{L^2} + \|\mathbb{P} \operatorname{div}(\mu(\rho)\mathcal{M}(u))\|_{L^2} + \|\mathbb{Q} \operatorname{div}(\mu(\rho)\mathcal{M}(u)) - \nabla \Pi\|_{L^2} dt \lesssim \sqrt{K} + K,$$

$$(3.8) \quad \int_0^\infty (t+e)^{1+2\beta(p)-\varepsilon} \left( \|u_t\|_{L^2} + \|\mathbb{P} \operatorname{div}(\mu(\rho)\mathcal{M}(u))\|_{L^2} + \|\mathbb{Q} \operatorname{div}(\mu(\rho)\mathcal{M}(u)) - \nabla \Pi\|_{L^2} \right)^2 dt \lesssim (K + K^7) \exp(K^2),$$

where

$$(3.9) \quad K = \left( \|u_0\|_{L^p}^2 + \|u_0\|_{H^1}^2 + \|\rho_0 - 1\|_{L^2}^2 + \|u_0\|_{L^p}^2 \|u_0\|_{H^1}^2 + (1 + \|\rho_0 - 1\|_{L^2}^2) \|u_0\|_{H^1}^4 \right) \exp\{C\|u_0\|_{L^2}^4\}.$$

*Proof.* We get the standard energy estimate to (1.1) that

$$\frac{d}{dt} \|\sqrt{\rho}u(t)\|_{L^2}^2 + \|\nabla u(t)\|_{L^2}^2 \leq 0.$$

Using Schonbek's strategy, we obtain

$$(3.10) \quad \frac{d}{dt} \|\sqrt{\rho}u(t)\|_{L^2}^2 + g^2(t) \|\sqrt{\rho}u(t)\|_{L^2}^2 \leq M g^2(t) \int_{S(t)} |\hat{u}(t, \xi)|^2 d\xi,$$

where  $S(t) \stackrel{def}{=} \{\xi : |\xi| \leq \sqrt{\frac{M}{2}}g(t)\}$  and  $g(t)$  satisfying  $g(t) \lesssim (1+t)^{-\frac{1}{2}}$ . We rewrite the momentum equation of (1.1) as

$$u(t) = e^{\mu_0 t \Delta} u_0 + \int_0^t e^{\mu_0(t-s)\Delta} \mathbb{P} \left( \operatorname{div}((\mu(\rho) - \mu_0)\mathcal{M}(u)) + (1 - \rho)u_t - \rho u \nabla u \right) (s) ds.$$

Taking Fourier transform with respect to  $x$  variables leads to

$$|\hat{u}(t, \xi)| \lesssim e^{-\mu_0 t |\xi|^2} |\hat{u}_0(\xi)| + \int_0^t e^{-\mu_0(t-s)|\xi|^2} \left[ |\xi| |\mathcal{F}_x((\mu(\rho) - \mu_0)\mathcal{M}(u))| + |\mathcal{F}_x((1 - \rho)u_t - \rho u \nabla u)| \right] ds,$$

which implies that

$$(3.11) \quad \int_{S(t)} |\hat{u}(t, \xi)|^2 d\xi \lesssim \int_{S(t)} e^{-2\mu_0 t |\xi|^2} |\hat{u}_0(\xi)|^2 d\xi + g^4(t) \left( \int_0^t \|\mathcal{F}_x((\mu(\rho) - \mu_0)\mathcal{M}(u))\|_{L_\xi^\infty} ds \right)^2 + g^2(t) \left( \int_0^t \|\mathcal{F}_x((1 - \rho)u_t - \rho u \nabla u)\|_{L_\xi^\infty} ds \right)^2.$$

Note that  $u_0 \in L^p$  for  $1 < p < 2$ , one has

$$(3.12) \quad \int_{S(t)} e^{-2\mu_0 t |\xi|^2} |\hat{u}_0(\xi)|^2 d\xi \lesssim \left( \int_{S(t)} e^{-2\mu_0 q t |\xi|^2} d\xi \right)^{\frac{1}{q}} \|\hat{u}_0(\xi)\|_{L^{p'}}^2 \lesssim \|u_0\|_{L^p}^2 (1+t)^{-2\beta(p)},$$

where  $\frac{1}{q} = \frac{2}{p} - 1$ ,  $\frac{1}{p} + \frac{1}{p'} = 1$ .

Note that  $u \in L^\infty(L^2) \cap L^2(\dot{H}^1)$  and  $u_t \in L^2(L^2)$ , we have

$$\begin{aligned} \left( \int_0^t \|\mathcal{F}_x((\mu(\rho) - \mu_0)\mathcal{M}(u))\|_{L_\xi^\infty} ds \right)^2 &\leq \left( \int_0^t \|(\mu(\rho) - \mu_0)\mathcal{M}(u)\|_{L^1} ds \right)^2 \\ &\leq \|\mu(\rho) - \mu_0\|_{L_\xi^\infty(L^2)}^2 \left( \int_0^t \|\nabla u\|_{L^2} ds \right)^2 \\ &\leq C \|\rho_0 - 1\|_{L^2}^2 \|u_0\|_{L^2}^2 (1+t), \end{aligned}$$

$$\left(\int_0^t \|\mathcal{F}_x((1-\rho)u_t)\|_{L_\xi^\infty} ds\right)^2 \leq \|1-\rho\|_{L_t^\infty(L^2)}^2 \left(\int_0^t \|u_t\|_{L^2} ds\right)^2 \leq C\|\rho_0-1\|_{L^2}^2 \|\nabla u_0\|_{L^2}^2 (1+t),$$

$$\left(\int_0^t \|\mathcal{F}_x(\rho u \nabla u)\|_{L_\xi^\infty} ds\right)^2 \leq \|\rho u\|_{L_t^\infty(L^2)}^2 \left(\int_0^t \|\nabla u\|_{L^2} ds\right)^2 \leq C\|u_0\|_{L^2}^4 (1+t).$$

Then we deduce from (3.10) to (3.12) that

$$\begin{aligned} & \frac{d}{dt} \|\sqrt{\rho}u(t)\|_{L^2}^2 + g^2(t) \|\sqrt{\rho}u(t)\|_{L^2}^2 \\ & \lesssim g^2(t) \|u_0\|_{L^p}^2 (1+t)^{-2\beta(p)} + g^6(t) \|\rho_0-1\|_{L^2}^2 \|u_0\|_{L^2}^2 (1+t) \\ & \quad + g^4(t) (\|\rho_0-1\|_{L^2}^2 \|\nabla u_0\|_{L^2}^2 + \|u_0\|_{L^2}^4) (1+t) \\ & \lesssim g^2(t) \|u_0\|_{L^p}^2 (1+t)^{-2\beta(p)} + g^4(t) (\|\rho_0-1\|_{L^2}^2 \|u_0\|_{H^1}^2 + \|u_0\|_{L^2}^4) (1+t). \end{aligned}$$

