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GLOBAL SOLUTIONS TO 2-D INHOMOGENEOUS NAVIER-STOKES SYSTEM WITH GENERAL VELOCITY

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Abstract. In this paper, we are concerned with the global wellposedness of 2-D density-dependent incompressible Navier-Stokes equations (1.1) with variable viscosity, in a critical functional framework which is invariant by the scaling of the equations and under a non-linear smallness condition on fluctuation of the initial density which has to be doubly exponential small compared with the size of the initial velocity. In the second part of the paper, we apply our methods combined with the techniques in [10] to prove the global existence of solutions to (1.1) with piecewise constant initial density which has small jump at the interface and is away from vacuum. In particular, this latter result removes the smallness condition for the initial velocity in a corresponding theorem of [10].

Keywords: Inhomogeneous Navier-Stokes Equations, Littlewood-Paley Theory, Wellposedness

AMS Subject Classification (2000): 35Q30, 76D03

1. Introduction

In this paper, we consider the global existence of solutions to the following 2-D incompressible inhomogeneous Navier-Stokes equations with initial data in the scaling invariant Besov spaces and without size restriction for the initial velocity:

$$\begin{cases} \partial_t \rho + u\cdot\nabla\rho = 0, \qquad (t,x)\in\mathbb{R}^+\times\mathbb{R}^2,\\ \partial_t(\rho u) + \mathrm{div}(\rho u\otimes u) - \mathrm{div}(\mu(\rho)\mathcal{M}) + \nabla\Pi = 0,\\ \mathrm{div} u = 0, \end{cases} \tag{1.1}$$

where $\rho, u = (u_1, u_2)$ stand for the density and velocity of the fluid respectively, $\mathcal{M} = \frac{1}{2}(\partial_i u_j + \partial_j u_i)$, Π is a scalar pressure function, and the viscosity coefficient $\mu(\rho)$ is a smooth, positive function on $[0,\infty)$. Such system describes a fluid which is obtained by mixing two immiscible fluids that are incompressible and that have different densities. It may also describe a fluid containing a melted substance.

There is a wide literatures devoted to the mathematical study of the incompressible Navier-Stokes equations in the homogeneous case (where the density is a constant) or in the more physical case of inhomogeneous fluids. In the homogeneous case, the celebrated theorem of Leray [20] on the existence of global weak solutions with finite energy in any space dimension is now a classical result. Moreover, in the two dimensional space, it is also classical that the Leray weak solution is in fact a global strong solution. In dimension larger than two, the Fujita-Kato theorem [13] allows to construct global strong solutions under a smallness condition on the initial data comparing with the viscosity of the fluid. To obtain those types of results in the inhomogeneous case are the topics of many recent works dedicated to this system [1, 2, 3, 4, 5, 8, 9, 10, 11, 12, 15, 16, 21]... Our main goal in this paper is to provide a global wellposedness result for the density-dependent incompressible Navier-Stokes equations with variable viscosity, in a critical functional framework which is invariant by the scaling of the equations and under a non-linear smallness condition on fluctuation of the initial density which has to be doubly exponential small compared with the size of the initial velocity. In the second part of the paper, we apply our methods combined with the

Date: 9/9/2012.

techniques in [10] to prove the global existence of the solution to (1.1) with piecewise constant initial density, which is away from vacuum and has small jumps at the interface. This latter problem is of a great interest from physical point of view as it represents the case of a immiscible mixture of fluids with different densities. We give in this manner a partial response of a question raised by Lions [21] concerning the propagation of the regularity of the boundary to a "density-patches".

We briefly describe in this paragraph some of the classical results for the inhomogeneous Navier-Stokes system. When the viscous coefficient equals some positive constant, Ladyženskaja and Solonnikov [19] first established the unique resolvability of (1.1) in a bounded domain Ω with homogeneous Dirichlet boundary condition for u; similar result was obtained by Danchin [9] in $\mathbb{R}^d$ with initial data in the almost critical Sobolev spaces; Simon [26] proved the global existence of weak solutions. In general, the global existence of weak solutions with finite energy to (1.1) with variable viscosity was proved by Lions in [21] (see also the references therein, and the monograph [5]). Yet the regularity and uniqueness of such weak solutions is a big open question in the field of mathematical fluid mechanics, even in two space dimensions when the viscosity depends on the density. Except under the assumptions:

$$\rho_0 \in L^\infty(\mathbb{T}^2), \quad \inf_{c>0}\Big\|\frac{\mu(\rho_0)}{c}-1\Big\|_{L^\infty(\mathbb{T}^2)} \le \epsilon, \quad \text{and} \quad u_0 \in H^1(\mathbb{T}^2),$$

Desjardins [12] proved that Lions weak solution (ρ, u) satisfies $u \in L^\infty((0,T); H^1(\mathbb{T}^2))$ and $\rho \in L^\infty((0,T)\times\mathbb{T}^2)$ for any $T<\infty$. Moreover, with additional assumption on the initial density, he could also prove that $u \in L^2((0,\tau); H^2(\mathbb{T}^2))$ for some short time τ. To understand this problem further, the third author to this paper proved the global wellposedness to a modified 2-D model problem of (1.1), which coincides with the 2-D inhomogeneous Navier-Stokes system with constant viscosity, with general initial data in [27]. Gui and Zhang [16] proved the global wellposedness of (1.1) with initial data satisfying $\|\rho_0-1\|_{H^{s+1}}$ being sufficiently small and $u_0 \in H^s(\mathbb{R}^2)\cap \dot{H}^{-\varepsilon}(\mathbb{R}^2)$ for some $s>2$ and $0<\varepsilon<1$. However, the exact size of $\|\rho_0-1\|_{H^{s+1}}$ was not given in [16].

Very recently, Danchin and Mucha [11] proved that: given initial density ρ_0 in $L^\infty(\Omega)$ with a positive lower bound and initial velocity $u_0 \in H^2(\Omega)$ for some bounded smooth domain of $\mathbb{R}^d$, the system (1.1) with constant viscosity has a unique local solution. Furthermore, with the initial density being close enough to some positive constant, for any initial velocity in two space dimensions, and sufficiently small velocity in three space dimensions, they also proved its global wellposedness. We remark that the Lagrangian formulation for the describing the flow plays a key role in the analysis in [11]. To prove the 2-D global result, they first applied energy method to obtain $L^\infty(\mathbb{R}^+; H^1(\Omega))$ estimate for the velocity field u and $L^2(\mathbb{R}^+; L^2(\Omega))$ estimate for $\partial_t u$. Then the authors employed the classical maximal $L^p_T(L^q)$ estimate for the linear Stokes operator to obtain the second order space derivative estimate for the velocity. Notice that when $\mu(\rho)$ depends on ρ, and the initial density is sufficiently close to some positive constant in $L^\infty(\mathbb{R}^2)$, one can recover $L^\infty(\mathbb{R}^+; H^1(\mathbb{R}^2))$ estimate for the velocity u and $L^2(\mathbb{R}^+; L^2(\mathbb{R}^2))$ estimate for $\partial_t u$ by using Desjardins' techniques from [12]. Yet we do not know then how to recover the second order space derivatives of the velocity. Therefore, I think it is a very challenging problem to prove Danchin and Mucha [11] type results for (1.1) with variable viscosity.

When the density ρ is away from zero, we denote by $a \stackrel{\text{def}}{=} \frac{1}{\rho}-1$ and $\widetilde{\mu}(a) \stackrel{\text{def}}{=} \mu(\frac{1}{1+a})$, then the system (1.1) can be equivalently reformulated as

$$\begin{cases} \partial_t a + u\cdot\nabla a = 0, \qquad (t,x)\in\mathbb{R}^+\times\mathbb{R}^2, \\ \partial_t u + u\cdot\nabla u + (1+a)(\nabla\Pi - \operatorname{div}(\widetilde{\mu}(a)\mathcal{M}) = 0, \\ \operatorname{div} u = 0. \end{cases} \tag{1.2}$$

Notice that just as the classical Navier-Stokes system (which corresponds to $a=0$ in (1.2)), the inhomogeneous Navier-Stokes system (1.2) also has a scaling. In fact, if (a,u) solves (1.2) with initial data (a_0, u_0), then for any $\ell > 0$,

$$(a,u)_\ell \stackrel{\text{def}}{=} (a(\ell^2\cdot,\ell\cdot),\ell u(\ell^2\cdot,\ell\cdot)) \quad \text{and} \quad (a_0,u_0)_\ell \stackrel{\text{def}}{=} (a_0(\ell\cdot),\ell u_0(\ell\cdot)) \tag{1.3}$$

$(a,u)_\ell$ is also a solution of (1.2) with initial data $(a_0,u_0)_\ell$.

