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► **To cite this version:**

Jingchi Huang, Marius Paicu, Ping Zhang. Global wellposedness to incompressible inhomogeneous fluid system with bounded density and non-Lipschitz velocity. 2012. hal-00765697

**HAL Id: hal-00765697**

**<https://hal.science/hal-00765697>**

Preprint submitted on 16 Dec 2012

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# GLOBAL WELLPOSEDNESS TO INCOMPRESSIBLE INHOMOGENEOUS FLUID SYSTEM WITH BOUNDED DENSITY AND NON-LIPSCHITZ VELOCITY

JINGCHI HUANG, MARIUS PAICU, AND PING ZHANG

ABSTRACT. In this paper, we first prove the global existence of weak solutions to the d-dimensional incompressible inhomogeneous Navier-Stokes equations with initial data  $a_0 \in L^\infty(\mathbb{R}^d)$ ,  $u_0 = (u_0^h, u_0^d) \in \dot{B}_{p,r}^{-1+\frac{d}{p}}(\mathbb{R}^d)$ , which satisfies  $(\mu\|a_0\|_{L^\infty} + \|u_0^h\|_{\dot{B}_{p,r}^{-1+\frac{d}{p}}}) \exp(C_r\mu^{-2r}\|u_0^d\|_{\dot{B}_{p,r}^{-1+\frac{d}{p}}}^{2r}) \leq c_0\mu$  for some positive constants  $c_0, C_r$  and  $1 < p < d, 1 < r < \infty$ . The regularity of the initial velocity is critical to the scaling of this system and is general enough to generate non-Lipschitz velocity field. Furthermore, with additional regularity assumption on the initial velocity or on the initial density, we can also prove the uniqueness of such solution. We should mention that the classical maximal  $L^p(L^q)$  regularity theorem for the heat kernel plays an essential role in this context.

*Keywords:* Inhomogeneous Navier-Stokes equations, maximal  $L^p(L^q)$  regularity for heat kernel, Littlewood-Paley theory.

*AMS Subject Classification (2000):* 35Q30, 76D03

## 1. INTRODUCTION

In this paper, we consider the global wellposedness to the following d-dimensional incompressible inhomogeneous Navier-Stokes equations with the regularity of the initial velocity being almost critical and the initial density being a bounded positive function, which satisfies some nonlinear smallness condition,

$$(1.1) \quad \begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0, & (t, x) \in \mathbb{R}^+ \times \mathbb{R}^d, \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) - \mu \Delta u + \nabla \Pi = 0, \\ \operatorname{div} u = 0, \\ \rho|_{t=0} = \rho_0, \quad \rho u|_{t=0} = \rho_0 u_0, \end{cases}$$

where  $\rho, u = (u^h, u^d)$  stand for the density and velocity of the fluid respectively,  $\Pi$  is a scalar pressure function, and  $\mu$  the viscosity coefficient. Such system describes a fluid which is obtained by mixing two immiscible fluids that are incompressible and that have different densities. It may also describe a fluid containing a melted substance. We remark that our hypothesis on the density is of physical interest, which corresponds to the case of a mixture of immiscible fluids with different and bounded densities.

In particular, we shall focus on the global wellposedness of (1.1) with small homogeneity for the initial density function in  $L^\infty(\mathbb{R}^d)$  and small horizontal components of the velocity compared with its vertical component. This approach was already applied by Paicu and Zhang [26, 27] for 3-D anisotropic Navier-Stokes equations and for inhomogeneous Navier-Stokes system in the framework of Besov spaces. The main novelty of the present paper is to consider the initial density function in  $L^\infty(\mathbb{R}^d)$ , which is close enough to some positive constant. Then in order to handle the nonlinear terms appearing in (1.1), we need to use the maximal regularity effect for the classical heat equation. We should mention that the initial data have scaling invariant regularities and the

global weak solutions obtained here, under a nonlinear-type smallness condition, also belong to the critical spaces. Moreover, the regularity of the velocity field obtained in this paper is general enough to include the case of non-Lipschitz vector fields.

When  $\rho_0$  is bounded away from 0, Kazhikov [22] proved that: the inhomogeneous Navier-Stokes equations (1.1) has at least one global weak solutions in the energy space. In addition, he also proved the global existence of strong solutions to this system for small data in three space dimensions and all data in two dimensions. However, the uniqueness of both type weak solutions has not be solved. Ladyženskaja and Solonnikov [23] first addressed the question of unique resolvability of (1.1). More precisely, they considered the system (1.1) in a bounded domain  $\Omega$  with homogeneous Dirichlet boundary condition for  $u$ . Under the assumption that  $u_0 \in W^{2-\frac{2}{p},p}(\Omega)$  ( $p > d$ ) is divergence free and vanishes on  $\partial\Omega$  and that  $\rho_0 \in C^1(\Omega)$  is bounded away from zero, then they [23] proved

- Global well-posedness in dimension  $d = 2$ ;
- Local well-posedness in dimension  $d = 3$ . If in addition  $u_0$  is small in  $W^{2-\frac{2}{p},p}(\Omega)$ , then global well-posedness holds true.

Similar results were obtained by Danchin [11] in  $\mathbb{R}^d$  with initial data in the almost critical Sobolev spaces. Abidi, Gui and Zhang [3] investigated the large time decay and stability to any given global smooth solutions of (1.1), which in particular implies the global wellposedness of 3-D inhomogeneous Navier-Stokes equations with axi-symmetric initial data provided that there is no swirl part for the initial velocity field and the initial density is close enough to a positive constant. In general, when the viscosity coefficient,  $\mu(\rho)$ , depends on  $\rho$ , Lions [24] proved the global existence of weak solutions to (1.1) in any space dimensions.

In the case when the density function  $\rho$  is away from zero, we denote by  $a \stackrel{\text{def}}{=} \frac{1}{\rho} - 1$ , then the system (1.1) can be equivalently reformulated as

$$(1.2) \quad \begin{cases} \partial_t a + u \cdot \nabla a = 0, & (t, x) \in \mathbb{R}^+ \times \mathbb{R}^d, \\ \partial_t u + u \cdot \nabla u + (1+a)(\nabla \Pi - \mu \Delta u) = 0, \\ \operatorname{div} u = 0, \\ (a, u)|_{t=0} = (a_0, u_0). \end{cases}$$

Notice that just as the classical Navier-Stokes system, the inhomogeneous Navier-Stokes system (1.2) also has a scaling. More precisely, if  $(a, u)$  solves (1.2) with initial data  $(a_0, u_0)$ , then for  $\forall \ell > 0$ ,

$$(1.3) \quad (a, u)_\ell \stackrel{\text{def}}{=} (a(\ell^2 \cdot, \ell \cdot), \ell u(\ell^2 \cdot, \ell \cdot)) \quad \text{and} \quad (a_0, u_0)_\ell \stackrel{\text{def}}{=} (a_0(\ell \cdot), \ell u_0(\ell \cdot))$$

$(a, u)_\ell$  is also a solution of (1.2) with initial data  $(a_0, u_0)_\ell$ .

In [10], Danchin studied in general space dimension  $d$  the unique solvability of the system (1.2) in scaling invariant homogeneous Besov spaces, which generalized the celebrated results by Fujita and Kato [16] devoted to the classical Navier-Stokes system. In particular, the norm of  $(a, u) \in \dot{B}_{2,\infty}^{\frac{d}{2}}(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d) \times \dot{B}_{2,1}^{\frac{d}{2}-1}(\mathbb{R}^d)$  is scaling invariant under the change of scale of (1.3). In this case, Danchin proved that if the initial data  $(a_0, u_0) \in \dot{B}_{2,\infty}^{\frac{d}{2}}(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d) \times \dot{B}_{2,1}^{\frac{d}{2}-1}(\mathbb{R}^d)$  with  $a_0$  sufficiently small in  $\dot{B}_{2,\infty}^{\frac{d}{2}}(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ , then the system (1.2) has a unique local-in-time solution. Abidi [1] proved that if  $1 < p < 2d$ ,  $0 < \underline{\mu} < \tilde{\mu}(a)$ ,  $u_0 \in \dot{B}_{p,1}^{\frac{d}{p}-1}(\mathbb{R}^d)$  and  $a_0 \in \dot{B}_{p,1}^{\frac{d}{p}}(\mathbb{R}^d)$ , then (1.2) has a global solution provided that  $\|a_0\|_{\dot{B}_{p,1}^{\frac{d}{p}}} + \|u_0\|_{\dot{B}_{p,1}^{\frac{d}{p}-1}} \leq c_0$  for some  $c_0$  sufficiently small. Furthermore, such a solution is unique if  $1 < p \leq d$ . This result generalized the corresponding results in [10, 11] and was improved by Abidi and Paicu in [2] when  $\tilde{\mu}(a)$  is a positive constant by using different Lebesgue indices for the density and for the velocity. More precisely, for  $a_0 \in \dot{B}_{q,1}^{\frac{d}{q}}(\mathbb{R}^d)$  and  $u_0 \in \dot{B}_{p,1}^{-1+\frac{d}{p}}(\mathbb{R}^d)$

with  $|\frac{1}{p} - \frac{1}{q}| < \frac{1}{d}$  and  $\frac{1}{p} + \frac{1}{q} > \frac{1}{d}$ , they obtained the existence of solutions to (1.2) and under a more restrictive condition:  $\frac{1}{p} + \frac{1}{q} \geq \frac{2}{d}$ , they proved the uniqueness of this solution. In particular, with a well prepared regularity for the density function, this result implies the global existence of solutions to (1.2) for any  $1 < p < \infty$  and the uniqueness of such solution when  $1 < p < 2d$ . Very recently, Danchin and Mucha [13] filled the gap for the uniqueness result in [1] with  $p \in (d, 2d)$  through Langrange approach, and Abidi, Gui and Zhang relaxed the smallness condition for  $a_0$  in [4, 5].

On the other hand, when the initial density  $\rho_0 \in L^\infty(\Omega)$  with positive lower bound and  $u_0 \in H^2(\Omega)$ , Danchin and Mucha [14] proved the local wellposedness of (1.1). They also proved the global wellposedness result provided that the fluctuation of the initial density is sufficiently small, and initial velocity is small in  $B_{q,p}^{2-\frac{2}{q}}(\Omega)$  for  $1 < p < \infty, d < q < \infty$  in 3-D and any velocity in  $B_{4,2}^1(\Omega) \cap L^2(\Omega)$  in 2-D. Motivated by Proposition 2.1 below concerning the alternative definition of Besov spaces (see Definition A.1) with negative indices and [17], where Kato solved the local (resp. global) wellposedness of 3-D classical Navier-Stokes system through elementary  $L^p$  approach, we shall investigate the global existence of weak solutions to (1.2) with initial data  $a_0 \in L^\infty(\mathbb{R}^d)$  and  $u_0 \in \dot{B}_{p,r}^{-1+\frac{d}{p}}(\mathbb{R}^d)$  for  $p \in (1, d)$  and  $r \in (1, \infty)$ , which satisfies the nonlinear smallness condition (1.6). Furthermore, if we assume, in addition,  $u_0 \in \dot{B}_{p,r}^{-1+\frac{d}{p}+\varepsilon}(\mathbb{R}^d)$  for some sufficiently small  $\varepsilon > 0$ , we can also prove the uniqueness of such solution.

**Definition 1.1.** We call  $(a, u, \nabla \Pi)$  a global weak solution of (1.2) if

- for any test function  $\phi \in C_c^\infty([0, \infty) \times \mathbb{R}^d)$ , there holds

$$(1.4) \quad \begin{aligned} \int_0^\infty \int_{\mathbb{R}^d} a(\partial_t \phi + u \cdot \nabla \phi) dx dt + \int_{\mathbb{R}^d} \phi(0, x) a_0(x) dx &= 0, \\ \int_0^\infty \int_{\mathbb{R}^d} \operatorname{div} u \phi dx dt &= 0, \end{aligned}$$

- for any vector valued function  $\Phi = (\Phi^1, \dots, \Phi^d) \in C_c^\infty([0, \infty) \times \mathbb{R}^d)$ , one has

$$(1.5) \quad \int_0^\infty \int_{\mathbb{R}^d} \left\{ u \cdot \partial_t \Phi - (u \cdot \nabla u) \cdot \Phi + (1+a)(\mu \Delta u - \nabla \Pi) \cdot \Phi \right\} dx dt + \int_{\mathbb{R}^d} u_0 \cdot \Phi(0, x) dx = 0.$$

We denote the vector field by  $u = (u^h, u^d)$  where  $u^h = (u^1, u^2, \dots, u^{d-1})$ . Our first main result in this paper is as follows:

**Theorem 1.1.** Let  $p \in (1, d)$  and  $r \in (1, \infty)$ . Let  $a_0 \in L^\infty(\mathbb{R}^d)$  and  $u_0 \in \dot{B}_{p,r}^{-1+\frac{d}{p}}(\mathbb{R}^d)$ . Then there exist positive constants  $c_0, C_r$  so that if

$$(1.6) \quad \eta \stackrel{\text{def}}{=} \left( \mu \|a_0\|_{L^\infty} + \|u_0^h\|_{\dot{B}_{p,r}^{-1+\frac{d}{p}}} \right) \exp \left\{ C_r \mu^{-2r} \|u_0^d\|_{\dot{B}_{p,r}^{-1+\frac{d}{p}}}^{2r} \right\} \leq c_0 \mu,$$

(1.2) has a global weak solution  $(a, u)$  in the sense of Definition 1.1, which satisfies (1) when  $p \in (1, \frac{dr}{3r-2}]$ ,

$$(1.7) \quad \begin{aligned} \mu^{\frac{1}{r}} \|\Delta u^h\|_{L^r(\mathbb{R}^+; L^{\frac{dr}{3r-2}})} + \mu^{\frac{1}{2r}} \|\nabla u^h\|_{L^{2r}(\mathbb{R}^+; L^{\frac{dr}{2r-1}})} &\leq C\eta, \\ \mu^{\frac{1}{r}} \|\Delta u^d\|_{L^r(\mathbb{R}^+; L^{\frac{dr}{3r-2}})} + \mu^{\frac{1}{2r}} \|\nabla u^d\|_{L^{2r}(\mathbb{R}^+; L^{\frac{dr}{2r-1}})} &\leq C \|u_0^d\|_{\dot{B}_{p,r}^{-1+\frac{d}{p}}} + c\mu, \\ \mu^{\frac{1}{r}} \|\nabla \Pi\|_{L^r(\mathbb{R}^+; L^{\frac{dr}{3r-2}})} &\leq C\eta \left( \|u_0^d\|_{\dot{B}_{p,r}^{-1+\frac{d}{p}}} + c\mu \right); \end{aligned}$$

(2) when  $p \in (\frac{dr}{3r-2}, d)$ ,

$$(1.8) \quad \begin{aligned} & \mu^{\frac{1}{2}(3-\frac{d}{p_1})} \|t^{\alpha_1} \Delta u^h\|_{L^{2r}(\mathbb{R}^+; L^{p_1})} + \mu^{1-\frac{d}{2p_2}} (\|t^{\beta_1} \nabla u^h\|_{L^{2r}(\mathbb{R}^+; L^{p_2})} + \|t^{\beta_2} \nabla u^h\|_{L^\infty(\mathbb{R}^+; L^{p_2})}) \\ & \quad + \mu^{\frac{1}{2}(1-\frac{d}{p_3})} (\|t^{\gamma_1} u^h\|_{L^\infty(\mathbb{R}^+; L^{p_3})} + \|t^{\gamma_2} u^h\|_{L^{2r}(\mathbb{R}^+; L^{p_3})}) \leq C\eta, \\ & \mu^{\frac{1}{2}(3-\frac{d}{p_1})} \|t^{\alpha_1} \Delta u^d\|_{L^{2r}(\mathbb{R}^+; L^{p_1})} + \mu^{1-\frac{d}{2p_2}} (\|t^{\beta_1} \nabla u^d\|_{L^{2r}(\mathbb{R}^+; L^{p_2})} + \|t^{\beta_2} \nabla u^d\|_{L^\infty(\mathbb{R}^+; L^{p_2})}) \\ & \quad + \mu^{\frac{1}{2}(1-\frac{d}{p_3})} (\|t^{\gamma_1} u^d\|_{L^\infty(\mathbb{R}^+; L^{p_3})} + \|t^{\gamma_2} u^d\|_{L^{2r}(\mathbb{R}^+; L^{p_3})}) \leq 2C \|u_0^d\|_{\dot{B}_{p,r}^{-1+\frac{d}{p}}} + c\mu, \end{aligned}$$

and

$$(1.9) \quad \begin{aligned} & \mu^{\frac{1}{2}(3-\frac{d}{p_1})} \|t^{\alpha_2} \Delta u\|_{L^r(\mathbb{R}^+; L^{p_1})} \leq C (\|u_0\|_{\dot{B}_{p,r}^{-1+\frac{d}{p}}} + \frac{\eta}{\mu} (\|u_0^d\|_{\dot{B}_{p,r}^{-1+\frac{d}{p}}} + c\mu + \eta)), \\ & \mu^{\frac{1}{2}(3-\frac{d}{p_1})} (\|t^{\alpha_1} \nabla \Pi\|_{L^{2r}(\mathbb{R}^+; L^{p_1})} + \|t^{\alpha_2} \nabla \Pi\|_{L^r(\mathbb{R}^+; L^{p_1})}) \leq C\eta (\|u_0^d\|_{\dot{B}_{p,r}^{-1+\frac{d}{p}}} + c\mu), \end{aligned}$$

for some small enough constant  $c$ , where  $p_1, p_2, p_3$  satisfy  $\max(p, \frac{dr}{2r-1}) < p_1 < d$ , and  $\frac{dr}{r-1} < p_3 < \infty$  so that  $\frac{1}{p_2} + \frac{1}{p_3} = \frac{1}{p_1}$ , the indices  $\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2$  are determined by

$$(1.10) \quad \begin{aligned} \alpha_1 &= \frac{1}{2}(3 - \frac{d}{p_1}) - \frac{1}{2r}, & \beta_1 &= \frac{1}{2}(2 - \frac{d}{p_2}) - \frac{1}{2r} & \text{and} & \gamma_1 &= \frac{1}{2}(1 - \frac{d}{p_3}), \\ \alpha_2 &= \frac{1}{2}(3 - \frac{d}{p_1}) - \frac{1}{r}, & \beta_2 &= \frac{1}{2}(2 - \frac{d}{p_2}) & \text{and} & \gamma_2 &= \frac{1}{2}(1 - \frac{d}{p_3}) - \frac{1}{2r}. \end{aligned}$$

Furthermore, if we assume, in addition, that  $u_0 \in \dot{B}_{p,r}^{-1+\frac{d}{p}+\varepsilon}(\mathbb{R}^d)$  for  $0 < \varepsilon < \min\{\frac{1}{r}, 1 - \frac{1}{r}, \frac{d}{p} - 1\}$ , then such a global solution is unique.

**Remark 1.1.** The main idea to prove Theorem 1.1 is to use the maximal  $L^p(L^q)$  regularizing effect for heat kernel (see Lemma 2.1). In fact, similar to the classical Navier-Stokes equations ([17]), we first reformulate (1.2) as

$$(1.11) \quad u = e^{\mu t \Delta} u_0 + \int_0^t e^{\mu(t-s)\Delta} \{-u \cdot \nabla u + \mu a \Delta u - (1+a) \nabla \Pi\} ds.$$

Then we can prove appropriate approximate solutions to (1.11) satisfies the uniform estimate (1.7-1.9). With these estimates, the existence part of Theorem 1.1 follows by a compactness argument.

We remark that given initial data  $u_0 \in \dot{B}_{p,r}^{-1+\frac{d}{p}}(\mathbb{R}^d)$ , the maximal regularity we can expect for  $u$  is  $\tilde{L}_t^1(\dot{B}_{p,r}^{1+\frac{d}{p}})$ . With this regularity for  $u$  and  $a \in L^\infty(\mathbb{R}^+ \times \mathbb{R}^d)$ , we do not know how to define the product  $a \Delta u$  in the sense of distribution if  $p < d$ . This explains in some sense why we can only prove Theorem 1.1 for  $p \in (1, d)$ .

**Remark 1.2.** The smallness condition (1.6) is motivated by the one in [27] (see also [18, 26, 29] for the related works on 3-D incompressible anisotropic Navier-Stokes system), where we prove that: for  $1 < q \leq p < 6$  with  $\frac{1}{q} - \frac{1}{p} \leq \frac{1}{3}$ , given any data  $a_0 \in \dot{B}_{q,1}^{\frac{3}{q}}(\mathbb{R}^3)$  and  $u_0 = (u_0^h, u_0^3) \in \dot{B}_{p,1}^{-1+\frac{3}{p}}(\mathbb{R}^3)$  verifying

$$(1.12) \quad \eta \stackrel{\text{def}}{=} (\mu \|a_0\|_{\dot{B}_{q,1}^{\frac{3}{q}}} + \|u_0^h\|_{\dot{B}_{p,1}^{-1+\frac{3}{p}}}) \exp\left\{C_0 \|u_0^3\|_{\dot{B}_{p,1}^{-1+\frac{3}{p}}}^2 / \mu^2\right\} \leq c_0 \mu,$$

for some positive constants  $c_0$  and  $C_0$ , (1.2) has a unique global solution  $a \in C([0, \infty); \dot{B}_{q,1}^{\frac{3}{q}}(\mathbb{R}^3))$  and  $u \in C([0, \infty); \dot{B}_{p,1}^{-1+\frac{3}{p}}(\mathbb{R}^3)) \cap L^1(\mathbb{R}^+; \dot{B}_{p,1}^{1+\frac{3}{p}}(\mathbb{R}^3))$ . Similar wellposedness result ([20]) holds with

$\|a_0\|_{\dot{B}_{q,1}^{\frac{3}{p}}}$  in (1.12) being replaced by  $\|a_0\|_{\mathcal{M}(\dot{B}_{p,1}^{-1+\frac{3}{p}})}$ , the norm to the multiplier space of  $\dot{B}_{p,1}^{-1+\frac{3}{p}}(\mathbb{R}^3)$ .

We emphasize that our proof in [27, 20] uses in a fundamental way the algebraical structure of (1.2), namely,  $\operatorname{div} u = 0$ , which will also be one of the key ingredients in the proof of Theorem 1.1 and Theorem 1.2 below.

**Remark 1.3.** We should also mention the recent interesting result by Danchin and Mucha [14] that with more regularity assumption on the initial velocity field, namely,  $m \leq \rho_0 < M$  and  $u_0 \in H^2(\mathbb{R}^d)$  for  $d = 2, 3$ , they can prove the local wellposedness of (1.1) for large data and global wellposedness for small data. We emphasize that here we work our initial velocity field in the critical space  $\dot{B}_{p,r}^{-1+\frac{d}{p}}(\mathbb{R}^d)$  for  $p \in (1, d)$  and  $r \in (1, \infty)$  and also the fact that Theorem 1.1 remains to be valid in the case of bounded smooth domain with Dirichlet boundary conditions for the velocity field. Moreover, our uniqueness result in Theorem 1.1 is strongly inspired by the Lagrangian approach in [14], but with an almost critical regularity for the velocity, the proof here will be much more complicated. In fact, the small extra regularity compared to the scaling (1.3), namely,  $u_0 \in B_{p,r}^{-1+\frac{d}{p}+\varepsilon}(\mathbb{R}^d)$  for some small  $\varepsilon > 0$ , is useful to obtain that  $\Delta u \in L_{loc}^1(L^{d+\eta})$  for some  $\eta > 0$ , which combined with  $\Delta u \in L_{loc}^1(L^{p_1})$  with  $p_1 < d$  (see (1.8)) implies that  $u \in L_{loc}^1(Lip)$  and this allows us to reformulate (1.2) in the Lagrangian coordinates. One may check Theorem 4.1 One may check Theorem 4.1 below for more information about this solution.

A different approach to recover the uniqueness of the solution is to impose more regularity on the density function. Indeed, if the density is such that  $a_0 \in B_{q,\infty}^{\frac{d}{q}+\varepsilon}(\mathbb{R}^d)$ , for some small positive  $\varepsilon$ , we can also prove the global wellposedness of (1.2) under the nonlinear smallness condition (1.13):

**Theorem 1.2.** Let  $r \in (1, \infty)$ ,  $1 < q \leq p < 2d$  with  $\frac{1}{q} - \frac{1}{p} \leq \frac{1}{d}$  and  $\varepsilon \in (0, \frac{2d}{p} - 1)$  be any positive real number. Let  $a_0 \in L^\infty(\mathbb{R}^d) \cap B_{q,\infty}^{\frac{d}{q}+\varepsilon}(\mathbb{R}^d)$  and  $u_0 \in \dot{B}_{p,r}^{-1+\frac{d}{p}-\varepsilon}(\mathbb{R}^d) \cap \dot{B}_{p,r}^{-1+\frac{d}{p}}(\mathbb{R}^d)$ . There exist positive constants  $c_0, C_{r,\varepsilon}$  so that if

$$(1.13) \quad \delta \stackrel{\text{def}}{=} \left( \mu \|a_0\|_{L^\infty \cap B_{q,\infty}^{\frac{d}{q}+\varepsilon}} + \|u_0^h\|_{\dot{B}_{p,r}^{-1+\frac{d}{p}-\varepsilon} \cap \dot{B}_{p,r}^{-1+\frac{d}{p}}} \right) \exp \left\{ C_{r,\varepsilon} \mu^{-2r} \|u_0^d\|_{\dot{B}_{p,r}^{-1+\frac{d}{p}-\varepsilon} \cap \dot{B}_{p,r}^{-1+\frac{d}{p}}}^{2r} \right\} \leq c_0 \mu,$$

(1.2) has a global solution  $(a, u)$  so that

$$\begin{aligned} a &\in C([0, \infty); L^\infty(\mathbb{R}^d) \cap B_{q,\infty}^{\frac{d}{q}+\frac{\varepsilon}{2}}(\mathbb{R}^d)), \\ u &\in C([0, \infty); \dot{B}_{p,r}^{-1+\frac{d}{p}-\varepsilon}(\mathbb{R}^d) \cap \dot{B}_{p,r}^{-1+\frac{d}{p}}(\mathbb{R}^d)) \cap \tilde{L}^1(\mathbb{R}^+; \dot{B}_{p,r}^{1+\frac{d}{p}-\varepsilon}(\mathbb{R}^d) \cap \dot{B}_{p,r}^{1+\frac{d}{p}}(\mathbb{R}^d)), \end{aligned}$$

and there holds

$$(1.14) \quad \begin{aligned} &\|u^h\|_{\tilde{L}^\infty(\mathbb{R}^+; \dot{B}_{p,r}^{-1+\frac{d}{p}-\varepsilon})} + \|u^h\|_{\tilde{L}^\infty(\mathbb{R}^+; \dot{B}_{p,r}^{-1+\frac{d}{p}})} + \mu \left( \|a\|_{\tilde{L}^\infty(\mathbb{R}^+; B_{q,\infty}^{\frac{d}{q}+\frac{\varepsilon}{2}})} \right. \\ &\quad \left. + \|u^h\|_{\tilde{L}^1(\mathbb{R}^+; \dot{B}_{p,r}^{1+\frac{d}{p}-\varepsilon})} + \|u^h\|_{\tilde{L}^1(\mathbb{R}^+; \dot{B}_{p,r}^{1+\frac{d}{p}})} \right) \leq C\delta, \\ &\|u^d\|_{\tilde{L}^\infty(\mathbb{R}^+; \dot{B}_{p,r}^{-1+\frac{d}{p}-\varepsilon})} + \|u^d\|_{\tilde{L}^\infty(\mathbb{R}^+; \dot{B}_{p,r}^{-1+\frac{d}{p}})} + \mu \left( \|u^d\|_{\tilde{L}^1(\mathbb{R}^+; \dot{B}_{p,r}^{1+\frac{d}{p}-\varepsilon})} \right. \\ &\quad \left. + \|u^d\|_{\tilde{L}^1(\mathbb{R}^+; \dot{B}_{p,r}^{1+\frac{d}{p}})} \right) \leq 2 \|u_0^d\|_{\dot{B}_{p,r}^{-1+\frac{d}{p}-\varepsilon} \cap \dot{B}_{p,r}^{-1+\frac{d}{p}}} + c_2 \mu, \end{aligned}$$

for some  $c_2$  sufficiently small, and the norm  $\|\cdot\|_{X \cap Y} = \|\cdot\|_X + \|\cdot\|_Y$ . Furthermore, this solution is unique if  $\frac{1}{q} + \frac{1}{p} \geq \frac{2}{d}$ .

**Remark 1.4.** We point out that Haspot [19] proved the local well-posedness of (1.2) under similar conditions of Theorem 1.2. Our novelty here is the global existence of solutions to (1.2) under the smallness condition (1.13). We should also mention that: to overcome the difficulty that one can

not use Gronwall's inequality in the framework of Chemin-Lerner spaces, motivated by [26, 27], we introduced the weighted Chemin-Lerner type Besov norms in Definition A.3, which will be one of key ingredients used in the proof of Theorem 1.2.

**Remark 1.5.** We remark that in the previous works on the global wellposedness of (1.2), the third index,  $r$ , of the Besov spaces, to which the initial data belong, always equals to 1. In this case, the regularizing effect of heat equation allows the velocity field to be in  $L^1(\mathbb{R}^+, \text{Lip}(\mathbb{R}^d))$ , which is very useful to solve the transport equation without losing any derivative of the initial data. In both Theorem 1.1 and Theorem 1.2, the regularity of the velocity field is general enough to include non-Lipschitz vector-fields.

**The organization of the paper.** In the second section, we present the proof to the existence part of Theorem 1.1 in the case when  $1 < p \leq \frac{dr}{3r-2}$ . In Section 3, we shall present the proof to the existence part of Theorem 1.1 for the remaining case:  $\frac{dr}{3r-2} < p < d$ . In Section 4, we shall present the proof to the uniqueness part of Theorem 1.1 with an extra regularity on the initial velocity. In Section 5, we prove Theorem 1.2 which gives the uniqueness in the case where we have an additional regularity on the density. Finally in the appendix, we collect some basic facts on Littlewood-Paley theory and Besov spaces, which have been used throughout this paper.

