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► **To cite this version:**

Jingchi Huang, Marius Paicu. Decay estimates of global solutions to 2D incompressible inhomogeneous Navier-Stokes equations with variable viscosity. 2012. hal-00765698v1

HAL Id: hal-00765698

<https://hal.science/hal-00765698v1>

Preprint submitted on 16 Dec 2012 (v1), last revised 20 Jan 2013 (v3)

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DECAY ESTIMATES OF GLOBAL SOLUTIONS TO 2D INCOMPRESSIBLE INHOMOGENEOUS NAVIER-STOKES EQUATIONS WITH VARIABLE VISCOSITY

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ABSTRACT. In this paper, we investigate the time decay behavior to Lions weak solution of 2D incompressible density-dependent Navier-Stokes equations with variable viscosity.

Keywords: Inhomogeneous Navier-Stokes equations, Decay estimates.

AMS Subject Classification (2000): 35Q30, 76D03

1. INTRODUCTION

The main purpose of this paper is to investigate the decay estimates for the global solutions of the following two-dimensional incompressible inhomogeneous Navier-Stokes equations with viscous coefficient depending on the density

$$(1.1) \quad \begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0, & (t, x) \in \mathbb{R}^+ \times \mathbb{R}^2, \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) - \operatorname{div}(\mu(\rho)\mathcal{M}(u)) + \nabla \Pi = 0, \\ \operatorname{div} u = 0, \end{cases}$$

where $\rho, u = (u_1, u_2)$ stand for the density and velocity of the fluid respectively, $\mathcal{M}(u) = \nabla u + \nabla^T u$, Π is a scalar pressure function, and in general, the viscosity coefficient $\mu(\rho)$ is a smooth positive function on $[0, \infty)$. Such system describes a fluid which is obtained by mixing two immiscible fluids that are incompressible and that have different densities. It may also describe a fluid containing a melted substance. One may check [9] for the detailed derivation.

When $\mu(\rho)$ is independent of ρ , i.e. μ is a positive constant, and the initial density has a positive lower bound, Ladyženskaja and Solonnikov [8] first addressed the question of unique solvability of (1.1). More precisely, they considered the system (1.1) in a bounded domain Ω with homogeneous Dirichlet boundary condition for u . Under the assumption that $u_0 \in W^{2-\frac{2}{p}, p}(\Omega)$ ($p > d$) is divergence free and vanishes on $\partial\Omega$ and that $\rho_0 \in C^1(\Omega)$ is bounded away from zero, then they [8] proved

- Global well-posedness in dimension $d = 2$;
- Local well-posedness in dimension $d = 3$. If in addition u_0 is small in $W^{2-\frac{2}{p}, p}(\Omega)$, then global well-posedness holds true.

Danchin [3] proved similar well-posedness result of (1.1) in the whole space case and the initial data in the almost critical spaces. In particular, in two dimension, he proved the global well-posedness of (1.1) provided the initial data (ρ_0, u_0) satisfying $\rho_0 - 1 \in H^{1+\alpha}(\mathbb{R}^2)$, $\rho_0 \geq m > 0$, and $u_0 \in H^\beta(\mathbb{R}^2)$ for any $\alpha \in (0, 1)$ and $\beta \in (0, 1]$.

In general, Lions [9] (see also the references therein) proved the global existence of weak solutions to (1.1) with finite energy. More precisely, given $0 \leq \rho_0 \in L^\infty(\mathbb{R}^d)$, $0 < \mu_0 \leq \mu(\rho)$ and u_0 satisfying

$\operatorname{div} u_0 = 0$, $\sqrt{\rho_0}u_0 \in L^2(\mathbb{R}^d)$, Lions proved that (1.1) has a global weak solution so that

$$\frac{1}{2} \|\sqrt{\rho(t)}u(t)\|_{L^2}^2 + \mu_0 \int_0^t \|\nabla u(\tau)\|_{L^2}^2 d\tau \leq \frac{1}{2} \|\sqrt{\rho_0}u_0\|_{L^2}^2.$$

Moreover, for any α and β , the Lebesgue measure

$$\mu\{x \in \mathbb{R}^d; \alpha \leq \rho(t, x) \leq \beta\} \text{ is independent of } t.$$

In dimension two and under the additional assumption that ρ_0 is close enough to a positive constant and $\nabla u_0 \in L^2(\mathbb{R}^2)$, smoother weak solutions may be built. Their existence stems from a quasi-conservation law involving the norm of $\nabla u \in L^\infty((0, T); L^2(\mathbb{R}^2))$ and of $\partial_t u, \nabla p \in L^2((0, T); L^2(\mathbb{R}^2))$ for any $T < \infty$. For both types of weak solutions however, the problem of uniqueness and regularities of such weak solutions are big open questions even in two space dimensions.

Under the additional assumptions that

$$(1.2) \quad \|\mu(\rho_0) - 1\|_{L^\infty(\mathbb{T}^2)} \leq \varepsilon \quad \text{and} \quad u_0 \in H^1(\mathbb{T}^2),$$

Desjardins [4] proved the following theorem.

Theorem 1.1. *Let $\rho_0 \in L^\infty(\mathbb{T}^2)$ and $\operatorname{div} u_0 = 0$. Then there exists $\varepsilon > 0$ such that under the assumption (1.2), Lions weak solutions ([9]) to (1.1) satisfy the following regularity properties hold for all $T > 0$:*

- (1) $u \in L^\infty((0, T); H^1(\mathbb{T}^2))$ and $\sqrt{\rho}\partial_t u \in L^2((0, T) \times \mathbb{T}^2)$,
- (2) ρ and $\mu(\rho) \in L^\infty((0, T) \times \mathbb{T}^2) \cap C([0, T]; L^p(\mathbb{T}^2))$ for all $p \in [1, \infty)$,
- (3) $\nabla(\Pi - \mathcal{R}_i \mathcal{R}_j(\mu \mathcal{M}(u)_{ij}))$ and $\nabla(\mathbb{P} \otimes \mathbb{Q}(\mu \mathcal{M}(u)_{ij})) \in L^2((0, T) \times \mathbb{T}^2)$,
- (4) Π may be renormalized in such a way that for some universal constant $C_0 > 0$, Π and $\nabla u \in L^2((0, T); L^p(\mathbb{T}^2))$ for all $p \in [4, p^*)$, where $\frac{1}{p^*} = 2C_0 \|\mu(\rho_0) - 1\|_{L^\infty}$.

In which, we denote \mathcal{R} as the Riesz transform: $\mathcal{R} = \nabla \Delta^{-\frac{1}{2}}$. $\mathbb{Q} = \nabla \Delta^{-1} \operatorname{div}$ and $\mathbb{P} = \mathbb{I} - \mathbb{Q}$ respectively denote the projection on the space of curl-free and divergence-free vector fields.

In order to investigate the global well-posedness of thus solutions, we first need to study the global-in-time type estimates. However, because of the difficulties of the continuity equation in (1.1) being of hyperbolic nature and the estimate of the diffusion term in the momentum equation, we shall first study the time decay of the solutions, which is very much motivated by [5, 10, 12].