Taking  $g^2(t) = \frac{2}{(e+t)\ln(e+t)}$ , then  $e^{\int_0^t g^2(s) ds} = \ln^2(t+e)$  and

$$\begin{aligned} & \ln^2(t+e) \|u(t)\|_{L^2}^2 \\ & \lesssim \|u_0\|_{L^2}^2 + \int_0^t [\|u_0\|_{L^p}^2 \frac{\ln(s+e)}{(s+e)^{1+2\beta(p)}} + (\|\rho_0-1\|_{L^2}^2 \|u_0\|_{H^1}^2 + \|u_0\|_{L^2}^4) (s+e)^{-1}] ds \\ & \lesssim \left( \|u_0\|_{L^p}^2 + (1 + \|\rho_0-1\|_{L^2}^2) \|u_0\|_{H^1}^2 + \|u_0\|_{L^2}^4 \right) \ln(t+e), \end{aligned}$$

which gives

$$(3.13) \quad \|u(t)\|_{L^2}^2 \lesssim \left( \|u_0\|_{L^p}^2 + (1 + \|\rho_0-1\|_{L^2}^2) \|u_0\|_{H^1}^2 + \|u_0\|_{L^2}^4 \right) \ln^{-1}(t+e).$$

Now we improve the estimate (3.13).

We choose  $f(t) = t+e$  in (3.2), then we have

$$\sup_{0 < t < \infty} (t+e) \|\nabla u(t)\|_{L^2}^2 + \int_0^\infty (t+e) \|u_t\|_{L^2}^2 dt \leq C\|u_0\|_{H^1}^2 \exp\{C\|u_0\|_{L^2}^4\},$$

so that

$$\begin{aligned} \left(\int_0^t \|\mathcal{F}_x((1-\rho)u_t)\|_{L_\xi^\infty} ds\right)^2 & \leq \|1-\rho\|_{L_t^\infty(L^2)}^2 \left(\int_0^t \|u_t\|_{L^2} ds\right)^2 \\ & \leq C\|\rho_0-1\|_{L^2}^2 \int_0^t (s+e) \|u_t\|_{L^2}^2 ds \int_0^t (s+e)^{-1} ds \\ & \leq C\|\rho_0-1\|_{L^2}^2 \|u_0\|_{H^1}^2 \exp\{C\|u_0\|_{L^2}^4\} \ln(t+e). \end{aligned}$$

$$\begin{aligned} \left(\int_0^t \|\mathcal{F}_x(\rho u \nabla u)\|_{L_\xi^\infty} ds\right)^2 & \leq \left(\int_0^t \|u(s)\|_{L^2} \|\nabla u(s)\|_{L^2} ds\right)^2 \\ & \leq C \left( \|u_0\|_{L^p}^2 + (1 + \|\rho_0-1\|_{L^2}^2) \|u_0\|_{H^1}^2 \right) \|u_0\|_{H^1}^2 \exp\{C\|u_0\|_{L^2}^4\} (t+e) \ln^{-1}(t+e). \end{aligned}$$

We plug these estimate into (3.11) and take  $g^2(t) = \frac{3}{(e+t)\ln(e+t)}$ , then  $e^{\int_0^t g^2(s) ds} = \ln^3(t+e)$  and

$$\begin{aligned} \ln^3(t+e) \|u(t)\|_{L^2}^2 & \lesssim K \int_0^t \left[ \frac{\ln^2(s+e)}{(s+e)^{1+2\beta(p)}} + \frac{\ln^2(s+e)}{(s+e)^2} + (s+e)^{-1} \right] ds \\ & \lesssim K \ln(t+e), \end{aligned}$$

which implies

$$(3.14) \quad \|u(t)\|_{L^2}^2 \lesssim K \ln^{-2}(t+e).$$

So that

$$\int_0^\infty (t+e)^{-1} \|u\|_{L^2}^2 dt \lesssim K.$$

We choose  $f'(t) = \ln(t+e)$  in (3.3), then get

$$\begin{aligned} & \sup_{0 < t < \infty} \ln(t+e) \|u(t)\|_{L^2}^2 + \int_0^\infty \ln(t+e) \|\nabla u\|_{L^2}^2 dt \\ & \leq C(\|u_0\|_{L^2}^2 + \int_0^\infty (t+e)^{-1} \|u\|_{L^2}^2 dt) \\ & \lesssim K. \end{aligned}$$

Consequently, we take  $f(t) = (t+e) \ln(t+e)$  in (3.2), obtain that

$$\begin{aligned} & \sup_{0 < t < \infty} (t+e) \ln(t+e) \|\nabla u(t)\|_{L^2}^2 + \int_0^\infty (t+e) \ln(t+e) \|u_t\|_{L^2}^2 dt \\ & \leq C(\|\nabla u_0\|_{L^2}^2 + \int_0^\infty (\ln(t+e) + 1) \|\nabla u\|_{L^2}^2 dx dt) \exp\{C\|u_0\|_{L^2}^4\} \\ & \lesssim K, \end{aligned}$$

which implies

$$(3.15) \quad \|\nabla u(t)\|_{L^2}^2 \lesssim K(t+e)^{-1} \ln^{-1}(t+e).$$

Combining (3.14) and (3.15), we get the revised estimates,

$$\begin{aligned} \left(\int_0^t \|u\|_{L^2} \|\nabla u\|_{L^2} ds\right)^2 & \lesssim K^2 \left(\int_0^t (s+e)^{-\frac{1}{2}} \ln^{-\frac{3}{2}}(s+e) ds\right)^2 \\ & \lesssim K^2 (t+e) \ln^{-3}(t+e), \end{aligned}$$

$$\begin{aligned} \left(\int_0^t \|1-\rho\|_{L^2} \|u_t\|_{L^2} ds\right)^2 & \lesssim K \left(\int_0^t (s+e) \ln(s+e) \|u_t\|_{L^2}^2 ds\right) \left(\int_0^t (s+e)^{-1} \ln^{-1}(s+e) ds\right) \\ & \lesssim K^2 \ln(\ln(t+e)). \end{aligned}$$