Let us complete this section by the notations we shall use in this context:

**Notation.** For  $X$  a Banach space and  $I$  an interval of  $\mathbb{R}$ , we denote by  $C(I; X)$  the set of continuous functions on  $I$  with values in  $X$ , and by  $L^q(I; X)$  stands for the set of measurable functions on  $I$  with values in  $X$ , such that  $t \mapsto \|f(t)\|_X$  belongs to  $L^q(I)$ . For  $a \lesssim b$ , we mean that there is a uniform constant  $C$ , which may be different on different lines, such that  $a \leq Cb$ . We shall denote by  $(c_{j,r})_{j \in \mathbb{Z}}$  to be a generic element of  $\ell^r(\mathbb{Z})$  so that  $c_{j,r} \geq 0$  and  $\sum_{j \in \mathbb{Z}} c_{j,r}^r = 1$ .

## 2. PROOF TO THE EXISTENCE PART OF THEOREM 1.1 FOR $1 < p \leq \frac{dr}{3r-2}$

One of the key ingredients used in the proof to the existence part of Theorem 1.1 lies in Proposition 2.1 below:

**Proposition 2.1** (Theorem 2.34 of [6]). *Let  $s$  be a negative real number and  $(p, r) \in [1, \infty]^2$ . A constant  $C$  exists such that*

$$C^{-1} \mu^{\frac{s}{2}} \|f\|_{\dot{B}_{p,r}^s} \leq \| \|t^{-\frac{s}{2}} e^{t\Delta} f \|_{L^p} \|_{L^r(\mathbb{R}^+; \frac{dt}{t})} \leq C \mu^{\frac{s}{2}} \|f\|_{\dot{B}_{p,r}^s}.$$

In particular, for  $r \geq 1$ , we deduce from Proposition 2.1 that  $f \in \dot{B}_{p,r}^{-\frac{2}{r}}(\mathbb{R}^d)$  is equivalent to  $e^{\mu t \Delta} f \in L^r(\mathbb{R}^+; L^p(\mathbb{R}^d))$ .

Notice that given  $u_0 \in \dot{B}_{p,r}^{-1+\frac{d}{p}}(\mathbb{R}^d)$  with  $1 < p \leq \frac{dr}{3r-2}$  and  $r \in (1, \infty)$ ,  $\Delta u_0 \in \dot{B}_{p,r}^{-3+\frac{d}{p}}(\mathbb{R}^d)$ , we can always find some  $q_1 \geq p$  and  $r_1 \geq r$  such that

$$-3 + \frac{d}{p} \geq -3 + \frac{d}{q_1} = -\frac{2}{r_1} \geq -\frac{2}{r}.$$

Choosing  $r_1 = r$  in the above inequality leads to  $q_1 = \frac{dr}{3r-2}$ . Then  $\Delta u_0 \in \dot{B}_{\frac{dr}{3r-2}, r}^{-\frac{2}{r}}(\mathbb{R}^d)$ , we infer that  $\Delta e^{\mu t \Delta} u_0 \in L^r(\mathbb{R}^+; L^{\frac{dr}{3r-2}}(\mathbb{R}^d))$ . Similarly, we can choose some  $q_2 \geq p$  and  $r_2 \geq r$  with  $-2 + \frac{d}{q_2} = -\frac{2}{r_2}$ , so that  $\nabla u_0 \in \dot{B}_{p,r}^{-2+\frac{d}{p}}(\mathbb{R}^d) \hookrightarrow \dot{B}_{q_2, r_2}^{-2+\frac{d}{q_2}}(\mathbb{R}^d)$ . Choosing  $r_2 = 2r$  gives rise to  $q_2 = \frac{dr}{2r-1} > \frac{dr}{3r-2} \geq p$  and  $\nabla e^{\mu t \Delta} u_0 \in L^{2r}(\mathbb{R}^+; L^{\frac{dr}{2r-1}}(\mathbb{R}^d))$ . And Sobolev embedding ensures that  $e^{t\Delta} u_0 \in L^{2r}(\mathbb{R}^+; L^{\frac{dr}{r-1}}(\mathbb{R}^d))$  in this case.

The other key ingredient used in the proof of Theorem 1.1 is the following lemma (see [25] for instance), which is called maximal  $L^p(L^q)$  regularity for the heat kernel.

**Lemma 2.1.** (Lemma 7.3 of [25]) The operator  $\mathcal{A}$  defined by  $f(t, x) \mapsto \int_0^t \Delta e^{\mu(t-s)\Delta} f ds$  is bounded from  $L^p((0, T); L^q(\mathbb{R}^d))$  to  $L^p((0, T); L^q(\mathbb{R}^d))$  for every  $T \in (0, \infty]$  and  $1 < p, q < \infty$ . Moreover, there holds

$$\|\mathcal{A}f\|_{L_T^p(L^q)} \leq \frac{C}{\mu} \|f\|_{L_T^p(L^q)}.$$

**Lemma 2.2.** Let  $1 < r < \infty$ . The operator  $\mathcal{B}$  defined by  $f(t, x) \mapsto \int_0^t \nabla e^{\mu(t-s)\Delta} f ds$  is bounded from  $L^{2r}((0, T); L^{\frac{dr}{2r-1}}(\mathbb{R}^d))$  to  $L^r((0, T); L^{\frac{dr}{3r-2}}(\mathbb{R}^d))$  for every  $T \in (0, \infty]$ , and there holds

$$(2.1) \quad \left\| \int_0^t \nabla e^{\mu(t-s)\Delta} f ds \right\|_{L_T^{2r}(L^{\frac{dr}{2r-1}})} \leq \frac{C}{\mu^{\frac{2r-1}{2r}}} \|f\|_{L_T^r(L^{\frac{dr}{3r-2}})}.$$

*Proof.* Notice that

$$(2.2) \quad \begin{aligned} \nabla e^{\mu(t-s)\Delta} f(s, x) &= \frac{\sqrt{\pi}}{(4\pi\mu(t-s))^{\frac{d+1}{2}}} \int_{\mathbb{R}^d} \frac{(x-y)}{2\sqrt{\mu(t-s)}} \exp\left\{-\frac{|x-y|^2}{4\mu(t-s)}\right\} f(s, y) dy \\ &\stackrel{\text{def}}{=} \frac{\sqrt{\pi}}{(4\pi\mu(t-s))^{\frac{d+1}{2}}} K\left(\frac{\cdot}{\sqrt{4\mu(t-s)}}\right) * f(s, x). \end{aligned}$$

Applying Young's inequality in the space variables yields

$$\begin{aligned} &\|\nabla e^{\mu(t-s)\Delta} (1_{[0,T]}(s)f)(s, \cdot)\|_{L^{\frac{dr}{2r-1}}} \\ &\leq C(\mu(t-s))^{-\frac{d+1}{2}} \|K\left(\frac{\cdot}{\sqrt{4\pi\mu(t-s)}}\right)\|_{L^{\frac{dr}{dr-r+1}}} \|1_{[0,T]}(s)f(s)\|_{L^{\frac{dr}{3r-2}}} \\ &\leq C(\mu(t-s))^{-\frac{2r-1}{2r}} \|1_{[0,T]}(s)f(s)\|_{L^{\frac{dr}{3r-2}}}, \end{aligned}$$

where  $1_{[0,t]}(s)$  denotes the characteristic function on  $[0, t]$ , from which and Hardy-Littlewood-Sobolev inequality, we conclude the proof of (2.1).  $\square$

In what follows, we shall seek a solution  $(a, u)$  of (1.1) in the following space:

$$(2.3) \quad X \stackrel{\text{def}}{=} \{(a, u) : a \in L^\infty(\mathbb{R}^+ \times \mathbb{R}^d), \nabla u \in L^{2r}(\mathbb{R}^+; L^{\frac{dr}{2r-1}}(\mathbb{R}^d)), \Delta u \in L^r(\mathbb{R}^+; L^{\frac{dr}{3r-2}}(\mathbb{R}^d))\}.$$

We first mollify the initial data  $(a_0, u_0)$ , and then construct the approximate solutions  $(a_n, u_n)$  via

$$(2.4) \quad \begin{cases} \partial_t a_n + u_n \cdot \nabla a_n = 0, \\ \partial_t u_n + u_n \cdot \nabla u_n - \mu \Delta u_n + \nabla \Pi_n = a_n(\mu \Delta u_n - \nabla \Pi_n) \\ \operatorname{div} u_n = 0 \\ (a_n, u_n)|_{t=0} = (S_{N+n}a_0, S_{N+n}u_0), \end{cases}$$

where  $N$  is a large enough positive integer, and  $S_{N+n}a_0$  denotes the partial sum of  $a_0$  (see the Appendix for its definition).

We have the following proposition concerning the uniform bounds of  $(a_n, u_n)$ .

**Proposition 2.2.** Under the assumptions of Theorem 1.1, (2.4) has a unique global smooth solution  $(a_n, u_n, \nabla \Pi_n)$  which satisfies

$$(2.5) \quad \begin{aligned} \mu^{\frac{1}{r}} \|\Delta u_n^h\|_{L^r(\mathbb{R}^+; L^{\frac{dr}{3r-2}})} \mu^{\frac{1}{2r}} \|\nabla u_n^h\|_{L^{2r}(\mathbb{R}^+; L^{\frac{dr}{2r-1}})} &\leq C\eta, \\ \mu^{\frac{1}{r}} \|\Delta u_n^d\|_{L^r(\mathbb{R}^+; L^{\frac{dr}{3r-2}})} + \mu^{\frac{1}{2r}} \|\nabla + u_n^d\|_{L^{2r}(\mathbb{R}^+; L^{\frac{dr}{2r-1}})} &\leq C \|u_0^d\|_{\dot{B}_{p,r}^{-1+\frac{d}{p}}} + c\mu, \end{aligned}$$

and

$$(2.6) \quad \mu^{\frac{1}{r}} \|\nabla \Pi_n\|_{L^r(\mathbb{R}^+; L^{\frac{dr}{3r-2}})} \leq C\eta (\|u_0^d\|_{\dot{B}_{p,r}^{-1+\frac{d}{p}}} + c\mu)$$



for some small enough constant  $c$  and  $\eta$  given by (1.6).

*Proof.* For  $N$  large enough, it is easy to prove that (2.4) has a unique local smooth solution  $(a_n, u_n, \nabla \Pi_n)$  on  $[0, T_n^*)$  for some positive time  $T_n^*$ . Without loss of generality, we may assume that  $T_n^*$  is the lifespan to this solution. It is easy to observe that

$$(2.7) \quad \|a_n\|_{L^\infty((0, T_n^*) \times \mathbb{R}^d)} \leq \|a_0\|_{L^\infty}.$$

Next, for  $\lambda_1, \lambda_2 > 0$ , we denote

$$(2.8) \quad \begin{aligned} f_{1,n}(t) &\stackrel{\text{def}}{=} \|\nabla u_n^d(t)\|_{L^{\frac{dr}{2r-1}}}^{2r}, & f_{2,n}(t) &\stackrel{\text{def}}{=} \|\Delta u_n^d(t)\|_{L^{\frac{dr}{3r-2}}}^r, \\ u_{\lambda,n}(t, x) &\stackrel{\text{def}}{=} u_n(t, x) \exp\left\{-\int_0^t (\lambda_1 f_{1,n}(t') + \lambda_2 f_{2,n}(t')) dt'\right\}, \end{aligned}$$

and similar notation for  $\Pi_{\lambda,n}(t, x)$ . To deal with the pressure function  $\Pi_n$  in (2.4), we get, by taking divergence to the momentum equation of (2.4), that

$$-\Delta \Pi_{\lambda,n} = \operatorname{div}(a_n \nabla \Pi_{\lambda,n}) - \mu \operatorname{div}(a_n \Delta u_{\lambda,n}) + \operatorname{div}(u_n \cdot \nabla u_{\lambda,n}),$$

from which, (2.7), and the fact  $\operatorname{div} u_n = 0$ , so that

$$\begin{aligned} \operatorname{div}(u_n \cdot \nabla u_{\lambda,n}) &= \operatorname{div}_h(u_n^h \cdot \nabla_h u_{\lambda,n}^h) + \operatorname{div}_h(u_n^d \partial_d u_{\lambda,n}^h) \\ &\quad + \partial_d(u_{\lambda,n}^h \cdot \nabla_h u_n^d) - \partial_d(u_n^h \operatorname{div}_h u_{\lambda,n}^h), \end{aligned}$$

we deduce that

$$(2.9) \quad \begin{aligned} \|\nabla \Pi_{\lambda,n}(t)\|_{L^{\frac{dr}{3r-2}}} &\leq C \left\{ \|a_0\|_{L^\infty} \|\nabla \Pi_{\lambda,n}(t)\|_{L^{\frac{dr}{3r-2}}} + \mu \|a_0\|_{L^\infty} \|\Delta u_{\lambda,n}\|_{L^{\frac{dr}{3r-2}}} \right. \\ &\quad + (\|u_n^h\|_{L^{\frac{dr}{r-1}}} + \|u_n^d\|_{L^{\frac{dr}{r-1}}}) \|\nabla u_{\lambda,n}^h\|_{L^{\frac{dr}{2r-1}}} \\ &\quad \left. + \|u_{\lambda,n}^h\|_{L^{\frac{dr}{r-1}}} \|\nabla u_n^d\|_{L^{\frac{dr}{2r-1}}} \right\}. \end{aligned}$$

In particular, if  $\eta$  in (1.6) is so small that  $C\|a_0\|_{L^\infty} \leq \frac{1}{2}$ , we infer from (2.9) that

$$(2.10) \quad \begin{aligned} \|\nabla \Pi_{\lambda,n}(t)\|_{L^{\frac{dr}{3r-2}}} &\leq C \left\{ \mu \|a_0\|_{L^\infty} \|\Delta u_{\lambda,n}\|_{L^{\frac{dr}{3r-2}}} + \|u_n^h\|_{L^{\frac{dr}{r-1}}} \|\nabla u_{\lambda,n}^h\|_{L^{\frac{dr}{2r-1}}} \right. \\ &\quad \left. + \|u_n^d\|_{L^{\frac{dr}{r-1}}} \|\nabla u_{\lambda,n}^h\|_{L^{\frac{dr}{2r-1}}} + \|u_{\lambda,n}^h\|_{L^{\frac{dr}{r-1}}} \|\nabla u_n^d\|_{L^{\frac{dr}{2r-1}}} \right\}. \end{aligned}$$

Notice on the other hand that we can also equivalently reformulate the momentum equation of (2.4) as

$$(2.11) \quad u_n^i = u_{n,L}^i + \int_0^t e^{\mu(t-s)\Delta} \{-u_n \nabla u_n^i + \mu a_n \Delta u_n^i - (1 + a_n) \partial_i \Pi_n\} ds \quad \text{for } i = 1, 2, 3,$$

and  $u_{n,L} \stackrel{\text{def}}{=} e^{\mu t \Delta} S_{n+N} u_0$ , from which and (2.8), we write

$$(2.12) \quad \begin{aligned} u_{\lambda,n}^h &= u_{n,L}^h \exp\left\{-\int_0^t (\lambda_1 f_{1,n}(t') + \lambda_2 f_{2,n}(t')) dt'\right\} \\ &\quad + \int_0^t e^{\mu(t-s)\Delta} \exp\left\{-\int_s^t (\lambda_1 f_{1,n}(t') + \lambda_2 f_{2,n}(t')) dt'\right\} \\ &\quad \times \{-u_n \nabla u_{\lambda,n}^h + \mu a_n \Delta u_{\lambda,n}^h - (1 + a_n) \nabla_h \Pi_{\lambda,n}\} ds. \end{aligned}$$

Applying Lemma 2.1, Lemma 2.2, (2.7), and (2.9) leads to

$$\begin{aligned}
(2.13) \quad & \mu^{1-\frac{1}{2r}} \|\nabla u_{\lambda,n}^h\|_{L_t^{2r}(L^{\frac{dr}{2r-1}})} + \mu \|\Delta u_{\lambda,n}^h\|_{L_t^r(L^{\frac{dr}{3r-2}})} \\
& \leq \mu^{1-\frac{1}{2r}} \|\nabla u_{n,L}^h\|_{L_t^{2r}(L^{\frac{dr}{2r-1}})} + \mu \|\Delta u_L^h\|_{L_t^r(L^{\frac{dr}{3r-2}})} + C \left\{ \mu \|a_0\|_{L^\infty} \|\Delta u_{\lambda,n}^h\|_{L_t^r(L^{\frac{dr}{3r-2}})} \right. \\
& \quad + \mu \|a_0\|_{L^\infty} \|\Delta u_{\lambda,n}^d\|_{L_t^r(L^{\frac{dr}{3r-2}})} + \|u_n^h\|_{L_t^{2r}(L^{\frac{dr}{r-1}})} \|\nabla u_{\lambda,n}^h\|_{L_t^{2r}(L^{\frac{dr}{2r-1}})} \\
& \quad \left. + \left( \int_0^t e^{-r \int_s^t (\lambda_1 f_{1,n}(t') + \lambda_2 f_{2,n}(t')) dt'} (\|u_n^d(s)\|_{L^{\frac{dr}{r-1}}}^r \|\nabla u_{\lambda,n}^h(s)\|_{L^{\frac{dr}{2r-1}}}^r \right. \right. \\
& \quad \left. \left. + \|u_{\lambda,n}^h\|_{L^{\frac{dr}{r-1}}}^r \|\nabla u_n^d(s)\|_{L^{\frac{dr}{2r-1}}}^r) ds \right)^{\frac{1}{r}} \right\} \quad \text{for } t \leq T_n^*.
\end{aligned}$$

Then as  $C\|a_0\|_{L^\infty} \leq \frac{1}{2}$ , and

$$\begin{aligned}
\|u_n\|_{L_t^{2r}(L^{\frac{dr}{r-1}})} & \leq C \|\nabla u_n\|_{L_t^{2r}(L^{\frac{dr}{2r-1}})}, \quad \|\Delta u_{\lambda,n}^d\|_{L_t^r(L^{\frac{dr}{3r-2}})} \leq \frac{1}{(\lambda_2 r)^{\frac{1}{r}}}, \\
\left( \int_0^t e^{-\lambda_1 r \int_s^t f_{1,n}(t') dt'} \|u_{n-1}^d(s)\|_{L^{\frac{dr}{r-1}}}^r \|\nabla u_{\lambda,n}^h(s)\|_{L^{\frac{dr}{2r-1}}}^r ds \right)^{\frac{1}{r}} & \leq \frac{1}{(\lambda_1 r)^{\frac{1}{2r}}} \|\nabla u_{\lambda,n}^h(s)\|_{L_t^{2r}(L^{\frac{dr}{2r-1}})},
\end{aligned}$$

so that we infer from (2.13) that

$$\begin{aligned}
& \mu^{1-\frac{1}{2r}} \|\nabla u_{\lambda,n}^h\|_{L_t^{2r}(L^{\frac{dr}{2r-1}})} + \mu \|\Delta u_{\lambda,n}^h\|_{L_t^r(L^{\frac{dr}{3r-2}})} \\
& \leq C \mu^{1-\frac{1}{r}} \|u_0^h\|_{\dot{B}_{p,r}^{-1+\frac{d}{p}}} + C \left\{ \frac{\mu}{(\lambda_2 r)^{\frac{1}{r}}} \|a_0\|_{L^\infty} + \|u_n^h\|_{L_t^{2r}(L^{\frac{dr}{r-1}})} \|\nabla u_{\lambda,n}^h\|_{L_t^{2r}(L^{\frac{dr}{2r-1}})} \right. \\
& \quad \left. + \frac{1}{(\lambda_1 r)^{\frac{1}{2r}}} \|\nabla u_{\lambda,n}^h(s)\|_{L_t^{2r}(L^{\frac{dr}{2r-1}})} \right\}.
\end{aligned}$$

Taking  $\lambda_1 = \frac{(4C)^{2r}}{\mu^{2r-1}r}$  and  $\lambda_2 = \frac{C^r}{r\mu^{r-1}}$  in the above inequality, we obtain

$$\begin{aligned}
(2.14) \quad & \frac{3}{4} \mu^{1-\frac{1}{2r}} \|\nabla u_{\lambda,n}^h\|_{L_t^{2r}(L^{\frac{dr}{2r-1}})} + \mu \|\Delta u_{\lambda,n}^h\|_{L_t^r(L^{\frac{dr}{3r-2}})} \\
& \leq C \mu^{1-\frac{1}{r}} \|u_0^h\|_{\dot{B}_{p,r}^{-1+\frac{d}{p}}} + \mu^{2-\frac{1}{r}} \|a_0\|_{L^\infty} + \|u_n^h\|_{L_t^{2r}(L^{\frac{dr}{r-1}})} \|\nabla u_{\lambda,n}^h\|_{L_t^{2r}(L^{\frac{dr}{2r-1}})}.
\end{aligned}$$

Let  $c_1$  be a small enough positive constant, which will be determined later on, we denote

$$(2.15) \quad \bar{T}_n \stackrel{\text{def}}{=} \sup \left\{ T \leq T_n^*; \mu^{\frac{1}{2r}} \|\nabla u_n^h\|_{L_T^{2r}(L^{\frac{dr}{r-1}})} + \mu^{\frac{1}{r}} \|\Delta u_n^h\|_{L_T^r(L^{\frac{dr}{3r-2}})} \leq c_1 \mu \right\}.$$

Then it follows from (2.8) and (2.14) that for  $t \leq \bar{T}_n$

$$\begin{aligned}
(2.16) \quad & \frac{1}{2} \mu^{\frac{1}{2r}} \|\nabla u_n^h\|_{L_t^{2r}(L^{\frac{dr}{2r-1}})} + \mu^{\frac{1}{r}} \|\Delta u_n^h\|_{L_t^r(L^{\frac{dr}{3r-2}})} \\
& \leq C (\|u_0^h\|_{\dot{B}_{p,r}^{-1+\frac{d}{p}}} + \mu \|a_0\|_{L^\infty}) \\
& \quad \times \exp \left\{ -C_r \int_0^t (\mu^{1-2r} \|\nabla u_n^d(s)\|_{L^{\frac{dr}{2r-1}}}^{2r} + \mu^{1-r} \|\Delta u_n^d(s)\|_{L^{\frac{dr}{3r-2}}}^r) ds \right\}.
\end{aligned}$$

On the other hand, it follows from a similar derivation of (2.13) that

$$\begin{aligned}
& \mu^{1-\frac{1}{2r}} \|\nabla u_n^d\|_{L_t^{2r}(L^{\frac{dr}{2r-1}})} + \mu \|\Delta u_n^d\|_{L_t^r(L^{\frac{dr}{3r-2}})} \\
& \leq \mu^{1-\frac{1}{2r}} \|\nabla u_{n,L}^d\|_{L_t^{2r}(L^{\frac{dr}{2r-1}})} + \mu \|\Delta u_{n,L}^d\|_{L^r(L^{\frac{dr}{3r-2}})} + C \left\{ \mu \|a_0\|_{L^\infty} \|\Delta u_n\|_{L_t^r(L^{\frac{dr}{3r-2}})} \right. \\
& \quad + \left( \|u_n^h\|_{L_t^{2r}(L^{\frac{dr}{r-1}})} + \|u_n^d\|_{L_t^{2r}(L^{\frac{dr}{r-1}})} \right) \|\nabla u_n^h\|_{L_t^{2r}(L^{\frac{dr}{2r-1}})} \\
& \quad \left. + \|u_n^h\|_{L_t^{2r}(L^{\frac{dr}{r-1}})} \|\nabla u_n^d\|_{L_t^{2r}(L^{\frac{dr}{2r-1}})} \right\} \quad \text{for } t \leq T_n^*,
\end{aligned}$$

from which and (2.15), we deduce that

$$\begin{aligned}
(2.17) \quad & \mu^{\frac{1}{r}} \|\Delta u_n^d\|_{L_t^r(L^{\frac{dr}{3r-2}})} + \mu^{\frac{1}{2r}} \|\nabla u_n^d\|_{L_t^{2r}(L^{\frac{dr}{2r-1}})} \\
& \leq 2C \|u_0^d\|_{\dot{B}_{p,r}^{-1+\frac{d}{p}}} + 2C c_1 \mu (1 + \|a_0\|_{L^\infty}) \quad \text{for } t \leq \bar{T}_n.
\end{aligned}$$

Substituting (2.17) into (2.16) leads to

$$\begin{aligned}
(2.18) \quad & \mu^{\frac{1}{r}} \|\Delta u_n^h\|_{L_t^r(L^{\frac{dr}{3r-2}})} + \frac{1}{2} \mu^{\frac{1}{2r}} \|\nabla u_n^h\|_{L_t^{2r}(L^{\frac{dr}{2r-1}})} \\
& \leq C \left( \|u_0^h\|_{\dot{B}_{p,r}^{-1+\frac{d}{p}}} + \mu \|a_0\|_{L^\infty} \right) \exp \left\{ C_r \mu^{-2r} \|u_0^d\|_{\dot{B}_{p,r}^{-1+\frac{d}{p}}}^{2r} \right\} \leq \frac{c_1}{2} \mu \quad \text{for } t \leq \bar{T}_n,
\end{aligned}$$

as long as  $C_r$  is sufficiently large and  $c_0$  is small enough in (1.6). This shows that  $\bar{T}_n = T_n^*$ . Then thanks to (2.17), (2.18) and Theorem 1.3 in [21], we conclude that  $T_n^* = \infty$  and there holds (2.5). By virtue of (2.5) and (2.9), we infer

$$\begin{aligned}
\|\nabla \Pi_n\|_{L_t^r(L^{\frac{dr}{3r-2}})} & \leq C \left\{ \mu \|a_0\|_{L^\infty} \|\Delta u_n\|_{L_t^r(L^{\frac{dr}{3r-2}})} + \left( \|\nabla u_n^h\|_{L_t^{2r}(L^{\frac{dr}{2r-1}})} \right. \right. \\
& \quad \left. \left. + \|\nabla u_n^d\|_{L_t^{2r}(L^{\frac{dr}{2r-1}})} \right) \|\nabla u_n^h\|_{L_t^{2r}(L^{\frac{dr}{2r-1}})} \right\} \\
& \leq C \mu^{-\frac{1}{r}} \eta \left( \|u_0^d\|_{\dot{B}_{p,r}^{-1+\frac{d}{p}}} + \eta + c\mu \right),
\end{aligned}$$

for some small constant  $c$ , which leads to (2.6). This completes the proof of the proposition.  $\square$

We now present the proof to the existence part of Theorem 1.1 in the case when  $1 < p \leq \frac{dr}{3r-2}$ .

**Proof to the existence part of Theorem 1.1 for  $1 < p \leq \frac{dr}{3r-2}$ .** Indeed thanks to (2.4) and (2.5), (2.6), we infer that  $\{\partial_t u_n\}$  is uniformly bounded in  $L^r(\mathbb{R}^+; L^{\frac{dr}{3r-2}}(\mathbb{R}^d))$ , from which, (2.5), (2.6), and Ascoli-Arzelà Theorem, we conclude that there exists a subsequence of  $\{a_n, u_n, \nabla \Pi_n\}$ , which we still denote by  $\{a_n, u_n, \nabla \Pi_n\}$  and some  $(a, u, \nabla \Pi)$  with  $a \in L^\infty(\mathbb{R}^+ \times \mathbb{R}^d)$ ,  $\nabla u \in L^{2r}(\mathbb{R}^+; L^{\frac{dr}{2r-1}}(\mathbb{R}^d))$  and  $\Delta u, \nabla \Pi \in L^r(\mathbb{R}^+; L^{\frac{dr}{3r-2}}(\mathbb{R}^d))$ , such that

$$\begin{aligned}
(2.19) \quad & a_n \rightharpoonup a \quad \text{weak } * \text{ in } L_{loc}^\infty(\mathbb{R}^+ \times \mathbb{R}^d), \\
& u_n \rightarrow u \quad \text{strongly in } L_{loc}^{2r}(\mathbb{R}^+; L_{loc}^{\frac{dr}{r-1}}(\mathbb{R}^d)), \\
& \nabla u_n \rightarrow \nabla u \quad \text{strongly in } L_{loc}^{2r}(\mathbb{R}^+; L_{loc}^{\frac{dr}{r-1}}(\mathbb{R}^d)), \\
& \Delta u_n \rightharpoonup \Delta u \quad \text{and } \nabla \Pi_n \rightharpoonup \nabla \Pi \quad \text{weakly in } L^r(\mathbb{R}^+; L^{\frac{dr}{3r-2}}(\mathbb{R}^d)).
\end{aligned}$$

Obviously,  $(a_n, u_n, \nabla \Pi_n)$  satisfies

$$(2.20) \quad \begin{aligned} & \int_0^\infty \int_{\mathbb{R}^d} a_n (\partial_t \phi + u_n \cdot \nabla \phi) dx dt + \int_{\mathbb{R}^d} S_{n+N} a_0(x) \phi(0, x) dx = 0, \\ & \int_0^\infty \int_{\mathbb{R}^d} \operatorname{div} u_n \phi dx dt = 0 \quad \text{and} \\ & \int_0^\infty \int_{\mathbb{R}^d} \left\{ u_n \cdot \partial_t \Phi - (u_n \cdot \nabla u_n) \cdot \Phi + (1 + a_n) (\mu \Delta u_n - \nabla \Pi_n) \cdot \Phi \right\} dx dt \\ & \quad + \int_{\mathbb{R}^d} S_{n+N} u_0 \cdot \Phi(0, x) dx = 0, \end{aligned}$$

for all the test function  $\phi, \Phi$  given by Definition 1.1.