Theorem 1.2. *For $1 < p < 2$, let $u_0 \in L^p(\mathbb{R}^2) \cap H^1(\mathbb{R}^2)$, $\rho_0 - 1 \in L^2(\mathbb{R}^2)$ and $\rho_0 \in L^\infty(\mathbb{R}^2)$ with a positive lower bound. We assume that $(\rho, u, \nabla p)$ is a given Lions weak solution of (1.1) with initial data (ρ_0, u_0) . Denote $\mu(1) = \mu_0$, then under the assumption*

$$(1.3) \quad \|\mu(\rho) - \mu_0\|_{L^\infty(\mathbb{R}^+; L^\infty(\mathbb{R}^2))} \leq \varepsilon_0,$$

for a small constant ε_0 , there exists a constant C_1 which depends on $\|\rho_0 - 1\|_{L^2}$, $\|u_0\|_{L^p}$ and $\|u_0\|_{H^1}$ such that there hold

$$(1.4) \quad \|u(t)\|_{L^2}^2 \leq C_1(t+e)^{-2\beta(p)}, \quad \|\nabla u(t)\|_{L^2}^2 \leq C_1(t+e)^{-1-2\beta(p)+\varepsilon},$$

$$(1.5) \quad \int_0^\infty \|u_t\|_{L^2} + \|\operatorname{div}(\mu(\rho)\mathcal{M}(u))\|_{L^2} + \|\nabla \Pi\|_{L^2} dt \leq C_1,$$

$$(1.6) \quad \int_0^\infty (t+e)^{1+2\beta(p)-\varepsilon} \left(\|u_t\|_{L^2} + \|\operatorname{div}(\mu(\rho)\mathcal{M}(u))\|_{L^2} + \|\nabla \Pi\|_{L^2} \right)^2 dt \leq C_1,$$

with $\beta(p) = \frac{1}{2}(\frac{2}{p} - 1)$ and any $\varepsilon > 0$.

Remark 1.1. *The first estimate of (1.4) coincides with the L^2 -norm decay result in [10, 12] for the weak solutions of the two-dimensional classical Navier-Stokes system, and also coincides with the result in [5] for (1.1). When $\mu(\rho)$ be a constant, we can get optimal decay of $\|\nabla u\|_{L^2}^2$ with the order $-1-2\beta(p)$, see [6]. Notice the main ingredients of the proof in [6, 10, 12] are the usual energy estimates and the phase space analysis. In our case, due to the additional difficulties mentioned above, we not only need to apply phase space analysis, but also need more explicit energy estimates, see Proposition 2.1 below. We note also that the 3D case with constant viscosity was studied in [1]. Using energy estimates with weight in time and the Fourier splitting method of Schonbek [10] we can generalize this decay in time estimates to the 3D case with variable viscosity.*

Motivated by Proposition 2.1, we have a more general result. Indeed, using interpolation argument we obtain a similar decay rate of the solution, under a weaker assumption on the initial velocity.

Theorem 1.3. *For $1 < p < 2$ and $0 < \alpha < 1$, let $u_0 \in L^p(\mathbb{R}^2) \cap H^\alpha(\mathbb{R}^2)$, $\rho_0 - 1 \in L^2(\mathbb{R}^2)$ and $\rho_0 \in L^\infty(\mathbb{R}^2)$ with a positive lower bound. We assume that $(\rho, u, \nabla p)$ is a given Lions weak solution of (1.1) with initial data (ρ_0, u_0) . Then under the assumption (1.3), there exists a constant C_α which depends on $\|\rho_0 - 1\|_{L^2}$, $\|u_0\|_{L^p}$ and $\|u_0\|_{H^\alpha}$ such that there hold*

$$(1.7) \quad \|u(t)\|_{L^2}^2 \leq C_\alpha(t+e)^{-2\beta(p)}, \quad \|\nabla u(t)\|_{L^2}^2 \leq C_\alpha(t+e)^{-1-2\beta(p)+\varepsilon},$$

$$(1.8) \quad \int_0^\infty \|u_t\|_{L^2} + \|\operatorname{div}(\mu(\rho)\mathcal{M}(u))\|_{L^2} + \|\nabla \Pi\|_{L^2} dt \leq C_\alpha,$$

$$(1.9) \quad \int_0^\infty t^{1-r}(t+e)^{r+2\beta(p)-\varepsilon} \left(\|u_t\|_{L^2} + \|\operatorname{div}(\mu(\rho)\mathcal{M}(u))\|_{L^2} + \|\nabla \Pi\|_{L^2} \right)^2 dt \leq C_\alpha,$$

with any $\varepsilon > 0$ and $0 < r < \alpha$.

Remark 1.2. *We note also that the 3D case with constant viscosity was studied in [1]. Using energy estimates with weight in time and the Fourier splitting method of Schonbek [10] it is possible to generalize this decay in time estimates to the 3D case with variable viscosity.*

The organization of the paper. In the second section, we shall present the proof of Theorem 1.2. In Section 3, we shall prove Theorem 1.3.

2. THE PROOF OF THEOREM 1.2

First, we need some calculus inequalities which can be found in [12].

Lemma 2.1. *Let $m \in \mathbb{R}^+$, $0 \leq \alpha < 1$ and $\beta > 0$. Then*

- (1) $\int_0^t (s+e)^{-1} \ln(s+e)^{-m} ds \leq \frac{1}{m-1}$ for $m > 1$,
- (2) there is some $\gamma_m > 0$ such that $\int_0^t (s+e)^{-1-\beta} \ln(s+e)^m ds \leq \gamma_m \beta^{-(m+1)}$,
- (3) there is some $\gamma_{m,\alpha} > 0$ such that, for all $t > 0$,
 $\int_0^t (s+e)^{-\alpha} \ln(s+e)^{-m} ds \leq \gamma_{m,\alpha} (t+e)^{1-\alpha} \ln(t+e)^{-m}$.

In this section, we will prove Theorem 1.2. First, we have some energy estimates.

Proposition 2.1. *Let $v \in L^\infty(\mathbb{R}^+; L^2) \cap L^2(\mathbb{R}^+; \dot{H}^1)$, $\operatorname{div} v = 0$. Assume that $u_0 \in H^1(\mathbb{R}^2)$ and $\rho_0 \in L^\infty(\mathbb{R}^2)$ with positive lower bound. $f(t)$ be a positive second-order differentiable function satisfies $f'(t) \geq 0$ and $f''(t) \geq 0$. (ρ, u) be the global weak solution of the linear system:*

$$(2.1) \quad \begin{cases} \partial_t \rho + v \nabla \rho = 0, \\ \rho \partial_t u + \rho v \nabla u - \operatorname{div}(\mu(\rho)\mathcal{M}(u)) + \nabla \Pi = 0, \\ \operatorname{div} u = 0, \\ (\rho, u)|_{t=0} = (\rho_0, u_0). \end{cases}$$

Then under the assumption (1.3), we have the following estimates:

$$(2.2) \quad \sup_{0 < t < \infty} f(t) \int_{\mathbb{R}^2} \mu(\rho) |\nabla u|^2(t) dx + \int_0^\infty f(t) \int_{\mathbb{R}^2} \bar{C} |\sqrt{\rho} u_t|^2 + |\operatorname{div}(\mu(\rho) \mathcal{M}(u))|^2 + |\nabla \Pi|^2 dx dt \\ \leq C(f(0) \|\nabla u_0\|_{L^2}^2 + \int_0^\infty f'(t) \int_{\mathbb{R}^2} \mu(\rho) |\nabla u|^2 dx dt) \exp\{C(1 + \|v\|_{L^\infty(L^2)}) \|\nabla v\|_{L^2(L^2)}^2\},$$

(2.3)

$$\sup_{0 < t < \infty} f'(t) \int_{\mathbb{R}^2} \rho |u|^2(t) dx + \int_0^\infty f'(t) \int_{\mathbb{R}^2} \mu(\rho) |\nabla u|^2 dx dt \leq C(f'(0) \|u_0\|_{L^2}^2 + \int_0^\infty f''(t) \int_{\mathbb{R}^2} \rho |u|^2 dx dt),$$

where \bar{C} be a large enough constant.