Substituting these two estimates in (3.11), and taking  $g^2(t) = \frac{5}{(t+e)\ln(t+e)}$ , then  $e^{\int_0^t g^2(s) ds} = \ln^5(t+e)$  and

$$\begin{aligned} \ln^5(t+e) \|u(t)\|_{L^2}^2 & \lesssim \|u_0\|_{L^2}^2 + \int_0^t \left[ \|u_0\|_{L^p}^2 \frac{\ln^4(s+e)}{(s+e)^{1+2\beta(p)}} + K^2 \frac{\ln^3(s+e) \ln(\ln(t+e))}{(s+e)^2} + \frac{K^2}{s+e} \right] ds \\ & \lesssim (K + K^2) \ln(t+e), \end{aligned}$$

from which, we obtain

$$(3.16) \quad \|u(t)\|_{L^2}^2 \lesssim (K + K^2) \ln^{-4}(t+e).$$

We choose  $f'(t) = \ln^2(t+e)$  in (3.3), then

$$\begin{aligned} & \sup_{0 < t < \infty} \ln^2(t+e) \|u(t)\|_{L^2}^2 + \int_0^\infty \ln^2(t+e) \|\nabla u\|_{L^2}^2 dt \\ & \leq C(\|u_0\|_{L^2}^2 + \int_0^\infty (t+e)^{-1} \ln(t+e) \|u(t)\|_{L^2}^2 dt) \\ & \leq C(\|u_0\|_{L^2}^2 + (K + K^2) \int_0^\infty (t+e)^{-1} \ln^{-3}(t+e) dt) \\ & \lesssim K + K^2. \end{aligned}$$

Finally, we take  $f(t) = (t + e) \ln^2(t + e)$  in (3.2) to get that

$$\begin{aligned} & \sup_{0 < t < \infty} (t + e) \ln^2(t + e) \|\nabla u(t)\|_{L^2}^2 + \int_0^\infty (t + e) \ln^2(t + e) \|u_t\|_{L^2}^2 dt \\ & \leq C \left( \|\nabla u_0\|_{L^2}^2 + \int_0^\infty (\ln(t + e) + \ln^2(t + e)) \|\nabla u\|_{L^2}^2 dt \right) \exp\{C\|u_0\|_{L^2}^4\} \\ & \lesssim K + K^2. \end{aligned}$$

Consequently, we obtain

$$\left( \int_0^\infty \|u_t\|_{L^2} dt \right)^2 \leq \left( \int_0^\infty (t + e) \ln^2(t + e) \|u_t\|_{L^2}^2 dt \right) \left( \int_0^\infty (t + e)^{-1} \ln^{-2}(t + e) dt \right) \lesssim K + K^2.$$

Which is the same for  $\mathbb{P} \operatorname{div}(\mu(\rho)\mathcal{M}(u))$ ,  $\mathbb{Q} \operatorname{div}(\mu(\rho)\mathcal{M}(u)) - \nabla \Pi \in L^1(\mathbb{R}^+; L^2)$ , and gives (3.7).

Moreover

$$\int_0^\infty \|u \nabla u\|_{L^2} dt \leq \int_0^\infty (\|u\|_{L^2} \|\nabla u\|_{L^2}^2 + \|u_t\|_{L^2}) dt \lesssim \sqrt{K} + K,$$

and

$$\begin{aligned} \left( \int_0^t \|u\|_{L^2} \|\nabla u\|_{L^2} ds \right)^2 & \leq \left( \int_0^t \ln^{-2}(s + e) \|u\|_{L^2}^2 ds \right) \left( \int_0^t \ln^2(s + e) \|\nabla u\|_{L^2}^2 ds \right) \\ & \lesssim (K + K^2) \int_0^t \ln^{-2}(s + e) \|u\|_{L^2}^2 ds. \end{aligned}$$

Substituting these estimates into (3.11), noting that  $2\beta(p) \in (0, 1)$ , and taking  $g^2(t) = \frac{\alpha}{t+e}$  with any positive  $\alpha \in (2\beta(p), 1)$ , then we get

$$\begin{aligned} (t + e)^\alpha \|u(t)\|_{L^2}^2 & \lesssim \|u_0\|_{L^2}^2 + (K + K^2) \int_0^t (s + e)^{\alpha-2} \int_0^s \ln^{-2}(\tau + e) \|u(\tau)\|_{L^2}^2 d\tau ds \\ & + K \int_0^t (s + e)^{\alpha-1-2\beta(p)} ds + (K^2 + K^3) \int_0^t (s + e)^{\alpha-2} ds \\ & \lesssim (K + K^3)(t + e)^{\alpha-2\beta(p)} + (K + K^2) \int_0^t (s + e)^{\alpha-2} \int_0^s \ln^{-2}(\tau + e) \|u(\tau)\|_{L^2}^2 d\tau ds. \end{aligned}$$

For  $t \geq 1$ , we define

$$\begin{aligned} y(t) & \stackrel{\text{def}}{=} \int_{t-1}^t (s + e)^\alpha \|u(s)\|_{L^2}^2 ds, \quad Y(t) \stackrel{\text{def}}{=} \max\{y(s) : 1 \leq s \leq t\}, \\ I(t) & \stackrel{\text{def}}{=} \int_0^t \ln^{-2}(s + e) \|u(s)\|_{L^2}^2 ds. \end{aligned}$$

Then recall that  $\alpha < 1$ , one has

$$\begin{aligned} I(t) & = \int_0^{t-[t]} \ln^{-2}(s + e) \|u(s)\|_{L^2}^2 ds + \int_{t-[t]}^t \ln^{-2}(s + e) \|u(s)\|_{L^2}^2 ds \\ & \lesssim K + \sum_{j=0}^{[t]-1} \int_{t-j-1}^{t-j} \|u(s)\|_{L^2}^2 (s + e)^\alpha (s + e)^{-\alpha} \ln^{-2}(s + e) ds \\ (3.17) \quad & \lesssim K + Y(t) \sum_{j=0}^{[t]-1} (t - j)^{-\alpha} \ln^{-2}(t - j) \lesssim K + Y(t)(t + e)^{1-\alpha} \ln^{-2}(t + e), \end{aligned}$$

from which, we infer that

$$y(t) \lesssim (K + K^3)(t + e)^{\alpha-2\beta(p)} + (K + K^2) \int_0^t (s + e)^{-1} \ln^{-2}(s + e) Y(s) ds.$$

Then, applying Gronwall's inequality (2.3), we have

(3.18)

$$\begin{aligned} Y(t) &\lesssim (K + K^3)(t + e)^{\alpha - 2\beta(p)} \\ &\quad + (K^2 + K^5) \int_0^t (s + e)^{\alpha - 2\beta(p) - 1} \ln^{-2}(s + e) \exp\left\{(K + K^2) \int_s^t (\tau + e)^{-1} \ln^{-2}(\tau + e) d\tau\right\} ds \\ &\lesssim (K + K^3)(t + e)^{\alpha - 2\beta(p)} + (K^2 + K^5) \exp(K^2) \int_0^t (s + e)^{\alpha - 2\beta(p) - 1} \ln^{-2}(s + e) ds \\ &\lesssim (K + K^5) \exp(K^2)(t + e)^{\alpha - 2\beta(p)}. \end{aligned}$$

Plunging (3.18) into (3.17) gives rise to  $I(t) \lesssim (K + K^5) \exp(K^2)(t + e)^{1 - 2\beta(p)} \ln^{-2}(t + e)$ , we obtain

$$\begin{aligned} (t + e)^\alpha \|u(t)\|_{L^2}^2 &\lesssim (K + K^3)(t + e)^{\alpha - 2\beta(p)} + (K^2 + K^7) \exp(K^2) \int_0^t (s + e)^{\alpha - 2\beta(p) - 1} \ln^{-2}(s + e) ds \\ &\lesssim (K + K^7) \exp(K^2)(t + e)^{\alpha - 2\beta(p)}, \end{aligned}$$

which gives the first inequality of (3.6).