Therefore thanks to (2.19), to prove that  $(a, u, \nabla \Pi)$  obtained in (2.19) is indeed a global weak solution of (1.2) in the sense of Definition 1.1, we only need to show that  $\frac{1}{1+a_n} \rightarrow \frac{1}{1+a}$  almost everywhere in  $\mathbb{R}^+ \times \mathbb{R}^d$  as  $n \rightarrow \infty$ . Toward this, we shall follow the compactness argument in [24] to prove that  $\{a_n\}$  strongly converges to  $a$  in  $L_{loc}^m(\mathbb{R}^+ \times \mathbb{R}^d)$  for any  $m < \infty$ . In fact, it is easy to observe from the transport equation of (2.4) that

$$\partial_t a_n^2 + \operatorname{div}(u_n a_n^2) = 0,$$

from which, we deduce that

$$(2.21) \quad \partial_t \overline{a^2} + \operatorname{div}(u \overline{a^2}) = 0,$$

where we denote  $\overline{a^2}$  to be the weak  $*$  limit of  $\{a_n^2\}$ .

While thanks to (2.19) and (2.20), there holds

$$\partial_t a + \operatorname{div}(ua) = 0$$

in the sense of distributions. Moreover, as  $\nabla u \in L^r(\mathbb{R}^+; L^{\frac{dr}{2r-1}}(\mathbb{R}^d))$ , we infer by a mollifying argument as that in [15] that

$$(2.22) \quad \partial_t a^2 + \operatorname{div}(ua^2) = 0.$$

Subtracting (2.22) from (2.21), we obtain

$$(2.23) \quad \partial_t (\overline{a^2} - a^2) + \operatorname{div}(u(\overline{a^2} - a^2)) = 0.$$

Notice that  $\{S_{n+N} a_0\}$  converges to  $a_0$  almost everywhere in  $\mathbb{R}^d$ , which implies  $(\overline{a^2} - a^2)(0, x) = 0$  for a. e.  $x \in \mathbb{R}^d$ . Whence it follows from the uniqueness theorem for the transport equation in [15] that

$$(\overline{a^2} - a^2)(t, x) = 0 \quad \text{for a. e. } (t, x) \in \mathbb{R}^+ \times \mathbb{R}^d,$$

which together with  $\|a_n\|_{L^\infty} \leq \|a_0\|_{L^\infty}$  implies that

$$(2.24) \quad a_n \rightarrow a \quad \text{strongly in } L_{loc}^m(\mathbb{R}^+ \times \mathbb{R}^d) \quad \text{for any } m < \infty.$$

Thanks to (2.19) and (2.24), we can take  $n \rightarrow \infty$  in (2.20) to verify that  $(a, u, \nabla \Pi)$  obtained in (2.19) satisfies (1.4) and (1.5). Moreover, thanks to (2.5) and (2.6), there holds (1.7). This completes the proof to the existence part of Theorem 1.1 for  $p \in (1, \frac{dr}{3r-2}]$ .  $\blacksquare$

3. PROOF TO THE EXISTENCE PART OF THEOREM 1.1 FOR  $\frac{dr}{3r-2} < p < d$ 

In this case, as  $-3 + \frac{d}{p} < -\frac{2}{r}$ , it is impossible to find some  $p_1 \geq p$  and  $r_1 \geq r$  so that  $-3 + \frac{d}{p_1} = -\frac{2}{r_1}$ . However, notice that for all  $p_1 \geq p$  and  $r_1 \geq r$ ,  $\Delta u_0 \in \dot{B}_{p_1, r_1}^{-3 + \frac{d}{p_1}}(\mathbb{R}^d)$ , then we deduce from Proposition 2.1 that  $t^{\frac{1}{2}(3 - \frac{d}{p_1}) - \frac{1}{r_1}} \Delta e^{t\Delta} u_0 \in L^{r_1}(\mathbb{R}^+; L^{p_1}(\mathbb{R}^d))$ . Similarly, we choose  $p_2 > \frac{d}{2}$  and  $p_3 > d$  with  $\frac{1}{p_2} + \frac{1}{p_3} = \frac{1}{p_1}$ , there holds  $t^{\frac{1}{2}(2 - \frac{d}{p_2}) - \frac{1}{r_1}} \nabla e^{t\Delta} u_0 \in L^{r_1}(\mathbb{R}^+; L^{p_2}(\mathbb{R}^d))$  and  $t^{\frac{1}{2}(1 - \frac{d}{p_3})} e^{t\Delta} u_0 \in L^\infty(\mathbb{R}^+; L^{p_3}(\mathbb{R}^d))$ . With these time weights before  $e^{t\Delta} u_0$ ,  $\nabla e^{t\Delta} u_0$  and  $\Delta e^{t\Delta} u_0$ , to prove the existence part of Theorem 1.1 for  $\frac{dr}{3r-2} < p < d$ , we need to use the following time-weighted version of maximal  $L^p(L^q)$  regularizing effect for the heat kernel:

**Lemma 3.1.** *Let  $1 < p, q < \infty$  and  $\alpha$  a non-negative real number satisfying  $\alpha + \frac{1}{p} < 1$ . Let  $\mathcal{A}$  be the operator defined by Lemma 2.1. Then if  $t^\alpha f \in L^p((0, T); L^q(\mathbb{R}^d))$  for some  $T \in (0, \infty]$ ,  $t^\alpha \mathcal{A}f \in L^p((0, T); L^q(\mathbb{R}^d))$ , and there holds*

$$(3.1) \quad \|t^\alpha \mathcal{A}f\|_{L_T^p(L^q)} \leq \frac{C}{\mu} \|t^\alpha f\|_{L_T^p(L^q)}.$$

*Proof.* We first split the operator  $\mathcal{A}$  as

$$(3.2) \quad \mathcal{A}f = \left( \int_0^{\frac{t}{2}} + \int_{\frac{t}{2}}^t \right) \Delta e^{\mu(t-s)\Delta} f ds \stackrel{\text{def}}{=} \mathcal{A}_1 f + \mathcal{A}_2 f.$$

Note that when  $s \in [\frac{t}{2}, t]$ ,  $t^\alpha$  is comparable to  $s^\alpha$ , so that it follows from the proof of Lemma 7.3 in [25] that

$$\|t^\alpha \mathcal{A}_2 f\|_{L_T^p(L^q)} \leq \frac{C}{\mu} \|t^\alpha f\|_{L_T^p(L^q)}.$$

To handle  $\mathcal{A}_1 f$  in (3.2), we write

$$t^\alpha \mathcal{A}_1 f(t) = \int_0^{\frac{t}{2}} \frac{t^\alpha}{\mu(t-s)s^\alpha} \mu(t-s) \Delta e^{\mu(t-s)\Delta} (s^\alpha f) ds.$$

As  $\mu t \Delta e^{\mu t \Delta}$  is a bounded operator from  $L^q(\mathbb{R}^d)$  to  $L^q(\mathbb{R}^d)$ , we have

$$\|t^\alpha \mathcal{A}_1 f(t)\|_{L^q(\mathbb{R}^d)} \leq \frac{C}{\mu} \int_0^{\frac{t}{2}} \frac{t^\alpha}{(t-s)s^\alpha} \|s^\alpha f(s)\|_{L^q(\mathbb{R}^d)} ds.$$

Let  $F_\alpha(s) \stackrel{\text{def}}{=} \|s^\alpha f(s)\|_{L^q(\mathbb{R}^d)}$  and using the change of variable that  $s = t\tau$ , we obtain

$$\|t^\alpha \mathcal{A}_1 f(t)\|_{L^q(\mathbb{R}^d)} \leq \frac{C}{\mu} \int_0^{\frac{1}{2}} \frac{1}{(1-\tau)\tau^\alpha} F_\alpha(t\tau) d\tau,$$

from which and Minkowski inequality, we infer

$$\|t^\alpha \mathcal{A}_1 f\|_{L_T^p(L^q)} \leq \frac{C}{\mu} \int_0^{\frac{1}{2}} \frac{1}{(1-\tau)\tau^{\alpha + \frac{1}{p}}} d\tau \|F_\alpha\|_{L_T^p},$$

which along with the fact:  $\alpha + \frac{1}{p} < 1$ , concludes the proof of (3.1).  $\square$

In order to deal with the estimate of  $u$  and  $\nabla u$ , we need the following lemmas, which will be used in the proof of both the existence part and uniqueness part of Theorem 1.1.

**Lemma 3.2.** *Let the operator  $\mathcal{B}$  be defined by  $f(t, x) \mapsto \int_0^t \nabla e^{\mu(t-s)\Delta} f ds$  and  $\mathcal{C}$  by  $f(t, x) \mapsto \int_0^t e^{\mu(t-s)\Delta} f ds$ . Let  $\varepsilon \geq 0$  be a small enough number,  $r_1 > 1$ , and  $q_1, q_2, q_3$  satisfy  $\frac{dr_1}{2r_1-2} < q_1 < \frac{d}{1-\varepsilon}$ ,  $d < q_3 < \infty$  and  $\frac{1}{q_2} + \frac{1}{q_3} = \frac{1}{q_1}$ . We denote*

$$(3.3) \quad \tilde{\alpha}^\varepsilon = \frac{1}{2} \left( 3 - \frac{d}{q_1} - \varepsilon \right) - \frac{1}{r_1}, \quad \tilde{\beta}_1^\varepsilon = \frac{1}{2} \left( 2 - \frac{d}{q_2} - \varepsilon \right) - \frac{1}{r_1} \quad \text{and} \quad \tilde{\gamma}_1^\varepsilon = \frac{1}{2} \left( 1 - \frac{d}{q_3} - \varepsilon \right).$$

*Then if  $t^{\tilde{\alpha}^\varepsilon} f \in L^{r_1}((0, T); L^{q_1}(\mathbb{R}^d))$  for some  $T \in (0, \infty]$ ,  $t^{\tilde{\beta}_1^\varepsilon} \mathcal{B}f \in L^{r_1}((0, T); L^{q_2}(\mathbb{R}^d))$  and  $t^{\tilde{\gamma}_1^\varepsilon} \mathcal{C}f \in L^\infty((0, T); L^{q_3}(\mathbb{R}^d))$ , and there holds*

$$(3.4) \quad \|t^{\tilde{\beta}_1^\varepsilon} \mathcal{B}f\|_{L_T^{r_1}(L^{q_2})} \leq \frac{C_\varepsilon}{\mu^{\frac{1}{2} + \frac{d}{2q_3}}} \|t^{\tilde{\alpha}^\varepsilon} f\|_{L_T^{r_1}(L^{q_1})},$$

and

$$(3.5) \quad \|t^{\tilde{\gamma}_1^\varepsilon} \mathcal{C}f\|_{L_T^\infty(L^{q_3})} \leq \frac{C_\varepsilon}{\mu^{\frac{d}{2q_2}}} \|t^{\tilde{\alpha}^\varepsilon} f\|_{L_T^{r_1}(L^{q_1})}.$$

*Proof.* We first get, by applying Young's inequality to (2.2), that

$$(3.6) \quad \begin{aligned} \|\nabla e^{\mu(t-s)\Delta} f(s, \cdot)\|_{L^{q_2}} &\leq C(\mu(t-s))^{-\frac{d+1}{2}} \|K(\frac{\cdot}{\sqrt{4\mu(t-s)}})\|_{L^{\frac{q_3}{q_3-1}}} \|f(s)\|_{L^{q_1}} \\ &\leq C(\mu(t-s))^{-\left(\frac{1}{2} + \frac{d}{2q_3}\right)} \|f(s)\|_{L^{q_1}}. \end{aligned}$$

Then let  $F^\varepsilon(s) \stackrel{\text{def}}{=} \|s^{\tilde{\alpha}^\varepsilon} f(s)\|_{L^{q_1}}$ , we have

$$t^{\tilde{\beta}_1^\varepsilon} \|\mathcal{B}f(t)\|_{L^{q_2}} \leq C\mu^{-\left(\frac{1}{2} + \frac{d}{2q_3}\right)} t^{\tilde{\beta}_1^\varepsilon} \int_0^t (t-s)^{-\left(\frac{1}{2} + \frac{d}{2q_3}\right)} s^{-\tilde{\alpha}^\varepsilon} F^\varepsilon(s) ds.$$

Using change of variable with  $s = t\tau$  and the fact that  $\tilde{\beta}_1^\varepsilon - \left(\frac{1}{2} + \frac{d}{2q_3}\right) - \tilde{\alpha}^\varepsilon + 1 = 0$ , we obtain

$$t^{\tilde{\beta}_1^\varepsilon} \|\mathcal{B}f(t)\|_{L^{q_2}} \leq C\mu^{-\left(\frac{1}{2} + \frac{d}{2q_3}\right)} \int_0^1 (1-\tau)^{-\left(\frac{1}{2} + \frac{d}{2q_3}\right)} \tau^{-\tilde{\alpha}^\varepsilon} F^\varepsilon(t\tau) d\tau.$$

Applying Minkowski inequality leads to

$$\begin{aligned} \|t^{\tilde{\beta}_1^\varepsilon} \mathcal{B}f\|_{L_T^{r_1}(L^{q_2})} &\leq C\mu^{-\left(\frac{1}{2} + \frac{d}{2q_3}\right)} \int_0^1 (1-\tau)^{-\left(\frac{1}{2} + \frac{d}{2q_3}\right)} \tau^{-\tilde{\alpha}^\varepsilon - \frac{1}{r_1}} d\tau \|F^\varepsilon\|_{L_T^{r_1}} \\ &\leq C\mu^{-\left(\frac{1}{2} + \frac{d}{2q_3}\right)} \int_0^1 (1-\tau)^{-\left(\frac{1}{2} + \frac{d}{2q_3}\right)} \tau^{-\frac{1}{2} \left(3 - \frac{d}{q_1} - \varepsilon\right)} d\tau \|t^{\tilde{\alpha}^\varepsilon} f\|_{L_T^{r_1}(L^{q_1})}. \end{aligned}$$

Note that the assumption:  $q_3 > d$ , which together with  $q_1 < \frac{d}{1-\varepsilon}$  implies that  $0 < \frac{1}{2} + \frac{d}{2q_3}, \frac{1}{2} \left(3 - \frac{d}{q_1} - \varepsilon\right) < 1$ . This proves (3.4).

To deal with  $\mathcal{C}f$ , we write

$$e^{\mu(t-s)\Delta} f = (\mu(t-s))^{-\frac{d}{2q_2}} e^{-\mu(t-s)\Delta} (-\mu(t-s)\Delta)^{\frac{d}{2q_2}} (-\Delta)^{-\frac{d}{2q_2}} f.$$

Observing that for any  $\delta > 0$ ,  $e^{\Delta}(-\Delta)^\delta$  is a bounded linear operator from  $L^{q_3}(\mathbb{R}^d)$  to  $L^{q_3}(\mathbb{R}^d)$ , and  $(-\Delta)^{-\delta} f = |\cdot|^{-(d-2\delta)} * f$  for  $0 < 2\delta < d$ , so that applying Hardy-Littlewood-Sobolev inequality gives rise to

$$(3.7) \quad \begin{aligned} \|e^{\mu(t-s)\Delta} f(s)\|_{L^{q_3}} &\leq C(\mu(t-s))^{-\frac{d}{2q_2}} \|(-\Delta)^{-\frac{d}{2q_2}} f(s)\|_{L^{q_3}} \\ &\leq C(\mu(t-s))^{-\frac{d}{2q_2}} \|f(s)\|_{L^{q_1}}. \end{aligned}$$

Let  $r'_1$  be the conjugate index of  $r_1$ , we get, by applying Hölder's inequality, that

$$(3.8) \quad \begin{aligned} \|\mathcal{C}f(t)\|_{L^{q_3}} &\leq C\mu^{-\frac{d}{2q_2}} \int_0^t (t-s)^{-\frac{d}{2q_2}} s^{-\tilde{\alpha}^\varepsilon} F^\varepsilon(s) ds \\ &\leq C\mu^{-\frac{d}{2q_2}} \left( \int_0^t (t-s)^{-\frac{d}{2q_2}r'_1} s^{-\tilde{\alpha}^\varepsilon r'_1} ds \right)^{\frac{1}{r'_1}} \|t^{\tilde{\alpha}^\varepsilon} f\|_{L_T^{r_1}(L^{q_1})}. \end{aligned}$$

While using once again the change of variable that  $s = t\tau$ , we obtain

$$\begin{aligned} \int_0^t (t-s)^{-\frac{d}{2q_2}r'_1} s^{-\tilde{\alpha}^\varepsilon r'_1} ds &= t^{1-\frac{d}{2q_2}r'_1-\tilde{\alpha}^\varepsilon r'_1} \int_0^1 (1-\tau)^{-\frac{d}{2q_2}r'_1} \tau^{-\tilde{\alpha}^\varepsilon r'_1} d\tau \\ &= t^{-\tilde{\gamma}_1^\varepsilon r'_1} \int_0^1 (1-\tau)^{-\frac{d}{2q_2}r'_1} \tau^{-\tilde{\alpha}^\varepsilon r'_1} d\tau. \end{aligned}$$

Recalling the assumptions that  $q_2 > q_1 > \frac{dr_1}{2r_1-2}$  and  $q_1 < \frac{d}{1-\varepsilon}$ , which imply  $\frac{d}{2p_2}r'_1 < 1$  and  $\tilde{\alpha}^\varepsilon r'_1 < 1$ . This together with (3.8) ensures that

$$\|\mathcal{C}f(t)\|_{L^{q_3}} \leq C_\varepsilon \mu^{-\frac{d}{2q_2}} t^{-\tilde{\gamma}_1^\varepsilon} \|t^{\tilde{\alpha}^\varepsilon} f\|_{L_T^{r_1}(L^{q_1})}.$$

This concludes the proof of (3.5) and Lemma 3.2.  $\square$

In order to deal with the term  $u^d \partial_d u$  in (2.11), we need also the following lemma.

**Lemma 3.3.** *Let  $\varepsilon, r_1, q_1, \tilde{\alpha}^\varepsilon$  and the operators  $\mathcal{B}, \mathcal{C}$  be given by Lemma 3.2. Let  $r_1 > 2$  and  $q_2, q_3$  satisfy  $\frac{1}{q_2} + \frac{1}{q_3} = \frac{1}{q_1}$ ,  $\frac{dr_1}{r_1-2} < q_3 < \infty$ , we denote*

$$(3.9) \quad \tilde{\beta}_2^\varepsilon = \frac{1}{2}(2 - \frac{d}{q_2} - \varepsilon) \quad \text{and} \quad \tilde{\gamma}_2^\varepsilon = \frac{1}{2}(1 - \frac{d}{q_3} - \varepsilon) - \frac{1}{r_1}.$$

Then if  $t^{\tilde{\alpha}^\varepsilon} f \in L^{r_1}((0, T); L^{q_1}(\mathbb{R}^d))$  for some  $T \in (0, \infty]$ ,  $t^{\tilde{\beta}_2^\varepsilon} \mathcal{B}f \in L^\infty((0, T); L^{q_2}(\mathbb{R}^d))$ ,  $t^{\tilde{\gamma}_2^\varepsilon} \mathcal{C}f \in L^{r_1}((0, T); L^{q_3}(\mathbb{R}^d))$ , and there holds

$$(3.10) \quad \|t^{\tilde{\beta}_2^\varepsilon} \mathcal{B}f\|_{L_T^\infty(L^{q_2})} \leq \frac{C_\varepsilon}{\mu^{\frac{1}{2} + \frac{d}{2q_3}}} \|t^{\tilde{\alpha}^\varepsilon} f\|_{L_T^{r_1}(L^{q_1})},$$

and

$$(3.11) \quad \|t^{\tilde{\gamma}_2^\varepsilon} \mathcal{C}f\|_{L_T^{r_1}(L^{q_3})} \leq \frac{C_\varepsilon}{\mu^{\frac{d}{2q_2}}} \|t^{\tilde{\alpha}^\varepsilon} f\|_{L_T^{r_1}(L^{q_1})}.$$

*Proof.* We first get, by a similar derivation of (3.6), that

$$\begin{aligned} \|\mathcal{B}f(t)\|_{L^{q_2}} &\leq C\mu^{-(\frac{1}{2} + \frac{d}{2q_3})} \int_0^t (t-s)^{-(\frac{1}{2} + \frac{d}{2q_3})} s^{-\tilde{\alpha}^\varepsilon} F^\varepsilon(s) ds \\ &\leq C\mu^{-(\frac{1}{2} + \frac{d}{2q_3})} \left( \int_0^t (t-s)^{-(\frac{1}{2} + \frac{d}{2q_3})r'_1} s^{-\tilde{\alpha}^\varepsilon r'_1} ds \right)^{\frac{1}{r'_1}} \|t^{\tilde{\alpha}^\varepsilon} f\|_{L_T^{r_1}(L^{q_1})}. \end{aligned}$$

Whereas using the change of variable:  $s = t\tau$ , leads to

$$\begin{aligned} \int_0^t (t-s)^{-(\frac{1}{2} + \frac{d}{2q_3})r'_1} s^{-\tilde{\alpha}^\varepsilon r'_1} ds &= t^{1 - (\frac{1}{2} + \frac{d}{2q_3})r'_1 - \tilde{\alpha}^\varepsilon r'_1} \int_0^1 (1-\tau)^{-(\frac{1}{2} + \frac{d}{2q_3})r'_1} \tau^{-\tilde{\alpha}^\varepsilon r'_1} d\tau \\ &= t^{-\tilde{\beta}_2^\varepsilon r'_1} \int_0^1 (1-\tau)^{-(\frac{1}{2} + \frac{d}{2q_3})r'_1} \tau^{-\tilde{\alpha}^\varepsilon r'_1} d\tau. \end{aligned}$$

By virtue of the assumptions:  $q_3 > \frac{dr_1}{r_1-2}$  and  $q_1 < \frac{d}{1-\varepsilon}$ , we have  $0 < (\frac{1}{2} + \frac{d}{2q_3})r'_1$ ,  $\tilde{\alpha}^\varepsilon r'_1 < 1$ . As a consequence, we obtain

$$\|\mathcal{B}f(t)\|_{L^{q_2}} \leq C_\varepsilon \mu^{-(\frac{1}{2} + \frac{d}{2q_3})} t^{-\tilde{\beta}_2^\varepsilon} \|t^{\tilde{\alpha}^\varepsilon} f\|_{L_T^{r_1}(L^{q_1})},$$

which yields (3.10).

On the other hand, it follows the same line of (3.7) that

$$\|e^{\mu(t-s)\Delta} f(s)\|_{L^{q_3}} \leq C(\mu(t-s))^{-\frac{d}{2q_2}} \|f(s)\|_{L^{q_1}},$$

which implies

$$t^{\tilde{\gamma}_2^\varepsilon} \|\mathcal{C}f(t)\|_{L^{q_3}} \leq C\mu^{-\frac{d}{2q_2}} t^{\tilde{\gamma}_2^\varepsilon} \int_0^t (t-s)^{-\frac{d}{2q_2}} s^{-\tilde{\alpha}^\varepsilon} F^\varepsilon(s) ds.$$

Using changing of variable with  $s = t\tau$  and the fact:  $\tilde{\gamma}_2^\varepsilon - \frac{d}{2q_2} - \tilde{\alpha}^\varepsilon + 1 = 0$ , we obtain

$$t^{\tilde{\gamma}_2^\varepsilon} \|\mathcal{C}f(t)\|_{L^{q_3}} \leq C\mu^{-\frac{d}{2q_2}} \int_0^1 (1-\tau)^{-\frac{d}{2q_2}} \tau^{-\tilde{\alpha}^\varepsilon} F^\varepsilon(t\tau) d\tau,$$

from which, we deduce

$$\begin{aligned} \|t^{\tilde{\gamma}_2^\varepsilon} \mathcal{C}f\|_{L_T^{r_1}(L^{q_3})} &\leq C\mu^{-\frac{d}{2q_2}} \int_0^1 (1-\tau)^{-\frac{d}{2q_2}} \tau^{-\tilde{\alpha}^\varepsilon - \frac{1}{r_1}} d\tau \|F^\varepsilon\|_{L_T^{r_1}} \\ &\leq C\mu^{-\frac{d}{2q_2}} \int_0^1 (1-\tau)^{-\frac{d}{2q_2}} \tau^{-\frac{1}{2}(3-\frac{d}{q_1}-\varepsilon)} d\tau \|t^{\tilde{\alpha}^\varepsilon} f\|_{L_T^{r_1}(L^{q_1})}, \end{aligned}$$

which along with the facts:  $q_2 > q_1 > \frac{dr_1}{2r_1-2} > \frac{d}{2}$ , ensures the integral above is finite. This gives (3.11) and we complete the proof of the lemma.  $\square$

In what follows, we take  $r_1 = 2r > 2$  and  $p_1, p_2$  and  $p_3$  satisfying  $\max(p, \frac{dr}{2r-1}) < p_1 < d$ , and  $\frac{dr}{r-1} < p_3 < \infty$  so that  $\frac{1}{p_2} + \frac{1}{p_3} = \frac{1}{p_1}$ . We shall seek a solution  $(a, u, \nabla\Pi)$  of (1.2) in the following functional space:

$$(3.12) \quad X = \left\{ u : a \in L^\infty(\mathbb{R}^+ \times \mathbb{R}^d), \quad t^{\gamma_1} u \in L^\infty(\mathbb{R}^+; L^{p_3}(\mathbb{R}^d)), \right. \\ \left. t^{\gamma_2} u \in L^{2r}(\mathbb{R}^+; L^{p_3}(\mathbb{R}^d)), \quad t^{\beta_1} \nabla u \in L^{2r}(\mathbb{R}^+; L^{p_2}(\mathbb{R}^d)), \right. \\ \left. t^{\beta_2} \nabla u \in L^\infty(\mathbb{R}^+; L^{p_2}(\mathbb{R}^d)) \quad t^{\alpha_1} \Delta u \in L^{2r}(\mathbb{R}^+; L^{p_1}(\mathbb{R}^d)) \right\},$$

with the indices  $\alpha_1, \beta_1, \beta_2, \gamma_1, \gamma_2$  being determined by (1.10).

We construct the approximate solution  $(a_n, u_n, \nabla\Pi_n)$  through (2.4). Similar to Proposition 2.2, we have the following proposition concerning the uniform time-weighted bounds of  $(a_n, u_n)$ .

**Proposition 3.1.** *Under the assumptions of Theorem 1.1, (2.4) has a unique global smooth solution  $(a_n, u_n, \nabla\Pi_n)$  which satisfies*

$$(3.13) \quad \begin{aligned} &\mu^{\frac{1}{2}(3-\frac{d}{p_1})} \|t^{\alpha_1} \Delta u_n^h\|_{L^{2r}(\mathbb{R}^+; L^{p_1})} + \mu^{1-\frac{d}{2p_2}} (\|t^{\beta_1} \nabla u_n^h\|_{L^{2r}(\mathbb{R}^+; L^{p_2})} + \|t^{\beta_2} \nabla u_n^h\|_{L^\infty(\mathbb{R}^+; L^{p_2})}) \\ &\quad + \mu^{\frac{1}{2}(1-\frac{d}{p_3})} (\|t^{\gamma_1} u_n^h\|_{L^\infty(\mathbb{R}^+; L^{p_3})} + \|t^{\gamma_2} u_n^h\|_{L^{2r}(\mathbb{R}^+; L^{p_3})}) \leq C\eta, \\ &\mu^{\frac{1}{2}(3-\frac{d}{p_1})} \|t^{\alpha_1} \Delta u_n^d\|_{L^{2r}(\mathbb{R}^+; L^{p_1})} + \mu^{1-\frac{d}{2p_2}} (\|t^{\beta_1} \nabla u_n^d\|_{L^{2r}(\mathbb{R}^+; L^{p_2})} + \|t^{\beta_2} \nabla u_n^d\|_{L^\infty(\mathbb{R}^+; L^{p_2})}) \\ &\quad + \mu^{\frac{1}{2}(1-\frac{d}{p_3})} (\|t^{\gamma_1} u_n^d\|_{L^\infty(\mathbb{R}^+; L^{p_3})} + \|t^{\gamma_2} u_n^d\|_{L^{2r}(\mathbb{R}^+; L^{p_3})}) \leq C\|u_0^d\|_{\dot{B}_{p,r}^{-1+\frac{d}{p}}} + c\mu, \end{aligned}$$

and

$$(3.14) \quad \begin{aligned} &\mu^{\frac{1}{2}(3-\frac{d}{p_1})} \|t^{\alpha_2} \Delta u_n\|_{L^r(\mathbb{R}^+; L^{p_1})} \leq C(\|u_0\|_{\dot{B}_{p,r}^{-1+\frac{d}{p}}} + \frac{\eta}{\mu} (\|u_0^d\|_{\dot{B}_{p,r}^{-1+\frac{d}{p}}} + c\mu + \eta)), \\ &\mu^{\frac{1}{2}(3-\frac{d}{p_1})} (\|t^{\alpha_1} \nabla\Pi_n\|_{L^{2r}(\mathbb{R}^+; L^{p_1})} + \|t^{\alpha_2} \nabla\Pi_n\|_{L^r(\mathbb{R}^+; L^{p_1})}) \leq C\eta(\|u_0^d\|_{\dot{B}_{p,r}^{-1+\frac{d}{p}}} + c\mu) \end{aligned}$$

for some small enough constant  $c, \eta$  being given by (1.6) and  $\alpha_2$  by (1.10).