Proof. First, we follow the line of the proof of Theorem 1.1, see [4]. By taking L^2 inner product of the momentum equation of (2.1) with $f(t)u_t$ and using integration by parts, we deduce that

$$f(t) \int_{\mathbb{R}^2} |\sqrt{\rho} u_t|^2 dx + f(t) \int_{\mathbb{R}^2} (\rho v \nabla u) \cdot u_t dx + f(t) \int_{\mathbb{R}^2} \mu(\rho) \nabla u : \nabla u_t dx = 0.$$

Note that

$$f(t) \int_{\mathbb{R}^2} \mu(\rho) \nabla u : \nabla u_t dx = \frac{1}{2} \partial_t [f(t) \int_{\mathbb{R}^2} \mu(\rho) |\nabla u|^2 dx] - \frac{1}{2} f'(t) \int_{\mathbb{R}^2} \mu(\rho) |\nabla u|^2 dx \\ - \frac{1}{2} f(t) \int_{\mathbb{R}^2} \partial_t \mu(\rho) |\nabla u|^2 dx,$$

and from the derivation of (29) in [4] that

$$- \int_{\mathbb{R}^2} \partial_t \mu(\rho) |\nabla u|^2 dx = \int_{\mathbb{R}^2} \operatorname{div}(\mu(\rho) v) |\nabla u|^2 dx \\ = \int_{\mathbb{R}^2} (v \nabla) u \cdot \operatorname{div}(\mu(\rho) \mathcal{M}(u)) dx + \int_{\mathbb{R}^2} \mu(\rho) \operatorname{tr}(\nabla v \nabla u \mathcal{M}(u)) dx \\ = \int_{\mathbb{R}^2} (v \nabla) u \cdot (\rho u_t + \rho v \nabla u + \nabla \Pi) dx + \int_{\mathbb{R}^2} \mu(\rho) \operatorname{tr}(\nabla v \nabla u \mathcal{M}(u)) dx,$$

we have

$$\frac{d}{dt} [f(t) \int_{\mathbb{R}^2} \mu(\rho) |\nabla u|^2 dx] + f(t) \int_{\mathbb{R}^2} |\sqrt{\rho} u_t|^2 dx \\ \lesssim f'(t) \int_{\mathbb{R}^2} \mu(\rho) |\nabla u|^2 dx + f(t) \int_{\mathbb{R}^2} |\sqrt{\rho} v \nabla u|^2 dx \\ + f(t) \int_{\mathbb{R}^2} \mu(\rho) |\nabla v| |\nabla u|^2 dx + f(t) \left| \int_{\mathbb{R}^2} \Pi \partial_i v_j \partial_j u_i dx \right|.$$

Recall that

$$-\mu_0 \Delta u = \operatorname{div}((\mu(\rho) - \mu_0) \mathcal{M}(u)) - \operatorname{div}(\mu(\rho) \mathcal{M}(u)),$$

so that we have

$$\mu_0 \partial_i u_j = \mathcal{R}_i \mathbb{P}_j \mathcal{R}((\mu(\rho) - \mu_0) \mathcal{M}(u)) - \mathcal{R}_i \mathbb{P}_j \mathcal{R}(\mu(\rho) \mathcal{M}(u)).$$

Estimating it in the $L^4(\mathbb{R}^2)$ and using the Gagliardo-Nirenberg inequality, we can write

$$\|\nabla u\|_{L^4} \lesssim \|\mathbb{P} \otimes \mathbb{Q}((\mu(\rho) - \mu_0) \mathcal{M}(u))\|_{L^4} + \|\mathbb{P} \otimes \mathbb{Q}(\mu(\rho) \mathcal{M}(u))\|_{L^4} \\ \lesssim \|\mu(\rho) - \mu_0\|_{L^\infty(\mathbb{R}^+; L^\infty)} \|\nabla u\|_{L^4} + \|\mathbb{P} \otimes \mathbb{Q}(\mu(\rho) \mathcal{M}(u))\|_{L^2}^{\frac{1}{2}} \|\nabla(\mathbb{P} \otimes \mathbb{Q}(\mu(\rho) \mathcal{M}(u)))\|_{L^2}^{\frac{1}{2}}$$

Finally, using (1.3) and the conservation of the momentum, we obtain that

$$\|\nabla u\|_{L^4} \lesssim \|\nabla u\|_{L^2}^{\frac{1}{2}} \|\mathbb{P}(\rho u_t + \rho v \nabla u)\|_{L^2}^{\frac{1}{2}},$$

Now letting $(-\Delta)^{-\frac{1}{2}}\mathcal{R}$ operate on the equation of momentum, we get that

$$\Pi = \mathcal{R}_i \mathcal{R}_j (\mu(\rho)(\partial_i u_j + \partial_j u_i)) + (-\Delta)^{-\frac{1}{2}} \mathcal{R}(\rho u_t + \rho v \nabla u).$$

It follows that

$$\|\Pi - \mathcal{R}_i \mathcal{R}_j (\mu(\rho)\mathcal{M}(u))\|_{BMO} \lesssim \|\nabla(\Pi - \mathcal{R}_i \mathcal{R}_j (\mu(\rho)\mathcal{M}(u)))\|_{L^2} \lesssim \|\rho u_t + \rho v \nabla u\|_{L^2}.$$

We obtain that

$$\begin{aligned} \left| \int_{\mathbb{R}^2} \Pi \partial_i v_j \partial_j u_i dx \right| &\leq \|\nabla v\|_{L^2} \|\nabla u\|_{L^4}^2 + \|\Pi - \mathcal{R}_i \mathcal{R}_j (\mu(\rho)\mathcal{M}(u))\|_{BMO} \|\partial_i v_j \partial_j u_i\|_{\mathcal{H}^1} \\ &\leq \|\nabla v\|_{L^2} \|\nabla u\|_{L^2} \|\rho u_t + \rho v \nabla u\|_{L^2}, \end{aligned}$$

so that

$$f(t) \left| \int_{\mathbb{R}^2} \Pi \partial_i u_j \partial_j u_i dx \right| \leq C_\varepsilon f(t) \|\nabla v\|_{L^2}^2 \|\nabla u\|_{L^2}^2 + \varepsilon f(t) (\|\sqrt{\rho} u_t\|_{L^2}^2 + \|v \nabla u\|_{L^2}^2).$$

$$\begin{aligned} f(t) \int_{\mathbb{R}^2} \mu(\rho) |\nabla v| |\nabla u|^2 dx &\leq C f(t) \|\nabla v\|_{L^2} \|\nabla u\|_{L^4}^2 \\ &\leq C_\varepsilon f(t) \|\nabla v\|_{L^2}^2 \|\nabla u\|_{L^2}^2 + \varepsilon f(t) (\|\sqrt{\rho} u_t\|_{L^2}^2 + \|v \nabla u\|_{L^2}^2). \end{aligned}$$