It is easy to check that the norm of $B^{\frac{d}{p}}_{p,1}(\mathbb{R}^d)\times B^{-1+\frac{d}{p}}_{p,1}(\mathbb{R}^d)$ is scaling invariant under the scaling transformation $(a_0,u_0)_\ell$ given by (1.3). In [1], Abidi proved in general space dimension d that: if $1<p<2d$, $0<\underline{\mu}<\mu(\rho)$, given $a_0\in B^{\frac{d}{p}}_{p,1}(\mathbb{R}^d)$ and $u_0\in B^{-1+\frac{d}{p}}_{p,1}(\mathbb{R}^d)$, (1.2) has a global solution provided that $\|a_0\|_{B^{\frac{d}{p}}_{p,1}}+\|u_0\|_{B^{-1+\frac{d}{p}}_{p,1}}\leq c_0$ for some sufficiently small c_0. Moreover, this solution is unique if $1<p\leq d$. This result generalized the corresponding results in [8, 9] and was improved by Abidi and Paicu in [2] with $a_0\in B^{\frac{d}{q}}_{q,1}(\mathbb{R}^d)$ and $u_0\in B^{-1+\frac{d}{p}}_{p,1}(\mathbb{R}^d)$ for p,q satisfying some technical assumptions. Abidi, Gui and Zhang removed the smallness condition for a_0 in [3, 4]. Notice that the main feature of the density space is to be a multiplier on the velocity space and this allows to define the nonlinear terms in the system (1.2). Recently, Danchin and Mucha [10] proved a more general wellposedness result of (1.1) with $\mu(\rho)=\mu>0$ by considering very rough densities in some multiplier spaces on the Besov spaces $B^{-1+\frac{d}{p}}_{p,1}(\mathbb{R}^d)$ for $1<p<2d$, which in particular completes the uniqueness result in [1] for $p\in(d,2d)$ in the case when $\mu(\rho)=\mu>0$.

On the other hand, motivated by [17, 23, 28] concerning the global wellposedness of 3-D incompressible anisotropic Navier-Stokes system with the third component of the initial velocity field being large, we [24] proved that: given $a_0\in B^{\frac{3}{q}}_{q,1}(\mathbb{R}^3)$ and $u_0=(u_0^h,u_0^3)\in B^{-1+\frac{3}{p}}_{p,1}(\mathbb{R}^3)$ for $1<q\leq p<6$ and $\frac{1}{q}-\frac{1}{p}\leq\frac{1}{3}$, (1.2) with $\widetilde{\mu}(a)=\mu>0$ has a unique global solution as long as

$$\big(\mu\|a_0\|_{B^{\frac{3}{q}}_{q,1}}+\|u_0^h\|_{B^{-1+\frac{3}{p}}_{p,1}}\big)\exp\Big\{C_0\|u_0^3\|^2_{B^{-1+\frac{3}{p}}_{p,1}}\Big/\mu^2\Big\}\leq c_0\mu$$

for some sufficiently small c_0. We emphasize that our proof in [24] used in a fundamental way the algebraical structure of (1.2), namely $\operatorname{div} u=0$.

The first object of this paper is to improve the global wellposedness result in [16] so that given initial data in the scaling invariant Besov spaces, for any initial velocity, (1.2) has a global solution provided that the fluctuation of the initial density is sufficiently small, furthermore, its explicit dependence on the initial velocity will be given here.

Theorem 1.1. *Let $1<q\leq p<4$, and $\frac{1}{q}-\frac{1}{p}\leq\frac{1}{2}$. Let $a_0\in B^{\frac{2}{q}}_{q,1}(\mathbb{R}^2)$ and $u_0\in B^{-1+\frac{2}{p}}_{p,1}(\mathbb{R}^2)$ be a solenoidal vector field. Then there exist positive constants c_0 and C_0, which depend on $\|\widetilde{\mu}'\|_{L^\infty(-1,1)}$, such that if*

$$\eta\stackrel{\text{def}}{=}\|a_0\|_{B^{\frac{2}{q}}_{q,1}}\exp\Big\{C_0\big(1+\widetilde{\mu}^2(0)\big)\exp\big(\frac{C_0}{\widetilde{\mu}^2(0)}\|u_0\|^2_{B^{-1+\frac{2}{p}}_{p,1}}\big)\Big\}\leq\frac{c_0\widetilde{\mu}(0)}{1+\widetilde{\mu}(0)}, \tag{1.4}$$

(1.2) has a global solution $a\in\mathcal{C}([0,\infty);B^{\frac{2}{q}}_{q,1}(\mathbb{R}^2))\cap\widetilde{L}^\infty(\mathbb{R}^+;B^{\frac{2}{q}}_{q,1}(\mathbb{R}^2))$ and $u\in\mathcal{C}([0,\infty);B^{-1+\frac{2}{p}}_{p,1}(\mathbb{R}^2))\cap\widetilde{L}^\infty(\mathbb{R}^+;B^{-1+\frac{2}{p}}_{p,1}(\mathbb{R}^2))\cap L^1(\mathbb{R}^+;B^{1+\frac{2}{p}}_{p,1}(\mathbb{R}^2))$. If $\frac{1}{p}+\frac{1}{q}\geq 1$, this solution is unique.

Remark 1.1.
- *The definitions of the functional spaces will be presented in Subsection 2.1.*
- *We remark that compared with the finite energy solutions constructed in [11], our solution here is not of finite energy and belongs to the critical spaces related to (1.2). While for the classical 2-D Navier-Stokes system, large infinite energy solution was proved by Gallagher and Planchon [14] and Germain [15].*

It turns out that we can apply the main idea to prove Theorem 1.1 combined with the techniques in [10] to remove the smallness condition for initial velocity in [10] when the space dimension equals to 2. Toward this, we first recall the definition of multiplier spaces to Besov spaces from [22]:

Definition 1.1. *The multiplier space $\mathcal{M}(B^s_{p,1}(\mathbb{R}^d))$ of $B^s_{p,1}(\mathbb{R}^d)$ is the set of distributions f such that $f\psi \in B^s_{p,1}(\mathbb{R}^d)$ whenever $\psi \in B^s_{p,1}(\mathbb{R}^d)$. We endow this space with the norm*

$$\|f\|_{\mathcal{M}(B^s_{p,1})} \stackrel{def}{=} \sup_{\psi\in B^s_{p,1}(\mathbb{R}^d):\ \|\psi\|_{B^s_{p,1}}\leq 1} \|\psi f\|_{B^s_{p,1}}.$$

In [10], Danchin and Mucha proved the following global wellposedness for (1.1) with constant viscosity:

Theorem 1.2. *(Theorem 1 and Theorem 3 of* [10]*) Let $p \in [1, 2d)$ and u_0 be a divergence-free vector field in $B^{-1+\frac{d}{p}}_{p,1}(\mathbb{R}^d)$. Assume that the initial density ρ_0 belongs to the multiplier space $\mathcal{M}(B^{-1+\frac{d}{p}}_{p,1}(\mathbb{R}^d))$. There exists a constant c depending only on d such that if*

$$\|\rho_0 - 1\|_{\mathcal{M}(B^{-1+\frac{d}{p}}_{p,1})} + \mu^{-1}\|u_0\|_{B^{-1+\frac{d}{p}}_{p,1}} \leq c,$$

system (1.1) *with $\mu(\rho) = \mu > 0$ has a unique global solution (ρ, u) with $\rho \in L^\infty(\mathbb{R}^+; \mathcal{M}(B^{-1+\frac{d}{p}}_{p,1}(\mathbb{R}^d)))$ and $u \in \mathcal{C}([0,\infty); B^{-1+\frac{d}{p}}_{p,1}(\mathbb{R}^d)) \cap L^1(\mathbb{R}^+; B^{1+\frac{d}{p}}_{p,1}(\mathbb{R}^d))$.*

Motivated by [10] and the proof of Theorem 1.1, here we consider similar global wellposedness of (1.2), which does not require any smallness assumption for u_0.