*Proof.* For  $N$  large enough, it is easy to prove that (2.4) has a unique local smooth solution  $(a_n, u_n)$  on  $[0, T_n^*)$ . Without loss of generality, we may assume that  $T_n^*$  is the lifespan to this solution. For  $\lambda_1, \lambda_2, \lambda_3 > 0$ , we denote

$$(3.15) \quad \begin{aligned} f_{1,n}(t) &\stackrel{\text{def}}{=} \|t^{\beta_1} \nabla u_n^d(t)\|_{L^{p_2}}^{2r}, & f_{2,n}(t) &\stackrel{\text{def}}{=} \|t^{\gamma_2} u_n^d(t)\|_{L^{p_3}}^{2r}, & f_{3,n}(t) &\stackrel{\text{def}}{=} \|t^{\alpha_1} \Delta u_n^d(t)\|_{L^{p_1}}^{2r}, \\ u_{\lambda,n}(t, x) &\stackrel{\text{def}}{=} u_n(t, x) \exp\left\{-\int_0^t (\lambda_1 f_{1,n}(t') + \lambda_2 f_{2,n}(t') + \lambda_3 f_{3,n}(t')) dt'\right\}, \end{aligned}$$

and similar notation for  $\Pi_{\lambda,n}(t, x)$ . Then we deduce by a similar derivation of (2.9) that

$$\begin{aligned} \|\nabla \Pi_{\lambda,n}(t)\|_{L^{p_1}} &\leq C \left\{ \|a_0\|_{L^\infty} \|\nabla \Pi_{\lambda,n}(t)\|_{L^{p_1}} + \mu \|a_0\|_{L^\infty} \|\Delta u_{\lambda,n}\|_{L^{p_1}} + \|u_n^h\|_{L^{p_3}} \|\nabla u_{\lambda,n}^h\|_{L^{p_2}} \right. \\ &\quad \left. + \|u_n^d\|_{L^{p_3}} \|\nabla u_{\lambda,n}^h\|_{L^{p_2}} + \|u_{\lambda,n}^h\|_{L^{p_3}} \|\nabla u_n^d\|_{L^{p_2}} \right\}. \end{aligned}$$

In particular, if  $\eta$  in (1.6) is so small that  $C\|a_0\|_{L^\infty} \leq \frac{1}{2}$ , we obtain

$$(3.16) \quad \begin{aligned} \|\nabla \Pi_{\lambda,n}(t)\|_{L^{p_1}} &\leq C \left\{ \mu \|a_0\|_{L^\infty} \|\Delta u_{\lambda,n}\|_{L^{p_1}} + \|u_n^h\|_{L^{p_3}} \|\nabla u_{\lambda,n}^h\|_{L^{p_2}} \right. \\ &\quad \left. + \|u_n^d\|_{L^{p_3}} \|\nabla u_{\lambda,n}^h\|_{L^{p_2}} + \|u_{\lambda,n}^h\|_{L^{p_3}} \|\nabla u_n^d\|_{L^{p_2}} \right\}. \end{aligned}$$

While it follows from (2.11) and (3.15) that

$$(3.17) \quad \begin{aligned} u_{\lambda,n}^h &= u_{n,L}^h \exp\left\{-\int_0^t (\lambda_1 f_{1,n}(t') + \lambda_2 f_{2,n}(t') + \lambda_3 f_{3,n}(t')) dt'\right\} \\ &\quad + \int_0^t e^{\mu(t-s)\Delta} \exp\left\{-\int_s^t (\lambda_1 f_{1,n}(t') + \lambda_2 f_{2,n}(t') + \lambda_3 f_{3,n}(t')) dt'\right\} \\ &\quad \times (-u_n \nabla u_{\lambda,n}^h + \mu a_n \Delta u_{\lambda,n}^h - (1 + a_n) \nabla_h \Pi_{\lambda,n}) ds. \end{aligned}$$

Applying Lemma 3.2 for  $\varepsilon = 0$  and  $r_1 = 2r$  gives

$$\begin{aligned} &\mu^{\frac{d}{2p_2}} \|t^{\gamma_1} u_{\lambda,n}^h\|_{L_t^\infty(L^{p_3})} + \mu^{\frac{1}{2} + \frac{d}{2p_3}} \|t^{\beta_1} \nabla u_{\lambda,n}^h\|_{L_t^{2r}(L^{p_2})} \\ &\leq \mu^{\frac{d}{2p_2}} \|t^{\gamma_1} u_{n,L}^h\|_{L_t^\infty(L^{p_3})} + \mu^{\frac{1}{2} + \frac{d}{2p_3}} \|t^{\beta_1} \nabla u_{n,L}^h\|_{L_t^{2r}(L^{p_2})} \\ &\quad + \left\| \exp\left\{-\int_s^t (\lambda_1 f_{1,n}(t') + \lambda_2 f_{2,n}(t') + \lambda_3 f_{3,n}(t')) dt'\right\} \right. \\ &\quad \left. \times s^{\alpha_1} (-u_n \nabla u_{\lambda,n}^h + \mu a_n \Delta u_{\lambda,n}^h - (1 + a_n) \nabla_h \Pi_{\lambda,n}) \right\|_{L_t^{2r}(L^{p_1})}, \end{aligned}$$

for indices  $\beta_1, \gamma_1$  given by (1.10).

Similarly, it follows from Lemma 3.1 and Lemma 3.3 for  $\varepsilon = 0$  and  $r_1 = 2r$  that

$$\begin{aligned} &\mu^{\frac{d}{2p_2}} \|t^{\gamma_2} u_{\lambda,n}^h\|_{L_t^{2r}(L^{p_3})} + \mu^{\frac{1}{2} + \frac{d}{2p_3}} \|t^{\beta_2} \nabla u_{\lambda,n}^h\|_{L_t^\infty(L^{p_2})} + \mu \|t^{\alpha_1} \Delta u_{\lambda,n}^h\|_{L_t^{2r}(L^{p_1})} \\ &\leq \mu^{\frac{d}{2p_2}} \|t^{\gamma_2} u_{n,L}^h\|_{L_t^{2r}(L^{p_3})} + \mu^{\frac{1}{2} + \frac{d}{2p_3}} \|t^{\beta_2} \nabla u_{n,L}^h\|_{L_t^\infty(L^{p_2})} + \mu \|t^{\alpha_1} \Delta u_{n,L}^h\|_{L_t^{2r}(L^{p_1})} \\ &\quad + \left\| \exp\left\{-\int_s^t (\lambda_1 f_{1,n}(t') + \lambda_2 f_{2,n}(t') + \lambda_3 f_{3,n}(t')) dt'\right\} \right. \\ &\quad \left. \times s^{\alpha_1} (-u_n \nabla u_{\lambda,n}^h + \mu a_n \Delta u_{\lambda,n}^h - (1 + a_n) \nabla_h \Pi_{\lambda,n}) \right\|_{L_t^{2r}(L^{p_1})}, \end{aligned}$$

for indices  $\alpha_1, \beta_2, \gamma_2$  given by (1.10).

As a consequence, we obtain, by using (2.7), that

$$\begin{aligned}
& \mu^{\frac{d}{2p_2}} (\|t^{\gamma_1} u_{\lambda,n}^h\|_{L_t^\infty(L^{p_3})} + \|t^{\gamma_2} u_{\lambda,n}^h\|_{L_t^{2r}(L^{p_3})}) + \mu^{\frac{1}{2} + \frac{d}{2p_3}} (\|t^{\beta_1} \nabla u_{\lambda,n}^h\|_{L_t^{2r}(L^{p_2})} \\
& \quad + \|t^{\beta_2} \nabla u_{\lambda,n}^h\|_{L_t^\infty(L^{p_2})}) + \mu \|t^{\alpha_1} \Delta u_{\lambda,n}^h\|_{L_t^{2r}(L^{p_1})} \\
(3.18) \quad & \leq \mu^{-\frac{1}{2}(1-\frac{d}{p_1})} \|u_0^h\|_{\dot{B}_{p_1,2r}^{-1+\frac{d}{p_1}}} + C \left\{ \mu \|a_0\|_{L^\infty} (\|t^{\alpha_1} \Delta u_{\lambda,n}^h\|_{L_t^{2r}(L^{p_1})} \right. \\
& \quad + \|t^{\alpha_1} \Delta u_{\lambda,n}^d\|_{L_t^{2r}(L^{p_1})}) + \|t^{\gamma_1} u_n^h\|_{L_t^\infty(L^{p_3})} \|t^{\beta_1} \nabla u_{\lambda,n}^h\|_{L_t^{2r}(L^{p_2})} \\
& \quad \left. + \left( \int_0^t e^{-2r \int_s^t (\lambda_1 f_{1,n}(t') + \lambda_2 f_{2,n}(t')) dt'} (\|s^{\gamma_2} u_n^d(s)\|_{L^{p_3}}^{2r} \|s^{\beta_2} \nabla u_{\lambda,n}^h(s)\|_{L^{p_2}}^{2r} \right. \right. \\
& \quad \left. \left. + \|s^{\gamma_1} u_{\lambda,n}^h\|_{L^{p_3}}^{2r} \|s^{\beta_1} \nabla u_n^d(s)\|_{L^{p_2}}^{2r} ds \right)^{\frac{1}{2r}} \right\}.
\end{aligned}$$

Then as  $C \|a_0\|_{L^\infty} \leq \frac{1}{2}$ , and

$$\begin{aligned}
& \|s^{\alpha_1} \Delta u_{\lambda,n}^d\|_{L_t^{2r}(L^{p_1})} \leq \frac{1}{(2\lambda_3 r)^{\frac{1}{2r}}}, \\
& \left( \int_0^t e^{-2\lambda_1 r \int_s^t f_{1,n}(t') dt'} \|s^{\gamma_1} u_{\lambda,n}^h\|_{L^{p_3}}^{2r} \|t^{\beta_1} \nabla u_n^d(s)\|_{L^{p_2}}^{2r} ds \right)^{\frac{1}{2r}} \leq \frac{1}{(2\lambda_1 r)^{\frac{1}{2r}}} \|s^{\gamma_1} u_{\lambda,n}^h\|_{L_t^\infty(L^{p_3})}, \\
& \left( \int_0^t e^{-2\lambda_2 r \int_s^t f_{2,n}(t') dt'} \|s^{\gamma_2} u_n^d(s)\|_{L^{p_3}}^{2r} \|t^{\beta_2} \nabla u_{\lambda,n}^h(s)\|_{L^{p_2}}^{2r} ds \right)^{\frac{1}{2r}} \leq \frac{1}{(2\lambda_2 r)^{\frac{1}{2r}}} \|s^{\beta_2} \nabla u_{\lambda,n}^h\|_{L_t^\infty(L^{p_2})},
\end{aligned}$$

we infer from (3.18) that

$$\begin{aligned}
& \mu^{\frac{d}{2p_2}} (\|t^{\gamma_1} u_{\lambda,n}^h\|_{L_t^\infty(L^{p_3})} + \|t^{\gamma_2} u_{\lambda,n}^h\|_{L_t^{2r}(L^{p_3})}) + \mu^{\frac{1}{2} + \frac{d}{2p_3}} (\|t^{\beta_2} \nabla u_{\lambda,n}^h\|_{L_t^\infty(L^{p_2})} \\
& \quad + \|t^{\beta_1} \nabla u_{\lambda,n}^h\|_{L_t^{2r}(L^{p_2})}) + \mu \|t^{\alpha_1} \Delta u_{\lambda,n}^h\|_{L_t^{2r}(L^{p_1})} \\
& \leq \mu^{-\frac{1}{2}(1-\frac{d}{p_1})} \|u_0\|_{\dot{B}_{p_1,r}^{-1+\frac{d}{p_1}}} + C \left\{ \frac{\mu}{(2\lambda_3 r)^{\frac{1}{2r}}} \|a_0\|_{L^\infty} + \|t^{\gamma_1} u_n^h\|_{L_t^\infty(L^{p_3})} \|t^{\beta_1} \nabla u_{\lambda,n}^h\|_{L_t^{2r}(L^{p_2})} \right. \\
& \quad \left. + \frac{1}{(2\lambda_1 r)^{\frac{1}{2r}}} \|t^{\gamma_1} u_{\lambda,n}^h\|_{L_t^\infty(L^{p_3})} + \frac{1}{(2\lambda_2 r)^{\frac{1}{2r}}} \|t^{\beta_2} \nabla u_{\lambda,n}^h\|_{L_t^\infty(L^{p_2})} \right\}.
\end{aligned}$$

Taking  $\lambda_1 = \frac{(4C)^{2r}}{2r\mu^{\frac{d}{p_2}}}$ ,  $\lambda_2 = \frac{(4C)^{2r}}{2r\mu^{(1+\frac{d}{p_3})r}}$  and  $\lambda_3 = \frac{C^{2r}}{2r\mu^{(\frac{d}{p_1}-1)r}}$  in the above inequality results in

$$\begin{aligned}
(3.19) \quad & \mu^{\frac{d}{2p_2}} \left( \frac{3}{4} \|t^{\gamma_1} u_{\lambda,n}^h\|_{L_t^\infty(L^{p_3})} + \|t^{\gamma_2} u_{\lambda,n}^h\|_{L_t^{2r}(L^{p_3})} \right) + \mu^{\frac{1}{2} + \frac{d}{2p_3}} (\|t^{\beta_1} \nabla u_{\lambda,n}^h\|_{L_t^{2r}(L^{p_2})} \\
& \quad + \frac{3}{4} \|t^{\beta_2} \nabla u_{\lambda,n}^h\|_{L_t^\infty(L^{p_2})}) + \mu \|t^{\alpha_1} \Delta u_{\lambda,n}^h\|_{L_t^{2r}(L^{p_1})} \\
& \leq C \mu^{\frac{1}{2}(\frac{d}{p_1}-1)} (\|u_0^h\|_{\dot{B}_{p,r}^{-1+\frac{d}{p}}} + \mu \|a_0\|_{L^\infty}) + C \|t^{\gamma_1} u_n^h\|_{L_t^\infty(L^{p_3})} \|t^{\beta_1} \nabla u_{\lambda,n}^h\|_{L_t^{2r}(L^{p_2})}.
\end{aligned}$$

Let  $c_1$  be a small enough positive constant, which will be determined later on, we denote

$$\begin{aligned}
(3.20) \quad \bar{T}_n \stackrel{\text{def}}{=} \sup \left\{ T \leq T_n^*; \quad \mu^{\frac{1}{2} - \frac{d}{2p_3}} \|t^{\gamma_1} u_n^h\|_{L_T^\infty(L^{p_3})} + \mu^{1 - \frac{d}{2p_2}} \|t^{\beta_1} \nabla u_n^h\|_{L_T^{2r}(L^{p_2})} \right. \\
\left. + \mu^{\frac{3}{2}(1-\frac{d}{p_1})} \|t^{\alpha_1} \Delta u_n^h\|_{L_T^{2r}(L^{p_1})} \leq c_1 \mu \right\}.
\end{aligned}$$

We shall prove that  $\bar{T}_n = T_n^* = \infty$  as long as we take  $C_r$  sufficiently large and  $c_0$  sufficiently small in (1.6). In fact, if  $\bar{T}_n < T_n^*$ , it follows from (3.15) and (3.19) that

$$\begin{aligned}
& \mu^{\frac{1}{2}(1-\frac{d}{p_3})} (\|t^{\gamma_1} u_n^h\|_{L_t^\infty(L^{p_3})} + \|t^{\gamma_2} u_n^h\|_{L_t^{2r}(L^{p_3})}) + \mu^{1-\frac{d}{2p_2}} (\|t^{\beta_1} \nabla u_n^h\|_{L_t^{2r}(L^{p_2})} \\
& \quad + \|t^{\beta_2} \nabla u_n^h\|_{L_t^\infty(L^{p_2})}) + \mu^{\frac{1}{2}(3-\frac{d}{p_1})} \|t^{\alpha_1} \Delta u_n^h\|_{L_t^{2r}(L^{p_1})} \\
(3.21) \quad & \leq C (\|u_0^h\|_{\dot{B}_{p,r}^{-1+\frac{d}{p}}} + \mu \|a_0\|_{L^\infty}) \exp \left\{ -C_r \int_0^t (\mu^{-\frac{d}{p_2}r} \|s^{\beta_1} \nabla u_n^d(s)\|_{L^{2r}}^{2r} \right. \\
& \quad \left. + \mu^{-(1+\frac{d}{p_3})r} \|s^{\gamma_2} u_n^d(s)\|_{L^{p_3}}^{2r} + \mu^{(1-\frac{d}{p_1})r} \|s^\alpha \Delta u_n^d(s)\|_{L^{p_1}}^{2r} ds \right\}.
\end{aligned}$$

On the other hand, it follows from a similar derivation of (3.18) that

$$\begin{aligned}
& \mu^{\frac{d}{2p_2}} (\|t^{\gamma_1} u_n^d\|_{L_t^\infty(L^{p_3})} + \|t^{\gamma_2} u_n^d\|_{L_t^{2r}(L^{p_3})}) + \mu^{\frac{1}{2}+\frac{d}{2p_3}} (\|t^{\beta_1} \nabla u_n^d\|_{L_t^{2r}(L^{p_2})} \\
& \quad + \|t^{\beta_2} \nabla u_n^d\|_{L_t^\infty(L^{p_2})}) + \mu \|t^{\alpha_1} \Delta u_n^d\|_{L_t^{2r}(L^{p_1})} \\
(3.22) \quad & \leq \mu^{-\frac{1}{2}(1-\frac{d}{p_1})} \|u_0^d\|_{\dot{B}_{p,r}^{-1+\frac{d}{p}}} + C \left\{ \mu \|a_0\|_{L^\infty} (\|t^{\alpha_1} \Delta u_n^h\|_{L_t^{2r}(L^{p_1})} + \|t^{\alpha_1} \Delta u_n^d\|_{L_t^{2r}(L^{p_1})}) \right. \\
& \quad + (\|t^{\gamma_1} u_n^h\|_{L_t^\infty(L^{p_3})} + \|t^{\gamma_1} u_n^d\|_{L_t^\infty(L^{p_3})}) \|t^{\beta_1} \nabla u_n^h\|_{L_t^{2r}(L^{p_2})} \\
& \quad \left. + \|t^{\gamma_1} u_n^h\|_{L_t^\infty(L^{p_3})} \|t^{\beta_1} \nabla u_n^d\|_{L_t^{2r}(L^{p_2})} \right\} \quad \text{for } t < T_n^*,
\end{aligned}$$

from which, (3.20) and taking  $c_0$  small enough in (1.6) so that  $C\|a_0\|_{L^\infty} \leq \frac{1}{2}$ , we deduce

$$\begin{aligned}
& \mu^{\frac{d}{2p_2}} (\|t^{\gamma_1} u_n^d\|_{L_t^\infty(L^{p_3})} + \|t^{\gamma_2} u_n^d\|_{L_t^{2r}(L^{p_3})}) + \mu^{\frac{1}{2}+\frac{d}{2p_3}} (\|t^{\beta_1} \nabla u_n^d\|_{L_t^{2r}(L^{p_2})} \\
& \quad + \|t^{\beta_2} \nabla u_n^d\|_{L_t^\infty(L^{p_2})}) + \mu \|t^{\alpha_1} \Delta u_n^d\|_{L_t^{2r}(L^{p_1})} \\
& \leq 2C \mu^{-\frac{1}{2}(1-\frac{d}{p_1})} \|u_0^d\|_{\dot{B}_{p,r}^{-1+\frac{d}{p}}} + 2C c_1 \mu^{\frac{1}{2}(1+\frac{d}{p_1})} (1 + \|a_0\|_{L^\infty}) \quad \text{for } t \leq \bar{T}_n.
\end{aligned}$$

which implies

$$\begin{aligned}
& \mu^{\frac{1}{2}(1-\frac{d}{p_3})} (\|t^{\gamma_1} u_n^d\|_{L_t^\infty(L^{p_3})} + \|t^{\gamma_2} u_n^d\|_{L_t^{2r}(L^{p_3})}) + \mu^{1-\frac{d}{2p_2}} (\|t^{\beta_1} \nabla u_n^d\|_{L_t^{2r}(L^{p_2})} \\
(3.23) \quad & \quad + \|t^{\beta_2} \nabla u_n^d\|_{L_t^\infty(L^{p_2})}) + \mu^{\frac{1}{2}(3-\frac{d}{p_1})} \|t^{\alpha_1} \Delta u_n^d\|_{L_t^{2r}(L^{p_1})} \\
& \leq 2C \|u_0^d\|_{\dot{B}_{p,r}^{-1+\frac{d}{p}}} + c\mu \quad \text{for } t \leq \bar{T}_n.
\end{aligned}$$

Substituting (3.23) into (3.21) leads to

$$\begin{aligned}
& \mu^{\frac{1}{2}(1-\frac{d}{p_3})} (\|t^{\gamma_1} u_n^h\|_{L_t^\infty(L^{p_3})} + \|t^{\gamma_2} u_n^h\|_{L_t^{2r}(L^{p_3})}) + \mu^{1-\frac{d}{2p_2}} (\|t^{\beta_1} \nabla u_n^h\|_{L_t^{2r}(L^{p_2})} \\
(3.24) \quad & \quad + \|t^{\beta_2} \nabla u_n^h\|_{L_t^\infty(L^{p_2})}) + \mu^{\frac{1}{2}(3-\frac{d}{p_1})} \|t^{\alpha_1} \Delta u_n^h\|_{L_t^{2r}(L^{p_1})} \\
& \leq C (\|u_0^h\|_{\dot{B}_{p,r}^{-1+\frac{d}{p}}} + \mu \|a_0\|_{L^\infty}) \exp \left\{ C_r \mu^{-2r} \|u_0^d\|_{\dot{B}_{p,r}^{-1+\frac{d}{p}}}^{2r} \right\} \leq \frac{c_1}{2} \mu \quad \text{for } t \leq \bar{T}_n,
\end{aligned}$$

as long as  $C_r$  is sufficiently large and  $c_0$  small enough in (1.6). This contradicts with (3.20) and it in turn shows that  $\bar{T}_n = T_n^*$ . Then thanks to (3.23), (3.24) and Theorem 1.3 in [21], we conclude that  $T_n^* = \infty$  and there holds (3.13). It remains to prove (3.14). Indeed, similar to (3.22), we get,

by applying Lemma 3.1, that

$$\begin{aligned} \mu \|t^{\alpha_2} \Delta u_n\|_{L^r(\mathbb{R}^+; L^{p_1})} &\leq \mu^{-\frac{1}{2}(1-\frac{d}{p_1})} \|u_0\|_{\dot{B}_{p,r}^{-1+\frac{d}{p}}} + C \left\{ \mu \|a_0\|_{L^\infty} \|t^{\alpha_2} \Delta u_n\|_{L^r(\mathbb{R}^+; L^{p_1})} \right. \\ &\quad + \left( \|t^{\gamma_2} u_n^h\|_{L^{2r}(\mathbb{R}^+; L^{p_3})} + \|t^{\gamma_2} u_n^d\|_{L^{2r}(\mathbb{R}^+; L^{p_3})} \right) \|t^{\beta_1} \nabla u_n^h\|_{L^{2r}(\mathbb{R}^+; L^{p_2})} \\ &\quad \left. + \|t^{\gamma_2} u_n^h\|_{L^{2r}(\mathbb{R}^+; L^{p_3})} \|t^{\beta_1} \nabla u_n^d\|_{L^{2r}(L^{p_2})} \right\}, \end{aligned}$$

which along with (3.13) and the fact:  $C \|a_0\|_{L^\infty} \leq \frac{1}{2}$ , gives rise to the first inequality of (3.14). Along the same line, one gets the estimate of  $\|t^{\alpha_2} \nabla \Pi_n\|_{L^r(\mathbb{R}^+; L^{p_1})}$  in (3.14). Whereas it follows from and (3.16) that

$$\begin{aligned} \|t^{\alpha_1} \nabla \Pi_n\|_{L^{2r}(\mathbb{R}^+; L^{p_1})} &\leq C \left\{ \mu \|a_0\|_{L^\infty} \|t^{\alpha_1} \Delta u_n\|_{L^{2r}(\mathbb{R}^+; L^{p_1})} \right. \\ &\quad \left. + \|t^{\gamma_1} u_n^h\|_{L^\infty(\mathbb{R}^+; L^{p_3})} \|t^{\beta_1} \nabla u_n\|_{L^{2r}(\mathbb{R}^+; L^{p_2})} + \|t^{\gamma_2} u_n^d\|_{L^{2r}(\mathbb{R}^+; L^{p_3})} \|t^{\beta_2} \nabla u_n^h\|_{L^\infty(\mathbb{R}^+; L^{p_2})} \right\}, \end{aligned}$$

from which and (3.13), we obtain the estimate of  $\|t^{\alpha_1} \nabla \Pi_n\|_{L^{2r}(\mathbb{R}^+; L^{p_1})}$  in (3.14). This completes the proof of the proposition.  $\square$

Now we are in a position to complete the proof to the existence part of Theorem 1.1 for the remaining case.

**Proof to the existence part of Theorem 1.1 for  $p \in (\frac{dr}{3r-2}, d)$ .** Notice that  $p_1 < d$  ensures  $\alpha_2 r' < 1$ , so that for any  $T > 0$ , we deduce from (3.14) that  $\Delta u_n = t^{-\alpha_2} (t^{\alpha_2} \Delta u_n)$  is uniformly bounded in  $L^{\tau_1}((0, T); L^{p_1}(\mathbb{R}^d))$  for some  $\tau_1 \in (1, \infty)$ . Similarly we infer from (1.10) and (3.13) that  $\{\nabla u_n\}$  is uniformly bounded in  $L^{\tau_2}((0, T); L^{p_2}(\mathbb{R}^d))$  for any  $\tau_2 < \frac{2p_2}{2p_2-d}$ , and  $\{u_n\}$  is uniformly bounded in  $L^{\tau_3}((0, T); L^{p_3}(\mathbb{R}^d))$  for any  $\tau_3 < \frac{2p_3}{p_3-d}$ . Moreover as  $\frac{2p_2-d}{2p_2} + \frac{p_3-d}{2p_3} = \frac{3}{2} - \frac{d}{2p_1}$  and  $p_1 < d$ , we can choose  $\tau_2$  and  $\tau_3$  so that  $\frac{1}{\tau_2} + \frac{1}{\tau_3} = \frac{1}{\tau_4} < 1$ . This together with (2.4) implies that  $\{\partial_t u_n\}$  is uniformly bounded in  $L^{\tau_1}((0, T); L^{p_1}(\mathbb{R}^d)) + L^{\tau_4}((0, T); L^{p_1}(\mathbb{R}^d))$  for any  $T < \infty$ , from which, Ascoli-Arzelà Theorem and  $p_2 < \frac{dp_1}{d-p_1}$ , we conclude that there exists a subsequence of  $\{a_n, u_n, \nabla \Pi_n\}$ , which we still denote by  $\{a_n, u_n, \nabla \Pi_n\}$  and some  $(a, u, \nabla \Pi)$  with  $a \in L^\infty(\mathbb{R}^+ \times \mathbb{R}^d)$ ,  $u \in L_{loc}^{\tau_3}(\mathbb{R}^+; L^{p_3}(\mathbb{R}^d))$  with  $\nabla u \in L_{loc}^{\tau_2}(\mathbb{R}^+; L^{p_2}(\mathbb{R}^d))$  and  $\Delta u, \nabla \Pi \in L_{loc}^{\tau_1}(\mathbb{R}^+; L^{p_1}(\mathbb{R}^d))$ , such that

$$\begin{aligned} (3.25) \quad &a_n \rightharpoonup a \quad \text{weak } * \text{ in } L_{loc}^\infty(\mathbb{R}^+ \times \mathbb{R}^d), \\ &u_n \rightarrow u \quad \text{strongly in } L_{loc}^{\tau_3}(\mathbb{R}^+; L^{p_3}(\mathbb{R}^d)), \\ &\nabla u_n \rightarrow \nabla u \quad \text{strongly in } L_{loc}^{\tau_2}(\mathbb{R}^+; L^{p_2}(\mathbb{R}^d)), \\ &\Delta u_n \rightharpoonup \Delta u \quad \text{and} \quad \nabla \Pi_n \rightharpoonup \nabla \Pi \quad \text{weakly in } L_{loc}^{\tau_1}(\mathbb{R}^+; L^{p_1}(\mathbb{R}^d)). \end{aligned}$$

With (3.13), (3.14) and (3.25), we can repeat the argument at the end of Section 2 to complete the proof to the existence part of Theorem 1.1 for the case when  $\frac{dr}{3r-2} < p < d$ .  $\blacksquare$

#### 4. THE UNIQUENESS PART OF THEOREM 1.1

With a little bit more regularity on the initial velocity, namely,  $u_0 \in \dot{B}_{p,r}^{-1+\frac{d}{p}+\varepsilon}(\mathbb{R}^d)$  for some small enough  $\varepsilon > 0$ , we can prove the uniqueness of the solution constructed in the last two sections. This result is strongly inspired by the Lagrangian approach in [13, 14]. Nevertheless, with an almost critical regularity for the initial velocity field, the proof here is more challenging. The main result is listed as follows:

**Theorem 4.1.** *Let  $r \in (1, \infty)$ ,  $p \in (1, d)$  and  $0 < \varepsilon < \min\{\frac{1}{r}, 1 - \frac{1}{r}, \frac{d}{p} - 1\}$ . Let  $a_0 \in L^\infty(\mathbb{R}^d)$  and  $u_0 \in \dot{B}_{p,r}^{-1+\frac{d}{p}}(\mathbb{R}^d) \cap \dot{B}_{p,r}^{-1+\frac{d}{p}+\varepsilon}(\mathbb{R}^d)$ , which satisfies the nonlinear smallness condition (1.6). Then*