Also

$$\begin{aligned} \|v \nabla u\|_{L^2}^2 &\leq \|v\|_{L^4}^2 \|\nabla u\|_{L^4}^2 \leq \|v\|_{L^2} \|\nabla v\|_{L^2} \|\nabla u\|_{L^2} \|\rho u_t + \rho v \nabla u\|_{L^2}, \\ f(t) \int_{\mathbb{R}^2} |\sqrt{\rho} v \nabla u|^2 dx &\leq C_\varepsilon f(t) \|v\|_{L^2}^2 \|\nabla v\|_{L^2}^2 \|\nabla u\|_{L^2}^2 + \varepsilon f(t) \|\sqrt{\rho} u_t\|_{L^2}^2. \end{aligned}$$

Consequently,

$$\begin{aligned} (2.4) \quad &\frac{d}{dt} [f(t) \int_{\mathbb{R}^2} \mu(\rho) |\nabla u|^2 dx] + f(t) \int_{\mathbb{R}^2} |\sqrt{\rho} u_t|^2 dx \\ &\lesssim f'(t) \int_{\mathbb{R}^2} \mu(\rho) |\nabla u|^2 dx + f(t) \|\nabla u\|_{L^2}^2 \|\nabla v\|_{L^2}^2 (1 + \|v\|_{L^2}^2). \end{aligned}$$

Second, we multiply $-\frac{1}{\rho} f(t) \operatorname{div}(\mu(\rho)\mathcal{M}(u))$ with the momentum equation of (2.1) and integrate in \mathbb{R}^2 to get that

$$\begin{aligned} &f(t) \int_{\mathbb{R}^2} \mu(\rho) \nabla u : \nabla u_t dx + f(t) \int_{\mathbb{R}^2} |\operatorname{div}(\mu(\rho)\mathcal{M}(u))|^2 dx \\ = &f(t) \int_{\mathbb{R}^2} v \nabla u \operatorname{div}(\mu(\rho)\mathcal{M}(u)) dx + f(t) \int_{\mathbb{R}^2} \frac{1}{\rho} \nabla \Pi \operatorname{div}(\mu(\rho)\mathcal{M}(u)) dx. \end{aligned}$$

The first term of both side have been dealt in previous. For the last term, we have

$$\begin{aligned} \left| \int_{\mathbb{R}^2} \frac{1}{\rho} \nabla \Pi \operatorname{div}(\mu(\rho)\mathcal{M}(u)) dx \right| &\lesssim \int_{\mathbb{R}^2} |\nabla \Pi - \operatorname{div}(\mu(\rho)\mathcal{M}(u))|^2 dx \\ &\lesssim \int_{\mathbb{R}^2} |\rho u_t + \rho v \nabla u|^2 dx. \end{aligned}$$

So we get

$$\begin{aligned} &\frac{d}{dt} [f(t) \int_{\mathbb{R}^2} \mu(\rho) |\nabla u|^2 dx] + f(t) \int_{\mathbb{R}^2} |\operatorname{div}(\mu(\rho)\mathcal{M}(u))|^2 dx \\ &\lesssim f'(t) \int_{\mathbb{R}^2} \mu(\rho) |\nabla u|^2 dx + f(t) \|\sqrt{\rho} u_t\|_{L^2}^2 + f(t) \|\nabla u\|_{L^2}^2 \|\nabla v\|_{L^2}^2 (1 + \|v\|_{L^2}^2), \end{aligned}$$

along with (2.4), for a large constant \bar{C}

$$\begin{aligned} & \frac{d}{dt}[f(t) \int_{\mathbb{R}^2} \mu(\rho)|\nabla u|^2 dx] + f(t) \int_{\mathbb{R}^2} \bar{C}|\sqrt{\rho}u_t|^2 + |\operatorname{div}(\mu(\rho)\mathcal{M}(u))|^2 dx \\ & \leq C(f'(t) \int_{\mathbb{R}^2} \mu(\rho)|\nabla u|^2 dx + f(t)\|\nabla u\|_{L^2}^2\|\nabla v\|_{L^2}^2(1 + \|v\|_{L^2}^2)). \end{aligned}$$

Recall that

$$\nabla \Pi = \operatorname{div}(\mu(\rho)\mathcal{M}(u)) - \rho u_t - \rho v \nabla u,$$

and note that $v \in L^\infty(L^2) \cap L^2(\dot{H}^1)$, so that

$$\int_0^\infty (1 + \|v\|_{L^2}^2)\|\nabla v\|_{L^2}^2 dt \leq (1 + \|v\|_{L^\infty(L^2)}^2)\|\nabla v\|_{L^2(L^2)}^2,$$

and (2.2) holds.

The same strategy can be held for $f'(t)u$, we have

$$\frac{1}{2} \frac{d}{dt}[f'(t) \int_{\mathbb{R}^2} |\sqrt{\rho}u|^2 dx] + f'(t) \int_{\mathbb{R}^2} \mu(\rho)|\nabla u|^2 dx = \frac{1}{2} f''(t) \int_{\mathbb{R}^2} |\sqrt{\rho}u|^2 dx,$$

so that

$$\sup_{0 < t < \infty} f'(t) \int_{\mathbb{R}^2} \rho|u|^2(t) dx + \int_0^\infty f'(t) \int_{\mathbb{R}^2} \mu(\rho)|\nabla u|^2 dx dt \leq C(f'(0)\|u_0\|_{L^2}^2 + \int_0^\infty f''(t) \int_{\mathbb{R}^2} \rho|u|^2 dx dt).$$

□

According to these two energy estimates, letting $v = u$, we can prove Theorem 1.2.

Proof of Theorem 1.2: We get the standard energy estimate to (1.1) that

$$\frac{d}{dt}\|\sqrt{\rho}u(t)\|_{L^2}^2 + \|\nabla u(t)\|_{L^2}^2 \leq 0.$$

Using Schonbek's strategy, we obtain

$$(2.5) \quad \frac{d}{dt}\|\sqrt{\rho}u(t)\|_{L^2}^2 + g^2(t)\|\sqrt{\rho}u(t)\|_{L^2}^2 \leq M g^2(t) \int_{S(t)} |\hat{u}(t, \xi)|^2 d\xi,$$

where $S(t) \stackrel{\text{def}}{=} \{\xi : |\xi| \leq \sqrt{\frac{M}{2}}g(t)\}$ and $g(t)$ satisfying $g(t) \lesssim (1+t)^{-\frac{1}{2}}$. We rewrite the momentum equation of (1.1) as

$$u(t) = e^{\mu_0 t \Delta} u_0 + \int_0^t e^{\mu_0(t-s)\Delta} \mathbb{P} \left(\operatorname{div}((\mu(\rho) - \mu_0)\mathcal{M}(u)) + (1 - \rho)u_t - \rho u \nabla u \right) (s) ds.$$