Theorem 1.3. *Let $p \in (2, 4)$, $a_0 \in \mathcal{M}(B^{-1+\frac{2}{p}}_{p,1}(\mathbb{R}^2))$ with $\widetilde{\mu}(a_0) \in \mathcal{M}(B^{\frac{2}{p}}_{p,1}(\mathbb{R}^2))$, and $u_0 \in B^{-1+\frac{2}{p}}_{p,1}(\mathbb{R}^2)$. Then there exist positive constants c_0 and C_0 such that if*

$$\begin{aligned}(1.5)\qquad &\big(\mu\|a_0\|_{\mathcal{M}(B^{-1+\frac{2}{p}}_{p,1})} + \|\widetilde{\mu}(a_0) - \widetilde{\mu}(0)\|_{\mathcal{M}(B^{\frac{2}{p}}_{p,1})}\big)\\ &\qquad\times \exp\Big\{C_0\big(1+\widetilde{\mu}^2(0)\big)\exp\big(\frac{C_0}{\widetilde{\mu}^2(0)}\|u_0\|^2_{B^{-1+\frac{2}{p}}_{p,1}}\big)\Big\} \leq c_0\widetilde{\mu}(0),\end{aligned}$$

(1.2) *has a unique global solution (a, u) with $a \in L^\infty(\mathbb{R}^+; \mathcal{M}(B^{-1+\frac{2}{p}}_{p,1}(\mathbb{R}^2))$ and $u \in \mathcal{C}([0,\infty); B^{-1+\frac{2}{p}}_{p,1}(\mathbb{R}^2)) \cap L^1(\mathbb{R}^+; B^{1+\frac{2}{p}}_{p,1}(\mathbb{R}^2))$.*

Notice from [10] that: let Ω_0 be a bounded C^1 domain of $\mathbb{R}^2$ and $\rho_0 = 1 + \sigma\chi_{\Omega_0}$ for some sufficiently small constant σ, $a_0 = \frac{1}{\rho_0} - 1 = -\frac{\sigma}{1+\sigma}\chi_{\Omega_0}$ and $\widetilde{\mu}(a_0) - \widetilde{\mu}(0) = (\mu(1+\sigma) - \mu(1))\chi_{\Omega_0}$ belong to $\mathcal{M}(B^{-1+\frac{2}{p}}_{p,1}(\mathbb{R}^2))$ for $2 \leq p < 4$ and their $\mathcal{M}(B^{-1+\frac{2}{p}}_{p,1}(\mathbb{R}^2))$ norm are small as long as $|\sigma|$ is small. This together with Theorem 1.3 implies that

Corollary 1.1. *Let $p \in (2, 4)$ and $u_0 \in B^{-1+\frac{2}{p}}_{p,1}(\mathbb{R}^2)$ be a solenoidal vector field. Let Ω_0 be a bounded C^1 domain of $\mathbb{R}^2$ and $\rho_0 = 1 + \sigma\chi_{\Omega_0}$ for some small enough constant σ (compared to $\|u_0\|_{B^{-1+\frac{2}{p}}_{p,1}}$). Then* (1.1) *has a unique global solution (ρ, u) with $u \in \mathcal{C}([0,\infty); B^{-1+\frac{2}{p}}_{p,1}(\mathbb{R}^2)) \cap L^1(\mathbb{R}^+; B^{1+\frac{2}{p}}_{p,1}(\mathbb{R}^2))$ and*

$$\rho(t) = 1 + \sigma\chi_{\Omega_t} \quad \text{for} \quad \Omega_t = X_u(t, \Omega_0),$$

where $X_u(t,y)$ is determined by

$$X_u(t,y)=y+\int_0^t u(\tau,X_u(\tau,y))d\tau. \tag{1.6}$$

Besides, the measure of Ω_t and the C^1 regularity of $\partial\Omega_t$ are preserved for all time.

Remark 1.2.
- *We have considered here the physical case of a density given by a discontinuous function (immiscible fluids) and of a viscous coefficient depending on the density of the fluid. In particular, our Corollary 1.1 removes the smallness condition for the initial velocity field in Corollary 1 of* [10]. *In fact, for Ω_0 being a bounded C^2 domain of $\mathbb{R}^2$ and $u_0\in L^2(\Omega)\cap B^1_{4,2}(\Omega)$ (which is above the critical regularity of* (1.1)*), Danchin and Mucha* [11] *can prove a similar global wellposedness result for* (1.1) *with constant viscosity.*
- *Given initial data a_0,u_0 in the scaling invariant spaces: $a_0\in L^\infty(\mathbb{R}^d)$ and $u_0\in B_{p,r}^{-1+\frac{d}{p}}(\mathbb{R}^d)$ for $1<p<d, 1<r<\infty$, and which satisfies some nonlinear smallness condition, we* [18] *proved that* (1.2) *with $\widetilde{\mu}(a)=\mu>0$ has a global weak solution. And the uniqueness of such solution is in progress.*

Scheme of the proof and organization of the paper. The strategy to the proof of both Theorem 1.1 and Theorem 1.3 is to seek a solution of (1.2) with the form $u=v+w$ with (w,p) solving the classical Navier-Stokes system

$$\begin{cases} \partial_t w+w\cdot\nabla w-\mu\Delta w+\nabla p=0, \qquad (t,x)\in\mathbb{R}^+\times\mathbb{R}^2,\\ \mathrm{div} w=0,\\ w|_{t=0}=u_0, \end{cases} \tag{1.7}$$

and (a,v,Π_1) solving

$$\begin{cases} \partial_t a+(v+w)\cdot\nabla a=0, \qquad (t,x)\in\mathbb{R}^+\times\mathbb{R}^2,\\ \partial_t v+v\cdot\nabla v+w\cdot\nabla v+v\cdot\nabla w-(1+a)\,\mathrm{div}(\tilde{\mu}(a)\mathcal{M}(v))+(1+a)\nabla\Pi_1\\ =(1+a)\,\mathrm{div}[(\tilde{\mu}(a)-\mu)\mathcal{M}(w)]+\mu a\Delta w-a\nabla p\overset{\text{déf}}{=}F,\\ \mathrm{div}\, v=0,\\ (a,v)|_{t=0}=(a_0,0), \end{cases} \tag{1.8}$$

where and in what follows, we shall always denote $\widetilde{\mu}(0)$ by μ.

In Section 2, we shall first collect some basic facts on Littlewood-Paley theory, and then present the estimates to the free transport equation and the pressure function determined by (1.8); in Section 3, we solve (1.7) for w with $u_0\in B_{p,1}^{-1+\frac{2}{p}}(\mathbb{R}^2)$ for $1<p<4$. We should mention that because of the restriction to the index p in $(1,4)$, the proof here is much simpler than that in [14, 15]. Then we prove Theorem 1.1 in Section 4. Finally along the same lines to the proof of Theorem 1.1, we present the proof of Theorem 1.3 in the last section.

Let us complete this introduction by the notations we shall use in this context.

Notation. Let A,B be two operators, we denote $[A;B]=AB-BA$, the commutator between A and B. For $a\lesssim b$, we mean that there is a uniform constant C, which may be different on different lines, such that $a\leq Cb$. We shall denote by $(a\mid b)$ the L^2 inner product of a and b. $(d_j)_{j\in\mathbb{Z}}$ will be a generic element of $\ell^1(\mathbb{Z})$ so that $\sum_{j\in\mathbb{Z}}d_j=1$. For X a Banach space and I an interval of $\mathbb{R}$, we denote by $\mathcal{C}(I;X)$ the set of continuous functions on I with values in X, and by $L^q(I;X)$ stands for the set of measurable functions on I with values in X, such that $t\longmapsto\|f(t)\|_X$ belongs to $L^q(I)$.

2. Preliminary Estimates

2.1. **Some Basic Facts on Littlewood-Paley Theory.** For the convenience of the readers, we recall the following basic facts on Littlewood-Paley theory from [6]: for $a \in \mathcal{S}'(\mathbb{R}^2)$, we set

$$\Delta_j a \stackrel{\text{def}}{=} \mathcal{F}^{-1}(\varphi(2^{-j}|\xi|)\widehat{a}), \qquad S_j a \stackrel{\text{def}}{=} \mathcal{F}^{-1}(\chi(2^{-j}|\xi|)\widehat{a}), \tag{2.1}$$

where $\mathcal{F}a$ and $\widehat{a}$ denote the Fourier transform of the distribution a, $\varphi(\tau)$ and $\chi(\tau)$ are smooth functions such that

$$\text{Supp } \varphi \subset \Big\{\tau \in \mathbb{R} \ / \ \frac{3}{4} \le |\tau| \le \frac{8}{3}\Big\} \quad \text{and} \quad \forall \tau > 0\,, \ \sum_{j\in\mathbb{Z}} \varphi(2^{-j}\tau) = 1,$$

$$\text{Supp } \chi \subset \Big\{\tau \in \mathbb{R} \ / \ \ |\tau| \le \frac{4}{3}\Big\} \qquad \text{and} \qquad \chi(\tau) + \sum_{j\ge 0} \varphi(2^{-j}\tau) = 1.$$