(1.2) has a unique global weak solution  $(a, u)$  which satisfies (1.7-1.9) and

$$(4.1) \quad \begin{aligned} & \mu^{\frac{1}{2}(3-\frac{d}{q_1}-\varepsilon)} \|t^{\alpha_1^\varepsilon} \Delta u\|_{L^{2r}(\mathbb{R}^+; L^{q_1})} + \mu^{1-\frac{d}{2q_2}-\frac{\varepsilon}{2}} (\|t^{\beta_1^\varepsilon} \nabla u\|_{L^{2r}(\mathbb{R}^+; L^{q_2})} + \|t^{\beta_2^\varepsilon} \nabla u\|_{L^\infty(\mathbb{R}^+; L^{q_2})}) \\ & + \mu^{\frac{1}{2}(1-\frac{d}{q_3}-\varepsilon)} (\|t^{\gamma_1^\varepsilon} u\|_{L^\infty(\mathbb{R}^+; L^{q_3})} + \|t^{\gamma_2^\varepsilon} u\|_{L^{2r}(\mathbb{R}^+; L^{q_3})}) \\ & \leq C \|u_0\|_{\dot{B}_{p,r}^{-1+\frac{d}{p}+\varepsilon}} \exp\left(C_r \mu^{-2r} \|u_0^d\|_{\dot{B}_{p,r}^{-1+\frac{d}{p}}}^{2r}\right), \end{aligned}$$

and

$$(4.2) \quad \begin{aligned} & \mu^{\frac{1}{2}(3-\frac{d}{q_1}-\varepsilon)} \|t^{\alpha_2^\varepsilon} \Delta u\|_{L^r(\mathbb{R}^+; L^{q_1})} \leq C \|u_0\|_{\dot{B}_{p,r}^{-1+\frac{d}{p}+\varepsilon}} \exp\left(C_r \mu^{-2r} \|u_0^d\|_{\dot{B}_{p,r}^{-1+\frac{d}{p}}}^{2r}\right), \\ & \mu^{\frac{1}{2}(3-\frac{d}{q_1}-\varepsilon)} (\|t^{\alpha_1^\varepsilon} \nabla \Pi\|_{L^{2r}(\mathbb{R}^+; L^{q_1})} + \|t^{\alpha_2^\varepsilon} \nabla \Pi\|_{L^r(\mathbb{R}^+; L^{q_1})}) \\ & \leq C \eta \|u_0\|_{\dot{B}_{p,r}^{-1+\frac{d}{p}+\varepsilon}} \exp\left(C_r \mu^{-2r} \|u_0^d\|_{\dot{B}_{p,r}^{-1+\frac{d}{p}}}^{2r}\right), \end{aligned}$$

where  $q_1, q_2, q_3$  satisfy  $\max(p, \frac{dr}{2r-1}) < q_1 < \frac{d}{1-\varepsilon}$ , and  $\frac{dr}{r-1} < q_3 < \infty$  so that  $\frac{1}{q_2} + \frac{1}{q_3} = \frac{1}{q_1}$ , the indices  $\alpha_1^\varepsilon, \alpha_2^\varepsilon, \beta_1^\varepsilon, \beta_2^\varepsilon, \gamma_1^\varepsilon, \gamma_2^\varepsilon$  are determined by

$$(4.3) \quad \begin{aligned} \alpha_1^\varepsilon &= \frac{1}{2}\left(3 - \frac{d}{q_1} - \varepsilon\right) - \frac{1}{2r}, & \beta_1^\varepsilon &= \frac{1}{2}\left(2 - \frac{d}{q_2} - \varepsilon\right) - \frac{1}{2r}, & \gamma_1^\varepsilon &= \frac{1}{2}\left(1 - \frac{d}{q_3} - \varepsilon\right), \\ \alpha_2^\varepsilon &= \frac{1}{2}\left(3 - \frac{d}{q_1} - \varepsilon\right) - \frac{1}{r}, & \beta_2^\varepsilon &= \frac{1}{2}\left(2 - \frac{d}{q_2} - \varepsilon\right), & \gamma_2^\varepsilon &= \frac{1}{2}\left(1 - \frac{d}{q_3} - \varepsilon\right) - \frac{1}{2r}. \end{aligned}$$

**Remark 4.1.** To prove the uniqueness part of Theorem 4.1, we shall choose  $p_1 = \frac{d}{1+\varepsilon}$  in (1.10). This choice of  $p_1$  satisfies  $\max(p, \frac{dr}{2r-1}) < p_1 < d$  with  $0 < \varepsilon < \min(1 - \frac{1}{r}, \frac{d}{p} - 1)$ . Then by virtue of (1.8),  $\Delta u = t^{-\alpha_2} (t^{\alpha_2} \Delta u) \in L^{\tau_1}((0, T); L^{p_1}(\mathbb{R}^d))$  for any  $T < \infty$  and  $\tau_1$  satisfying  $\alpha_2 < \frac{1}{\tau_1} - \frac{1}{r}$ , which implies  $\tau_1 < \frac{2}{2-\varepsilon}$ , we thus take  $\tau_1 = \frac{8}{8-\varepsilon}$ . While as  $q_1 < \frac{d}{1-\varepsilon}$  in Theorem 4.1, we can choose  $q_1 > d$  in order to get  $\nabla u \in L_{loc}^1(L^\infty)$  (see Lemma 4.1 below), and this is the reason we need an additional regularity on  $u_0$  for the uniqueness part of Theorem 1.1. We shall choose  $q_1 = \frac{2d}{2-\varepsilon} < \frac{d}{1-\varepsilon}$  in (4.3) later on. Then in order that  $\Delta u \in L^{\tau_1}((0, T); L^{p_1}(\mathbb{R}^d))$ , we have  $\alpha_2^\varepsilon < \frac{1}{\tau_1} - \frac{1}{r}$ , and hence we take  $\tau_1^\varepsilon = \frac{8}{8-\varepsilon} < \frac{4}{4-\varepsilon}$ .

It follows from the existence proof of Theorem 1.1 that: in order to prove the solution constructed in the last two sections satisfies (4.1) and (4.2), we only need to prove that the same inequalities hold for the approximate solutions  $(a_n, u_n, \nabla \Pi_n)$  determined by (2.4).

We now turn to the uniform estimate of  $(u_n, \nabla \Pi_n)_{n \in \mathbb{N}}$  when the initial velocity  $u_0 \in \dot{B}_{p,r}^{-3+\frac{d}{p}+\varepsilon}(\mathbb{R}^d)$  for some  $\varepsilon \in (0, \min\{\frac{1}{r}, 1 - \frac{d}{r} - 1\})$ . Notice that for all  $q_1 \geq p$  and  $r_1 \geq r$ ,  $\Delta u_0 \in \dot{B}_{q_1, r_1}^{-3+\frac{d}{q_1}+\varepsilon}(\mathbb{R}^d)$ , then we deduce from Proposition 2.1 that  $t^{\frac{1}{2}(3-\frac{d}{q_1}-\varepsilon)-\frac{1}{r_1}} \Delta e^{t\Delta} u_0 \in L^{r_1}(\mathbb{R}^+; L^{q_1}(\mathbb{R}^d))$ . Similarly, we choose  $q_2 > \frac{d}{2-\varepsilon}$  and  $q_3 > \frac{d}{1-\varepsilon}$  with  $\frac{1}{q_2} + \frac{1}{q_3} = \frac{1}{q_1}$ , there holds  $t^{\frac{1}{2}(2-\frac{d}{q_2}-\varepsilon)-\frac{1}{r_1}} \nabla e^{t\Delta} u_0 \in L^{r_1}(\mathbb{R}^+; L^{q_2}(\mathbb{R}^d))$  and  $t^{\frac{1}{2}(1-\frac{d}{q_3}-\varepsilon)} e^{t\Delta} u_0 \in L^\infty(\mathbb{R}^+; L^{q_3}(\mathbb{R}^d))$ . We thus take  $r_1 = 2r > 2$  and  $q_1, q_2$  and  $q_3$  satisfying  $\max(p, \frac{dr}{2r-1}) < q_1 < \frac{d}{1-\varepsilon}$ ,  $\frac{dr}{r-1} < q_3 < \infty$ , and  $\frac{1}{q_2} + \frac{1}{q_3} = \frac{1}{q_1}$ . We shall investigate the uniform estimate to the solutions  $(a_n, u_n, \nabla \Pi_n)$  of (2.4) in the following functional space:

$$(4.4) \quad \begin{aligned} X = \left\{ u : a \in L^\infty(\mathbb{R}^+ \times \mathbb{R}^d), \quad t^{\gamma_1^\varepsilon} u \in L^\infty(\mathbb{R}^+; L^{q_3}(\mathbb{R}^d)), \right. \\ \left. t^{\gamma_2^\varepsilon} u \in L^{2r}(\mathbb{R}^+; L^{q_3}(\mathbb{R}^d)), \quad t^{\beta_1^\varepsilon} \nabla u \in L^{2r}(\mathbb{R}^+; L^{q_2}(\mathbb{R}^d)), \right. \\ \left. t^{\beta_2^\varepsilon} \nabla u \in L^\infty(\mathbb{R}^+; L^{q_2}(\mathbb{R}^d)) \quad t^{\alpha_1^\varepsilon} \Delta u \in L^{2r}(\mathbb{R}^+; L^{q_1}(\mathbb{R}^d)) \right\}, \end{aligned}$$

with the indices  $\alpha_1^\varepsilon, \beta_1^\varepsilon, \beta_2^\varepsilon, \gamma_1^\varepsilon, \gamma_2^\varepsilon$  being determined by (4.3). Note that  $\alpha_1^\varepsilon \neq \beta_i^\varepsilon + \gamma_i^\varepsilon$ , but  $\alpha_1^\varepsilon = \bar{\beta}_1 + \gamma_1^\varepsilon = \beta_2^\varepsilon + \bar{\gamma}_2$  with  $\bar{\beta}_1 = \frac{1}{2}(1 - \frac{d}{q_3}) - \frac{1}{2r}$  and  $\bar{\gamma}_2 = \frac{1}{2}(2 - \frac{d}{q_2}) - \frac{1}{2r}$ . Then we can apply Proposition 3.1 to prove the following proposition concerning the uniform time-weighted bounds of  $(a_n, u_n, \nabla \Pi_n)$ .

**Proposition 4.1.** *Under the assumptions of Theorem 4.1, (2.4) has a unique global smooth solution  $(a_n, u_n, \nabla \Pi_n)$  which satisfies*

$$(4.5) \quad \begin{aligned} & \mu^{\frac{1}{2}(3 - \frac{d}{q_1} - \varepsilon)} \|t^{\alpha_1^\varepsilon} \Delta u_n\|_{L^{2r}(\mathbb{R}^+; L^{q_1})} + \mu^{1 - \frac{d}{2q_2} - \frac{\varepsilon}{2}} (\|t^{\beta_1^\varepsilon} \nabla u_n\|_{L^{2r}(\mathbb{R}^+; L^{q_2})} + \|t^{\beta_2^\varepsilon} \nabla u_n\|_{L^\infty(\mathbb{R}^+; L^{q_2})}) \\ & + \mu^{\frac{1}{2}(1 - \frac{d}{q_3} - \varepsilon)} (\|t^{\gamma_1^\varepsilon} u_n\|_{L^\infty(\mathbb{R}^+; L^{q_3})} + \|t^{\gamma_2^\varepsilon} u_n\|_{L^{2r}(\mathbb{R}^+; L^{q_3})}) \\ & \leq C \|u_0\|_{\dot{B}_{p,r}^{-1 + \frac{d}{p} + \varepsilon}} \exp\left(C_r \mu^{-2r} \|u_0^d\|_{\dot{B}_{p,r}^{-1 + \frac{d}{p}}}\right)^{2r}, \end{aligned}$$

and

$$(4.6) \quad \begin{aligned} & \mu^{\frac{1}{2}(3 - \frac{d}{q_1} - \varepsilon)} \|t^{\alpha_2^\varepsilon} \Delta u_n\|_{L^r(\mathbb{R}^+; L^{q_1})} \leq C \|u_0\|_{\dot{B}_{p,r}^{-1 + \frac{d}{p} + \varepsilon}} \exp\left(C_r \mu^{-2r} \|u_0^d\|_{\dot{B}_{p,r}^{-1 + \frac{d}{p}}}\right)^{2r}, \\ & \mu^{\frac{1}{2}(3 - \frac{d}{q_1} - \varepsilon)} (\|t^{\alpha_1^\varepsilon} \nabla \Pi_n\|_{L^{2r}(\mathbb{R}^+; L^{q_1})} + \|t^{\alpha_2^\varepsilon} \nabla \Pi_n\|_{L^r(\mathbb{R}^+; L^{q_1})}) \\ & \leq C \eta \|u_0\|_{\dot{B}_{p,r}^{-1 + \frac{d}{p} + \varepsilon}} \exp\left(C_r \mu^{-2r} \|u_0^d\|_{\dot{B}_{p,r}^{-1 + \frac{d}{p}}}\right)^{2r}, \end{aligned}$$

for some constant  $C$  and  $\alpha_i^\varepsilon, \beta_i^\varepsilon, \gamma_i^\varepsilon, i = 1, 2$ , given by (4.3).

*Proof.* For  $N$  large enough, we already proved in Proposition 3.1 that (2.4) has a unique global smooth solution  $(a_n, u_n, \nabla \Pi_n)$ . It remains to prove (4.5) and (4.6). In order to do so, for  $\lambda_1, \lambda_2 > 0$ , we denote

$$(4.7) \quad \begin{aligned} g_{1,n}(t) & \stackrel{\text{def}}{=} \|t^{\bar{\beta}_1} \nabla u_n^d(t)\|_{L^{q_2}}^{2r}, \quad g_{2,n}(t) \stackrel{\text{def}}{=} \|t^{\bar{\gamma}_2} u_n^d(t)\|_{L^{q_3}}^{2r}, \\ u_{\lambda,n}(t, x) & \stackrel{\text{def}}{=} u_n(t, x) \exp\left\{-\int_0^t (\lambda_1 g_{1,n}(t') + \lambda_2 g_{2,n}(t')) dt'\right\}, \end{aligned}$$

and similar notation for  $\Pi_{\lambda,n}(t, x)$ . Then it follows from a similar derivation of (2.10) that

$$(4.8) \quad \begin{aligned} \|\nabla \Pi_{\lambda,n}(t)\|_{L^{q_1}} & \leq C \left\{ \mu \|a_0\|_{L^\infty} \|\Delta u_{\lambda,n}\|_{L^{q_1}} + \|u_{\lambda,n}^h\|_{L^{q_3}} \|\nabla u_n^h\|_{L^{q_2}} \right. \\ & \left. + \|u_n^d\|_{L^{q_3}} \|\nabla u_{\lambda,n}^h\|_{L^{q_2}} + \|u_{\lambda,n}^h\|_{L^{q_3}} \|\nabla u_n^d\|_{L^{q_2}} \right\}. \end{aligned}$$

Whereas by virtue of (2.11) and (4.7), we write

$$(4.9) \quad \begin{aligned} u_{\lambda,n} & = u_{n,L} \exp\left\{-\int_0^t (\lambda_1 g_{1,n}(t') + \lambda_2 g_{2,n}(t')) dt'\right\} \\ & + \int_0^t e^{\mu(t-s)\Delta} \exp\left\{-\int_s^t (\lambda_1 g_{1,n}(t') + \lambda_2 g_{2,n}(t')) dt'\right\} \\ & \quad \times (-u_n \cdot \nabla u_{\lambda,n} + \mu a_n \Delta u_{\lambda,n} - (1 + a_n) \nabla \Pi_{\lambda,n}) ds. \end{aligned}$$

For  $\gamma_1^\varepsilon, \beta_1^\varepsilon$  given by (4.3), we get, by applying Lemma 3.2 for  $r_1 = 2r$ , that

$$\begin{aligned} & \mu^{\frac{d}{2q_2}} \|t^{\gamma_1^\varepsilon} u_{\lambda,n}\|_{L_t^\infty(L^{q_3})} + \mu^{\frac{1}{2} + \frac{d}{2q_3}} \|t^{\beta_1^\varepsilon} \nabla u_{\lambda,n}\|_{L_t^{2r}(L^{q_2})} \\ & \leq \mu^{\frac{d}{2q_2}} \|t^{\gamma_1^\varepsilon} u_{n,L}\|_{L_t^\infty(L^{q_3})} + \mu^{\frac{1}{2} + \frac{d}{2q_3}} \|t^{\beta_1^\varepsilon} \nabla u_{n,L}\|_{L_t^{2r}(L^{q_2})} \\ & + \left\| \exp\left\{-\int_s^t (\lambda_1 g_{1,n}(t') + \lambda_2 g_{2,n}(t')) dt'\right\} \right. \\ & \quad \left. \times s^{\alpha_1^\varepsilon} (-u_n \cdot \nabla u_{\lambda,n} + \mu a_n \Delta u_{\lambda,n} - (1 + a_n) \nabla \Pi_{\lambda,n}) \right\|_{L_t^{2r}(L^{q_1})}. \end{aligned}$$

Similarly for  $\gamma_2^\varepsilon, \beta_2^\varepsilon$  and  $\alpha_1^\varepsilon$ , we deduce from Lemma 2.1 and Lemma 3.3 for  $r_1 = 2r$  that

$$\begin{aligned} & \mu^{\frac{d}{2q_2}} \|t^{\gamma_2^\varepsilon} u_{\lambda,n}\|_{L_t^{2r}(L^{q_3})} + \mu^{\frac{1}{2} + \frac{d}{2q_3}} \|t^{\beta_2^\varepsilon} \nabla u_{\lambda,n}\|_{L_t^\infty(L^{q_2})} + \mu \|t^{\alpha_1^\varepsilon} \Delta u_{\lambda,n}\|_{L_t^{2r}(L^{q_1})} \\ & \leq \mu^{\frac{d}{2q_2}} \|t^{\gamma_2^\varepsilon} u_{n,L}\|_{L_t^{2r}(L^{q_3})} + \mu^{\frac{1}{2} + \frac{d}{2q_3}} \|t^{\beta_2^\varepsilon} \nabla u_{n,L}\|_{L_t^\infty(L^{q_2})} + \mu \|t^{\alpha_1^\varepsilon} \Delta u_{n,L}\|_{L_t^{2r}(L^{q_1})} \\ & \quad + \|\exp\{-\int_s^t (\lambda_1 g_{1,n}(t') + \lambda_2 g_{2,n}(t')) dt'\} \\ & \quad \times s^{\alpha_1^\varepsilon} (-u_n \cdot \nabla u_{\lambda,n} + \mu a_n \Delta u_{\lambda,n} - (1 + a_n) \nabla \Pi_{\lambda,n})\|_{L_t^{2r}(L^{q_1})}. \end{aligned}$$

Hence, by using (2.7), Proposition 2.1 and (4.8), we obtain

$$\begin{aligned} & \mu^{\frac{d}{2q_2}} (\|t^{\gamma_1^\varepsilon} u_{\lambda,n}\|_{L_t^\infty(L^{q_3})} + \|t^{\gamma_2^\varepsilon} u_{\lambda,n}\|_{L_t^{2r}(L^{q_3})}) + \mu^{\frac{1}{2} + \frac{d}{2q_3}} (\|t^{\beta_1^\varepsilon} \nabla u_{\lambda,n}\|_{L_t^{2r}(L^{q_2})} \\ & \quad + \|t^{\beta_2^\varepsilon} \nabla u_{\lambda,n}\|_{L_t^\infty(L^{q_2})}) + \mu \|t^{\alpha_1^\varepsilon} \Delta u_{\lambda,n}\|_{L_t^{2r}(L^{q_1})} \\ (4.10) \quad & \leq C \left\{ \mu^{-\frac{1}{2}(1-\frac{d}{q_1}-\varepsilon)} \|u_0\|_{\dot{B}_{q_1,2r}^{-1+\frac{d}{q_1}+\varepsilon}} + \mu \|a_0\|_{L^\infty} \|t^{\alpha_1^\varepsilon} \Delta u_{\lambda,n}\|_{L_t^{2r}(L^{q_1})} \right. \\ & \quad + \|t^{\gamma_1^\varepsilon} u_{\lambda,n}^h\|_{L_t^\infty(L^{q_3})} \|t^{\beta_1^\varepsilon} \nabla u_n^h\|_{L_t^{2r}(L^{q_2})} \\ & \quad \left. + \left( \int_0^t e^{-2r \int_s^t (\lambda_1 g_{1,n}(t') + \lambda_2 g_{2,n}(t')) dt'} (\|s^{\tilde{\gamma}_2} u_n^d(s)\|_{L^{q_3}}^{2r} \|s^{\beta_2^\varepsilon} \nabla u_{\lambda,n}^h(s)\|_{L^{q_2}}^{2r} \right. \right. \\ & \quad \left. \left. + \|s^{\tilde{\gamma}_1} u_{\lambda,n}^h(s)\|_{L^{q_3}}^{2r} \|s^{\beta_1^\varepsilon} \nabla u_n^d(s)\|_{L^{q_2}}^{2r} ds \right)^{\frac{1}{2r}} \right\}. \end{aligned}$$

Then as  $C \|a_0\|_{L^\infty} \leq \frac{1}{2}$ , and

$$\begin{aligned} & \left( \int_0^t e^{-2\lambda_1 r \int_s^t g_{1,n}(t') dt'} \|s^{\tilde{\gamma}_1} u_{\lambda,n}^h\|_{L^{q_3}}^{2r} \|t^{\beta_1^\varepsilon} \nabla u_n^d(s)\|_{L^{q_2}}^{2r} ds \right)^{\frac{1}{2r}} \leq \frac{1}{(2\lambda_1 r)^{\frac{1}{2r}}} \|s^{\tilde{\gamma}_1} u_{\lambda,n}^h\|_{L_t^\infty(L^{q_3})}, \\ & \left( \int_0^t e^{-2\lambda_2 r \int_s^t g_{2,n}(t') dt'} \|s^{\tilde{\gamma}_2} u_n^d(s)\|_{L^{q_3}}^{2r} \|t^{\beta_2^\varepsilon} \nabla u_{\lambda,n}^h(s)\|_{L^{q_2}}^{2r} ds \right)^{\frac{1}{2r}} \leq \frac{1}{(2\lambda_2 r)^{\frac{1}{2r}}} \|s^{\beta_2^\varepsilon} \nabla u_{\lambda,n}^h\|_{L_t^\infty(L^{q_2})}, \end{aligned}$$

we infer from (4.10) that

$$\begin{aligned} & \mu^{\frac{d}{2q_2}} (\|t^{\gamma_1^\varepsilon} u_{\lambda,n}\|_{L_t^\infty(L^{q_3})} + \|t^{\gamma_2^\varepsilon} u_{\lambda,n}\|_{L_t^{2r}(L^{q_3})}) + \mu^{\frac{1}{2} + \frac{d}{2q_3}} (\|t^{\beta_2^\varepsilon} \nabla u_{\lambda,n}\|_{L_t^\infty(L^{q_2})} \\ & \quad + \|t^{\beta_1^\varepsilon} \nabla u_{\lambda,n}\|_{L_t^{2r}(L^{q_2})}) + \mu \|t^{\alpha_1^\varepsilon} \Delta u_{\lambda,n}\|_{L_t^{2r}(L^{q_1})} \\ (4.11) \quad & \leq C \left\{ \mu^{-\frac{1}{2}(1-\frac{d}{q_1}-\varepsilon)} \|u_0\|_{\dot{B}_{q_1,r}^{-1+\frac{d}{q_1}+\varepsilon}} + \|t^{\gamma_1^\varepsilon} u_{\lambda,n}^h\|_{L_t^\infty(L^{q_3})} \|t^{\beta_1^\varepsilon} \nabla u_n^h\|_{L_t^{2r}(L^{q_2})} \right. \\ & \quad \left. + \frac{1}{(2\lambda_1 r)^{\frac{1}{2r}}} \|t^{\gamma_1^\varepsilon} u_{\lambda,n}^h\|_{L_t^\infty(L^{q_3})} + \frac{1}{(2\lambda_2 r)^{\frac{1}{2r}}} \|t^{\beta_2^\varepsilon} \nabla u_{\lambda,n}^h\|_{L_t^\infty(L^{q_2})} \right\}. \end{aligned}$$

Recalling from (3.13) that

$$\|t^{\beta_1^\varepsilon} \nabla u_n^h\|_{L_t^{2r}(L^{q_2})} \leq C c_0 \mu^{\frac{d}{2q_2}},$$

so that as long as  $c_0$  is small enough in (1.6), taking  $\lambda_1 = \frac{(4C)^{2r}}{2r\mu^{\frac{d}{q_2}r}}$ ,  $\lambda_2 = \frac{(4C)^{2r}}{2r\mu^{(1+\frac{d}{q_3})r}}$  in (4.11) results in

$$\begin{aligned} & \mu^{\frac{d}{2q_2}} \left( \frac{1}{2} \|t^{\gamma_1^\varepsilon} u_{\lambda,n}^h\|_{L_t^\infty(L^{q_3})} + \|t^{\gamma_2^\varepsilon} u_{\lambda,n}^h\|_{L_t^{2r}(L^{q_3})} \right) + \mu^{\frac{1}{2} + \frac{d}{2q_3}} (\|t^{\beta_1^\varepsilon} \nabla u_{\lambda,n}^h\|_{L_t^{2r}(L^{q_2})} \\ (4.12) \quad & \quad + \frac{3}{4} \|t^{\beta_2^\varepsilon} \nabla u_{\lambda,n}^h\|_{L_t^\infty(L^{q_2})}) + \mu \|t^{\alpha_1^\varepsilon} \Delta u_{\lambda,n}^h\|_{L_t^{2r}(L^{q_1})} \\ & \leq C \mu^{\frac{1}{2}(\frac{d}{q_1}-1-\varepsilon)} \|u_0\|_{\dot{B}_{p,r}^{-1+\frac{d}{p}+\varepsilon}}. \end{aligned}$$

From (3.13), (4.7), and (4.12), we infer

$$\begin{aligned}
& \mu^{\frac{d}{2q_2}} \left( \|t^{\gamma_1^\varepsilon} u_n\|_{L_t^\infty(L^{q_3})} + \|t^{\gamma_2^\varepsilon} u_n\|_{L_t^{2r}(L^{q_3})} \right) + \mu^{\frac{1}{2} + \frac{d}{2q_3}} \left( \|t^{\beta_1^\varepsilon} \nabla u_n\|_{L_t^{2r}(L^{q_2})} \right. \\
& \quad \left. + \|t^{\beta_2^\varepsilon} \nabla u_n\|_{L_t^\infty(L^{q_2})} \right) + \mu \|t^{\alpha_1^\varepsilon} \Delta u_n\|_{L_t^{2r}(L^{q_1})} \\
& \leq C \mu^{\frac{1}{2}(\frac{d}{q_1} - 1 - \varepsilon)} \|u_0\|_{\dot{B}_{p,r}^{-1 + \frac{d}{p} + \varepsilon}} \exp \left\{ C_r \int_0^t \mu^{-\frac{d}{q_2} r} \|s^{\beta_1} \nabla u_n^d(s)\|_{L^{q_3}}^{2r} \right. \\
& \quad \left. + \mu^{-(1 + \frac{d}{q_3})r} \|s^{\tilde{\gamma}_2} u_n^d(s)\|_{L^{q_3}}^{2r} ds \right\} \\
& \leq C \mu^{\frac{1}{2}(\frac{d}{q_1} - 1 - \varepsilon)} \|u_0\|_{\dot{B}_{p,r}^{-1 + \frac{d}{p} + \varepsilon}} \exp \left\{ C_r \mu^{-2r} \|u_0^d\|_{\dot{B}_{p,r}^{-1 + \frac{d}{p}}}^{2r} \right\},
\end{aligned}$$

which implies (4.5). It remains to prove (4.6). In fact, we get, by applying Lemma 2.1 and (4.8), that

$$\begin{aligned}
\mu \|t^{\alpha_2^\varepsilon} \Delta u_n\|_{L^r(\mathbb{R}^+; L^{q_1})} & \leq \mu^{-\frac{1}{2}(1 - \frac{d}{q_1} - \varepsilon)} \|u_0\|_{\dot{B}_{p,r}^{-1 + \frac{d}{p} + \varepsilon}} + C \left\{ \mu \|a_0\|_{L^\infty} \|t^{\alpha_2^\varepsilon} \Delta u_n\|_{L^r(\mathbb{R}^+; L^{q_1})} \right. \\
& \quad + \|t^{\gamma_2^\varepsilon} u_n\|_{L^{2r}(\mathbb{R}^+; L^{q_3})} \|t^{\beta_1} \nabla u_n^h\|_{L^{2r}(\mathbb{R}^+; L^{q_2})} \\
& \quad \left. + \|t^{\tilde{\gamma}_2} u_n^h\|_{L^{2r}(\mathbb{R}^+; L^{q_3})} \|t^{\beta_1^\varepsilon} \nabla u_n^d\|_{L^{2r}(\mathbb{R}^+; L^{q_2})} \right\},
\end{aligned}$$

which along with (3.13), (4.5) and the fact:  $C \|a_0\|_{L^\infty} \leq \frac{1}{2}$ , gives rise to the first inequality of (4.6). The second inequality of (4.6) follows along the same line. This completes the proof of Proposition 3.1.  $\square$

To prove the uniqueness part of Theorem 4.1, we need the following lemma:

**Proposition 4.2.** *Let  $\alpha_1, \beta_1, \beta_2, \gamma_1, \gamma_2$ , and  $p_1, p_2, p_3$  be given by Theorem 1.1, if  $t^{\alpha_1} f, t^{\alpha_1} \nabla g, t^{\alpha_1} R \in L^{2r}((0, T); L^{p_1}(\mathbb{R}^d))$ . Then the system*

$$(4.13) \quad \begin{cases} \partial_t v - \Delta v + \nabla P = f, \\ \operatorname{div} v = g, \\ \partial_t g = \operatorname{div} R, \\ v|_{t=0} = 0, \end{cases}$$

has a unique solution  $(v, \nabla P)$  so that

$$(4.14) \quad \begin{aligned}
& \|t^{\gamma_1} v\|_{L_T^\infty(L^{p_3})} + \|t^{\gamma_2} v\|_{L_T^{2r}(L^{p_3})} + \|t^{\beta_1} \nabla v\|_{L_T^{2r}(L^{p_2})} + \|t^{\beta_2} \nabla v\|_{L_T^\infty(L^{p_2})} \\
& + \|(t^{\alpha_1} \partial_t v, t^{\alpha_1} \nabla^2 v, t^{\alpha_1} \nabla P)\|_{L_T^{2r}(L^{p_1})} \leq C \|(t^{\alpha_1} f, t^{\alpha_1} \nabla g, t^{\alpha_1} R)\|_{L_T^{2r}(L^{p_1})}.
\end{aligned}$$