Taking Fourier transform with respect to x variables leads to

$$|\hat{u}(t, \xi)| \lesssim e^{-\mu_0 t |\xi|^2} |\hat{u}_0(\xi)| + \int_0^t e^{-\mu_0(t-s)|\xi|^2} [|\xi| |\mathcal{F}_x((\mu(\rho) - \mu_0)\mathcal{M}(u))| + |\mathcal{F}_x((1 - \rho)u_t - \rho u \nabla u)|] ds,$$

which implies that

$$\begin{aligned} \int_{S(t)} |\hat{u}(t, \xi)|^2 d\xi & \lesssim \int_{S(t)} e^{-2\mu_0 t |\xi|^2} |\hat{u}_0(\xi)|^2 d\xi + g^4(t) \left(\int_0^t \|\mathcal{F}_x((\mu(\rho) - \mu_0)\mathcal{M}(u))\|_{L^\infty_\xi} ds \right)^2 \\ (2.6) \quad & + g^2(t) \left(\int_0^t \|\mathcal{F}_x((1 - \rho)u_t - \rho u \nabla u)\|_{L^\infty_\xi} ds \right)^2. \end{aligned}$$

Note that $u_0 \in L^p$ for $1 < p < 2$, one has

$$(2.7) \quad \int_{S(t)} e^{-2\mu_0 t |\xi|^2} |\hat{u}_0(\xi)|^2 d\xi \lesssim \left(\int_{S(t)} e^{-2\mu_0 q t |\xi|^2} d\xi \right)^{\frac{1}{q}} \|\hat{u}_0(\xi)\|_{L^{p'}}^2 \lesssim \|u_0\|_{L^p}^2 (1+t)^{-2\beta(p)},$$

where $\frac{1}{q} = \frac{2}{p} - 1$, $\frac{1}{p} + \frac{1}{p'} = 1$.

Note that $u \in L^\infty(L^2) \cap L^2(\dot{H}^1)$ and $u_t \in L^2(L^2)$, we have

$$\begin{aligned} \left(\int_0^t \|\mathcal{F}_x((\mu(\rho) - \mu_0)\mathcal{M}(u))\|_{L_\xi^\infty} ds \right)^2 &\leq \left(\int_0^t \|(\mu(\rho) - \mu_0)\mathcal{M}(u)\|_{L^1} ds \right)^2 \\ &\leq \|\mu(\rho) - \mu_0\|_{L_t^\infty(L^2)}^2 \left(\int_0^t \|\nabla u\|_{L^2} ds \right)^2 \\ &\leq C\|\rho_0 - 1\|_{L^2}^2 \|u_0\|_{L^2}^2 (1+t), \end{aligned}$$

$$\left(\int_0^t \|\mathcal{F}_x((1-\rho)u_t)\|_{L_\xi^\infty} ds \right)^2 \leq \|1-\rho\|_{L_t^\infty(L^2)}^2 \left(\int_0^t \|u_t\|_{L^2} ds \right)^2 \leq C\|\rho_0 - 1\|_{L^2}^2 \|\nabla u_0\|_{L^2}^2 (1+t),$$

$$\left(\int_0^t \|\mathcal{F}_x(\rho u \nabla u)\|_{L_\xi^\infty} ds \right)^2 \leq \|\rho u\|_{L_t^\infty(L^2)}^2 \left(\int_0^t \|\nabla u\|_{L^2} ds \right)^2 \leq C\|u_0\|_{L^2}^4 (1+t).$$

Then we deduce from (2.5) to (2.7) that

$$\begin{aligned} &\frac{d}{dt} \|\sqrt{\rho}u(t)\|_{L^2}^2 + g^2(t) \|\sqrt{\rho}u(t)\|_{L^2}^2 \\ &\lesssim g^2(t)(1+t)^{-2\beta(p)} + g^6(t)(1+t) + g^4(t)(1+t) \\ &\lesssim g^2(t)(1+t)^{-2\beta(p)} + g^4(t)(1+t). \end{aligned}$$

Taking $g^2(t) = \frac{2}{(e+t)\ln(e+t)}$, then $e \int_0^t g^2(s) ds = \ln^2(t+e)$ and

$$\begin{aligned} &\ln^2(t+e) \|u(t)\|_{L^2}^2 \\ &\lesssim \|u_0\|_{L^2}^2 + \int_0^t \left[\frac{\ln(s+e)}{(s+e)^{1+2\beta(p)}} + (s+e)^{-1} \right] ds \\ &\lesssim \ln(t+e), \end{aligned}$$

which gives

$$(2.8) \quad \|u(t)\|_{L^2}^2 \lesssim \ln^{-1}(t+e).$$

Now we improve the estimate (2.8).

We choose $f(t) = t+e$ in (2.2), then we have

$$\sup_{0 < t < \infty} (t+e) \|\nabla u(t)\|_{L^2}^2 + \int_0^\infty (t+e) \|u_t\|_{L^2}^2 dt \leq C\|u_0\|_{H^1}^2 \exp\{C\|u_0\|_{L^2}^4\},$$

so that

$$\begin{aligned} \left(\int_0^t \|\mathcal{F}_x((1-\rho)u_t)\|_{L_\xi^\infty} ds \right)^2 &\leq \|1-\rho\|_{L_t^\infty(L^2)}^2 \left(\int_0^t \|u_t\|_{L^2} ds \right)^2 \\ &\leq C\|\rho_0 - 1\|_{L^2}^2 \int_0^t (s+e) \|u_t\|_{L^2}^2 ds \int_0^t (s+e)^{-1} ds \\ &\lesssim \ln(t+e). \end{aligned}$$

$$\left(\int_0^t \|\mathcal{F}_x(\rho u \nabla u)\|_{L_\xi^\infty} ds \right)^2 \leq \left(\int_0^t \|u(s)\|_{L^2} \|\nabla u(s)\|_{L^2} ds \right)^2 \lesssim (t+e) \ln^{-1}(t+e).$$

We plug these estimate into (2.6) and take $g^2(t) = \frac{3}{(e+t)\ln(e+t)}$, then $e \int_0^t g^2(s) ds = \ln^3(t+e)$ and

$$\begin{aligned} \ln^3(t+e) \|u(t)\|_{L^2}^2 &\lesssim \int_0^t \left[\frac{\ln^2(s+e)}{(s+e)^{1+2\beta(p)}} + \frac{\ln^2(s+e)}{(s+e)^2} + (s+e)^{-1} \right] ds \\ &\lesssim \ln(t+e), \end{aligned}$$

which implies

$$(2.9) \quad \|u(t)\|_{L^2}^2 \lesssim \ln^{-2}(t+e).$$

So that

$$\int_0^\infty (t+e)^{-1} \|u\|_{L^2}^2 dt \leq C_1.$$

We choose $f'(t) = \ln(t+e)$ in (2.3), then get

$$\begin{aligned} & \sup_{0 < t < \infty} \ln(t+e) \|u(t)\|_{L^2}^2 + \int_0^\infty \ln(t+e) \|\nabla u\|_{L^2}^2 dt \\ & \leq C(\|u_0\|_{L^2}^2 + \int_0^\infty (t+e)^{-1} \|u\|_{L^2}^2 dt) \\ & \leq C_1. \end{aligned}$$