Go back to (3.3), we choose  $f''(t)$  such that  $\int_0^\infty f''(t) \|u(t)\|_{L^2}^2 dt$  is finite. For example, we let  $f''(t) = (t + e)^{-1 + 2\beta(p) - \varepsilon}$  for any  $\varepsilon > 0$ , (or  $f''(t) = (t + e)^{-1 + 2\beta(p)} \ln^{-\alpha}(t + e)$  for any  $\alpha > 1$ ), then  $f(t) = (t + e)^{1 + 2\beta(p) - \varepsilon}$ . Finally, we get (3.8) and the second inequality of (3.6).  $\square$

#### 4. THE PROOF OF THEOREM 1.3

The proof of Theorem 1.3 is very similar to Theorem 1.2. We should estimate every term in terms of  $\|u_0\|_{H^\alpha}$  instead of  $\|u_0\|_{H^1}$ . First, we choose  $f(t) = t + e$  and  $t$  in (3.2), get that

$$\sup_{0 < t < \infty} (t + e) \|\nabla u(t)\|_{L^2}^2 + \int_0^\infty (t + e) \|u_t\|_{L^2}^2 dt \leq C \|u_0\|_{H^1}^2 \exp\{C(1 + \|v\|_{L^\infty(L^2)}^2) \|\nabla v\|_{L^2(L^2)}^2\},$$

and

$$\sup_{0 < t < \infty} t \|\nabla u(t)\|_{L^2}^2 + \int_0^\infty t \|u_t\|_{L^2}^2 dt \leq C \|u_0\|_{L^2}^2 \exp\{C(1 + \|v\|_{L^\infty(L^2)}^2) \|\nabla v\|_{L^2(L^2)}^2\}.$$

By interpolation, and let  $v = u$ , we get that

$$(4.1) \quad \sup_{0 < t < \infty} (t + e)^\alpha t^{1 - \alpha} \|\nabla u(t)\|_{L^2}^2 + \int_0^\infty (t + e)^\alpha t^{1 - \alpha} \|u_t\|_{L^2}^2 dt \leq C \|u_0\|_{H^\alpha}^2 \exp\{C \|u_0\|_{L^2}^4\}.$$

So that

$$\begin{aligned} \left(\int_0^t \|\mathcal{F}_x((1 - \rho)u_t)\|_{L^\infty_\xi} ds\right)^2 &\leq \|1 - \rho\|_{L^\infty_t(L^2)}^2 \left(\int_0^t \|u_t\|_{L^2} ds\right)^2 \\ &\leq C \|\rho_0 - 1\|_{L^2}^2 \int_0^t s^{1 - \alpha} (s + e)^\alpha \|u_t\|_{L^2}^2 ds \int_0^t s^{\alpha - 1} (s + e)^{-\alpha} ds \\ &\leq C_\alpha t^\alpha, \end{aligned}$$

$$\begin{aligned} \left(\int_0^t \|\mathcal{F}_x((\mu(\rho) - \mu_0)\mathcal{M}(u))\|_{L^\infty_\xi} ds\right)^2 &\leq \left(\int_0^t \|(\mu(\rho) - \mu_0)\mathcal{M}(u)\|_{L^1} ds\right)^2 \\ &\leq \|\mu(\rho) - \mu_0\|_{L^\infty_t(L^2)}^2 \left(\int_0^t \|\nabla u\|_{L^2} ds\right)^2 \\ &\leq C \|\rho_0 - 1\|_{L^2}^2 \|u_0\|_{L^2}^2 (1 + t), \end{aligned}$$

$$\left(\int_0^t \|\mathcal{F}_x(\rho u \nabla u)\|_{L^\infty_\xi} ds\right)^2 \leq \|\rho u\|_{L^\infty_t(L^2)}^2 \left(\int_0^t \|\nabla u\|_{L^2} ds\right)^2 \leq C \|u_0\|_{L^2}^4 (1 + t).$$

From which, we can deduce that

$$\begin{aligned} & \frac{d}{dt} \|\sqrt{\rho}u(t)\|_{L^2}^2 + g^2(t) \|\sqrt{\rho}u(t)\|_{L^2}^2 \\ & \leq C_\alpha \left( g^2(t)(1+t)^{-2\beta(p)} + g^6(t)(1+t) + g^4(t)(1+t) + g^4(t)t^\alpha \right) \\ & \leq C_\alpha \left( g^2(t)(1+t)^{-2\beta(p)} + g^4(t)(1+t) + g^4(t)t^\alpha \right). \end{aligned}$$

Taking  $g^2(t) = \frac{2}{(e+t)\ln(e+t)}$ , then  $e^{\int_0^t g^2(s) ds} = \ln^2(t+e)$  and

$$\begin{aligned} & \ln^2(t+e) \|u(t)\|_{L^2}^2 \\ & \leq C \|u_0\|_{L^2}^2 + C_\alpha \int_0^t \left[ \frac{\ln(s+e)}{(s+e)^{1+2\beta(p)}} + \frac{1}{s+e} + \frac{1}{(s+e)^{2-\alpha}} \right] ds \\ & \leq C_\alpha \ln(t+e), \end{aligned}$$

which gives

$$(4.2) \quad \|u(t)\|_{L^2}^2 \leq C_\alpha \ln^{-1}(t+e).$$

Now, for  $t > 1$ , we have

$$\begin{aligned} \left( \int_0^t \|\mathcal{F}_x(\rho u \nabla u)\|_{L_\xi^\infty} ds \right)^2 & \leq C \left( \int_0^t \|u\|_{L^2} \|\nabla u\|_{L^2} ds \right)^2 \\ & \leq C_\alpha \int_0^t s^{\frac{\alpha-1}{2}} (s+e)^{-\frac{\alpha}{2}} \ln^{-\frac{1}{2}}(s+e) ds \\ & \leq C_\alpha (1 + (t+e) \ln^{-1}(t+e)) \leq C_\alpha (t+e) \ln^{-1}(t+e). \end{aligned}$$