*Proof.* We first get, by taking space divergence on (4.13), that

$$\Delta P = \operatorname{div}(f + \nabla g - R),$$

which implies

$$\|\nabla P\|_{L^q(\mathbb{R}^d)} \leq C \|(f, \nabla g, R)\|_{L^q(\mathbb{R}^d)},$$

for any  $q \in (1, \infty)$ . On the other hand, we have

$$(4.15) \quad v = \int_0^t e^{(t-s)\Delta} (f - \nabla P) ds,$$

from which, (4.13), Lemma 3.1, and Lemma 3.2 and Lemma 3.3 for  $\varepsilon = 0$  and  $r_1 = 2r$ , we deduce (4.14).  $\square$



**Lemma 4.1.** *Let  $(a, u)$  be a global weak solution of (1.2), which satisfies (1.8-1.9) and (4.1-4.2). Then for any  $\varepsilon \in (0, \min\{1 - \frac{1}{r}, \frac{d}{p} - 1\})$ , one has*

$$(4.16) \quad \begin{aligned} & \|\nabla^2 u\|_{L_T^{\frac{8}{8-\varepsilon}}(L^d)} + \|\nabla u\|_{L_T^{\frac{8}{8-\varepsilon}}(L^\infty)} \leq C_{T,\varepsilon}, \\ & \|t^{\alpha_1} \nabla u\|_{L_T^{2r}(L^\infty)} + \|t^{\alpha_1} \nabla^2 u\|_{L_T^{2r}(L^d)} \leq C_{T,\varepsilon} \quad \text{for any } T < \infty. \end{aligned}$$

*Proof.* We first deduce from Remark 4.1 that  $\nabla^2 u \in L^{\frac{8}{8-\varepsilon}}((0, T); L^{\frac{d}{1+\varepsilon}}(\mathbb{R}^d)) \cap L^{\frac{8}{8-\varepsilon}}((0, T); L^{\frac{2d}{2-\varepsilon}}(\mathbb{R}^d))$  for any  $T < \infty$ . Then applying Hölder's inequality yields

$$(4.17) \quad \|\nabla^2 u\|_{L_T^{\frac{8}{8-\varepsilon}}(L^d)} \leq \|\nabla^2 u\|_{L_T^{\frac{8}{8-\varepsilon}}(L^{\frac{d}{1+\varepsilon}})}^{\frac{1}{3}} \|\nabla^2 u\|_{L_T^{\frac{8}{8-\varepsilon}}(L^{\frac{2d}{2-\varepsilon}})}^{\frac{2}{3}} \leq C_{T,\varepsilon}.$$

Whereas by virtue of Lemma A.1, one has

$$\begin{aligned} \|\nabla u\|_{L_T^{\frac{8}{8-\varepsilon}}(L^\infty)} & \lesssim \sum_{j \leq 0} \|\nabla \dot{\Delta}_j u\|_{L_T^{\frac{8}{8-\varepsilon}}(L^\infty)} + \sum_{j > 0} \|\nabla \dot{\Delta}_j u\|_{L_T^{\frac{8}{8-\varepsilon}}(L^\infty)} \\ & \lesssim \sum_{j \leq 0} 2^{j\varepsilon} \|\nabla^2 \dot{\Delta}_j u\|_{L_T^{\frac{8}{8-\varepsilon}}(L^{\frac{d}{1+\varepsilon}})} + \sum_{j > 0} 2^{-j\frac{\varepsilon}{2}} \|\nabla^2 \dot{\Delta}_j u\|_{L_T^{\frac{8}{8-\varepsilon}}(L^{\frac{2d}{2-\varepsilon}})} \\ & \lesssim \|\nabla^2 u\|_{L_T^{\frac{8}{8-\varepsilon}}(L^{\frac{d}{1+\varepsilon}})} + \|\nabla^2 u\|_{L_T^{\frac{8}{8-\varepsilon}}(L^{\frac{2d}{2-\varepsilon}})} \leq C_{T,\varepsilon}, \end{aligned}$$

which together with (4.17) proves the first line of (4.16). The second line of (4.17) follows along the same lines.  $\square$

Thanks to Lemma 4.1, we can taking  $T$  small enough so that

$$(4.18) \quad \int_0^T \|\nabla u(t)\|_{L^\infty} dt \leq \frac{1}{2}.$$

As in [13, 14], we shall prove the uniqueness part of Theorem 4.1 by using the Lagrangian formulation of (1.2). Toward this, we first recall some basic facts concerning Lagrangian coordinates from [13, 14]. By virtue of (4.18), for any  $y \in \mathbb{R}^d$ , the following ordinary differential equation has a unique solution on  $[0, T]$

$$(4.19) \quad \frac{dX(t, y)}{dt} = u(t, X(t, y)) \stackrel{\text{def}}{=} v(t, y), \quad X(t, y)|_{t=0} = y.$$

This leads to the following relation between the Eulerian coordinates  $x$  and the Lagrangian coordinates  $y$ :

$$(4.20) \quad x = X(t, y) = y + \int_0^t v(\tau, y) d\tau.$$

Let  $Y(t, \cdot)$  be the inverse mapping of  $X(t, \cdot)$ . Then  $D_x Y(t, x) = (D_y X(t, y))^{-1}$  for  $x = X(t, y)$ . Providing  $D_y X - Id$  is small enough, we have

$$D_x Y = (Id + (D_y X - Id))^{-1} = \sum_{k=0}^{\infty} (-1)^k \left( \int_0^t D_y v(\tau, y) d\tau \right)^k.$$

We denote  $A(t, y) \stackrel{\text{def}}{=} (D_y X(t, y))^{-1} = D_x Y(t, x)$ , then we have

$$(4.21) \quad \nabla_x u(t, x) = {}^T A(t, x) \nabla_y v(t, y) \quad \text{and} \quad \text{div } u(t, x) = \text{div}(A(t, y)v(t, y)).$$

By the chain rule, we also have

$$(4.22) \quad \text{div}_y (A \cdot) = {}^T A : \nabla_y.$$

Here and in what follows, we always denote  ${}^T A$  the transpose matrix of  $A$ .

As in [13, 14], we denote

$$(4.23) \quad \begin{aligned} \nabla_u &\stackrel{\text{def}}{=} A \cdot \nabla_y, \quad \operatorname{div}_u \stackrel{\text{def}}{=} \operatorname{div}(A \cdot) \quad \text{and} \quad \Delta_u \stackrel{\text{def}}{=} \operatorname{div}_u \nabla_u, \\ b(t, y) &\stackrel{\text{def}}{=} a(t, X(t, y)), \quad v(t, y) \stackrel{\text{def}}{=} u(t, X(t, y)) \quad \text{and} \quad P(t, y) \stackrel{\text{def}}{=} \Pi(t, X(t, y)). \end{aligned}$$

Notice that for any  $t > 0$ , the solution of (1.2) obtained in Theorem 4.1 satisfies the smoothness assumption of Proposition 2 in [14], so that  $(b, v, \nabla P)$  defined by (4.23) fulfils

$$(4.24) \quad \begin{cases} b_t = 0, \\ \partial_t v - (1 + b)(\mu \Delta_u v - \nabla_u P) = 0, \\ \operatorname{div}_u v = 0, \\ (b, v)|_{t=0} = (a_0, u_0), \end{cases}$$

which is the Lagrangian formulation of (1.2). For the sake of simplicity, we shall take  $\mu = 1$  in what follows.

We now present the proof of Theorem 4.1.

*Proof of Theorem 4.1.* We first deduce from the proof to the existence part of Theorem 1.1, (4.5) and (4.6) that the global weak solution  $(a, u, \nabla \Pi)$  constructed in Theorem 1.1 satisfies (4.1) and (4.2). It remains to prove the uniqueness part of Theorem 4.1.

Let  $(a_i, u_i, \Pi_i), i = 1, 2$ , be two solutions of (1.2) which satisfies (1.8-1.9) and (4.1-4.2). Let  $X_i, (v_i, P_i), A_i, i = 1, 2$  be given by (4.20) and (4.23). We denote

$$\delta v \stackrel{\text{def}}{=} v_2 - v_1, \quad \delta P \stackrel{\text{def}}{=} P_2 - P_1,$$

then  $(\delta v, \delta P)$  solves

$$(4.25) \quad \begin{cases} \partial_t \delta v - \Delta \delta v + \nabla \delta P = a_0(\Delta \delta v - \nabla \delta P) + \delta f_1 + \delta f_2, \\ \operatorname{div} \delta v = \delta g, \\ \partial_t \delta g = \operatorname{div} \delta R, \\ \delta v|_{t=0} = 0, \end{cases}$$

with

$$\begin{aligned} \delta f_1 &\stackrel{\text{def}}{=} (1 + a_0)[(Id - A_2) \nabla \delta P - \delta A \nabla P_1], \\ \delta f_2 &\stackrel{\text{def}}{=} \mu(1 + a_0) \operatorname{div}[(A_2^T A_2 - Id) \nabla \delta v + (A_2^T A_2 - A_1^T A_1) \nabla v_1], \\ \delta g &\stackrel{\text{def}}{=} (Id - A_2) : D \delta v - \delta A : D v_1, \\ \delta R &\stackrel{\text{def}}{=} \partial_t[(Id - A_2) \delta v] - \partial_t[\delta A v_1]. \end{aligned}$$

In what follows, we will use repeatedly the following fact (see [13] for instance) that

$$(4.26) \quad \delta A(t) = \left( \int_0^t D \delta v d\tau \right) \left( \sum_{k \geq 1} \sum_{0 \leq j < k} C_i^j C_2^{k-1-j} \right) \quad \text{with} \quad C_i(t) \stackrel{\text{def}}{=} \int_0^t D v_i d\tau.$$

Let the indices  $\alpha_1, \beta_1, \beta_2, \gamma_1$  and  $\gamma_2$  be given by (1.10). As in Remark 4.1, we take  $p_1 = \frac{d}{1+\varepsilon}$  and  $p_2, p_3$  satisfying  $\frac{dr}{r-1} < p_3 < \infty$  and  $\frac{1}{p_2} + \frac{1}{p_3} = \frac{1}{p_1}$ . We denote

$$(4.27) \quad \begin{aligned} G(t) &\stackrel{\text{def}}{=} \|t^{\gamma_1} \delta v\|_{L_t^\infty(L^{p_3})} + \|t^{\gamma_2} \delta v\|_{L_t^{2r}(L^{p_3})} + \|t^{\beta_1} \nabla \delta v\|_{L_t^{2r}(L^{p_2})} \\ &\quad + \|t^{\beta_2} \nabla \delta v\|_{L_t^\infty(L^{p_2})} + \|(t^{\alpha_1} \partial_t \delta v, t^{\alpha_1} \nabla^2 \delta v, t^{\alpha_1} \nabla \delta P)\|_{L_t^{2r}(L^{\frac{d}{1+\varepsilon}})}. \end{aligned}$$

Then we deduce from Proposition 4.2 and (4.25) that

$$(4.28) \quad G(t) \leq C \left( \|t^{\alpha_1} \delta f_1\|_{L_t^{2r}(L^{\frac{d}{1+\varepsilon}})} + \|t^{\alpha_1} \delta f_2\|_{L_t^{2r}(L^{\frac{d}{1+\varepsilon}})} + \|t^{\alpha_1} \nabla \delta g\|_{L_t^{2r}(L^{\frac{d}{1+\varepsilon}})} + \|t^{\alpha_1} \delta R\|_{L_t^{2r}(L^{\frac{d}{1+\varepsilon}})} \right)$$

as long as  $C\|a_0\|_{L^\infty} \leq \frac{1}{2}$ .

Let us now estimate term by term on the right-hand side of (4.28). We first get, by using (1.8) and (4.18), that

$$\begin{aligned} \|t^{\alpha_1} \delta f_1\|_{L_t^{2r}(L^{\frac{d}{1+\varepsilon}})} &\lesssim \|Id - A_2\|_{L_t^\infty(L^\infty)} \|t^{\alpha_1} \nabla \delta P\|_{L_t^{2r}(L^{\frac{d}{1+\varepsilon}})} + \|\delta A\|_{L_t^\infty(L^{\frac{d}{\varepsilon}})} \|t^{\alpha_1} \nabla P_1\|_{L_t^{2r}(L^d)} \\ &\lesssim \|\nabla v_2\|_{L_t^1(L^\infty)} G(t) + \|\nabla \delta v\|_{L_t^1(L^{\frac{d}{\varepsilon}})} (\|t^{\alpha_1} \nabla P_1\|_{L_t^{2r}(L^{p_1})})^\theta (\|t^{\alpha_1} \nabla P_1\|_{L_t^{2r}(L^{q_1})})^{1-\theta}, \end{aligned}$$

for  $\theta$  determined by  $\frac{1}{d} = \frac{\theta}{p_1} + \frac{1-\theta}{q_1}$ . However by virtue of Sobolev embedding theorem,  $W^{1, \frac{d}{1+\varepsilon}}(\mathbb{R}^d) \hookrightarrow L^{\frac{d}{\varepsilon}}(\mathbb{R}^d)$ , one has

$$(4.29) \quad \|\nabla \delta v\|_{L_t^1(L^{\frac{d}{\varepsilon}})} \lesssim \|\nabla^2 \delta v\|_{L_t^1(L^{\frac{d}{1+\varepsilon}})} \lesssim t^{\frac{1}{2}(\frac{d}{p_1}-1)} \|t^{\alpha_1} \nabla^2 \delta v\|_{L_t^{2r}(L^{\frac{d}{1+\varepsilon}})},$$

we obtain

$$(4.30) \quad \|t^{\alpha_1} \delta f_1\|_{L_t^{2r}(L^{\frac{d}{1+\varepsilon}})} \lesssim \eta(t) G(t),$$

for some positive continuous function  $\eta(t)$  which tends to 0 as  $t \rightarrow 0$ . Along the same line, we deduce

$$\begin{aligned} &\|t^{\alpha_1} \nabla((Id - A_2) : D\delta v)\|_{L_t^{2r}(L^{\frac{d}{1+\varepsilon}})} \\ &\lesssim \|DA_2 \otimes t^{\alpha_1} D\delta v\|_{L_t^{2r}(L^{\frac{d}{1+\varepsilon}})} + \|(Id - A_2) \otimes t^{\alpha_1} D^2 \delta v\|_{L_t^{2r}(L^{\frac{d}{1+\varepsilon}})} \\ &\lesssim \|\nabla^2 v_2\|_{L_t^1(L^d)} \|t^{\alpha_1} \nabla \delta v\|_{L_t^{2r}(L^{\frac{d}{\varepsilon}})} + \|\nabla v_2\|_{L^1(L^\infty)} \|t^{\alpha_1} \nabla^2 \delta v\|_{L_t^{2r}(L^{\frac{d}{1+\varepsilon}})} \\ &\lesssim \|t^{\alpha_1} \nabla^2 \delta v\|_{L_t^{2r}(L^{\frac{d}{1+\varepsilon}})} (\|\nabla^2 v_2\|_{L_t^1(L^d)} + \|\nabla v_2\|_{L^1(L^\infty)}), \end{aligned}$$

and

$$\begin{aligned} &\|t^{\alpha_1} \nabla(\delta A : Dv_1)\|_{L_t^{2r}(L^{\frac{d}{1+\varepsilon}})} \\ &\lesssim \|t^{\alpha_1} |\nabla v_1| \int_0^\tau |\nabla^2 \delta v| d\tau'\|_{L_t^{2r}(L^{\frac{d}{1+\varepsilon}})} + \|t^{\alpha_1} |\nabla^2 v_1| \int_0^\tau |\nabla \delta v| d\tau'\|_{L_t^{2r}(L^{\frac{d}{1+\varepsilon}})} \\ &\lesssim \|t^{\alpha_1} \nabla v_1\|_{L_t^{2r}(L^\infty)} \|\nabla^2 \delta v\|_{L_t^1(L^{\frac{d}{1+\varepsilon}})} + \|t^{\alpha_1} \nabla^2 v_1\|_{L_t^{2r}(L^d)} \|\nabla \delta v\|_{L_t^1(L^{\frac{d}{\varepsilon}})} \\ &\lesssim t^{\frac{1}{2}(\frac{d}{p_1}-1)} \|t^{\alpha_1} \nabla^2 \delta v\|_{L_t^{2r}(L^{\frac{d}{1+\varepsilon}})} (\|t^{\alpha_1} \nabla v_1\|_{L_t^{2r}(L^\infty)} + \|t^{\alpha_1} \nabla^2 v_1\|_{L_t^{2r}(L^d)}). \end{aligned}$$

So that for all  $t \in [0, T]$ , we get, by using (4.16), that

$$(4.31) \quad \|t^{\alpha_1} \nabla \delta g\|_{L_t^{2r}(L^{\frac{d}{1+\varepsilon}})} \lesssim \eta(t) G(t).$$

The estimate to the term  $\delta f_2$  can be handled along the same line.

To deal with  $\delta R$ , we denote  $Dv_{1,2}$  to be the components of  $Dv_1$  and  $Dv_2$ . Then we get, by using (4.26) once again, that

$$\begin{aligned} \|t^{\alpha_1} \partial_t((Id - A_2)\delta v)\|_{L_t^{2r}(L^{\frac{d}{1+\varepsilon}})} &\lesssim \|t^{\beta_1} Dv_2 t^{\gamma_1} \delta v\|_{L_t^{2r}(L^{\frac{d}{1+\varepsilon}})} + \|(Id - A_2) t^{\alpha_1} \partial_t \delta v\|_{L_t^{2r}(L^{\frac{d}{1+\varepsilon}})} \\ &\lesssim \|t^{\beta_1} \nabla v_2\|_{L_t^{2r}(L^{p_2})} \|t^{\gamma_1} \delta v\|_{L_t^\infty(L^{p_3})} \\ &\quad + \|\nabla v_2\|_{L_t^1(L^\infty)} \|t^{\alpha_1} \partial_t \delta v\|_{L_t^{2r}(L^{\frac{d}{1+\varepsilon}})}, \end{aligned}$$

and

$$\begin{aligned}
\|t^{\alpha_1} \partial_t(\delta A v_1)\|_{L_t^{2r}(L^{\frac{d}{1+\varepsilon}})} &\lesssim \|t^{\gamma_2} v_1 t^{\beta_2} D\delta v\|_{L_t^{2r}(L^{\frac{d}{1+\varepsilon}})} + \|\delta A t^{\alpha_1} \partial_t v_1\|_{L_t^{2r}(L^{\frac{d}{1+\varepsilon}})} \\
&\quad + \left\| \int_0^\tau |D\delta v| d\tau' t^{\beta_2} |Dv_{1,2}| t^{\gamma_2} |v_1| \right\|_{L_t^{2r}(L^{\frac{d}{1+\varepsilon}})} \\
&\lesssim \|t^{\beta_2} \nabla \delta v\|_{L_t^\infty(L^{p_2})} \|t^{\gamma_2} v_1\|_{L_t^{2r}(L^{p_3})} + \|\nabla \delta v\|_{L_t^1(L^{\frac{d}{\varepsilon}})} \|t^{\alpha_1} \partial_t v_1\|_{L_t^{2r}(L^d)} \\
&\quad + \|\nabla \delta v\|_{L_t^1(L^{\frac{d}{\varepsilon}})} \left( \|t^{\beta_2} \nabla v_{1,2}\|_{L_t^\infty(L^{p_2})} \|t^{\gamma_2} v_1\|_{L_t^{2r}(L^{p_3})} \right)^\theta \\
&\quad \times \left( \|t^{\beta_2} \nabla v_{1,2}\|_{L_t^\infty(L^{q_2})} \|t^{\gamma_2} v_1\|_{L_t^{2r}(L^{q_3})} \right)^{1-\theta},
\end{aligned}$$

for  $\theta$  given by  $\frac{1}{d} = \frac{\theta}{p_1} + \frac{1-\theta}{q_1}$ . Hence it follows from (1.8) and (4.1) that

$$(4.32) \quad \|t^{\alpha_1} \delta R\|_{L_t^{2r}(L^{\frac{d}{1+\varepsilon}})} \lesssim \eta(t)G(t).$$

Substituting (4.30-4.32) into (4.28) results in

$$(4.33) \quad G(t) \lesssim \eta(t)G(t),$$

which implies the uniqueness of the solution to (1.2) on a sufficiently small time interval. Then uniqueness part of Theorem 4.1 can be completed by a bootstrap method. This concludes the proof of Theorem 4.1.  $\square$

## 5. PROOF OF THEOREM 1.2

The goal of this section is to present the proof of Theorem 1.2. Indeed given  $a_0 \in L^\infty(\mathbb{R}^d) \cap B_{q,\infty}^{\frac{d}{q}+\varepsilon}(\mathbb{R}^d)$ ,  $u_0 \in \dot{B}_{p,r}^{-1+\frac{d}{p}-\varepsilon}(\mathbb{R}^d) \cap \dot{B}_{p,r}^{-1+\frac{d}{p}}(\mathbb{R}^d)$  with  $\|a_0\|_{L^\infty \cap B_{q,\infty}^{\frac{d}{q}+\varepsilon}}$  being sufficiently small and  $p, q, \varepsilon$  satisfying the conditions listed in Theorem 1.2, we deduce from [19] that there exists a positive time  $T$  so that (1.2) has a unique solution  $(a, u, \nabla \Pi)$  with

$$(5.1) \quad \begin{aligned} a &\in C([0, T]; L^\infty(\mathbb{R}^d) \cap B_{q,\infty}^{\frac{d}{q}+\frac{\varepsilon}{2}}(\mathbb{R}^d)), \quad u \in \tilde{L}_T^\infty(\dot{B}_{p,r}^{-1+\frac{d}{p}-s}) \cap \tilde{L}_T^1(\dot{B}_{p,r}^{1+\frac{d}{p}-s}) \\ &\text{and } \nabla \Pi \in \tilde{L}_T^1(\dot{B}_{p,r}^{-1+\frac{d}{p}-s}) \text{ for } s = 0 \text{ and } \varepsilon. \end{aligned}$$

We denote  $T^*$  to be the lifespan of this solution. Then the proof of Theorem 1.2 is reduced to prove that  $T^* = \infty$ .

**5.1. The estimate of the density.** Notice from (5.1) that  $u \in \tilde{L}_T^1(\dot{B}_{p,r}^{-1+\frac{d}{p}-\varepsilon}) \cap \tilde{L}_T^1(\dot{B}_{p,r}^{-1+\frac{d}{p}})$ , which is not Lipschitz in the space variables, the regularity of the solution  $a$  to (1.2) may be coarsen for positive time. In order to applying the losing derivative estimate in [6], we first need to prove that  $u \in L_T^1(C_\mu)$  for  $\mu(r) = r(-\log r)^\alpha$  and some  $\alpha \in (0, 1)$ . Indeed, let  $u \in \tilde{L}_T^1(\dot{B}_{p,r}^{1+\frac{d}{p}-\varepsilon}) \cap \tilde{L}_T^1(\dot{B}_{p,r}^{1+\frac{d}{p}})$  for some  $\varepsilon > 0$ , we denote  $\theta(t, x, y) \stackrel{\text{def}}{=} |u(t, x) - u(t, y)|$ . Then similar to the proof of Proposition 2.1 in [8], for any positive integer  $N$  and  $\varepsilon_1 > 0$ , one has

$$\begin{aligned}
\theta(t, x, y) &\leq |x - y|(2 + N)^{1-\frac{1}{r}+\varepsilon_1} \sum_{-1 \leq j \leq N} \frac{\|\nabla \Delta_j u(t)\|_{L^\infty}}{(2 + j)^{1-\frac{1}{r}+\varepsilon_1}} \\
&\quad + 2 \sum_{j > N} 2^{-(2+j)} (2 + j)^{1-\frac{1}{r}+\varepsilon_1} \frac{2^{2+j} \|\Delta_j u(t)\|_{L^\infty}}{(2 + j)^{1-\frac{1}{r}+\varepsilon_1}}.
\end{aligned}$$

Note that  $2^x x^\alpha$  with  $0 < \alpha < 1$  is a decreasing function, we get, by taking  $N = [1 - \log |x - y|] - 2$  in the above inequality, that

$$\theta(t, x, y) \leq C|x - y|(1 - \log |x - y|)^{1 - \frac{1}{r} + \varepsilon_1} \sum_{j \geq -1} \frac{\|\nabla \Delta_j u(t)\|_{L^\infty}}{(2 + j)^{1 - \frac{1}{r} + \varepsilon_1}},$$

from which, we infer

$$(5.2) \quad \int_0^T \sup_{0 < |x-y| < 1} \frac{\theta(t, x, y)}{|x - y|(1 - \log |x - y|)^{1 - \frac{1}{r} + \varepsilon_1}} dt \leq C \left( \sum_{j \geq -1} \|\nabla \Delta_j u\|_{L_T^1(L^\infty)}^r \right)^{\frac{1}{r}} \left( \sum_{j \geq -1} \frac{1}{(2 + j)^{(1 - \frac{1}{r} + \varepsilon_1)r'}} \right)^{\frac{1}{r'}} \leq C_{\varepsilon_1} \|\nabla u\|_{\tilde{L}_T^1(B_{\infty, r}^0)},$$

where  $r'$  denotes the conjugate number of  $r$ .

On the other hand, it follows from Lemma A.1 that

$$\|\nabla \Delta_{-1} u\|_{L_T^1(L^\infty)} \leq C_\varepsilon \|u\|_{\tilde{L}_T^1(\dot{B}_{p, r}^{1 + \frac{d}{p} - \varepsilon})},$$

so that

$$\|\nabla u\|_{\tilde{L}_T^1(B_{\infty, r}^0)} \leq C_\varepsilon \left( \|u\|_{\tilde{L}_T^1(\dot{B}_{p, r}^{1 + \frac{d}{p} - \varepsilon})} + \|u\|_{\tilde{L}_T^1(\dot{B}_{p, r}^{1 + \frac{d}{p}})} \right),$$

which together with (5.2) and Theorem 3.33 of [6] (see also [12]) implies that

$$(5.3) \quad \|a\|_{\tilde{L}_t^\infty(B_{q, \infty}^{\frac{d}{q} + \frac{\varepsilon}{2}})} \leq C_\varepsilon \|a_0\|_{B_{q, \infty}^{\frac{d}{q} + \varepsilon}} \exp \left\{ C_\varepsilon \left( \|u\|_{\tilde{L}_t^1(\dot{B}_{p, r}^{1 + \frac{d}{p} - \varepsilon})} + \|u\|_{\tilde{L}_t^1(\dot{B}_{p, r}^{1 + \frac{d}{p}})} \right) \right\},$$

for any  $t < T$ ,  $\frac{d}{q} < 1 + \frac{d}{p}$  and  $0 < \varepsilon < 1 + \frac{d}{p} - \frac{d}{q}$ .

**5.2. The estimate of the pressure.** We first get, by taking space divergence to the momentum equation of (1.2), that

$$-\Delta \Pi = \operatorname{div}(a \nabla \Pi) - \mu \operatorname{div}(a \Delta u) + \operatorname{div}_h \operatorname{div}_h(u^h \otimes u^h) + \operatorname{div}_h(u^d \partial_d u^h) + \partial_d(u^h \cdot \nabla_h u^d) + \partial_d^2(u^d)^2.$$

Thanks to  $\operatorname{div} u = 0$ , one has

$$\begin{aligned} \partial_d(u^h \cdot \nabla_h u^d) &= \partial_d u^h \cdot \nabla_h u^d + u^h \cdot \nabla_h \partial_d u^d \\ &= \operatorname{div}_h(u^d \partial_d u^h) - u^d \partial_d \operatorname{div}_h u^h + \operatorname{div}_h(u^h \partial_d u^d) - \operatorname{div}_h u^h \partial_d u^d \\ &= \operatorname{div}_h(u^d \partial_d u^h) - \partial_d(u^d \operatorname{div}_h u^h) - \operatorname{div}_h(u^h \operatorname{div}_h u^h), \end{aligned}$$

which gives rise to

$$(5.4) \quad \begin{aligned} -\Delta \Pi &= \operatorname{div}(a \nabla \Pi) - \mu \operatorname{div}(a \Delta u) + \operatorname{div}_h \operatorname{div}_h(u^h \otimes u^h) + 2 \operatorname{div}_h(u^d \partial_d u^h) \\ &\quad - 3 \partial_d(u^d \operatorname{div}_h u^h) - \operatorname{div}_h(u^h \operatorname{div}_h u^h). \end{aligned}$$

The following proposition concerning the estimate of the pressure will be the key ingredient used in the estimate of the horizontal component of the velocity.