Consequently, we take $f(t) = (t+e) \ln(t+e)$ in (2.2), obtain that

$$\begin{aligned} & \sup_{0 < t < \infty} (t+e) \ln(t+e) \|\nabla u(t)\|_{L^2}^2 + \int_0^\infty (t+e) \ln(t+e) \|u_t\|_{L^2}^2 dt \\ & \leq C(\|\nabla u_0\|_{L^2}^2 + \int_0^\infty (\ln(t+e) + 1) \|\nabla u\|_{L^2}^2 dt) \exp\{C\|u_0\|_{L^2}^4\} \\ & \leq C_1, \end{aligned}$$

which implies

$$(2.10) \quad \|\nabla u(t)\|_{L^2}^2 \lesssim (t+e)^{-1} \ln^{-1}(t+e).$$

Combining (2.9) and (2.10), we get the revised estimates,

$$\begin{aligned} \left(\int_0^t \|u\|_{L^2} \|\nabla u\|_{L^2} ds\right)^2 & \lesssim \left(\int_0^t (s+e)^{-\frac{1}{2}} \ln^{-\frac{3}{2}}(s+e) ds\right)^2 \\ & \lesssim (t+e) \ln^{-3}(t+e), \end{aligned}$$

$$\begin{aligned} \left(\int_0^t \|1-\rho\|_{L^2} \|u_t\|_{L^2} ds\right)^2 & \lesssim \left(\int_0^t (s+e) \ln(s+e) \|u_t\|_{L^2}^2 ds\right) \left(\int_0^t (s+e)^{-1} \ln^{-1}(s+e) ds\right) \\ & \lesssim \ln(\ln(t+e)). \end{aligned}$$

Substituting these two estimates in (2.6), and taking $g^2(t) = \frac{5}{(t+e)\ln(t+e)}$, then $e^{\int_0^t g^2(s) ds} = \ln^5(t+e)$ and

$$\begin{aligned} \ln^5(t+e) \|u(t)\|_{L^2}^2 & \lesssim \|u_0\|_{L^2}^2 + \int_0^t \left[\frac{\ln^4(s+e)}{(s+e)^{1+2\beta(p)}} + \frac{\ln^3(s+e) \ln(\ln(t+e))}{(s+e)^2} + \frac{1}{s+e} \right] ds \\ & \lesssim \ln(t+e), \end{aligned}$$

from which, we obtain

$$(2.11) \quad \|u(t)\|_{L^2}^2 \lesssim \ln^{-4}(t+e).$$

We choose $f'(t) = \ln^2(t+e)$ in (2.3), then

$$\begin{aligned} & \sup_{0 < t < \infty} \ln^2(t+e) \|u(t)\|_{L^2}^2 + \int_0^\infty \ln^2(t+e) \|\nabla u\|_{L^2}^2 dt \\ & \leq C(\|u_0\|_{L^2}^2 + \int_0^\infty (t+e)^{-1} \ln(t+e) \|u(t)\|_{L^2}^2 dt) \\ & \leq C(\|u_0\|_{L^2}^2 + \int_0^\infty (t+e)^{-1} \ln^{-3}(t+e) dt) \\ & \leq C_1. \end{aligned}$$

Finally, we take $f(t) = (t + e) \ln^2(t + e)$ in (2.2) to get that

$$\begin{aligned} & \sup_{0 < t < \infty} (t + e) \ln^2(t + e) \|\nabla u(t)\|_{L^2}^2 + \int_0^\infty (t + e) \ln^2(t + e) \|u_t\|_{L^2}^2 dt \\ & \leq C \left(\|\nabla u_0\|_{L^2}^2 + \int_0^\infty (\ln(t + e) + \ln^2(t + e)) \|\nabla u\|_{L^2}^2 dt \right) \exp\{C \|u_0\|_{L^2}^4\} \\ & \leq C_1. \end{aligned}$$

Consequently, we obtain

$$\left(\int_0^\infty \|u_t\|_{L^2} dt \right)^2 \leq \left(\int_0^\infty (t + e) \ln^2(t + e) \|u_t\|_{L^2}^2 dt \right) \left(\int_0^\infty (t + e)^{-1} \ln^{-2}(t + e) dt \right) \leq C_1.$$

Which is the same for $\nabla \Pi, \operatorname{div}(\mu(\rho)\mathcal{M}(u)) \in L^1(\mathbb{R}^+; L^2)$, and gives (1.5). Recall that

$$\left(\int_0^t \|u\|_{L^2} \|\nabla u\|_{L^2} ds \right)^2 \leq \left(\int_0^t \|u\|_{L^2}^2 ds \right) \left(\int_0^t \|\nabla u\|_{L^2}^2 ds \right) \lesssim \int_0^t \|u\|_{L^2}^2 ds,$$

substituting these estimates into (2.6), noting that $2\beta(p) \in (0, 1)$, and taking $g^2(t) = \frac{\alpha}{t+e}$ with α large enough, then we get

$$\begin{aligned} (t + e)^\alpha \|u(t)\|_{L^2}^2 & \lesssim \|u_0\|_{L^2}^2 + \int_0^t (s + e)^{\alpha-2} \int_0^s \|u(\tau)\|_{L^2}^2 d\tau ds \\ & \quad + \int_0^t (s + e)^{\alpha-1-2\beta(p)} ds + \int_0^t (s + e)^{\alpha-2} ds \\ & \lesssim (t + e)^{\alpha-2\beta(p)} + \int_0^t (s + e)^{\alpha-2} \int_0^s \|u(\tau)\|_{L^2}^2 d\tau ds. \end{aligned}$$

For $t \geq 1$, we define

$$\begin{aligned} y(t) & \stackrel{def}{=} \int_{t-1}^t (s + e)^\alpha \|u(s)\|_{L^2}^2 ds, & Y(t) & \stackrel{def}{=} \max\{y(s) : 1 \leq s \leq t\}, \\ I(t) & \stackrel{def}{=} \int_0^t \|u(s)\|_{L^2}^2 ds. \end{aligned}$$

Then one has

$$\begin{aligned} I(t) & = \int_0^{t-[t]} \|u(s)\|_{L^2}^2 ds + \int_{t-[t]}^t \|u(s)\|_{L^2}^2 ds \\ & \leq C + \sum_{j=0}^{[t]-1} \int_{t-j-1}^{t-j} \|u(s)\|_{L^2}^2 (s + e)^\alpha (s + e)^{-\alpha} ds \\ (2.12) \quad & \leq C + \sum_{j=0}^{[t]-1} Y(t) (t - j)^{-\alpha} \lesssim C + Y(t) (t + e)^{1-\alpha}, \end{aligned}$$

from which, we infer that

$$y(t) \lesssim (t + e)^{\alpha-2\beta(p)} + \int_0^t (s + e)^{-1} Y(s) ds.$$

Then, applying Gronwall's inequality, we have

$$(2.13) \quad Y(t) \lesssim (t + e)^{\alpha-2\beta(p)} + \int_0^t (s + e)^{\alpha-2\beta(p)-1} ds \lesssim (t + e)^{\alpha-2\beta(p)}.$$

Plunging (2.13) into (2.12) gives rise to $I(t) \lesssim (t+e)^{1-2\beta(p)}$, we obtain

$$\begin{aligned} (t+e)^\alpha \|u(t)\|_{L^2}^2 &\lesssim (t+e)^{\alpha-2\beta(p)} + \int_0^t (s+e)^{\alpha-2\beta(p)-1} ds \\ &\lesssim (t+e)^{\alpha-2\beta(p)}, \end{aligned}$$

which gives the first inequality of (1.4).