We have the formal decomposition

$$u = \sum_{j\in\mathbb{Z}} \Delta_j\, u, \quad \forall\, u \in \mathcal{S}'(\mathbb{R}^2)/\mathcal{P}[\mathbb{R}^2],$$

where $\mathcal{P}[\mathbb{R}^2]$ is the set of polynomials (see [25]). Moreover, the Littlewood-Paley decomposition satisfies the property of almost orthogonality:

$$\Delta_j\Delta_k u \equiv 0 \quad \text{if} \quad |j-k| \ge 2 \quad \text{and} \quad \Delta_j(S_{k-1}u\Delta_k v) \equiv 0 \quad \text{if} \quad |j-k| \ge 5. \tag{2.2}$$

Definition 2.1. *[Definition 2.15 of [6]] Let $(p,r) \in [1,+\infty]^2$, $s \in \mathbb{R}$. The homogeneous Besov space $B^s_{p,r}(\mathbb{R}^2)$ consists of those distributions $u \in \mathcal{S}'_h(\mathbb{R}^2)$, which means that $u \in \mathcal{S}'(\mathbb{R}^2)$ and $\lim_{j\to-\infty} \|S_j u\|_{L^\infty} = 0$ (see Definition 1.26 of [6]), such that*

$$\|u\|_{B^s_{p,r}} \stackrel{def}{=} \Big(2^{qs}\|\Delta_q u\|_{L^p}\Big)_{\ell^r(\mathbb{Z})} < \infty.$$

In order to obtain a better description of the regularizing effect to the transport-diffusion equation, we will use Chemin-Lerner type spaces $\widetilde{L}^\lambda_T(B^s_{p,r}(\mathbb{R}^2))$ (see [6] for instance).

Definition 2.2. *Let $(r,\lambda,p) \in [1,\,+\infty]^3$ and $T \in (0,\,+\infty]$. We define $\widetilde{L}^\lambda_T(B^s_{p\,r}(\mathbb{R}^2))$ as the completion of $C([0,T];\mathcal{S}(\mathbb{R}^2))$ by the norm*

$$\|f\|_{\widetilde{L}^\lambda_T(B^s_{p,r})} \stackrel{def}{=} \Big(\sum_{q\in\mathbb{Z}} 2^{qrs}\Big(\int_0^T \|\Delta_q\, f(t)\|^\lambda_{L^p}\, dt\Big)^{\frac{r}{\lambda}}\Big)^{\frac{1}{r}} < \infty,$$

with the usual change if $r = \infty$. For short, we just denote this space by $\widetilde{L}^\lambda_T(B^s_{p,r})$.

We also need the following form of functional framework, which is a sort of generalization to the weighted Chemin-Lerner type norm defined [23, 24]:

Definition 2.3. *Let $f(t) \in L^1_{loc}(\mathbb{R}^+)$, $f(t) \ge 0$ and X be a Banach space. We define*

$$\|u\|_{L^1_{T,f}(X)} \stackrel{def}{=} \int_0^T f(t)\|u(t)\|_X\, dt.$$

Lemma 2.1. *Let $\mathcal{B}$ be a ball of $\mathbb{R}^2$, and $\mathcal{C}$ be a ring of $\mathbb{R}^2$; let $1 \le p_2 \le p_1 \le \infty$. Then there hold: If the support of $\hat{a}$ is included in $2^k\mathcal{B}$, then*

$$\|\partial^\alpha_x a\|_{L^{p_1}} \lesssim 2^{k(|\alpha|+2(\frac{1}{p_2}-\frac{1}{p_1}))}\|a\|_{L^{p_2}}.$$

If the support of $\hat{a}$ is included in $2^k\mathcal{C}$, then

$$\|a\|_{L^{p_1}} \lesssim 2^{-kN} \sup_{|\alpha|=N} \|\partial^\alpha_x a\|_{L^{p_1}}.$$

Lemma 2.2. *Let θ be a smooth function supported in an annulus $\mathcal{C}$ of $\mathbb{R}^d$. There exists a constant C such that for any $C^{0,1}$ measure-preserving global diffeomorphism ψ over $\mathbb{R}^d$ with inverse ϕ, any tempered distribution u with $\hat{u}$ supported in $\lambda\mathcal{C}$, any $p \in [1,\infty]$ and any $(\lambda,\mu) \in (0,\infty)^2$, we have*

$$\|\theta(\mu^{-1}D)(u\circ\psi)\|_{L^p} \leq C\|u\|_{L^p} \min(\frac{\mu}{\lambda}\|\nabla\phi\|_{L^\infty}, \frac{\lambda}{\mu}\|\nabla\psi\|_{L^\infty}).$$

Lemma 2.3. *If the support of $\hat{u}$ is included in $\lambda\mathcal{C}$, then there exists a positive constant c, such that*

$$\|e^{t\Delta}u\|_{L^p} \lesssim e^{-c\lambda^2 t}\|u\|_{L^p} \quad \text{for any} \quad p \in [1,\infty].$$

Lemma 2.4. *Let $p_1 \geq p_2 \geq 1$, and $s_1 \leq \frac{2}{p_1}$, $s_2 \leq \frac{2}{p_2}$ with $s_1 + s_2 > 0$. Let $a \in B^{s_1}_{p_1,1}(\mathbb{R}^2)$, $b \in B^{s_2}_{p_2,1}(\mathbb{R}^2)$. Then $ab \in B^{s_1+s_2-\frac{2}{p_1}}_{p_1,1}(\mathbb{R}^2)$ and*

$$\|ab\|_{B^{s_1+s_2-\frac{2}{p_1}}_{p_1,1}} \lesssim \|a\|_{B^{s_1}_{p_1,1}}\|b\|_{B^{s_2}_{p_2,1}}.$$

Proposition 2.1. *Let $p \in (1,\infty)$, $r \in [1,\infty]$ and $s \in \mathbb{R}$. Let $u_0 \in B^s_{p,r}(\mathbb{R}^2)$ be a divergence-free field and $g \in \widetilde{L}^1_T(B^s_{p,r})$. Then the following system*

$$\begin{cases} \partial_t u - \nu\Delta u + \nabla\Pi = g, & (t,x) \in \mathbb{R}^+ \times \mathbb{R}^2, \\ \operatorname{div} u = 0, \\ u|_{t=0} = u_0, \end{cases}$$

has a unique solution $(u,\nabla\Pi)$ so that

$$\|u\|_{\widetilde{L}^\infty_T(B^s_{p,r})} + \mu\|u\|_{\widetilde{L}^1_T(B^{s+2}_{p,r})} + \|\nabla\Pi\|_{\widetilde{L}^1_T(B^s_{p,r})} \leq C\big(\|u_0\|_{B^s_{p,r}} + \|g\|_{\widetilde{L}^1_T(B^s_{p,r})}\big).$$

2.2. **Estimates of the transport equation.** The goal of this section is to investigate the transport equation in (1.8)

$$\partial_t a + (v+w)\cdot\nabla a = 0, \qquad a|_{t=0} = a_0. \tag{2.3}$$

More precisely, we shall prove the following proposition:

Proposition 2.2. *Let $1 < q \leq p$ with $\frac{1}{q} - \frac{1}{p} \leq \frac{1}{2}$. Let $v, w \in L^1((0,T), B^{1+\frac{2}{p}}_{p,1}(\mathbb{R}^2))$ be divergence free vector fields, and $a_0 \in B^{\frac{2}{q}}_{q,1}(\mathbb{R}^2)$. We denote $f(t) \stackrel{\text{def}}{=} \|w(t)\|_{B^{1+\frac{2}{p}}_{p,1}}$ and $a_\lambda \stackrel{\text{def}}{=} a\exp\{-\lambda\int_0^t f(\tau)d\tau\}$. Then (2.3) has a unique solution $a \in \mathcal{C}([0,T]; B^{\frac{2}{q}}_{q,1}(\mathbb{R}^2))$ so that*

$$\|a_\lambda\|_{\widetilde{L}^\infty_t(B^{\frac{2}{q}}_{q,1})} + \frac{\lambda}{2}\|a_\lambda\|_{L^1_{t,f}(B^{\frac{2}{q}}_{q,1})} \leq \|a_0\|_{B^{\frac{2}{q}}_{q,1}} + C\|v\|_{L^1_t(B^{1+\frac{2}{p}}_{p,1})}\|a_\lambda\|_{\widetilde{L}^\infty_t(B^{\frac{2}{q}}_{q,1})} \tag{2.4}$$

for any $t \in (0,T]$ and λ large enough, and where $\|a_\lambda\|_{L^1_{t,f}(B^{\frac{2}{q}}_{q,1})}$ is given by Definition 2.3.