We take  $g^2(t) = \frac{3}{(e+t)\ln(e+t)}$ , then  $e^{\int_0^t g^2(s) ds} = \ln^3(t+e)$  and

$$\begin{aligned} \ln^3(t+e) \|u(t)\|_{L^2}^2 & \leq C \|u_0\|_{L^2}^2 + C_\alpha \int_0^t \left[ \frac{\ln^2(s+e)}{(s+e)^{1+2\beta(p)}} + \frac{(s^\alpha+1)\ln(s+e)}{(s+e)^2} + \frac{1}{s+e} \right] ds \\ & \leq C_\alpha \ln(t+e), \end{aligned}$$

which implies

$$\|u(t)\|_{L^2}^2 \leq C_\alpha \ln^{-2}(t+e), \quad \text{for } t > 1.$$

And for  $0 < t < 1$ , it is obvious, so that

$$(4.3) \quad \|u(t)\|_{L^2}^2 \leq C_\alpha \ln^{-2}(t+e),$$

and

$$\int_0^\infty (t+e)^{-1} \|u\|_{L^2}^2 dt \leq C_\alpha \int_0^\infty (t+e)^{-1} \ln^{-2}(t+e) dt \leq C_\alpha.$$

We choose  $f'(t) = \ln(t+e)$  in (3.3), then get

$$\begin{aligned} & \sup_{0 < t < \infty} \ln(t+e) \|u(t)\|_{L^2}^2 + \int_0^\infty \ln(t+e) \|\nabla u\|_{L^2}^2 dt \\ & \leq C (\|u_0\|_{L^2}^2 + \int_0^\infty (t+e)^{-1} \|u\|_{L^2}^2 dt) \\ & \leq C_\alpha. \end{aligned}$$

Consequently, for any  $0 < r < \alpha$ , we take  $f(t) = t^{1-r}(t+e)^r \ln(t+e)$  in (3.2), obtain that

$$\begin{aligned} & \sup_{0 < t < \infty} t^{1-r}(t+e)^r \ln(t+e) \|\nabla u(t)\|_{L^2}^2 + \int_0^\infty t^{1-r}(t+e)^r \ln(t+e) \|u_t\|_{L^2}^2 dt \\ & \leq C \int_0^\infty \left[ \left(\frac{t}{t+e}\right)^{1-r} + \ln(t+e) \left(\left(\frac{t}{t+e}\right)^{1-r} + \left(\frac{t+e}{t}\right)^r\right) \right] \|\nabla u\|_{L^2}^2 dt \exp\{C\|u_0\|_{L^2}^4\}. \end{aligned}$$

Using (4.1), we get that

$$\int_0^1 t^{-r} \|\nabla u(t)\|_{L^2}^2 dt \leq C_\alpha \int_0^1 t^{\alpha-r-1} (t+e)^{-\alpha} dt \leq C_\alpha,$$

which implies

$$(4.4) \quad \|\nabla u(t)\|_{L^2}^2 \leq C_\alpha t^{r-1} (t+e)^{-r} \ln^{-1}(t+e).$$

Combining (4.3) and (4.4), for any  $t > 1$ , we get the revised estimates,

$$\begin{aligned} & \left( \int_0^t \|u\|_{L^2} \|\nabla u\|_{L^2} ds \right)^2 \\ & \leq C_\alpha \left( \int_0^1 s^{\frac{r-1}{2}} (s+e)^{-\frac{r}{2}} \ln^{-\frac{1}{2}}(s+e) ds \right)^2 + C_\alpha \left( \int_1^t s^{\frac{r-1}{2}} (s+e)^{-\frac{r}{2}} \ln^{-\frac{3}{2}}(s+e) ds \right)^2 \\ & \leq C_\alpha (1 + (t+e) \ln^{-3}(t+e)) \leq C_\alpha (t+e) \ln^{-3}(t+e), \end{aligned}$$

$$\begin{aligned} & \left( \int_0^t \|1 - \rho\|_{L^2} \|u_t\|_{L^2} ds \right)^2 \\ & \leq C_\alpha \left( \int_0^t s^{1-r} (s+e)^r \ln(s+e) \|u_t\|_{L^2}^2 ds \right) \left( \int_0^t s^{r-1} (s+e)^{-r} \ln^{-1}(s+e) ds \right) \\ & \leq C_\alpha \ln(\ln(t+e)). \end{aligned}$$

For  $t > 1$ , taking  $g^2(t) = \frac{5}{(t+e)\ln(t+e)}$ , then  $e^{\int_0^t g^2(s) ds} = \ln^5(t+e)$  and

$$\begin{aligned} & \ln^5(t+e) \|u(t)\|_{L^2}^2 \\ & \leq C \|u_0\|_{L^2}^2 + C_\alpha \int_0^t \left[ \frac{\ln^4(s+e)}{(s+e)^{1+2\beta(p)}} + \frac{\ln^3(s+e) \ln(\ln(t+e))}{(s+e)^2} + \frac{1}{s+e} \right] ds \\ & \leq C_\alpha \ln(t+e), \end{aligned}$$

from which, we obtain

$$(4.5) \quad \|u(t)\|_{L^2}^2 \leq C_\alpha \ln^{-4}(t+e).$$

We choose  $f'(t) = \ln^2(t+e)$  in (3.3), then

$$\begin{aligned} & \sup_{0 < t < \infty} \ln^2(t+e) \|u(t)\|_{L^2}^2 + \int_0^\infty \ln^2(t+e) \|\nabla u\|_{L^2}^2 dt \\ & \leq C (\|u_0\|_{L^2}^2 + \int_0^\infty (t+e)^{-1} \ln(t+e) \|u(t)\|_{L^2}^2 dt) \\ & \leq C_\alpha (1 + \int_0^1 (t+e)^{-1} dt + \int_1^\infty (t+e)^{-1} \ln^{-3}(t+e) dt) \\ & \leq C_\alpha. \end{aligned}$$

Finally, we take  $f(t) = t^{1-r}(t+e)^r \ln^2(t+e)$  in (3.2) to get that

$$\begin{aligned} & \sup_{0 < t < \infty} t^{1-r}(t+e)^r \ln^2(t+e) \|\nabla u(t)\|_{L^2}^2 + \int_0^\infty t^{1-r}(t+e)^r \ln^2(t+e) \|u_t\|_{L^2}^2 dt \\ & \leq C \int_0^\infty \left\{ \ln(t+e) \left(\frac{t}{t+e}\right)^{1-r} + \ln^2(t+e) \left[ \left(\frac{t}{t+e}\right)^{1-r} + \left(\frac{t+e}{t}\right)^r \right] \right\} \|\nabla u\|_{L^2}^2 dt \exp\{C\|u_0\|_{L^2}^4\} \\ & \leq C_\alpha. \end{aligned}$$

Consequently, we obtain

$$\begin{aligned} & \left( \int_0^\infty \|u_t\|_{L^2} dt \right)^2 \\ & \leq \left( \int_0^\infty t^{1-r}(t+e)^r \ln^2(t+e) \|u_t\|_{L^2}^2 dt \right) \left( \int_0^\infty t^{r-1}(t+e)^{-r} \ln^{-2}(t+e) dt \right) \\ & \leq C_\alpha. \end{aligned}$$

Which is the same for  $\mathbb{P} \operatorname{div}(\mu(\rho)\mathcal{M}(u))$ ,  $\mathbb{Q} \operatorname{div}(\mu(\rho)\mathcal{M}(u)) - \nabla \Pi \in L^1(\mathbb{R}^+; L^2)$ , and gives (1.8).