**Proposition 5.1.** *Let  $1 < q \leq p < 2d$ ,  $r > 1$  and  $0 < \varepsilon < \frac{2d}{p} - 1$ . let  $a \in L_T^\infty(L^\infty) \cap \tilde{L}_T^\infty(B_{q, \infty}^{\frac{d}{q}})$ ,  $u \in \tilde{L}_T^\infty(\dot{B}_{p, r}^{-1 + \frac{d}{p} - s}) \cap \tilde{L}_T^1(\dot{B}_{p, r}^{1 + \frac{d}{p} - s})$ , for  $s = 0$  and  $\varepsilon$ , be the unique local solution of (1.2) given by (5.1). We denote*

$$(5.5) \quad f(t) \stackrel{\text{def}}{=} \|u^d(t)\|_{\dot{B}_{p, r}^{-1 + \frac{d}{p} + \frac{1}{r}}}^{2r} \quad \text{and} \quad \Pi_\lambda \stackrel{\text{def}}{=} \Pi \exp \left\{ -\lambda \int_0^t f(t') dt' \right\} \quad \text{for} \quad \lambda > 0,$$

and similar notation for  $u_\lambda$ . Then (5.4) has a unique solution  $\nabla\Pi \in \widetilde{L}_T^1(\dot{B}_{p,r}^{-1+\frac{d}{p}-\varepsilon}) \cap \widetilde{L}_T^1(\dot{B}_{p,r}^{-1+\frac{d}{p}})$  which decays to zero when  $|x| \rightarrow \infty$  so that for all  $t \in [0, T]$ , there holds

$$(5.6) \quad \begin{aligned} \|\nabla\Pi_\lambda\|_{\widetilde{L}_t^1(\dot{B}_{p,r}^{-1+\frac{d}{p}-s})} &\leq \frac{C}{1-C\|a\|_{L_t^\infty(L^\infty) \cap \widetilde{L}_t^\infty(B_{q,\infty}^{\frac{d}{q}})}} \left\{ \|u_\lambda^h\|_{\widetilde{L}_t^1(\dot{B}_{p,r}^{-1+\frac{d}{p}-s})}^{1-\frac{1}{2r}} \|u_\lambda^h\|_{\widetilde{L}_{t,f}^1(\dot{B}_{p,r}^{-1+\frac{d}{p}-s})}^{\frac{1}{2r}} \right. \\ &\quad \left. (\mu\|a\|_{L_t^\infty(L^\infty) \cap \widetilde{L}_t^\infty(B_{q,\infty}^{\frac{d}{q}})} + \|u^h\|_{\widetilde{L}_t^\infty(\dot{B}_{p,r}^{-1+\frac{d}{p}})}) \|u_\lambda^h\|_{\widetilde{L}_t^1(\dot{B}_{p,r}^{-1+\frac{d}{p}-s})} \right. \\ &\quad \left. + \mu\|a\|_{L_t^\infty(L^\infty) \cap \widetilde{L}_t^\infty(B_{q,\infty}^{\frac{d}{q}})} \|u^d\|_{\widetilde{L}_t^1(\dot{B}_{p,r}^{-1+\frac{d}{p}-s})} \right\} \quad \text{for } s = 0, \varepsilon, \end{aligned}$$

provided that  $C\|a\|_{L_T^\infty(L^\infty) \cap \widetilde{L}_T^\infty(B_{q,\infty}^{\frac{d}{q}})} \leq \frac{1}{2}$ , and where the norms of  $\widetilde{L}_t^1(\dot{B}_{p,r}^{-1+\frac{d}{p}-s})$  and  $\widetilde{L}_{t,f}^1(\dot{B}_{p,r}^{-1+\frac{d}{p}-s})$  are given by Definitions A.2 and A.3 respectively.

The proof of this proposition will mainly be based on the following lemmas:

**Lemma 5.1.** For  $s < \frac{d}{p}$ , one has

$$\|gh\|_{\widetilde{L}_t^1(\dot{B}_{p,r}^{\frac{d}{p}-s})} \lesssim \|g\|_{\widetilde{L}_t^\infty(\dot{B}_{p,r}^{-1+\frac{d}{p}})} \|h\|_{\widetilde{L}_t^1(\dot{B}_{p,r}^{-1+\frac{d}{p}-s})} + \|g\|_{\widetilde{L}_t^1(\dot{B}_{p,r}^{-1+\frac{d}{p}})} \|h\|_{\widetilde{L}_t^\infty(\dot{B}_{p,r}^{-1+\frac{d}{p}-s})}.$$

*Proof.* Applying Bony's decomposition ([7]) gives

$$\dot{\Delta}_j(gh) = \sum_{j' \geq j - N_0} \dot{\Delta}_j(\dot{S}_{j'}g\dot{\Delta}_{j'}h + \dot{\Delta}_{j'}g\dot{S}_{j'+1}h),$$

from which and Lemma A.1, we infer

$$\begin{aligned} \|\dot{\Delta}_j(gh)\|_{L_t^1(L^p)} &\lesssim \sum_{j' \geq j - N_0} (\|\dot{S}_{j'}g\|_{L_t^\infty(L^\infty)} \|\dot{\Delta}_{j'}h\|_{L_t^1(L^p)} + \|\dot{\Delta}_{j'}g\|_{L_t^1(L^p)} \|\dot{S}_{j'+1}h\|_{L_t^\infty(L^\infty)}) \\ &\lesssim \sum_{j' \geq j - N_0} c_{j',r} 2^{-j'(\frac{d}{p}-s)} (\|g\|_{\widetilde{L}_t^\infty(\dot{B}_{p,r}^{-1+\frac{d}{p}})} \|h\|_{\widetilde{L}_t^1(\dot{B}_{p,r}^{-1+\frac{d}{p}-s})} \\ &\quad + \|g\|_{\widetilde{L}_t^1(\dot{B}_{p,r}^{-1+\frac{d}{p}})} \|h\|_{\widetilde{L}_t^\infty(\dot{B}_{p,r}^{-1+\frac{d}{p}-s})}) \\ &\lesssim c_{j,r} 2^{-j(\frac{d}{p}-s)} (\|g\|_{\widetilde{L}_t^\infty(\dot{B}_{p,r}^{-1+\frac{d}{p}})} \|h\|_{\widetilde{L}_t^1(\dot{B}_{p,r}^{-1+\frac{d}{p}-s})} + \|g\|_{\widetilde{L}_t^1(\dot{B}_{p,r}^{-1+\frac{d}{p}})} \|h\|_{\widetilde{L}_t^\infty(\dot{B}_{p,r}^{-1+\frac{d}{p}-s})}), \end{aligned}$$

where and in what follows, we always denote  $(c_{j,r})_{j \in \mathbb{Z}}$  as a generic element of  $\ell^r(\mathbb{Z})$  so that  $\sum_{j \in \mathbb{Z}} c_{j,r}^r = 1$ . Then by virtue of Definition A.2, we complete the proof of the lemma.  $\square$

**Lemma 5.2.** Let  $1 \leq p < 2d$ ,  $s \in (-\frac{1}{r}, \frac{2d}{p}-1)$  and  $f$  be given by (5.5). Then under the assumptions of Proposition 5.1, one has

$$(5.7) \quad \|u^d \nabla u^h\|_{\widetilde{L}_t^1(\dot{B}_{p,r}^{-1+\frac{d}{p}-s})} \lesssim \|u^h\|_{\widetilde{L}_t^1(\dot{B}_{p,r}^{-1+\frac{d}{p}-s})}^{1-\frac{1}{2r}} \|u^h\|_{\widetilde{L}_{t,f}^1(\dot{B}_{p,r}^{-1+\frac{d}{p}-s})}^{\frac{1}{2r}},$$

and

$$(5.8) \quad \|u^d \nabla u^h\|_{\widetilde{L}_t^1(\dot{B}_{p,r}^{-1+\frac{d}{p}-s})} \lesssim \|u^h\|_{\widetilde{L}_t^\infty(\dot{B}_{p,r}^{-1+\frac{d}{p}})} \|u^d\|_{\widetilde{L}_t^1(\dot{B}_{p,r}^{-1+\frac{d}{p}-s})} + \|u^d\|_{\widetilde{L}_t^\infty(\dot{B}_{p,r}^{-1+\frac{d}{p}-s})} \|u^h\|_{\widetilde{L}_t^1(\dot{B}_{p,r}^{-1+\frac{d}{p}})}.$$

*Proof.* We first get, by applying Bony's decomposition (A.5), that

$$(5.9) \quad u^d \nabla u^h = \dot{T}_{u^d} \nabla u^h + \dot{T}_{\nabla u^h} u^d + \dot{R}(u^d, \nabla u^h).$$

Applying Lemma A.1 gives

$$\begin{aligned} \|\dot{\Delta}_j(T_{u^d}\nabla u^h)(t')\|_{L^p} &\lesssim \sum_{|j'-j|\leq 5} 2^{j'} \|\dot{S}_{j'-1}u^d(t')\|_{L^\infty} \|\dot{\Delta}_{j'}u^h(t')\|_{L^p} \\ &\lesssim \sum_{|j'-j|\leq 5} 2^{j'(2-\frac{1}{r})} \|u^d(t')\|_{\dot{B}_{p,r}^{-1+\frac{d}{p}+\frac{1}{r}}} \|\dot{\Delta}_{j'}u^h(t')\|_{L^p}, \end{aligned}$$

integrating the above inequality over  $[0, t]$  and using Definition A.3, one has

$$\begin{aligned} &\|\dot{\Delta}_j(\dot{T}_{u^d}\nabla u^h)\|_{L_t^1(L^p)} \\ &\lesssim \sum_{|j'-j|\leq 5} 2^{j'(2-\frac{1}{r})} \left\{ \int_0^t \|u^d(t')\|_{\dot{B}_{p,r}^{-1+\frac{d}{p}+\frac{1}{r}}}^{2r} \|\dot{\Delta}_{j'}u^h(t')\|_{L^p} dt' \right\}^{\frac{1}{2r}} \|\dot{\Delta}_{j'}u^h(t')\|_{L_t^1(L^p)}^{1-\frac{1}{2r}} \\ &\lesssim c_{j,r} 2^{-j(-1+\frac{d}{p}-s)} \|u^h\|_{\tilde{L}_t^1(\dot{B}_{p,r}^{1+\frac{d}{p}-s})}^{1-\frac{1}{2r}} \|u^h\|_{\tilde{L}_{t,f}^1(\dot{B}_{p,r}^{-1+\frac{d}{p}-s})}^{\frac{1}{2r}}. \end{aligned}$$

It follows from the same line that

$$\begin{aligned} \|\dot{\Delta}_j(\dot{T}_{\nabla u^h}u^d)(t')\|_{L^p} &\lesssim \sum_{|j'-j|\leq 5} \|\dot{S}_{j'-1}\nabla u^h(t')\|_{L^\infty} \|\dot{\Delta}_{j'}u^d(t')\|_{L^p} \\ &\lesssim 2^{-j(-1+\frac{d}{p}+\frac{1}{r})} \sum_{\ell\leq j+4} 2^{\ell(1+\frac{d}{p})} \|u^d(t')\|_{\dot{B}_{p,r}^{-1+\frac{d}{p}+\frac{1}{r}}} \|\dot{\Delta}_\ell u^h(t')\|_{L^p}, \end{aligned}$$

from which and  $s > -\frac{1}{r}$ , we infer

$$\begin{aligned} &\|\dot{\Delta}_j(\dot{T}_{\nabla u^h}u^d)\|_{L_t^1(L^p)} \\ &\lesssim 2^{-j(-1+\frac{d}{p}+\frac{1}{r})} \sum_{\ell\leq j+4} 2^{\ell(1+\frac{d}{p})} \left\{ \int_0^t \|u^d(t')\|_{\dot{B}_{p,r}^{-1+\frac{d}{p}+\frac{1}{r}}}^{2r} \|\dot{\Delta}_\ell u^h(t')\|_{L^p} dt' \right\}^{\frac{1}{2r}} \|\dot{\Delta}_\ell u^h(t')\|_{L_t^1(L^p)}^{1-\frac{1}{2r}} \\ &\lesssim c_{j,r} 2^{-j(-1+\frac{d}{p}-s)} \|u^h\|_{\tilde{L}_t^1(\dot{B}_{p,r}^{1+\frac{d}{p}-s})}^{1-\frac{1}{2r}} \|u^h\|_{\tilde{L}_{t,f}^1(\dot{B}_{p,r}^{-1+\frac{d}{p}-s})}^{\frac{1}{2r}}. \end{aligned}$$

Finally, we deal with the remaining term in (5.9). Firstly for  $2 \leq p < 2d$ , as  $s < \frac{2d}{p} - 1$ , we get, by applying Lemma A.1, that

$$\begin{aligned} \|\dot{\Delta}_j(\dot{R}(u^d, \nabla u^h))\|_{L_t^1(L^p)} &\lesssim 2^{j\frac{d}{p}} \sum_{j'\geq j-N_0} 2^{j'} \int_0^t \|\tilde{\Delta}_{j'}u^d(t')\|_{L^p} \|\dot{\Delta}_{j'}u^h(t')\|_{L^p} dt' \\ &\lesssim 2^{j\frac{d}{p}} \sum_{j'\geq j-N_0} 2^{-j'(-2+\frac{d}{p}+\frac{1}{r})} \int_0^t \|u^d(t')\|_{\dot{B}_{p,r}^{-1+\frac{d}{p}+\frac{1}{r}}} \|\dot{\Delta}_{j'}u^h(t')\|_{L^p} dt' \\ &\lesssim c_{j,r} 2^{-j(-1+\frac{d}{p}-s)} \|u^h\|_{\tilde{L}_t^1(\dot{B}_{p,r}^{1+\frac{d}{p}-s})}^{1-\frac{1}{2r}} \|u^h\|_{\tilde{L}_{t,f}^1(\dot{B}_{p,r}^{-1+\frac{d}{p}-s})}^{\frac{1}{2r}}. \end{aligned}$$

For  $1 \leq p < 2$ , we get, by applying Lemma A.1 once again, that

$$\begin{aligned}
& \|\dot{\Delta}_j(\dot{R}(u^d, \nabla u^h))\|_{L_t^1(L^p)} \\
& \lesssim 2^{jd(1-\frac{1}{p})} \sum_{j' \geq j-N_0} 2^{j'} \int_0^t \|\tilde{\Delta}_{j'} u^d(t')\|_{L^{\frac{p}{p-1}}} \|\dot{\Delta}_{j'} u^h(t')\|_{L^p} dt' \\
(5.10) \quad & \lesssim 2^{jd(1-\frac{1}{p})} \sum_{j' \geq j-N_0} 2^{j'(2+\frac{d}{p}-d-\frac{1}{r})} \int_0^t \|u^d(t')\|_{\dot{B}_{p,r}^{-1+\frac{d}{p}+\frac{1}{r}}} \|\dot{\Delta}_{j'} u^h(t')\|_{L^p} dt' \\
& \lesssim c_{j,r} 2^{-j(-1+\frac{d}{p}-s)} \|u^h\|_{\tilde{L}_t^1(\dot{B}_{p,r}^{1+\frac{d}{p}-s})}^{1-\frac{1}{2r}} \|u^h\|_{\tilde{L}_{t,f}^1(\dot{B}_{p,r}^{-1+\frac{d}{p}-s})}^{\frac{1}{2r}}.
\end{aligned}$$

Whence thanks to (5.9), we finish the proof of (5.7).

On the other hand, it is easy to observe that

$$\begin{aligned}
& \|\dot{\Delta}_j(\dot{T}_{u^d} \nabla u^h + \dot{T}_{\nabla u^h} u^d)\|_{L_t^1(L^p)} \\
& \lesssim \sum_{|j'-j| \leq 5} (\|\dot{S}_{j'-1} u^d\|_{L_t^\infty(L^\infty)} \|\dot{\Delta}_{j'} \nabla u^h\|_{L_t^1(L^p)} + \|\dot{S}_{j'-1} \nabla u^h\|_{L_t^\infty(L^\infty)} \|\dot{\Delta}_{j'} u^d\|_{L_t^1(L^p)}) \\
& \lesssim c_{j,r} 2^{-j(-1+\frac{d}{p}-s)} (\|u^h\|_{\tilde{L}_t^\infty(\dot{B}_{p,r}^{-1+\frac{d}{p}})} \|u^d\|_{\tilde{L}_t^1(\dot{B}_{p,r}^{1+\frac{d}{p}-s})} + \|u^d\|_{\tilde{L}_t^\infty(\dot{B}_{p,r}^{-1+\frac{d}{p}-s})} \|u^h\|_{\tilde{L}_t^1(\dot{B}_{p,r}^{1+\frac{d}{p}-s})}).
\end{aligned}$$

Whereas as  $s < \frac{2d}{p} - 1$ , for  $2 \leq p < 2d$ , we get, by applying Lemma A.1, that

$$\begin{aligned}
\|\dot{\Delta}_j(\dot{R}(u^d, \nabla u^h))\|_{L_t^1(L^p)} & \lesssim 2^{j\frac{d}{p}} \sum_{j' \geq j-N_0} 2^{j'} \|\dot{\Delta}_{j'} u^d\|_{L_t^1(L^p)} \|\tilde{\Delta}_{j'} u^h\|_{L_t^\infty(L^p)} \\
& \lesssim c_{j,r} 2^{-j(-1+\frac{d}{p}-s)} \|u^h\|_{\tilde{L}_t^\infty(\dot{B}_{p,r}^{-1+\frac{d}{p}})} \|u^d\|_{\tilde{L}_t^1(\dot{B}_{p,r}^{1+\frac{d}{p}-s})}.
\end{aligned}$$

Along the same line to the proof of (5.10), we can prove the same estimate holds for  $1 \leq p < 2$ . This proves (5.8) and Lemma 5.2.  $\square$

**Lemma 5.3.** *Let  $1 < q \leq p < 2d$ ,  $s < \frac{d}{p} + \frac{d}{q} - 1$  and  $g \in \tilde{L}_T^1(\dot{B}_{p,r}^{-1+\frac{d}{p}-s})$ . Then under the assumptions of Proposition 5.1, one has*

$$\|ag\|_{\tilde{L}_t^1(\dot{B}_{p,r}^{-1+\frac{d}{p}-s})} \lesssim \|a\|_{L_t^\infty(L^\infty) \cap \tilde{L}_t^\infty(B_{q,\infty}^{\frac{d}{q}})} \|g\|_{\tilde{L}_t^1(\dot{B}_{p,r}^{-1+\frac{d}{p}-s})}.$$

*Proof.* Again thanks to Bony's decomposition (A.5), we have

$$(5.11) \quad ag = \dot{T}_a g + \dot{T}_g a + \dot{R}(a, g).$$

Applying Lemma A.1 gives

$$\begin{aligned}
\|\dot{\Delta}_j(\dot{T}_a g)\|_{L_t^1(L^p)} & \lesssim \sum_{|j'-j| \leq 5} \|\dot{S}_{j'-1} a\|_{L_t^\infty(L^\infty)} \|\dot{\Delta}_{j'} g\|_{L_t^1(L^p)} \\
& \lesssim c_{j,r} 2^{-j(-1+\frac{d}{p}-s)} \|a\|_{L_t^\infty(L^\infty)} \|g\|_{\tilde{L}_t^1(\dot{B}_{p,r}^{-1+\frac{d}{p}-s})}.
\end{aligned}$$

While as  $p \geq q$ , applying Lemma A.1 once again gives rise to

$$\begin{aligned}
\|\dot{\Delta}_j(\dot{T}_g a)\|_{L_t^1(L^p)} & \lesssim \sum_{|j'-j| \leq 5} \|\dot{S}_{j'-1} g\|_{L_t^1(L^\infty)} \|\dot{\Delta}_{j'} a\|_{L_t^\infty(L^p)} \\
& \lesssim c_{j,r} 2^{-j(-1+\frac{d}{p}-s)} \|a\|_{\tilde{L}_t^\infty(B_{q,\infty}^{\frac{d}{q}})} \|g\|_{\tilde{L}_t^1(\dot{B}_{p,r}^{-1+\frac{d}{p}-s})}.
\end{aligned}$$



Finally as  $s < \frac{d}{p} + \frac{d}{q} - 1$ , for  $\frac{1}{p} + \frac{1}{q} \leq 1$ , we get, by applying Lemma A.1, that

$$\begin{aligned} \|\dot{\Delta}_j(\dot{R}(a, g))\|_{L_t^1(L^p)} &\lesssim 2^{j\frac{d}{q}} \sum_{j' \geq j - N_0} \|\tilde{\Delta}_{j'} a\|_{L_t^\infty(L^q)} \|\dot{\Delta}_{j'} g\|_{L_t^1(L^p)} \\ &\lesssim 2^{j\frac{d}{q}} \sum_{j' \geq j - N_0} c_{j', r} 2^{j'(1 - \frac{d}{p} - \frac{d}{q} + s)} \|a\|_{\tilde{L}_t^\infty(B_{q, \infty}^{\frac{d}{q}})} \|g\|_{\tilde{L}_t^1(\dot{B}_{p, r}^{-1 + \frac{d}{p} - s})} \\ &\lesssim c_{j, r} 2^{-j(-1 + \frac{d}{p} - s)} \|a\|_{\tilde{L}_t^\infty(B_{q, \infty}^{\frac{d}{q}})} \|g\|_{\tilde{L}_t^1(\dot{B}_{p, r}^{-1 + \frac{d}{p} - s})}. \end{aligned}$$

For the case when  $\frac{1}{p} + \frac{1}{q} > 1$ , we have

$$\begin{aligned} \|\dot{\Delta}_j(\dot{R}(a, g))\|_{L_t^1(L^p)} &\lesssim 2^{jd(1 - \frac{1}{p})} \sum_{j' \geq j - N_0} \|\tilde{\Delta}_{j'} a\|_{L_t^\infty(L^{\frac{p}{p-1}})} \|\dot{\Delta}_{j'} g\|_{L_t^1(L^p)} \\ &\lesssim 2^{jd(1 - \frac{1}{p})} \sum_{j' \geq j - N_0} c_{j', r} 2^{j'(1 - d + s)} \|a\|_{\tilde{L}_t^\infty(B_{q, \infty}^{\frac{d}{q}})} \|g\|_{\tilde{L}_t^1(\dot{B}_{p, r}^{-1 + \frac{d}{p} - s})} \\ &\lesssim c_{j, r} 2^{-j(-1 + \frac{d}{p} - s)} \|a\|_{\tilde{L}_t^\infty(B_{q, \infty}^{\frac{d}{q}})} \|g\|_{\tilde{L}_t^1(\dot{B}_{p, r}^{-1 + \frac{d}{p} - s})}. \end{aligned}$$

Whence thanks to (5.11), we prove Lemma 5.3.  $\square$

We now turn to the proof of Proposition 5.1.

**Proof of Proposition 5.1.** Again as both the proof of the existence and uniqueness of solutions to (5.4) essentially follows from the estimates (5.6) for some appropriate approximate solutions. For the sake of simplicity, we just prove (5.6) for smooth enough solutions of (5.4). Indeed thanks to (5.4) and (5.5), we write

$$\begin{aligned} \nabla \Pi_\lambda &= \nabla(-\Delta)^{-1} [\operatorname{div}(a \nabla \Pi_\lambda) + \operatorname{div}_h \operatorname{div}_h(u^h \otimes u_\lambda^h) + 2 \operatorname{div}_h(u^d \partial_d u_\lambda^h) - 3 \partial_d(u^d \operatorname{div}_h u_\lambda^h) \\ &\quad - \operatorname{div}_h(u^h \operatorname{div}_h u_\lambda^h) - \mu \operatorname{div}_h(a \Delta u_\lambda^h) - \mu \partial_d(a \Delta u_\lambda^d)]. \end{aligned}$$

Acting  $\dot{\Delta}_j$  to the above equation and using Lemma A.1 leads to

$$(5.12) \quad \begin{aligned} \|\dot{\Delta}_j(\nabla \Pi_\lambda)\|_{L_t^1(L^p)} &\lesssim \|\dot{\Delta}_j(a \nabla \Pi_\lambda)\|_{L_t^1(L^p)} + 2^j \|\dot{\Delta}_j(u^h \otimes u_\lambda^h)\|_{L_t^1(L^p)} + \|\dot{\Delta}_j(u^h \operatorname{div}_h u_\lambda^h)\|_{L_t^1(L^p)} \\ &\quad + \|\dot{\Delta}_j(u^d \nabla u_\lambda^h)\|_{L_t^1(L^p)} + \mu \|\dot{\Delta}_j(a \Delta u_\lambda^h)\|_{L_t^1(L^p)} + \mu \|\dot{\Delta}_j(a \Delta u_\lambda^d)\|_{L_t^1(L^p)}, \end{aligned}$$

from which, and Lemma 5.1 to Lemma 5.3, we deduce that for  $s = 0$  and  $\varepsilon$

$$\begin{aligned} \|\dot{\Delta}_j(\nabla \Pi_\lambda)\|_{L_t^1(L^p)} &\lesssim c_{j, r} 2^{-j(-1 + \frac{d}{p} - s)} \left\{ \|a\|_{L_t^\infty(L^\infty) \cap \tilde{L}_t^\infty(B_{q, \infty}^{\frac{d}{q}})} (\mu \|u_\lambda^h\|_{\tilde{L}_t^1(\dot{B}_{p, r}^{1 + \frac{d}{p} - s})} + \mu \|u^d\|_{\tilde{L}_t^1(\dot{B}_{p, r}^{1 + \frac{d}{p} - s})}) \right. \\ &\quad \left. + \|\nabla \Pi_\lambda\|_{\tilde{L}_t^1(\dot{B}_{p, r}^{-1 + \frac{d}{p} - s})} + \|u^h\|_{\tilde{L}_t^\infty(\dot{B}_{p, r}^{-1 + \frac{d}{p}})} \|u_\lambda^h\|_{\tilde{L}_t^1(\dot{B}_{p, r}^{1 + \frac{d}{p} - s})} + \|u_\lambda^h\|_{\tilde{L}_t^1(\dot{B}_{p, r}^{1 + \frac{d}{p} - s})}^{1 - \frac{1}{2r}} \|u^h\|_{\tilde{L}_t^1(\dot{B}_{p, r}^{-1 + \frac{d}{p} - s})}^{\frac{1}{2r}} \right\}. \end{aligned}$$

Therefore (5.6) follows as long as  $C \|a\|_{L_T^\infty(L^\infty) \cap \tilde{L}_T^\infty(B_{q, \infty}^{\frac{d}{q}})} \leq \frac{1}{2}$ . This finishes the proof of Proposition 5.1.  $\blacksquare$

To deal with the estimate of  $u^d$ , we also need the following proposition:

**Proposition 5.2.** *Under the assumptions of Proposition 5.1, one has for  $s = 0$  and  $\varepsilon$*

$$(5.13) \quad \begin{aligned} \|\nabla \Pi\|_{\tilde{L}_t^1(\dot{B}_{p, r}^{-1 + \frac{d}{p} - s})} &\leq \frac{C}{1 - C \|a\|_{L_t^\infty(L^\infty) \cap \tilde{L}_t^\infty(B_{q, \infty}^{\frac{d}{q}})}} \left\{ \|u^h\|_{\tilde{L}_t^1(\dot{B}_{p, r}^{1 + \frac{d}{p}})} \|u^d\|_{\tilde{L}_t^\infty(\dot{B}_{p, r}^{-1 + \frac{d}{p} - s})} \right. \\ &\quad \left. + (\mu \|a\|_{L_t^\infty(L^\infty) \cap \tilde{L}_t^\infty(B_{q, \infty}^{\frac{d}{q}})} + \|u^h\|_{\tilde{L}_t^\infty(\dot{B}_{p, r}^{-1 + \frac{d}{p}})}) (\|u^h\|_{\tilde{L}_t^1(\dot{B}_{p, r}^{1 + \frac{d}{p} - s})} + \|u^d\|_{\tilde{L}_t^1(\dot{B}_{p, r}^{1 + \frac{d}{p} - s})}) \right\} \end{aligned}$$

for  $t \leq T$  provided that  $C\|a\|_{L_T^\infty(L^\infty) \cap \tilde{L}_T^\infty(B_{q,\infty}^{\frac{d}{q}})} \leq \frac{1}{2}$ .