Proof. As both the existence and uniqueness of solution to (2.3) basically follows from (2.4). For simplicity, we just present the *a priori* estimate (2.4) for smooth enough solutions of (2.3). Indeed thanks to (2.3), we have

$$\partial_t a_\lambda + \lambda f(t)a_\lambda + (v+w)\cdot\nabla a_\lambda = 0.$$

Applying Δ_j to the above equation and taking the L^2 inner product of the resulting equation with $|\Delta_j a_\lambda|^{q-2}\Delta_j a_\lambda$ (in the case when $q \in (1,2)$, we need a small modification to make this argument rigorous, which we omit here), we obtain

$$\frac{1}{q}\frac{d}{dt}\|\Delta_j a_\lambda(t)\|^q_{L^q} + \lambda f(t)\|\Delta_j a_\lambda(t)\|^q_{L^q} + \big(\Delta_j((v+w)\cdot\nabla a_\lambda) \mid |\Delta_j a_\lambda|^{q-2}\Delta_j a_\lambda\big) = 0. \tag{2.5}$$

Applying Bony's decomposition to $(v+w)\cdot\nabla a_\lambda$ gives rise to

$$(v+w)\cdot\nabla a_\lambda = T_{(v+w)}\nabla a_\lambda + T_{\nabla a_\lambda}(v+w) + R((v+w),\nabla a_\lambda).$$

One gets by using a standard commutator argument that

$$\begin{aligned}
&\big(\Delta_j(T_{(v+w)}\nabla a_\lambda)\;\big|\;|\Delta_j a_\lambda|^{q-2}\Delta_j a_\lambda\big)\\
&\quad=\sum_{|j-j'|\leq 5}\Big\{\big([\Delta_j;S_{j'-1}(v+w)]\Delta_{j'}\nabla a_\lambda\;\big|\;|\Delta_j a_\lambda|^{q-2}\Delta_j a_\lambda\big)\\
&\qquad+\big((S_{j'-1}(v+w)-S_{j-1}(v+w))\Delta_j\Delta_{j'}\nabla a_\lambda\;\big|\;|\Delta_j a_\lambda|^{q-2}\Delta_j a_\lambda\big)\Big\}\\
&\qquad+\big(S_{j-1}(v+w)\nabla\Delta_j a_\lambda\;\big|\;|\Delta_j a_\lambda|^{q-2}\Delta_j a_\lambda\big),
\end{aligned}$$

as $\operatorname{div} v=\operatorname{div} w=0$, the last term equals 0, from which and (2.5), we infer

$$\begin{aligned}
&\|\Delta_j a_\lambda(t)\|_{L^q}+\lambda\int_0^t f(\tau)\|\Delta_j a_\lambda(\tau)\|_{L^q}\,d\tau\\
&\leq\|\Delta_j a_0\|_{L^q}+C\Big\{\sum_{|j-j'|\leq 5}\big(\|[\Delta_j;S_{j'-1}(v+w)]\Delta_{j'}\nabla a_\lambda\|_{L^1_t(L^q)}\\
&\quad+\|(S_{j'-1}(v+w)-S_{j-1}(v+w))\Delta_j\Delta_{j'}\nabla a_\lambda\|_{L^1_t(L^q)}\big)\\
&\quad+\|T_{\nabla a_\lambda}(v+w)\|_{L^1_t(L^q)}+\|R((v+w),\nabla a_\lambda)\|_{L^1_t(L^q)}\Big\}.
\end{aligned}\tag{2.6}$$

We first get by applying the classical estimate on commutator (see [6] for instance) and Definition 2.3 that

$$\begin{aligned}
&\sum_{|j-j'|\leq 5}\|[\Delta_j;S_{j'-1}(v+w)]\Delta_{j'}\nabla a_\lambda\|_{L^1_t(L^q)}\\
&\lesssim\sum_{|j-j'|\leq 5}\Big(\|\nabla S_{j'-1}v\|_{L^1_t(L^\infty)}\|\Delta_{j'}a_\lambda\|_{L^\infty(L^q)}+\int_0^t\|\nabla S_{j'-1}w(\tau)\|_{L^\infty}\|\Delta_{j'}a_\lambda(\tau)\|_{L^q}\,d\tau\Big)\\
&\lesssim\sum_{|j-j'|\leq 5}\Big(d_{j'}2^{-j'\frac{2}{q}}\|v\|_{L^1_t(B^{1+\frac{2}{p}}_{p,1})}\|a_\lambda\|_{\widetilde{L}^\infty_t(B^{\frac{2}{q}}_{q,1})}+\int_0^t\|w(\tau)\|_{B^{1+\frac{2}{p}}_{p,1}}\|\Delta_{j'}a_\lambda(\tau)\|_{L^q}\,d\tau\Big)\\
&\lesssim d_j2^{-j\frac{2}{q}}\Big(\|v\|_{L^1_t(B^{1+\frac{2}{p}}_{p,1})}\|a_\lambda\|_{\widetilde{L}^\infty_t(B^{\frac{2}{q}}_{q,1})}+\|a_\lambda\|_{L^1_{t,f}(B^{\frac{2}{q}}_{q,1})}\Big).
\end{aligned}$$

Applying Lemma 2.1 leads to

$$\begin{aligned}
&\sum_{|j-j'|\leq 5}\|(S_{j'-1}(v+w)-S_{j-1}(v+w))\Delta_j\Delta_{j'}\nabla a_\lambda\|_{L^1_t(L^q)}\\
&\lesssim\sum_{|j-j'|\leq 5}\Big(\|S_{j'-1}\nabla v-S_{j-1}\nabla v\|_{L^1_t(L^\infty)}\|\Delta_j a_\lambda\|_{L^\infty(L^q)}\\
&\qquad+\int_0^t\|(S_{j'-1}\nabla w-S_{j-1}\nabla w)(\tau)\|_{L^\infty}\|\Delta_{j'}a_\lambda(\tau)\|_{L^q}\,d\tau\Big)\\
&\lesssim d_j2^{-j\frac{2}{q}}\|v\|_{L^1_t(B^{1+\frac{2}{p}}_{p,1})}\|a_\lambda\|_{\widetilde{L}^\infty_t(B^{\frac{2}{q}}_{q,1})}+\sum_{|j-j'|\leq 5}\int_0^t\|w(\tau)\|_{B^{1+\frac{2}{p}}_{p,1}}\|\Delta_{j'}a_\lambda(\tau)\|_{L^q}\,d\tau\\
&\lesssim d_j2^{-j\frac{2}{q}}\Big(\|v\|_{L^1_t(B^{1+\frac{2}{p}}_{p,1})}\|a_\lambda\|_{\widetilde{L}^\infty_t(B^{\frac{2}{q}}_{q,1})}+\|a_\lambda\|_{L^1_{t,f}(B^{\frac{2}{q}}_{q,1})}\Big).
\end{aligned}$$

On the other hand, as $q \le p$, let r be determined by $\frac{1}{r} = \frac{1}{q} - \frac{1}{p}$. Then we get that

$$\begin{aligned}\|T_{\nabla a_\lambda}(v+w)\|_{L^1_t(L^q)} \lesssim \sum_{|j-j'|\le 5} \big(& \|S_{j'-1}\nabla a_\lambda\|_{L^\infty_t(L^r)} \|\Delta_{j'} v\|_{L^1_t(L^p)} \\ & + \int_0^t \|S_{j'-1}\nabla a_\lambda(\tau)\|_{L^r} \|\Delta_{j'} w(\tau)\|_{L^p} d\tau \big),\end{aligned}$$

now as $\frac{1}{q} - \frac{1}{p} \le \frac{1}{2}$, one has

$$\begin{aligned}\|S_{j'-1}\nabla a_\lambda\|_{L^\infty_t(L^r)} &\lesssim \sum_{\ell \le j'-2} 2^{\ell(1+\frac{2}{p})} \|\Delta_\ell a_\lambda\|_{L^\infty_t(L^q)} \\ &\lesssim \sum_{l \le j'-2} d_\ell 2^{\ell(1+\frac{2}{p}-\frac{2}{q})} \|a_\lambda\|_{\widetilde{L}^\infty_t(B^{\frac{2}{q}}_{q,1})} \lesssim 2^{j'(1+\frac{2}{p}-\frac{2}{q})} \|a_\lambda\|_{\widetilde{L}^\infty_t(B^{\frac{2}{q}}_{q,1})}.\end{aligned}$$