Then follow the same line to the proof of Theorem 1.2, we get the first inequality of (1.7). We choose  $f'(t) = (t+e)^{2\beta-\varepsilon}$  in (3.3), obtain that

$$\begin{aligned} & \sup_{0 < t < \infty} (t+e)^{2\beta-\varepsilon} \|u(t)\|_{L^2}^2 + \int_0^\infty (t+e)^{2\beta-\varepsilon} \|\nabla u\|_{L^2}^2 dt \\ & \leq C(\|u_0\|_{L^2}^2 + \int_0^\infty (t+e)^{-1+2\beta-\varepsilon} \|u\|_{L^2}^2 dt) \\ & \leq C_\alpha. \end{aligned}$$

Then taking  $f(t) = t^{1-r}(t+e)^{r+2\beta-\varepsilon}$  in (3.2), we deduce that

$$\begin{aligned} & \sup_{0 < t < \infty} t^{1-r}(t+e)^{r+2\beta-\varepsilon} \|\nabla u(t)\|_{L^2}^2 + \int_0^\infty t^{1-r}(t+e)^{r+2\beta-\varepsilon} \|u_t\|_{L^2}^2 dt \\ & \leq C \int_0^\infty (t+e)^{2\beta-\varepsilon} \left[ \left(\frac{t}{t+e}\right)^{1-r} + \left(\frac{t+e}{t}\right)^r \right] \|\nabla u\|_{L^2}^2 dt \exp\{C\|u_0\|_{L^2}^4\} \\ & \leq C_\alpha, \end{aligned}$$

which implies (1.9) and the second inequality of (1.7). This completes the proof of Theorem 1.3.

## 5. APPLICATION: GLOBAL EXISTENCE OF (1.1)

First we present a polynomial relation between the velocity and the density, which is the case between exponential and linear cases. In general, we consider the case of non-Lipschitz velocity, the loss of regularity will occur. With the non-Lipschitz velocity and logarithms regular density, we have the following proposition.

**Proposition 5.1.** *For  $\eta > 0$ , let  $\rho_0 \in B_{\infty,1}^{(\eta+1)\ln}$  and  $\nabla u \in L^1(\mathbb{R}^+; B_{\infty,2}^0)$ . Then we have  $\rho \in L^\infty((0, \infty); B_{\infty,1}^{\eta \ln})$ , and*

$$(5.1) \quad \|\rho(t)\|_{B_{\infty,1}^{\eta \ln}} \leq C \|\rho_0\|_{B_{\infty,1}^{(\eta+1)\ln}} \left( \int_0^t \|\nabla u\|_{B_{\infty,2}^0} d\tau \right)^{\eta+1},$$

for any  $t > 0$ .

*Proof.* First, we observe the continuity equation as follow:  $\rho = \sum_{j \geq -1} \rho_j$ , where  $\rho_j$  satisfies

$$(5.2) \quad \begin{cases} \partial_t \rho_j + u \cdot \nabla \rho_j = 0, \\ \rho_j|_{t=0} = \Delta_j \rho_0. \end{cases}$$



Then we have

$$\|\rho_j(t)\|_{B_{\infty,\infty}^{\frac{1}{4}}} \leq C \|\rho_j(0)\|_{B_{\infty,\infty}^{\frac{1}{2}}} \exp\{C \int_0^t \|\nabla u\|_{B_{\infty^2}^0} d\tau\},$$

so that

$$\|\Delta_q \rho_j(t)\|_{L^\infty} \leq C 2^{-\frac{1}{2}(\frac{q}{2}-j)} \|\Delta_j \rho_0\|_{L^\infty} F(u),$$

where  $F(u) = \exp\{C \int_0^t \|\nabla u\|_{B_{\infty^2}^0} d\tau\}$ . If  $\frac{q}{2} - j > N$ , for a positive inter number  $N$  will be fixed later, we obtain that

$$\begin{aligned} \sum_q \sum_{j < \frac{q}{2} - N} \|\Delta_q \rho_j(t)\|_{L^\infty} (2+q)^\eta &\leq CF(u) \sum_q (2+q)^\eta \sum_{j < \frac{q}{2} - N} 2^{-\frac{1}{2}(\frac{q}{2}-j)} \|\Delta_j \rho_0\|_{L^\infty} \\ &= CF(u) \sum_j 2^{\frac{1}{2}j} \|\Delta_j \rho_0\|_{L^\infty} \sum_{q > 2(N+j)} (2+q)^\eta 2^{-\frac{q}{4}} \\ &\leq CF(u) \sum_j 2^{\frac{1}{2}j} \|\Delta_j \rho_0\|_{L^\infty} 2^{-\frac{1}{2}(N+j)} (2+2(N+j))^\eta \\ &\leq CF(u) 2^{-\frac{N}{2}} \sum_j \|\Delta_j \rho_0\|_{L^\infty} (1+N+j)^\eta \\ &\leq C 2^{-\frac{N}{2}} N^\eta \|\rho_0\|_{B_{\infty,1}^{\eta \ln}} F(u). \end{aligned}$$

On the other hand, we have

$$\|\rho_j(t)\|_{B_{\infty,\infty}^{-\frac{3}{4}}} \leq C \|\rho_j(0)\|_{B_{\infty,\infty}^{-\frac{1}{2}}} F(u),$$

which implies that

$$\|\Delta_q \rho_j(t)\|_{L^\infty} \leq C 2^{-\frac{1}{2}(j-\frac{3q}{2})} \|\Delta_j \rho_0\|_{L^\infty} F(u).$$

If  $j - \frac{3q}{2} > N$ , then we obtain

$$\begin{aligned} \sum_q \sum_{j > \frac{3q}{2} + N} \|\Delta_q \rho_j(t)\|_{L^\infty} (2+q)^\eta &\leq CF(u) \sum_j 2^{-\frac{1}{2}j} \|\Delta_j \rho_0\|_{L^\infty} \sum_{q < \frac{2}{3}(j-N)} (2+q)^\eta 2^{\frac{3q}{4}} \\ &\leq CF(u) \sum_j 2^{-\frac{1}{2}j} \|\Delta_j \rho_0\|_{L^\infty} 2^{\frac{1}{2}(j-N)} (2 + \frac{2}{3}(j-N))^\eta \\ &\leq C 2^{-\frac{N}{2}} N^\eta \|\rho_0\|_{B_{\infty,1}^{\eta \ln}} F(u). \end{aligned}$$