*Proof.* The proof of this proposition exactly follows from that of Proposition 5.1. In fact, taking  $\lambda = 0$  in (5.12), and then applying Lemma 5.1, Lemma 5.3, we arrive at

$$\begin{aligned} \|\nabla \Pi\|_{\tilde{L}_t^1(\dot{B}_{p,r}^{-1+\frac{d}{p}-s})} &\leq C \left\{ \|a\|_{L_t^\infty(L^\infty) \cap \tilde{L}_t^\infty(B_{q,\infty}^{\frac{d}{q}})} \|\nabla \Pi\|_{\tilde{L}_t^1(\dot{B}_{p,r}^{-1+\frac{d}{p}-s})} + \|u^h\|_{\tilde{L}_t^1(\dot{B}_{p,r}^{1+\frac{d}{p}})} \|u^d\|_{\tilde{L}_t^\infty(\dot{B}_{p,r}^{-1+\frac{d}{p}-s})} \right. \\ &\quad \left. + (\mu \|a\|_{L_t^\infty(L^\infty) \cap \tilde{L}_t^\infty(B_{q,\infty}^{\frac{d}{q}})} + \|u^h\|_{\tilde{L}_t^\infty(\dot{B}_{p,r}^{-1+\frac{d}{p}})}) \|u\|_{\tilde{L}_t^1(\dot{B}_{p,r}^{1+\frac{d}{p}-s})} \right\} \end{aligned}$$

for  $t \leq T$ , from which and the fact that  $C\|a\|_{L_T^\infty(L^\infty) \cap \tilde{L}_T^\infty(B_{q,\infty}^{\frac{d}{q}})} \leq \frac{1}{2}$ , we conclude the proof of (5.13).  $\square$

**5.3. The estimate of  $u^h$ .** We first deduce from the transport equation of (1.2) that

$$(5.14) \quad \|a\|_{L_t^\infty(L^\infty)} \leq \|a_0\|_{L^\infty} \quad \text{for } t < T^*.$$

Let  $f(t)$ ,  $u_\lambda$ ,  $\Pi_\lambda$  be given by (5.5). Then thanks to (1.2), we write

$$\partial_t u_\lambda^h + \lambda f(t) u_\lambda^h - \mu \Delta u_\lambda^h = -u \cdot \nabla u_\lambda^h - (1+a) \nabla_h \Pi_\lambda + \mu a \Delta u_\lambda^h.$$

Applying the operator  $\dot{\Delta}_j$  to the above equation and then taking the  $L^2$  inner product of the resulting equation with  $|\dot{\Delta}_j u_\lambda^h|^{p-2} \dot{\Delta}_j u_\lambda^h$  (in the case when  $p \in (1, 2)$ , we need to make some modification as that in [9]), we obtain

$$\begin{aligned} &\frac{1}{p} \frac{d}{dt} \|\dot{\Delta}_j u_\lambda^h(t)\|_{L^p}^p + \lambda f(t) \|\dot{\Delta}_j u_\lambda^h(t)\|_{L^p}^p - \mu \int_{\mathbb{R}^d} \Delta \dot{\Delta}_j u_\lambda^h \mid |\dot{\Delta}_j u_\lambda^h|^{p-2} \dot{\Delta}_j u_\lambda^h dx \\ &= - \int_{\mathbb{R}^d} (\dot{\Delta}_j(u \cdot \nabla u_\lambda^h) + \dot{\Delta}_j((1+a) \nabla_h \Pi_\lambda) - \mu \dot{\Delta}_j(a \Delta u_\lambda^h)) \mid |\dot{\Delta}_j u_\lambda^h|^{p-2} \dot{\Delta}_j u_\lambda^h dx. \end{aligned}$$

However thanks to [9] (see also [28]), there exists a positive constant  $\bar{c}$  so that

$$- \int_{\mathbb{R}^d} \Delta \dot{\Delta}_j u_\lambda^h \mid |\dot{\Delta}_j u_\lambda^h|^{p-2} \dot{\Delta}_j u_\lambda^h dx \geq \bar{c} 2^{2j} \|\dot{\Delta}_j u_\lambda^h\|_{L^p}^p,$$

whence a similar argument as that in [9] gives rise to

$$(5.15) \quad \begin{aligned} &\|\dot{\Delta}_j u_\lambda^h\|_{L_t^\infty(L^p)} + \lambda \int_0^t f(t') \|\dot{\Delta}_j u_\lambda^h(t')\|_{L^p} dt' + \bar{c} \mu 2^{2j} \|\dot{\Delta}_j u_\lambda^h\|_{L_t^1(L^p)} \\ &\leq \|\dot{\Delta}_j(u \cdot \nabla u_\lambda^h)\|_{L_t^1(L^p)} + \|\dot{\Delta}_j((1+a) \nabla_h \Pi_\lambda)\|_{L_t^1(L^p)} + \mu \|\dot{\Delta}_j(a \Delta u_\lambda^h)\|_{L_t^1(L^p)}. \end{aligned}$$

For  $s = 0$  and  $\varepsilon$ , applying Lemma 5.2 gives

$$\begin{aligned} \|\dot{\Delta}_j(u \cdot \nabla u_\lambda^h)\|_{L_t^1(L^p)} &\leq \|\dot{\Delta}_j(u^h \cdot \nabla_h u_\lambda^h)\|_{L_t^1(L^p)} + \|\dot{\Delta}_j(u^d \partial_d u_\lambda^h)\|_{L_t^1(L^p)} \\ &\leq C c_{j,r} 2^{-j(-1+\frac{d}{p}-s)} (\|u^h\|_{\tilde{L}_t^\infty(\dot{B}_{p,r}^{-1+\frac{d}{p}})} \|u_\lambda^h\|_{\tilde{L}_t^1(\dot{B}_{p,r}^{1+\frac{d}{p}-s})} \\ &\quad + \|u_\lambda^h\|_{\tilde{L}_t^1(\dot{B}_{p,r}^{1+\frac{d}{p}-s})}^{1-\frac{1}{2r}} \|u_\lambda^h\|_{\tilde{L}_{t,f}^1(\dot{B}_{p,r}^{-1+\frac{d}{p}-s})}^{\frac{1}{2r}}). \end{aligned}$$

And applying Lemma 5.3 and (5.14) yields

$$\|\dot{\Delta}_j(a \Delta u_\lambda^h)\|_{L_t^1(L^p)} \leq C c_{j,r} 2^{-j(-1+\frac{d}{p}-s)} (\|a_0\|_{L^\infty} + \|a\|_{\tilde{L}_t^\infty(B_{q,\infty}^{\frac{d}{q}})}) \|u_\lambda^h\|_{\tilde{L}_t^1(\dot{B}_{p,r}^{1+\frac{d}{p}-s})},$$

and

$$\|\dot{\Delta}_j((1+a) \nabla_h \Pi_\lambda)\|_{L_t^1(L^p)} \leq C c_{j,r} 2^{-j(-1+\frac{d}{p}-s)} (1 + \|a_0\|_{L^\infty} + \|a\|_{\tilde{L}_t^\infty(B_{q,\infty}^{\frac{d}{q}})}) \|\nabla_h \Pi_\lambda\|_{\tilde{L}_t^1(\dot{B}_{p,r}^{-1+\frac{d}{p}-s})}.$$

Now let  $c_1$  be a small enough positive constant, which will be determined later on, we define  $\mathfrak{T}$  by

$$(5.16) \quad \mathfrak{T} \stackrel{\text{def}}{=} \max \left\{ t \in [0, T^*) : \|u^h\|_{\tilde{L}_t^\infty(\dot{B}_{p,r}^{-1+\frac{d}{p}-\varepsilon})} + \|u^h\|_{\tilde{L}_t^\infty(\dot{B}_{p,r}^{-1+\frac{d}{p}})} + \mu(\|a_0\|_{L^\infty} + \|a\|_{\tilde{L}_t^\infty(B_{q,\infty}^{\frac{d}{q}})} + \|u^h\|_{\tilde{L}_t^1(\dot{B}_{p,r}^{1+\frac{d}{p}-\varepsilon})} + \|u^h\|_{\tilde{L}_t^1(\dot{B}_{p,r}^{1+\frac{d}{p}})}) \leq c_1 \mu \right\}.$$

In particular, (5.16) implies that  $\|a_0\|_{L^\infty} + \|a\|_{\tilde{L}_t^\infty(B_{q,\infty}^{\frac{d}{q}})} \leq c_1$  for  $t \leq \mathfrak{T}$ . Taking  $c_1$  so small that  $Cc_1 \leq \frac{1}{2}$ , (5.6) and (5.14) ensures that

$$\begin{aligned} \|\dot{\Delta}_j((1+a)\nabla_h \Pi \lambda)\|_{L_t^1(L^p)} &\leq Cc_{j,r} 2^{-j(-1+\frac{d}{p}-s)} \left\{ \|u_\lambda^h\|_{\tilde{L}_t^1(\dot{B}_{p,r}^{1+\frac{d}{p}-s})}^{1-\frac{1}{2r}} \|u_\lambda^h\|_{\tilde{L}_t^1(\dot{B}_{p,1}^{-1+\frac{d}{p}-s})}^{\frac{1}{2r}} \right. \\ &\quad + [\mu(\|a_0\|_{L^\infty} + \|a\|_{\tilde{L}_t^\infty(B_{q,\infty}^{\frac{d}{q}})}) + \|u^h\|_{\tilde{L}_t^\infty(\dot{B}_{p,r}^{-1+\frac{d}{p}})}] \|u_\lambda^h\|_{\tilde{L}_t^1(\dot{B}_{p,r}^{1+\frac{d}{p}-s})} \\ &\quad \left. + \mu(\|a_0\|_{L^\infty} + \|a\|_{\tilde{L}_t^\infty(B_{q,\infty}^{\frac{d}{q}})}) \|u^d\|_{\tilde{L}_t^1(\dot{B}_{p,r}^{1+\frac{d}{p}-s})} \right\} \quad \text{for } t \leq \mathfrak{T}. \end{aligned}$$

Substituting the above estimates into (5.15), we obtain for  $s = 0$  and  $\varepsilon$  that

$$\begin{aligned} &\|u_\lambda^h\|_{\tilde{L}_t^\infty(\dot{B}_{p,r}^{-1+\frac{d}{p}-s})} + \lambda \|u_\lambda^h\|_{\tilde{L}_t^1(\dot{B}_{p,r}^{-1+\frac{d}{p}-s})} + \bar{c}\mu \|u_\lambda^h\|_{\tilde{L}_t^1(\dot{B}_{p,r}^{1+\frac{d}{p}-s})} \\ &\leq \|u_0^h\|_{\dot{B}_{p,r}^{-1+\frac{d}{p}-s}} + \frac{\bar{c}\mu}{4} \|u_\lambda^h\|_{\tilde{L}_t^1(\dot{B}_{p,r}^{1+\frac{d}{p}-s})} + C \left\{ \frac{1}{\mu^{2r-1}} \|u_\lambda^h\|_{\tilde{L}_t^1(\dot{B}_{p,r}^{-1+\frac{d}{p}-s})} \right. \\ &\quad + [\mu(\|a_0\|_{L^\infty} + \|a\|_{\tilde{L}_t^\infty(B_{q,\infty}^{\frac{d}{q}})}) + \|u^h\|_{\tilde{L}_t^\infty(\dot{B}_{p,r}^{-1+\frac{d}{p}})}] \|u_\lambda^h\|_{\tilde{L}_t^1(\dot{B}_{p,r}^{1+\frac{d}{p}-s})} \\ &\quad \left. + \mu(\|a_0\|_{L^\infty} + \|a\|_{\tilde{L}_t^\infty(B_{q,\infty}^{\frac{d}{q}})}) \|u^d\|_{\tilde{L}_t^1(\dot{B}_{p,r}^{1+\frac{d}{p}-s})} \right\} \quad \text{for } t \leq \mathfrak{T}. \end{aligned}$$

Taking  $\lambda = \frac{2C}{\mu^{2r-1}}$  in the above inequality and thanks to (5.16), we get

$$(5.17) \quad \begin{aligned} &\|u_\lambda^h\|_{\tilde{L}_t^\infty(\dot{B}_{p,r}^{-1+\frac{d}{p}-s})} + \frac{\bar{c}}{2}\mu \|u_\lambda^h\|_{\tilde{L}_t^1(\dot{B}_{p,r}^{1+\frac{d}{p}-s})} \\ &\leq \|u_0^h\|_{\dot{B}_{p,r}^{-1+\frac{d}{p}-s}} + C\mu(\|a_0\|_{L^\infty} + \|a\|_{\tilde{L}_t^\infty(B_{q,\infty}^{\frac{d}{q}})}) \|u^d\|_{\tilde{L}_t^1(\dot{B}_{p,r}^{1+\frac{d}{p}-s})} \quad \text{for } t \leq \mathfrak{T}, \end{aligned}$$

provided that  $c_1$  in (5.16) is so small that  $Cc_1 \leq \frac{\bar{c}}{4}$ .

On the other hand, it is easy to observe from (5.5) and Definition A.3 that

$$\begin{aligned} &(\|u^h\|_{\tilde{L}_t^\infty(\dot{B}_{p,r}^{-1+\frac{d}{p}-s})} + \frac{\bar{c}}{2}\mu \|u^h\|_{\tilde{L}_t^1(\dot{B}_{p,r}^{1+\frac{d}{p}-s})}) \exp \left\{ - \int_0^t \lambda f(t') dt' \right\} \\ &\leq \|u_\lambda^h\|_{\tilde{L}_t^\infty(\dot{B}_{p,r}^{-1+\frac{d}{p}-s})} + \frac{\bar{c}}{2}\mu \|u_\lambda^h\|_{\tilde{L}_t^1(\dot{B}_{p,r}^{1+\frac{d}{p}-s})}, \end{aligned}$$

from which, and (5.3), (5.16), (5.17), we infer that for  $s = 0, \varepsilon$ , and  $t \leq \mathfrak{T}$ ,

$$(5.18) \quad \begin{aligned} &\|u^h\|_{\tilde{L}_t^\infty(\dot{B}_{p,r}^{-1+\frac{d}{p}-s})} + \frac{\mu}{2} (\|a\|_{\tilde{L}_t^\infty(B_{q,\infty}^{\frac{d}{q}+\frac{\varepsilon}{2}})} + \|u^h\|_{\tilde{L}_t^1(\dot{B}_{p,r}^{1+\frac{d}{p}-s})}) \\ &\leq C [\mu(\|a_0\|_{L^\infty} + \|a_0\|_{B_{q,\infty}^{\frac{d}{q}+\varepsilon}}) + \|u_0^h\|_{\dot{B}_{p,r}^{-1+\frac{d}{p}-s}}] \\ &\quad \times \exp \left\{ C_\varepsilon (\|u^d\|_{\tilde{L}_t^1(\dot{B}_{p,r}^{1+\frac{d}{p}-\varepsilon})} + \|u^d\|_{\tilde{L}_t^1(\dot{B}_{p,r}^{1+\frac{d}{p}})}) + \int_0^t \frac{1}{\mu^{2r-1}} \|u^d(t')\|_{\dot{B}_{p,r}^{-1+\frac{d}{p}+\frac{1}{r}}}^{2r} dt' \right\}. \end{aligned}$$

5.4. **The estimate of  $u^d$ .** By virtue of (1.2), we get, by a similar derivation of (5.15), that

$$(5.19) \quad \begin{aligned} & \|\dot{\Delta}_j u^d\|_{L_t^\infty(L^p)} + \bar{c}\mu 2^{2j} \|\dot{\Delta}_j u^d\|_{L_t^1(L^p)} \leq \|\dot{\Delta}_j u_0^d\|_{L^p} \\ & + C \left( \|\dot{\Delta}_j(u \cdot \nabla u^d)\|_{L_t^1(L^p)} + \|\dot{\Delta}_j((1+a)\partial_d \Pi)\|_{L_t^1(L^p)} + \mu \|\dot{\Delta}_j(a\Delta u^d)\|_{L_t^1(L^p)} \right). \end{aligned}$$

Applying Lemma 5.1 and Lemma 5.2 gives for  $s = 0$  and  $\varepsilon$  that

$$\begin{aligned} & \|\Delta_j(u \cdot \nabla u^d)\|_{L_t^1(L^p)} \lesssim 2^j \|\Delta_j(u^h u^d)\|_{L_t^1(L^p)} + \|\Delta_j(u^d \operatorname{div}_h u^h)\|_{L_t^1(L^p)} \\ & \lesssim c_{j,r} 2^{-j(-1+\frac{d}{p}-s)} \left( \|u^h\|_{\tilde{L}_t^\infty(\dot{B}_{p,r}^{-1+\frac{d}{p}})} \|u^d\|_{\tilde{L}_t^1(\dot{B}_{p,r}^{1+\frac{d}{p}-s})} + \|u^h\|_{\tilde{L}_t^1(\dot{B}_{p,r}^{1+\frac{d}{p}})} \|u^d\|_{\tilde{L}_t^\infty(\dot{B}_{p,r}^{-1+\frac{d}{p}-s})} \right). \end{aligned}$$

Whereas thanks to (5.16) and (5.14), we get, by applying Lemma 5.3 and Proposition 5.2, that

$$\begin{aligned} & \|\dot{\Delta}_j((1+a)\partial_d \Pi)\|_{L_t^1(L^p)} \leq C c_{j,r} 2^{-j(-1+\frac{d}{p}-s)} (1 + \|a_0\|_{L^\infty} + \|a\|_{\tilde{L}_t^\infty(B_{q,\infty}^{\frac{d}{q}})}) \|\partial_d \Pi\|_{\tilde{L}_t^1(\dot{B}_{p,r}^{-1+\frac{d}{p}-s})} \\ & \leq C c_{j,r} 2^{-j(-1+\frac{d}{p}-s)} \left\{ \|u^h\|_{\tilde{L}_t^1(\dot{B}_{p,r}^{1+\frac{d}{p}})} \|u^d\|_{\tilde{L}_t^\infty(\dot{B}_{p,r}^{-1+\frac{d}{p}-s})} \right. \\ & \quad \left. + [\mu(\|a_0\|_{L^\infty} + \|a\|_{\tilde{L}_t^\infty(B_{q,\infty}^{\frac{d}{q}})}) + \|u^h\|_{\tilde{L}_t^\infty(\dot{B}_{p,r}^{-1+\frac{d}{p}})}] (\|u^h\|_{\tilde{L}_t^1(\dot{B}_{p,r}^{1+\frac{d}{p}-s})} + \|u^d\|_{\tilde{L}_t^1(\dot{B}_{p,r}^{1+\frac{d}{p}-s})}) \right\}, \end{aligned}$$

for  $t \leq \mathfrak{T}$ . Substituting the above estimates into (5.19) leads to

$$(5.20) \quad \begin{aligned} & \|u^d\|_{\tilde{L}_t^\infty(\dot{B}_{p,r}^{-1+\frac{d}{p}-s})} + \bar{c}\mu \|u^d\|_{\tilde{L}_t^1(\dot{B}_{p,r}^{1+\frac{d}{p}-s})} \leq \|u_0^d\|_{\dot{B}_{p,r}^{-1+\frac{d}{p}-s}} + C \left\{ \|u^h\|_{\tilde{L}_t^1(\dot{B}_{p,r}^{1+\frac{d}{p}})} \|u^d\|_{\tilde{L}_t^\infty(\dot{B}_{p,r}^{-1+\frac{d}{p}-s})} \right. \\ & \quad \left. + [\mu(\|a_0\|_{L^\infty} + \|a\|_{\tilde{L}_t^\infty(B_{q,\infty}^{\frac{d}{q}})}) + \|u^h\|_{\tilde{L}_t^\infty(\dot{B}_{p,r}^{-1+\frac{d}{p}})}] (\|u^h\|_{\tilde{L}_t^1(\dot{B}_{p,r}^{1+\frac{d}{p}-s})} + \|u^d\|_{\tilde{L}_t^1(\dot{B}_{p,r}^{1+\frac{d}{p}-s})}) \right\}, \end{aligned}$$

for  $t \leq \mathfrak{T}$  and  $s = 0, \varepsilon$ .

In particular, if we take  $c_1 \leq \min\{\frac{1}{2C}, \frac{\bar{c}}{4C}\}$  in (5.16), we deduce from (5.20) that

$$(5.21) \quad \|u^d\|_{\tilde{L}_t^\infty(\dot{B}_{p,r}^{-1+\frac{d}{p}-s})} + \mu \bar{c} \|u^d\|_{\tilde{L}_t^1(\dot{B}_{p,r}^{1+\frac{d}{p}-s})} \leq 2 \|u_0^d\|_{\dot{B}_{p,r}^{-1+\frac{d}{p}-s}} + c_2 \mu$$

for  $t \leq \mathfrak{T}$  and  $s = 0, \varepsilon$ .

5.5. **The proof of Theorem 1.2.** According to the arguments at the beginning of this section, we only need to prove that  $\mathfrak{T} = \infty$  under the assumption of (1.13). Otherwise, if  $\mathfrak{T} < T^* < \infty$ , we first deduce from (5.21) that

$$(5.22) \quad \begin{aligned} & \|u^d\|_{L_t^{2r}(\dot{B}_{p,r}^{-1+\frac{d}{p}+\frac{1}{r}})} \leq C \|u^d\|_{\tilde{L}_t^{2r}(\dot{B}_{p,r}^{-1+\frac{d}{p}+\frac{1}{r}})} \leq C \|u^d\|_{\tilde{L}_t^1(\dot{B}_{p,r}^{1+\frac{d}{p}})}^{\frac{1}{2r}} \|u^d\|_{\tilde{L}_t^\infty(\dot{B}_{p,r}^{-1+\frac{d}{p}})}^{1-\frac{1}{2r}} \\ & \leq C \mu^{-\frac{1}{2r}} (\|u_0^d\|_{\dot{B}_{p,r}^{-1+\frac{d}{p}}} + c_2 \mu) \quad \text{for } t \leq \mathfrak{T}. \end{aligned}$$

Substituting (5.21) and (5.22) into (5.18) gives rise to

$$\begin{aligned} & \|u^h\|_{\tilde{L}_t^\infty(\dot{B}_{p,r}^{-1+\frac{d}{p}-\varepsilon})} + \|u^h\|_{\tilde{L}_t^\infty(\dot{B}_{p,r}^{-1+\frac{d}{p}})} + \mu (\|a\|_{\tilde{L}_t^\infty(B_{q,\infty}^{\frac{d}{q}+\frac{\varepsilon}{2}})} + \|u^h\|_{\tilde{L}_t^1(\dot{B}_{p,r}^{1+\frac{d}{p}-\varepsilon})} + \|u^h\|_{\tilde{L}_t^1(\dot{B}_{p,r}^{1+\frac{d}{p}})}) \\ & \leq \mathfrak{C}_1 (\mu \|a_0\|_{L^\infty \cap B_{q,\infty}^{\frac{d}{q}+\varepsilon}} + \|u_0^h\|_{\dot{B}_{p,r}^{-1+\frac{d}{p}-\varepsilon} \cap \dot{B}_{p,r}^{-1+\frac{d}{p}}}) \exp \left\{ \frac{\mathfrak{C}_2}{\mu^{2r}} (\|u_0^d\|_{\dot{B}_{p,r}^{-1+\frac{d}{p}-\varepsilon} \cap \dot{B}_{p,r}^{-1+\frac{d}{p}}})^{2r} \right\} \end{aligned}$$

for  $t \leq \mathfrak{T}$  and some positive constants  $\mathfrak{C}_1, \mathfrak{C}_2$  which depends on  $\bar{c}, c_1$  and  $\varepsilon$ . In particular, if we take  $C_{r,\varepsilon}$  large enough and  $c_0$  sufficiently small in (1.13), the above inequality implies that for  $\delta$  given by (1.13)

$$\begin{aligned} & \|u^h\|_{\tilde{L}_t^\infty(\dot{B}_{p,r}^{-1+\frac{d}{p}-\varepsilon})} + \|u^h\|_{\tilde{L}_t^\infty(\dot{B}_{p,r}^{-1+\frac{d}{p}})} + \mu (\|a_0\|_{L^\infty} + \|a\|_{\tilde{L}_t^\infty(B_{q,\infty}^{\frac{d}{q}+\frac{\varepsilon}{2}})}) \\ & \quad + \|u^h\|_{\tilde{L}_t^1(\dot{B}_{p,r}^{1+\frac{d}{p}-\varepsilon})} + \|u^h\|_{\tilde{L}_t^1(\dot{B}_{p,r}^{1+\frac{d}{p}})} \leq C \delta \leq \frac{c_1}{2} \mu \quad \text{for } t \leq \mathfrak{T}, \end{aligned}$$

which contradicts with (5.16). Whence we conclude that  $\mathfrak{T} = T^* = \infty$ , and there holds (1.14). This completes the proof of Theorem 1.2.  $\blacksquare$

#### APPENDIX A. LITTLEWOOD-PALEY ANALYSIS

The proof of Theorem 1.2 requires Littlewood-Paley decomposition. Let us briefly explain how it may be built in the case  $x \in \mathbb{R}^d$  (see e.g. [6]). Let  $\varphi$  be a smooth function supported in the ring  $\mathcal{C} \stackrel{\text{def}}{=} \{\xi \in \mathbb{R}^d, \frac{3}{4} \leq |\xi| \leq \frac{8}{3}\}$  and  $\chi$  be a smooth function supported in the ball  $\mathcal{B} \stackrel{\text{def}}{=} \{\xi \in \mathbb{R}^d, |\xi| \leq \frac{4}{3}\}$  such that

$$\chi(\xi) + \sum_{q \geq 0} \varphi(2^{-q}\xi) = 1 \quad \text{for all } \xi \in \mathbb{R}^d \quad \text{and} \quad \sum_{j \in \mathbb{Z}} \varphi(2^{-j}\xi) = 1 \quad \text{for } \xi \neq 0.$$

Then for  $u \in \mathcal{S}'(\mathbb{R}^d)$ , we set

$$(A.1) \quad \begin{aligned} \forall j \in \mathbb{Z}, \quad \dot{\Delta}_j u &\stackrel{\text{def}}{=} \varphi(2^{-j}D)u & \text{and} & \quad \dot{S}_j u \stackrel{\text{def}}{=} \chi(2^{-j}D)u, \\ \forall q \in \mathbb{N}, \quad \Delta_q u &= \varphi(2^{-q}D)u & \text{and} & \quad S_q u = \sum_{\ell \leq q-1} \Delta_\ell u, \quad \Delta_{-1} u = \chi(D)u. \end{aligned}$$

We have the formal decomposition

$$(A.2) \quad u = \sum_{j \in \mathbb{Z}} \dot{\Delta}_j u, \quad \forall u \in \mathcal{S}'(\mathbb{R}^d)/\mathcal{P}[\mathbb{R}^d] \quad \text{and} \quad u = \sum_{q \geq -1} \Delta_q u, \quad \forall u \in \mathcal{S}'(\mathbb{R}^d),$$

where  $\mathcal{P}[\mathbb{R}^d]$  is the set of polynomials. Moreover, the Littlewood-Paley decomposition satisfies the property of almost orthogonality:

$$(A.3) \quad \dot{\Delta}_j \dot{\Delta}_\ell u \equiv 0 \quad \text{if } |j - \ell| \geq 2 \quad \text{and} \quad \dot{\Delta}_j (\dot{S}_{\ell-1} u \dot{\Delta}_\ell u) \equiv 0 \quad \text{if } |j - \ell| \geq 5.$$

We recall now the definition of homogeneous Besov spaces and Bernstein type inequalities from [6].

**Definition A.1.** Let  $(p, r) \in [1, +\infty]^2$ ,  $s \in \mathbb{R}$  and  $u \in \mathcal{S}'_h(\mathbb{R}^3)$ , which means that  $u \in \mathcal{S}'(\mathbb{R}^3)$  and  $\lim_{j \rightarrow -\infty} \|\dot{S}_j u\|_{L^\infty} = 0$ , we define

$$\|u\|_{B_{p,r}^s} \stackrel{\text{def}}{=} \left( 2^{qs} \|\Delta_q u\|_{L^p} \right)_{\ell^r} \quad \text{and} \quad \dot{B}_{p,r}^s(\mathbb{R}^d) \stackrel{\text{def}}{=} \{u \in \mathcal{S}'_h(\mathbb{R}^d) \mid \|u\|_{\dot{B}_{p,r}^s} < \infty\}.$$

Inhomogeneous Besov spaces  $B_{p,r}^s(\mathbb{R}^d)$  can be defined in a similar way so that

$$\|u\|_{B_{p,r}^s} \stackrel{\text{def}}{=} \left( 2^{qs} \|\Delta_q u\|_{L^p} \right)_{\ell^r(\mathbb{N})}.$$

**Lemma A.1.** Let  $\mathcal{B}$  be a ball and  $\mathcal{C}$  a ring of  $\mathbb{R}^d$ . A constant  $C$  exists so that for any positive real number  $\delta$ , any non negative integer  $k$ , any smooth homogeneous function  $\sigma$  of degree  $m$ , and any couple of real numbers  $(a, b)$  with  $b \geq a \geq 1$ , there hold

$$(A.4) \quad \begin{aligned} \text{Supp } \hat{u} \subset \delta\mathcal{B} &\Rightarrow \sup_{|\alpha|=k} \|\partial^\alpha u\|_{L^b} \leq C^{k+1} \delta^{k+3(\frac{1}{a}-\frac{1}{b})} \|u\|_{L^a}, \\ \text{Supp } \hat{u} \subset \delta\mathcal{C} &\Rightarrow C^{-1-k} \delta^k \|u\|_{L^a} \leq \sup_{|\alpha|=k} \|\partial^\alpha u\|_{L^a} \leq C^{1+k} \delta^k \|u\|_{L^a}, \\ \text{Supp } \hat{u} \subset \delta\mathcal{C} &\Rightarrow \|\sigma(D)u\|_{L^b} \leq C_{\sigma,m} \delta^{m+3(\frac{1}{a}-\frac{1}{b})} \|u\|_{L^a}. \end{aligned}$$

We shall frequently use Bony's decomposition from [7] in the homogeneous context:

$$(A.5) \quad uv = \dot{T}_u v + \dot{\mathcal{R}}(u, v) = \dot{T}_u v + \dot{T}_v u + \dot{R}(u, v),$$

where

$$\begin{aligned} \dot{T}_u v &\stackrel{\text{def}}{=} \sum_{j \in \mathbb{Z}} \dot{S}_{j-1} u \dot{\Delta}_j v, & \dot{\mathcal{R}}(u, v) &\stackrel{\text{def}}{=} \sum_{j \in \mathbb{Z}} \dot{\Delta}_j u \dot{S}_{j+2} v, \\ \dot{R}(u, v) &\stackrel{\text{def}}{=} \sum_{j \in \mathbb{Z}} \dot{\Delta}_j u \tilde{\Delta}_j v & \text{with } \tilde{\Delta}_j v &\stackrel{\text{def}}{=} \sum_{|j'-j| \leq 1} \dot{\Delta}_{j'} v. \end{aligned}$$

In order to obtain a better description of the regularizing effect of the transport-diffusion equation, we need to use Chemin-Lerner type spaces  $\tilde{L}_T^\lambda(\dot{B}_{p,r}^s(\mathbb{R}^d))$ .

**Definition A.2.** Let  $(r, \lambda, p) \in [1, +\infty]^3$  and  $T \in (0, +\infty]$ . We define  $\tilde{L}_T^\lambda(\dot{B}_{p,r}^s(\mathbb{R}^d))$  as the completion of  $C([0, T]; \mathcal{S}(\mathbb{R}^d))$  by the norm

$$\|u\|_{\tilde{L}_T^\lambda(\dot{B}_{p,r}^s)} \stackrel{\text{def}}{=} \left( \sum_{j \in \mathbb{Z}} 2^{jrs} \left( \int_0^T \|\dot{\Delta}_j u(t)\|_{L^p}^\lambda dt \right)^{\frac{r}{\lambda}} \right)^{\frac{1}{r}} < \infty.$$

with the usual change if  $r = \infty$ . For short, we just denote this space by  $\tilde{L}_T^\lambda(\dot{B}_{p,r}^s)$ .

As one can not use Gronwall's inequality in the Chemin-Lerner type spaces, we [26, 27] introduced weighted Chemin-Lerner norm in the context of anisotropic Besov spaces. To prove 1.2, we need the following version of weighted Chemin-Lerner in the context of isentropic Besov spaces:

**Definition A.3.** Let  $(r, p) \in [1, +\infty]^2$  and  $T \in (0, +\infty]$ . Let  $0 \leq f(t) \in L^1(0, T)$ . We define the norm  $\tilde{L}_{T,f}^1(\dot{B}_{p,r}^s)$  as

$$\|u\|_{\tilde{L}_{T,f}^1(\dot{B}_{p,r}^s)} \stackrel{\text{def}}{=} \left( \sum_{j \in \mathbb{Z}} 2^{jrs} \|f(t) \dot{\Delta}_j u(t)\|_{L_T^1(L^p)}^r \right)^{\frac{1}{r}}.$$

**Acknowledgments.** Part of this work was done when M. Paicu was visiting Morningside Center of the Chinese Academy of Sciences in the Spring of 2012. We appreciate the hospitality of MCM and the financial support from the Chinese Academy of Sciences. P. Zhang is partially supported by NSF of China under Grant 10421101 and 10931007, the one hundred talents' plan from Chinese Academy of Sciences under Grant GJHZ200829 and innovation grant from National Center for Mathematics and Interdisciplinary Sciences.

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