Applying Lemma 2.1 and Definition 2.3 once again gives, if $\frac{1}{q} - \frac{1}{p} < \frac{1}{2}$

$$\begin{aligned}&\sum_{|j-j'|\le 5} \int_0^t \|S_{j'-1}\nabla a_\lambda(\tau)\|_{L^r} \|\Delta_{j'} w(\tau)\|_{L^p} d\tau \\ &\quad\lesssim 2^{-j(1+\frac{2}{p})} \sum_{|j-j'|\le 5} \sum_{\ell \le j'-2} 2^{\ell(1+\frac{2}{p})} \int_0^t \|\Delta_\ell a_\lambda(\tau)\|_{L^q} \|w(\tau)\|_{B^{1+\frac{2}{p}}_{p,1}} d\tau \\ &\quad\lesssim 2^{-j(1+\frac{2}{p})} \sum_{\ell \le j+3} d_\ell 2^{\ell(1+\frac{2}{p}-\frac{2}{q})} \|a_\lambda\|_{L^1_{t,f}(B^{\frac{2}{q}}_{q,1})} \\ &\quad\lesssim d_j 2^{-j\frac{2}{q}} \|a_\lambda\|_{L^1_{t,f}(B^{\frac{2}{q}}_{q,1})}.\end{aligned}$$

In the case when $\frac{1}{q} - \frac{1}{p} = \frac{1}{2}$, we have

$$\begin{aligned}&\sum_{j\in\mathbb{Z}} 2^{j\frac{2}{q}} \sum_{|j-j'|\le 5} \int_0^t \|S_{j'-1}\nabla a_\lambda(\tau)\|_{L^r} \|\Delta_{j'} w(\tau)\|_{L^p} d\tau \\ &\quad\lesssim \sum_{j\in\mathbb{Z}} \sum_{\ell \le j+3} 2^{\ell(1+\frac{2}{p})} \int_0^t d_j(\tau) \|\Delta_\ell a_\lambda(\tau)\|_{L^q} \|w(\tau)\|_{B^{1+\frac{2}{p}}_{p,1}} d\tau \\ &\quad\lesssim \sum_{\ell \le j+3} d_\ell \|a_\lambda\|_{L^1_{t,f}(B^{\frac{2}{q}}_{q,1})} \lesssim \|a_\lambda\|_{L^1_{t,f}(B^{\frac{2}{q}}_{q,1})}.\end{aligned}$$

As a consequence, we obtain

$$\|T_{\nabla a_\lambda}(v+w)\|_{L^1_t(L^q)} \lesssim d_j 2^{-j\frac{2}{q}} (\|v\|_{L^1_t(B^{1+\frac{2}{p}}_{p,1})} \|a_\lambda\|_{\widetilde{L}^\infty_t(B^{\frac{2}{q}}_{q,1})} + \|a_\lambda\|_{L^1_{t,f}(B^{\frac{2}{q}}_{q,1})}).$$

For the last term in (2.6), we deduce from Lemma 2.1 that

$$\begin{aligned}
\|R(v+w,\nabla a_\lambda)\|_{L^1_t(L^q)} \lesssim & 2^{\frac{2j}{p}} \sum_{j'\geq j-N_0} \bigl(\|\Delta_{j'}v\|_{L^1_t(L^p)}\|\tilde{\Delta}_{j'}\nabla a_\lambda\|_{L^\infty_t(L^q)} \\
& \qquad + \int_0^t \|\Delta_{j'}w(\tau)\|_{L^p}\|\tilde{\Delta}_{j'}\nabla a_\lambda(\tau)\|_{L^q}d\tau\bigr) \\
\lesssim & 2^{\frac{2j}{p}} \sum_{j'\geq j-N_0} \bigl(d_{j'}2^{-j'(\frac{2}{p}+\frac{2}{q})}\|v\|_{L^1_t(B^{1+\frac{2}{p}}_{p,1})}\|a_\lambda\|_{\widetilde{L}^\infty_t(B^{\frac{2}{q}}_{q,1})} \\
& \qquad + 2^{-j'\frac{2}{p}}\int_0^t \|w(\tau)\|_{B^{1+\frac{2}{p}}_{p,1}}\|\tilde{\Delta}_{j'}a_\lambda(\tau)\|_{L^q}d\tau\bigr) \\
\lesssim & 2^{\frac{2j}{p}} \sum_{j'\geq j-N_0} d_{j'}2^{-j'(\frac{2}{p}+\frac{2}{q})}\bigl(\|v\|_{L^1_t(B^{1+\frac{2}{p}}_{p,1})}\|a_\lambda\|_{\widetilde{L}^\infty_t(B^{\frac{2}{q}}_{q,1})} + \|a_\lambda\|_{L^1_{t,f}(B^{\frac{2}{q}}_{q,1})}\bigr) \\
\lesssim & d_j 2^{-j\frac{2}{q}}\bigl(\|v\|_{L^1_t(B^{1+\frac{2}{p}}_{p,1})}\|a_\lambda\|_{\widetilde{L}^\infty_t(B^{\frac{2}{q}}_{q,1})} + \|a_\lambda\|_{L^1_{t,f}(B^{\frac{2}{q}}_{q,1})}\bigr).
\end{aligned}$$

Substituting the above estimates into (2.6) and taking summation for $j\in\mathbb{Z}$, we arrive at

$$\|a_\lambda\|_{\widetilde{L}^\infty_t(B^{\frac{2}{q}}_{q,1})} + \lambda\|a_\lambda\|_{L^1_{t,f}(B^{\frac{2}{q}}_{q,1})} \leq \|a_0\|_{B^{\frac{2}{q}}_{q,1}} + C\bigl(\|v\|_{L^1_t(B^{1+\frac{2}{p}}_{p,1})}\|a_\lambda\|_{\widetilde{L}^\infty_t(B^{\frac{2}{q}}_{q,1})} + \|a_\lambda\|_{L^1_{t,f}(B^{\frac{2}{q}}_{q,1})}\bigr).$$

Taking $\lambda\geq 2C$ in the above inequality, we conclude the proof of (2.4). $\square$

2.3. **Estimates of the pressure function.** In this subsection, we aim at providing the *a priori* estimate for $\nabla\Pi_1$ determined by (1.8). We first get by taking div to the momentum equation of (1.8) that

$$\begin{aligned}
-\Delta\Pi_1 =& \operatorname{div}(a\nabla\Pi_1) + \operatorname{div} F - \operatorname{div}(v\cdot\nabla v + w\cdot\nabla v + v\cdot\nabla w) \\
&+ \operatorname{div}[(1+a)\operatorname{div}\bigl((\tilde{\mu}(a)-\mu)\mathcal{M}(v)\bigr)] + \mu\operatorname{div}(a\Delta v),
\end{aligned} \tag{2.7}$$

where F is given by (1.8).

Proposition 2.3. *Let $1<q\leq p<4$. Let $a\in\widetilde{L}^\infty_T(B^{\frac{2}{q}}_{q,1})$, $w,v\in L^1_T(B^{1+\frac{2}{p}}_{p,1})\cap\widetilde{L}^\infty_T(B^{-1+\frac{2}{p}}_{p,1})$ and $\nabla p\in L^1_T(B^{-1+\frac{2}{p}}_{p,1})$. For $\lambda_1,\lambda_2>0$, we denote*

$$\begin{aligned}
& f_1(t) \stackrel{\text{def}}{=} \|w(t)\|_{B^{1+\frac{2}{p}}_{p,1}} + \frac{1}{\mu}\|\nabla p(t)\|_{B^{-1+\frac{2}{p}}_{p,1}}, \quad f_2(t) \stackrel{\text{def}}{=} \|w(t)\|^2_{B^{\frac{2}{p}}_{p,1}} \quad \text{and} \\
& \Pi_{\bar{\lambda}} \stackrel{\text{def}}{=} \Pi_1\exp\Bigl\{-\lambda_1\int_0^t f_1(\tau)d\tau - \lambda_2\int_0^t f_2(\tau)d\tau\Bigr\},
\end{aligned} \tag{2.8}$$

and similar notations for $a_{\bar{\lambda}}$ and $v_{\bar{\lambda}}$. Then (2.7) has a unique solution $\nabla\Pi_1\in L^1_T(B^{-1+\frac{2}{p}}_{p,1})$ so that for any $\epsilon>0$, there holds