Note that  $\|\Delta_q \rho_j(t)\|_{L^\infty} \leq \|\rho_j(t)\|_{L^\infty} \leq \|\Delta_j \rho_0\|_{L^\infty}$ , for  $\frac{q}{2} - N \leq j \leq \frac{3q}{2} + N$ , we get that

$$\begin{aligned} \sum_q \sum_{\frac{q}{2} - N \leq j \leq \frac{3q}{2} + N} \|\Delta_q \rho_j(t)\|_{L^\infty} (2+q)^\eta &\leq \sum_j \|\Delta_j \rho_0\|_{L^\infty} \sum_{\frac{2}{3}(j-N) \leq q \leq 2(j+N)} (2+q)^\eta \\ &\leq C \sum_j \|\Delta_j \rho_0\|_{L^\infty} \frac{(2+2(j+N))^{\eta+1} - (2+\frac{2}{3}(j-N))^{\eta+1}}{\eta+1} \\ &\leq C N^{\eta+1} \|\rho_0\|_{B_{\infty,1}^{(\eta+1) \ln}}. \end{aligned}$$

Finally, we obtain that

$$\begin{aligned} \|\rho(t)\|_{B_{\infty,1}^{\eta \ln}} &\leq \sum_q \sum_j (2+q)^\eta \|\Delta_q \rho_j(t)\|_{L^\infty} \\ &\leq C N^\eta 2^{-\frac{N}{2}} \|\rho_0\|_{B_{\infty,1}^{\eta \ln}} F(u) + C N^{\eta+1} \|\rho_0\|_{B_{\infty,1}^{(\eta+1) \ln}} \\ &\leq C N^{\eta+1} \|\rho_0\|_{B_{\infty,1}^{(\eta+1) \ln}} (1 + 2^{-\frac{N}{2}} F(u)), \end{aligned}$$

where we use  $\|\rho_0\|_{B_{\infty,1}^{\eta \ln}} \leq \|\rho_0\|_{B_{\infty,1}^{(\eta+1) \ln}}$ . We choose  $2^{-\frac{N}{2}} F(u) \sim 1$ , i.e.  $N \sim \int_0^t \|\nabla u\|_{B_{\infty,2}^0} d\tau$ , then

$$\|\rho(t)\|_{B_{\infty,1}^{\eta \ln}} \leq C \|\rho_0\|_{B_{\infty,1}^{(\eta+1) \ln}} \left( \int_0^t \|\nabla u\|_{B_{\infty,2}^0} d\tau \right)^{\eta+1}.$$

□

Now, we present the product law with logarithms Besov space and the usual Besov space.

**Proposition 5.2.** *Let  $\eta > 1$ , and  $a \in B_{\infty,1}^{\eta \ln}$ ,  $b \in B_{\infty,2}^0$ . Then  $ab \in B_{\infty,2}^0$ , and*

$$(5.3) \quad \|ab\|_{B_{\infty,2}^0} \leq C \|a\|_{B_{\infty,1}^{\eta \ln}} \|b\|_{B_{\infty,2}^0}.$$

*Proof.* We use Bony's decomposition that

$$ab = T_a b + T_b a + R(a, b).$$

For the first term, we have

$$\begin{aligned} \|\Delta_j T_a b\|_{L^\infty} &\leq \sum_{|j-q| \leq N} \|S_{q-1} a\|_{L^\infty} \|\Delta_q b\|_{L^\infty} \\ &\leq \|b\|_{B_{\infty,2}^0} \sum_{|j-q| \leq N} c_{q,2} \sum_{k \leq q-2} \|\Delta_k a\|_{L^\infty} \\ &\leq \|a\|_{B_{\infty,1}^{\eta \ln}} \|b\|_{B_{\infty,2}^0} \sum_{|j-q| \leq N} c_{q,2} \sum_{k \leq q-2} c_{k,1} (2+k)^{-\eta} \\ &\lesssim c_{j,2} \|a\|_{B_{\infty,1}^{\eta \ln}} \|b\|_{B_{\infty,2}^0}, \end{aligned}$$

where we use  $\eta > 1$ .

To deal with  $T_b a$ , one has

$$\begin{aligned} \|\Delta_j T_b a\|_{L^\infty} &\leq \sum_{|j-q| \leq N} \|\Delta_q a\|_{L^\infty} \sum_{-1 \leq k \leq q-2} \|\Delta_k b\|_{L^\infty} \\ &\leq \|a\|_{B_{\infty,1}^{\eta \ln}} \|b\|_{B_{\infty,2}^0} \sum_{|j-q| \leq N} (2+q)^{-\eta} \sqrt{q} \\ &\lesssim \|a\|_{B_{\infty,1}^{\eta \ln}} \|b\|_{B_{\infty,2}^0} \sum_{|j-q| \leq N} (2+q)^{-(\eta-\frac{1}{2})} \\ &\lesssim c_{j,2} \|a\|_{B_{\infty,1}^{\eta \ln}} \|b\|_{B_{\infty,2}^0}, \end{aligned}$$

where we use again  $\eta > 1$  so that  $\sum_{q \geq -1} (2+q)^{-(2\eta-1)} < \infty$ .

For the last term, we obtain that

$$\begin{aligned}
\sum_{j \geq -1} \|\Delta_j R(a, b)\|_{L^\infty}^2 &\leq \sum_{j \geq -1} \left( \sum_{q \geq j-N} \|\Delta_q a\|_{L^\infty} \|\tilde{\Delta}_q b\|_{L^\infty} \right) 2 \\
&\leq \sum_{j \geq -1} \left( \sum_{q \geq j-N} (2+q)^{2\eta} \|\Delta_q a\|_{L^\infty}^2 \right) \left( \sum_{q \geq j-N} (2+q)^{-2\eta} \|\Delta_q b\|_{L^\infty}^2 \right) \\
&\leq \|a\|_{B_{\infty,1}^{\eta \ln}}^2 \sum_{j \geq -1} \sum_{q \geq j-N} (2+q)^{-2\eta} \|\Delta_q b\|_{L^\infty}^2 \\
&\leq \|a\|_{B_{\infty,1}^{\eta \ln}}^2 \|b\|_{B_{\infty,2}^0}^2 \sum_{q \geq -1} (2+q)^{-2\eta} c_{q,2}^2 \sum_{-1 \leq j \leq q+N} 1 \\
&\lesssim \|a\|_{B_{\infty,1}^{\eta \ln}}^2 \|b\|_{B_{\infty,2}^0}^2 \sum_{q \geq -1} (2+q)^{-(2\eta-1)} \\
&\lesssim \|a\|_{B_{\infty,1}^{\eta \ln}}^2 \|b\|_{B_{\infty,2}^0}^2.
\end{aligned}$$

By summing up the above estimates, we get (5.3).  $\square$

Now we are at the position to proof Theorem 1.4.