Go back to (2.3), we choose $f''(t)$ such that $\int_0^\infty f''(t) \|u(t)\|_{L^2}^2 dt$ is finite. For example, we let $f''(t) = (t+e)^{-1+2\beta(p)-\varepsilon}$ for any $\varepsilon > 0$, (or $f''(t) = (t+e)^{-1+2\beta(p)} \ln^{-\alpha}(t+e)$ for any $\alpha > 1$), then $f(t) = (t+e)^{1+2\beta(p)-\varepsilon}$. Finally, we get (1.6) and the second inequality of (1.4). \blacksquare

3. THE PROOF OF THEOREM 1.3

The proof of Theorem 1.3 is very similar to Theorem 1.2. We should estimate every term in terms of $\|u_0\|_{H^\alpha}$ instead of $\|u_0\|_{H^1}$. First, we choose $f(t) = t+e$ and t in (2.2), get that

$$\sup_{0 < t < \infty} (t+e) \|\nabla u(t)\|_{L^2}^2 + \int_0^\infty (t+e) \|u_t\|_{L^2}^2 dt \leq C \|u_0\|_{H^1}^2 \exp\{C(1 + \|v\|_{L^\infty(L^2)}^2) \|\nabla v\|_{L^2(L^2)}^2\},$$

and

$$\sup_{0 < t < \infty} t \|\nabla u(t)\|_{L^2}^2 + \int_0^\infty t \|u_t\|_{L^2}^2 dt \leq C \|u_0\|_{L^2}^2 \exp\{C(1 + \|v\|_{L^\infty(L^2)}^2) \|\nabla v\|_{L^2(L^2)}^2\}.$$

By interpolation, and let $v = u$, we get that

$$(3.1) \quad \sup_{0 < t < \infty} (t+e)^{\alpha t^{1-\alpha}} \|\nabla u(t)\|_{L^2}^2 + \int_0^\infty (t+e)^{\alpha t^{1-\alpha}} \|u_t\|_{L^2}^2 dt \leq C \|u_0\|_{H^\alpha}^2 \exp\{C \|u_0\|_{L^2}^4\}.$$

So that

$$\begin{aligned} \left(\int_0^t \|\mathcal{F}_x((1-\rho)u_t)\|_{L_\xi^\infty} ds\right)^2 &\leq \|1-\rho\|_{L_t^\infty(L^2)}^2 \left(\int_0^t \|u_t\|_{L^2} ds\right)^2 \\ &\leq C \|\rho_0 - 1\|_{L^2}^2 \int_0^t s^{1-\alpha} (s+e)^\alpha \|u_t\|_{L^2}^2 ds \int_0^t s^{\alpha-1} (s+e)^{-\alpha} ds \\ &\leq C_\alpha t^\alpha, \end{aligned}$$

$$\begin{aligned} \left(\int_0^t \|\mathcal{F}_x((\mu(\rho) - \mu_0)\mathcal{M}(u))\|_{L_\xi^\infty} ds\right)^2 &\leq \left(\int_0^t \|(\mu(\rho) - \mu_0)\mathcal{M}(u)\|_{L^1} ds\right)^2 \\ &\leq \|\mu(\rho) - \mu_0\|_{L_t^\infty(L^2)}^2 \left(\int_0^t \|\nabla u\|_{L^2} ds\right)^2 \\ &\leq C \|\rho_0 - 1\|_{L^2}^2 \|u_0\|_{L^2}^2 (1+t), \end{aligned}$$

$$\left(\int_0^t \|\mathcal{F}_x(\rho u \nabla u)\|_{L_\xi^\infty} ds\right)^2 \leq \|\rho u\|_{L_t^\infty(L^2)}^2 \left(\int_0^t \|\nabla u\|_{L^2} ds\right)^2 \leq C \|u_0\|_{L^2}^4 (1+t).$$

From which, we can deduce that

$$\begin{aligned} &\frac{d}{dt} \|\sqrt{\rho}u(t)\|_{L^2}^2 + g^2(t) \|\sqrt{\rho}u(t)\|_{L^2}^2 \\ &\leq C_\alpha \left(g^2(t)(1+t)^{-2\beta(p)} + g^6(t)(1+t) + g^4(t)(1+t) + g^4(t)t^\alpha \right) \\ &\leq C_\alpha \left(g^2(t)(1+t)^{-2\beta(p)} + g^4(t)(1+t) + g^4(t)t^\alpha \right). \end{aligned}$$

Taking $g^2(t) = \frac{2}{(e+t)\ln(e+t)}$, then $e^{\int_0^t g^2(s) ds} = \ln^2(t+e)$ and

$$\begin{aligned} & \ln^2(t+e)\|u(t)\|_{L^2}^2 \\ & \leq C\|u_0\|_{L^2}^2 + C_\alpha \int_0^t \left[\frac{\ln(s+e)}{(s+e)^{1+2\beta(p)}} + \frac{1}{s+e} + \frac{1}{(s+e)^{2-\alpha}} \right] ds \\ & \leq C_\alpha \ln(t+e), \end{aligned}$$

which gives

$$(3.2) \quad \|u(t)\|_{L^2}^2 \leq C_\alpha \ln^{-1}(t+e).$$

Now, for $t > 1$, we have

$$\begin{aligned} \left(\int_0^t \|\mathcal{F}_x(\rho u \nabla u)\|_{L^\infty_\xi} ds \right)^2 & \leq C \left(\int_0^t \|u\|_{L^2} \|\nabla u\|_{L^2} ds \right)^2 \\ & \leq C_\alpha \left(\int_0^t s^{\frac{\alpha-1}{2}} (s+e)^{-\frac{\alpha}{2}} \ln^{-\frac{1}{2}}(s+e) ds \right)^2 \\ & \leq C_\alpha (1 + (t+e) \ln^{-1}(t+e)) \leq C_\alpha (t+e) \ln^{-1}(t+e). \end{aligned}$$

We take $g^2(t) = \frac{3}{(e+t)\ln(e+t)}$, then $e^{\int_0^t g^2(s) ds} = \ln^3(t+e)$ and

$$\begin{aligned} \ln^3(t+e)\|u(t)\|_{L^2}^2 & \leq C\|u_0\|_{L^2}^2 + C_\alpha \int_0^t \left[\frac{\ln^2(s+e)}{(s+e)^{1+2\beta(p)}} + \frac{(s^\alpha+1)\ln(s+e)}{(s+e)^2} + \frac{1}{s+e} \right] ds \\ & \leq C_\alpha \ln(t+e), \end{aligned}$$

which implies

$$\|u(t)\|_{L^2}^2 \leq C_\alpha \ln^{-2}(t+e), \quad \text{for } t > 1.$$

And for $0 < t < 1$, it is obvious, so that

$$(3.3) \quad \|u(t)\|_{L^2}^2 \leq C_\alpha \ln^{-2}(t+e),$$

and

$$\int_0^\infty (t+e)^{-1} \|u\|_{L^2}^2 dt \leq C_\alpha \int_0^\infty (t+e)^{-1} \ln^{-2}(t+e) dt \leq C_\alpha.$$

We choose $f'(t) = \ln(t+e)$ in (2.3), then get

$$\begin{aligned} & \sup_{0 < t < \infty} \ln(t+e)\|u(t)\|_{L^2}^2 + \int_0^\infty \ln(t+e)\|\nabla u\|_{L^2}^2 dt \\ & \leq C(\|u_0\|_{L^2}^2 + \int_0^\infty (t+e)^{-1}\|u\|_{L^2}^2 dt) \\ & \leq C_\alpha. \end{aligned}$$