$$\begin{aligned}
\|\nabla\Pi_{\bar{\lambda}}\|_{L^1_t(B^{-1+\frac{2}{p}}_{p,1})} \leq & \frac{C}{1-C\|a\|_{\widetilde{L}^\infty_t(B^{\frac{2}{q}}_{q,1})}}\Bigl\{\epsilon\|v_{\bar{\lambda}}\|_{L^1_t(B^{1+\frac{2}{p}}_{p,1})} + \|v\|_{\widetilde{L}^\infty_t(B^{-1+\frac{2}{p}}_{p,1})}\|v_{\bar{\lambda}}\|_{L^1_t(B^{1+\frac{2}{p}}_{p,1})} \\
& + \|v_{\bar{\lambda}}\|_{L^1_{t,f_1}(B^{-1+\frac{2}{p}}_{p,1})} + \frac{1}{\epsilon}\|v_{\bar{\lambda}}\|_{L^1_{t,f_2}(B^{-1+\frac{2}{p}}_{p,1})} \\
& + \bigl(\mu+\mathfrak{C}(1+\|a\|_{\widetilde{L}^\infty_t(B^{\frac{2}{q}}_{q,1})})\bigr)\bigl(\|a_{\bar{\lambda}}\|_{L^1_{t,f_1}(B^{\frac{2}{q}}_{q,1})} + \|a\|_{\widetilde{L}^\infty_t(B^{\frac{2}{q}}_{q,1})}\|v_{\bar{\lambda}}\|_{L^1_t(B^{1+\frac{2}{p}}_{p,1})}\bigr)\Bigr\}
\end{aligned} \tag{2.9}$$

provided that $C\|a\|_{\widetilde{L}^\infty_t(B^{\frac{2}{q}}_{q,1})} \leq \frac{1}{2}$, *where* $\|v_{\bar\lambda}\|_{L^1_{t,f}(B^{-1+\frac{2}{p}}_{p,1})}$ *is given by Definition 2.3 and the positive constant* $\mathfrak{C}$ *depends on* $\|\widetilde{\mu}'\|_{L^\infty(-1,1)}$.

The proof of this proposition will mainly be based on the following lemmas:

Lemma 2.5. *Under the assumptions of Proposition 2.3, one has for any* $\epsilon > 0$

$$\|v\cdot\nabla w + w\cdot\nabla v\|_{L^1_t(B^{-1+\frac{2}{p}}_{p,1})} \lesssim \epsilon\|v\|_{L^1_t(B^{1+\frac{2}{p}}_{p,1})} + \|v\|_{L^1_{t,f_1}(B^{-1+\frac{2}{p}}_{p,1})} + \frac{1}{\epsilon}\|v\|_{L^1_{t,f_2}(B^{-1+\frac{2}{p}}_{p,1})}.$$

Proof. As $\operatorname{div} v = \operatorname{div} w = 0$, we have

$$\|v\cdot\nabla w + w\cdot\nabla v\|_{L^1_t(B^{-1+\frac{2}{p}}_{p,1})} \lesssim \|vw\|_{L^1_t(B^{\frac{2}{p}}_{p,1})}.$$

While we get by applying Bony's decomposition that

$$vw = T_v w + T_w v + R(v,w).$$

Notice that applying Lemma 2.1 leads to

$$\begin{aligned}
\|\Delta_j(T_v w)\|_{L^1_t(L^p)} &\lesssim \int_0^t \sum_{|k-j|\leq 5} \|S_{k-1}v\|_{L^\infty}\|\Delta_k w\|_{L^p}\,d\tau\\
&\lesssim \int_0^t d_j(\tau)2^{-j(1+\frac{2}{p})}\|w(\tau)\|_{B^{1+\frac{2}{p}}_{p,1}} \sum_{\ell\leq j} 2^{\frac{2}{p}\ell}\|\Delta_\ell v(\tau)\|_{L^p}\,d\tau\\
&\lesssim d_j 2^{-j\frac{2}{p}} \int_0^t \|v(\tau)\|_{B^{-1+\frac{2}{p}}_{p,1}}\|w(\tau)\|_{B^{1+\frac{2}{p}}_{p,1}}\,d\tau,
\end{aligned}$$

and

$$\begin{aligned}
\|\Delta_j(R(v,w))\|_{L^1_t(L^p)} &\lesssim 2^{j\frac{2}{p}} \int_0^t \sum_{k\geq j-N_0} \|\widetilde{\Delta}_k v\|_{L^p}\|\Delta_k w\|_{L^p}\,d\tau\\
&\lesssim 2^{j\frac{2}{p}} \int_0^t \sum_{k\geq j-N_0} d_k(\tau)2^{-k\frac{4}{p}}\|v(\tau)\|_{B^{-1+\frac{2}{p}}_{p,1}}\|w(\tau)\|_{B^{1+\frac{2}{p}}_{p,1}}\,d\tau\\
&\lesssim d_j 2^{-j\frac{2}{p}} \int_0^t \|v(\tau)\|_{B^{-1+\frac{2}{p}}_{p,1}}\|w(\tau)\|_{B^{1+\frac{2}{p}}_{p,1}}\,d\tau.
\end{aligned}$$

It follows from the same line that

$$\begin{aligned}
\|\Delta_j(T_w v)\|_{L^1_t(L^p)} &\lesssim \int_0^t \sum_{|k-j|\leq 5} \|S_{k-1}w\|_{L^\infty}\|\Delta_k v\|_{L^p}\,d\tau \lesssim \sum_{|k-j|\leq 5}\int_0^t \|w\|_{B^{\frac{2}{p}}_{p,1}}\|\Delta_k v\|_{L^p}\,d\tau\\
&\lesssim \Big(\sum_{|k-j|\leq 5}\int_0^t 2^{-k}\|w\|^2_{B^{\frac{2}{p}}_{p,1}}\|\Delta_k v\|_{L^p}\,d\tau\Big)^{\frac{1}{2}}\Big(\sum_{|k-j|\leq 5}\int_0^t 2^{k}\|\Delta_k v\|_{L^p}\,d\tau\Big)^{\frac{1}{2}}\\
&\lesssim d_j 2^{-j\frac{2}{p}}\Big(\frac{1}{\epsilon}\int_0^t \|w(\tau)\|^2_{B^{\frac{2}{p}}_{p,1}}\|v(\tau)\|_{B^{-1+\frac{2}{p}}_{p,1}}\,d\tau + \epsilon\int_0^t \|v(\tau)\|_{B^{1+\frac{2}{p}}_{p,1}}\,d\tau\Big).
\end{aligned}$$

The above estimates together with Definition 2.3 prove the lemma. □

Lemma 2.6. *Let* F *be determined by* (1.8)*. Then under the assumptions of Proposition 2.3, one has*

$$\|F\|_{L^1_t(B^{-1+\frac{2}{p}}_{p,1})} \leq \big(C\mu + \mathfrak{C}(1+\|a\|_{\widetilde{L}^\infty_t(B^{\frac{2}{q}}_{q,1})})\big)\|a\|_{L^1_{t,f_1}(B^{\frac{2}{q}}_{q,1})}.$$

for some positive constant $\mathfrak{C}$ *depending on* $\|\widetilde{\mu}'\|_{L^\infty(-1,1)}$.

Proof. Note that $q \le p$ and $\frac{1}{q}+\frac{1}{p}>\frac{1}{2}$, we deduce by the product laws in Besov space that

$$\begin{aligned}\|\mu a\Delta w - a\nabla p\|_{L^1_t(B^{-1+\frac{2}{p}}_{p,1})} \lesssim & \mu\int_0^t \|a\|_{B^{\frac{2}{q}}_{q,1}}\big(\|w\|_{B^{1+\frac{2}{p}}_{p,1}}+\frac{1}{\mu}\|\nabla p\|_{B^{-1+\frac{2}{p}}_{p,1}}\big)\,d\tau\\ \lesssim & \mu\|a\|_{L^1_{t,f_1}(B^{\frac{2}{q}}_{q,1})}.\end{aligned}$$

Along the same line, we get that

$$\begin{aligned}\|(1+a)\operatorname{div}[(\tilde\mu(a)-\mu)\mathcal{M}(w)]\|_{B^{-1+\frac{2}{p}}_{p,1}} &\lesssim (1+\|a\|_{B^{\frac{2}{q}}_{q,1}})\|(\tilde\mu(a)-\mu)\mathcal{M}(w)\|_{B^{\frac{2}{p}}_{p,1}}\\ &\lesssim (1+\|a\|_{B^{\frac{2}{q}}_{q,1}})\|\tilde\mu(a)-\mu\|_{B^{\frac{2}{q}}_{q,1}}\|\mathcal{M}(w)\|_{B^{\frac{2}{p}}_{p,1}}\\ &\le \mathfrak{C}(1+\|a\|_{B^{\frac{2}{q}}_{q,1}})\|a\|_{B^{\frac{2}{q}}_{q,1}}\|w\|_{B^{1+\frac{2}{p}}_{p,1}},\end{aligned}$$

for some positive constant $\mathfrak{C}$ depending on $\|\widetilde{\mu}'\|_{L^\infty(-1,1)}$ as long as $\|a\|_{L^\infty}\le 1$. This gives rise to

$$\begin{aligned}\|(1+a)\operatorname{div}[(\tilde\mu(a)-\mu)\mathcal{M}(w)]\|_{L^1_t(B^{-1+\frac{2}{p}}_{p,1})} &\le \mathfrak{C}(1+\|a\|_{\widetilde L^\infty_t(B^{\frac{2}{q}}_{q,1})})\int_0^t\|a\|_{B^{\frac{2}{q}}_{q,1}}\|w\|_{B^{1+\frac{2}{p}}_{p,1}}\,d\tau\\ &\le \mathfrak{C}(1+\|a\|_{\widetilde L^\infty_t(B^{\frac{2}{q}}_{q,1})})\|a\|_{L^1_{t,f_1}(B^{\frac{2}{q}}_{q,1})}.\end{aligned}$$

This finishes the proof of Lemma 2.6. □

Now let us turn to the proof of Proposition 2.3.