*Proof.* We rewrite the momentum equation of (1.1) as

$$\mu_0 \Delta u = \mathbb{P}(\rho u_t + \rho u \nabla u) - \mathbb{P} \operatorname{div}((\mu(\rho) - \mu_0) \mathcal{M}(u)),$$

from which, we get

$$\mu_0 (I - S_0) \nabla u = \nabla(-\Delta)^{-1} \mathbb{P} \operatorname{div}(I - S_0)((\mu(\rho) - \mu_0) \mathcal{M}(u)) - \nabla(-\Delta)^{-1} \mathbb{P}(I - S_0)(\rho u_t + \rho u \nabla u).$$

Now we can estimate  $\nabla u$  in the norm of  $L_t^1(B_{\infty,2}^0)$ . Note that

$$\|\nabla(-\Delta)^{-1}(I - S_0)f\|_{B_{\infty,2}^0} \leq C\|f\|_{L^2},$$

and recall (5.3), we obtain that

$$\begin{aligned}
\mu_0 \|(I - S_0) \nabla u\|_{L_t^1(B_{\infty,2}^0)} &\leq C \|\mu(\rho) - \mu_0\|_{L_t^\infty(B_{\infty,1}^{\eta \ln})} \|\nabla u\|_{L_t^1(B_{\infty,2}^0)} \\
&\quad + C(\|u_t\|_{L_t^1(L^2)} + \|u \nabla u\|_{L^1(L^2)}) \\
&\leq C \|\mu(\rho) - \mu_0\|_{L_t^\infty(B_{\infty,1}^{\eta \ln})} \left( \|(I - S_0) \nabla u\|_{L_t^1(B_{\infty,2}^0)} + \|S_0 \nabla u\|_{L_t^1(L^\infty)} \right) \\
&\quad + C(\|u_t\|_{L_t^1(L^2)} + \|u \nabla u\|_{L^1(L^2)}).
\end{aligned}$$

Let  $c_1$  be a small enough positive constant, which will be determined later on, we denote

$$(5.4) \quad \bar{T} \stackrel{\text{def}}{=} \sup \left\{ t; \|\mu(\rho) - \mu_0\|_{L_t^\infty(B_{\infty,1}^{\eta \ln})} \leq c_1 \mu_0 \right\}.$$

Then for any  $t \leq \bar{T}$ , the assumption (1.3) holds and

$$\mu_0 \|\nabla u\|_{L_t^1(B_{\infty,2}^0)} \leq C \left( \mu_0 \|S_0 \nabla u\|_{L_t^1(L^\infty)} + \|u_t\|_{L_t^1(L^2)} + \|u \nabla u\|_{L^1(L^2)} \right).$$

Note that  $p < \frac{4}{3}$ , we can find some positive  $\varepsilon$  such that  $\frac{1}{2} + 2\beta(p) - 2\varepsilon > 1$ . Then using interpolation (3.4), and decay estimates (3.6), (3.8), we obtain that

$$\begin{aligned} \|S_0 \nabla u\|_{L_t^1(L^\infty)} &\lesssim \|\nabla u\|_{L_t^1(L^4)} \lesssim \left( \int_0^t (s+e)^{\frac{1}{2}+2\beta(p)-2\varepsilon} \|\nabla u\|_{L^4}^2 ds \right)^{\frac{1}{2}} \left( \int_0^t (s+e)^{-(\frac{1}{2}+2\beta(p)-2\varepsilon)} ds \right)^{\frac{1}{2}} \\ &\lesssim \left( \int_0^t (s+e)^{\frac{1}{2}+2\beta(p)-2\varepsilon} \|\nabla u\|_{L^2} \|\rho u_t + \rho u \nabla u\|_{L^2} ds \right)^{\frac{1}{2}} \\ &\lesssim \left( (K^{\frac{1}{2}} + K^{\frac{7}{2}}) \exp(K^2) \int_0^t (s+e)^{\beta(p)-\varepsilon} \|\rho u_t + \rho u \nabla u\|_{L^2} ds \right)^{\frac{1}{2}} \\ &\lesssim \left( (K^{\frac{1}{2}} + K^{\frac{7}{2}}) \exp(K^2) \left( \int_0^t (s+e)^{1+2\beta(p)-\varepsilon} \|\rho u_t + \rho u \nabla u\|_{L^2}^2 ds \right)^{\frac{1}{2}} \left( \int_0^t (s+e)^{-1-\varepsilon} ds \right)^{\frac{1}{2}} \right)^{\frac{1}{2}} \\ &\lesssim (K^{\frac{1}{2}} + K^{\frac{7}{2}}) \exp(K^2). \end{aligned}$$

Combining (3.7), we get that

$$\mu_0 \|\nabla u\|_{L_t^1(B_{\infty,2}^0)} \leq C(\mu_0(K^{\frac{1}{2}} + K^{\frac{7}{2}}) \exp(K^2) + K^{\frac{1}{2}} + K).$$

Recall the definition of  $K$  and  $G(\rho_0, u_0)$ , we deduce that

$$(5.5) \quad \mu_0 \|\nabla u\|_{L_t^1(B_{\infty,2}^0)} \leq C(1 + \mu_0)G(\rho_0, u_0) \exp\{\exp(C\|u_0\|_{L^2}^4)\}.$$

Now, substituting (5.5) into (5.1) leads to

$$(5.6) \quad \begin{aligned} \|\mu(\rho) - \mu_0\|_{L_t^\infty(B_{\infty,1}^{\eta \ln})} &\leq C\|\mu(\rho_0) - \mu_0\|_{B_{\infty,1}^{(\eta+1) \ln}} \left( \|\nabla u\|_{L_t^1(B_{\infty,2}^0)} \right)^{\eta+1} \\ &\leq C\|\mu(\rho_0) - \mu_0\|_{B_{\infty,1}^{(\eta+1) \ln}} \left( \frac{(1 + \mu_0)G(\rho_0, u_0)}{\mu_0} \right)^{\eta+1} \exp\{(\eta + 1) \exp(C\|u_0\|_{L^2}^4)\} \\ &\leq \frac{c_1}{2} \mu_0, \end{aligned}$$

as long as  $C_0$  is sufficiently large and  $c_0$  small enough in (1.11). This contradicts with (5.4) and it in turn shows that  $\bar{T} = \infty$ . So the Theorem is proven.  $\square$

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