Consequently, for any $0 < r < \alpha$, we take $f(t) = t^{1-r}(t+e)^r \ln(t+e)$ in (2.2), obtain that

$$\begin{aligned} & \sup_{0 < t < \infty} t^{1-r}(t+e)^r \ln(t+e)\|\nabla u(t)\|_{L^2}^2 + \int_0^\infty t^{1-r}(t+e)^r \ln(t+e)\|u_t\|_{L^2}^2 dt \\ & \leq C \int_0^\infty \left[\left(\frac{t}{t+e}\right)^{1-r} + \ln(t+e) \left(\left(\frac{t}{t+e}\right)^{1-r} + \left(\frac{t+e}{t}\right)^r\right) \right] \|\nabla u\|_{L^2}^2 dt \exp\{C\|u_0\|_{L^2}^4\}. \end{aligned}$$

Using (3.1), we get that

$$\int_0^1 t^{-r}\|\nabla u(t)\|_{L^2}^2 dt \leq C_\alpha \int_0^1 t^{\alpha-r-1}(t+e)^{-\alpha} dt \leq C_\alpha,$$

which implies

$$(3.4) \quad \|\nabla u(t)\|_{L^2}^2 \leq C_\alpha t^{r-1}(t+e)^{-r} \ln^{-1}(t+e).$$

Combining (3.3) and (3.4), for any $t > 1$, we get the revised estimates,

$$\begin{aligned}
& \left(\int_0^t \|u\|_{L^2} \|\nabla u\|_{L^2} ds \right)^2 \\
& \leq C_\alpha \left(\int_0^1 s^{\frac{r-1}{2}} (s+e)^{-\frac{r}{2}} \ln^{-\frac{1}{2}}(s+e) ds \right)^2 + C_\alpha \left(\int_1^t s^{\frac{r-1}{2}} (s+e)^{-\frac{r}{2}} \ln^{-\frac{3}{2}}(s+e) ds \right)^2 \\
& \leq C_\alpha (1 + (t+e) \ln^{-3}(t+e)) \leq C_\alpha (t+e) \ln^{-3}(t+e), \\
& \left(\int_0^t \|1 - \rho\|_{L^2} \|u_t\|_{L^2} ds \right)^2 \\
& \leq C_\alpha \left(\int_0^t s^{1-r} (s+e)^r \ln(s+e) \|u_t\|_{L^2}^2 ds \right) \left(\int_0^t s^{r-1} (s+e)^{-r} \ln^{-1}(s+e) ds \right) \\
& \leq C_\alpha \ln(\ln(t+e)).
\end{aligned}$$

For $t > 1$, taking $g^2(t) = \frac{5}{(t+e) \ln(t+e)}$, then $e^{\int_0^t g^2(s) ds} = \ln^5(t+e)$ and

$$\begin{aligned}
& \ln^5(t+e) \|u(t)\|_{L^2}^2 \\
& \leq C \|u_0\|_{L^2}^2 + C_\alpha \int_0^t \left[\frac{\ln^4(s+e)}{(s+e)^{1+2\beta(p)}} + \frac{\ln^3(s+e) \ln(\ln(t+e))}{(s+e)^2} + \frac{1}{s+e} \right] ds \\
& \leq C_\alpha \ln(t+e),
\end{aligned}$$

from which, we obtain

$$(3.5) \quad \|u(t)\|_{L^2}^2 \leq C_\alpha \ln^{-4}(t+e).$$

We choose $f'(t) = \ln^2(t+e)$ in (2.3), then

$$\begin{aligned}
& \sup_{0 < t < \infty} \ln^2(t+e) \|u(t)\|_{L^2}^2 + \int_0^\infty \ln^2(t+e) \|\nabla u\|_{L^2}^2 dt \\
& \leq C (\|u_0\|_{L^2}^2 + \int_0^\infty (t+e)^{-1} \ln(t+e) \|u(t)\|_{L^2}^2 dt) \\
& \leq C_\alpha (1 + \int_0^1 (t+e)^{-1} dt + \int_1^\infty (t+e)^{-1} \ln^{-3}(t+e) dt) \\
& \leq C_\alpha.
\end{aligned}$$

Finally, we take $f(t) = t^{1-r}(t+e)^r \ln^2(t+e)$ in (2.2) to get that

$$\begin{aligned}
& \sup_{0 < t < \infty} t^{1-r}(t+e)^r \ln^2(t+e) \|\nabla u(t)\|_{L^2}^2 + \int_0^\infty t^{1-r}(t+e)^r \ln^2(t+e) \|u_t\|_{L^2}^2 dt \\
& \leq C \int_0^\infty \left\{ \ln(t+e) \left(\frac{t}{t+e} \right)^{1-r} + \ln^2(t+e) \left[\left(\frac{t}{t+e} \right)^{1-r} + \left(\frac{t+e}{t} \right)^r \right] \right\} \|\nabla u\|_{L^2}^2 dt \exp\{C \|u_0\|_{L^2}^4\} \\
& \leq C_\alpha.
\end{aligned}$$

Consequently, we obtain

$$\begin{aligned}
& \left(\int_0^\infty \|u_t\|_{L^2} dt \right)^2 \\
& \leq \left(\int_0^\infty t^{1-r}(t+e)^r \ln^2(t+e) \|u_t\|_{L^2}^2 dt \right) \left(\int_0^\infty t^{r-1}(t+e)^{-r} \ln^{-2}(t+e) dt \right) \\
& \leq C_\alpha.
\end{aligned}$$

Which is the same for $\nabla \Pi, \operatorname{div}(\mu(\rho)\mathcal{M}(u)) \in L^1(\mathbb{R}^+; L^2)$, and gives (1.8).

Then follow the same line to the proof of Theorem 1.2, we get the first inequality of (1.7). We choose $f'(t) = (t + e)^{2\beta - \varepsilon}$ in (2.3), obtain that

$$\begin{aligned} & \sup_{0 < t < \infty} (t + e)^{2\beta - \varepsilon} \|u(t)\|_{L^2}^2 + \int_0^\infty (t + e)^{2\beta - \varepsilon} \|\nabla u\|_{L^2}^2 dt \\ & \leq C(\|u_0\|_{L^2}^2 + \int_0^\infty (t + e)^{-1 + 2\beta - \varepsilon} \|u\|_{L^2}^2 dt) \\ & \leq C_\alpha. \end{aligned}$$

Then taking $f(t) = t^{1-r}(t + e)^{r + 2\beta - \varepsilon}$ in (2.2), we deduce that

$$\begin{aligned} & \sup_{0 < t < \infty} t^{1-r}(t + e)^{r + 2\beta - \varepsilon} \|\nabla u(t)\|_{L^2}^2 + \int_0^\infty t^{1-r}(t + e)^{r + 2\beta - \varepsilon} \|u_t\|_{L^2}^2 dt \\ & \leq C \int_0^\infty (t + e)^{2\beta - \varepsilon} \left[\left(\frac{t}{t + e}\right)^{1-r} + \left(\frac{t + e}{t}\right)^r \right] \|\nabla u\|_{L^2}^2 dt \exp\{C\|u_0\|_{L^2}^4\} \\ & \leq C_\alpha, \end{aligned}$$

which implies (1.9) and the second inequality of (1.7). This completes the proof of Theorem 1.3.

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