Proof of Proposition 2.3. As both the existence and uniqueness parts of Proposition 2.3 basically follows from the uniform estimate (2.9) for appropriate approximate solutions of (2.7). For simplicity, we just prove (2.9) for smooth enough solutions of (2.7). Indeed thanks to (2.7), we have

$$\begin{aligned}\nabla\Pi_{\bar\lambda} = \nabla(-\Delta)^{-1}\Big(&\operatorname{div}(a\nabla\Pi_{\bar\lambda})+\operatorname{div}F_{\bar\lambda}-\operatorname{div}(v\cdot\nabla v_{\bar\lambda}+v_{\bar\lambda}\cdot\nabla w+w\cdot\nabla v_{\bar\lambda})\\ &+\operatorname{div}\big((1+a)\operatorname{div}((\widetilde\mu(a)-\mu)\mathcal{M}(v_{\bar\lambda}))\big)+\mu\operatorname{div}(a\Delta v_{\bar\lambda})\Big),\end{aligned}$$

from which, we deduce that

$$\begin{aligned}\|\nabla\Pi_{\bar\lambda}\|_{L^1_t(B^{-1+\frac{2}{p}}_{p,1})} \le & C\Big\{\|a\nabla\Pi_{\bar\lambda}\|_{L^1_t(B^{-1+\frac{2}{p}}_{p,1})}+\|F_{\bar\lambda}\|_{L^1_t(B^{-1+\frac{2}{p}}_{p,1})}+\|v\cdot\nabla v_{\bar\lambda}\|_{L^1_t(B^{-1+\frac{2}{p}}_{p,1})}\\ &+\|v_{\bar\lambda}\cdot\nabla w+w\cdot\nabla v_{\bar\lambda}\|_{L^1_t(B^{-1+\frac{2}{p}}_{p,1})}+\mu\|a\Delta v_{\bar\lambda}\|_{L^1_t(B^{-1+\frac{2}{p}}_{p,1})}\\ &+\|(1+a)\operatorname{div}((\widetilde\mu(a)-\mu)\mathcal{M}(v_{\bar\lambda}))\|_{L^1_t(B^{-1+\frac{2}{p}}_{p,1})}\Big\}.\end{aligned}\tag{2.10}$$

However as $q\le p$ and $\frac{1}{q}+\frac{1}{p}>\frac{1}{2}$, applying standard product laws in Besov space leads to

$$\begin{aligned}&\|v\cdot\nabla v_{\bar\lambda}\|_{L^1_t(B^{-1+\frac{2}{p}}_{p,1})}\lesssim\|v\|_{\widetilde L^\infty_t(B^{-1+\frac{2}{p}}_{p,1})}\|v_{\bar\lambda}\|_{L^1_t(B^{1+\frac{2}{p}}_{p,1})},\\ &\|a\nabla\Pi_{\bar\lambda}\|_{L^1_t(B^{-1+\frac{2}{p}}_{p,1})}\lesssim\|a\|_{\widetilde L^\infty_t(B^{\frac{2}{q}}_{q,1})}\|\nabla\Pi_{\bar\lambda}\|_{L^1_t(B^{-1+\frac{2}{p}}_{p,1})},\\ &\mu\|a\Delta v_{\bar\lambda}\|_{L^1_t(B^{-1+\frac{2}{p}}_{p,1})}\lesssim\mu\|a\|_{\widetilde L^\infty_t(B^{\frac{2}{q}}_{q,1})}\|v_{\bar\lambda}\|_{L^1_t(B^{1+\frac{2}{p}}_{p,1})},\\ &\|(1+a)\operatorname{div}((\widetilde\mu(a)-\mu)\mathcal{M}(v_{\bar\lambda}))\|_{L^1_t(B^{-1+\frac{2}{p}}_{p,1})}\le\mathfrak{C}(1+\|a\|_{\widetilde L^\infty_t(B^{\frac{2}{q}}_{q,1})})\|a\|_{\widetilde L^\infty_t(B^{\frac{2}{q}}_{q,1})}\|v_{\bar\lambda}\|_{L^1_t(B^{1+\frac{2}{p}}_{p,1})},\end{aligned}$$

for some positive constant $\mathfrak{C}$ depending on $\|\widetilde{\mu}'\|_{L^\infty(-1,1)}$ as long as $\|a\|_{L^\infty}\le 1$. This along with Lemmas 2.5 to 2.6 implies Proposition 2.3 provided that $C\|a\|_{\widetilde L^\infty_t(B^{\frac{2}{q}}_{q,1})}\le\frac{1}{2}$. ∎

3. The global infinite energy solutions to classical 2-D Navier-Stokes system

In this section, we shall solve the global wellposedness of the classical Navier-Stokes system (1.7) with initial data $u_0 \in B^{-1+\frac{2}{p}}_{p,1}(\mathbb{R}^2)$ for $1 < p < 4$, which is not of finite energy. In general, the global wellposedness to 2-D classical Navier-Stokes system with initial data in the scaling invariant Besov spaces and of infinite energy was solved in [14, 15]. However considering the special structure of $B^{-1+\frac{2}{p}}_{p,1}(\mathbb{R}^2)$ for $1 < p < 4$, we shall provide a much simpler proof than that in [14, 15], furthermore, more detailed information to this solution will be given here. More precisely, we shall split the solution w to (1.7) as $w_L + \bar{w}$ with $w_L \stackrel{\text{def}}{=} e^{\mu t\Delta} u_0$. Then it follows from (1.7) and Lemma 2.3 that

$$\|w_L\|_{\widetilde{L}^\infty_t(B^{-1+\frac{2}{p}}_{p,1})} + \mu\|w_L\|_{L^1_t(B^{1+\frac{2}{p}}_{p,1})} \leq C\|u_0\|_{B^{-1+\frac{2}{p}}_{p,1}}, \tag{3.1}$$

and $\bar{w}$ solves

$$\begin{cases} \partial_t \bar{w} + \bar{w}\cdot\nabla\bar{w} + \bar{w}\cdot\nabla w_L + w_L\cdot\nabla\bar{w} + w_L\cdot\nabla w_L - \mu\Delta\bar{w} + \nabla p = 0, \\ \operatorname{div}\bar{w} = 0, \\ \bar{w}|_{t=0} = 0. \end{cases} \tag{3.2}$$

The main result of this section is as follows:

Proposition 3.1. *Given solenoidal vector filed $u_0 \in B^{-1+\frac{2}{p}}_{p,1}(\mathbb{R}^2)$ for $p \in (1,4)$, (1.7) has a unique solution w of the form: $w_L + \bar{w}$, with $\bar{w} \in \mathcal{C}([0,\infty); B^0_{2,1}(\mathbb{R}^2)) \cap \widetilde{L}^\infty(\mathbb{R}^+; B^0_{2,1}(\mathbb{R}^2)) \cap L^1(\mathbb{R}^+; B^2_{2,1}(\mathbb{R}^2))$, and there holds*

$$\begin{aligned} &\|w\|_{\widetilde{L}^\infty_t(B^{-1+\frac{2}{p}}_{p,1})} + \mu\|w\|_{L^1_t(B^{1+\frac{2}{p}}_{p,1})} + \|\nabla p\|_{L^1_t(B^{-1+\frac{2}{p}}_{p,1})} \\ \leq &C\|u_0\|_{B^{-1+\frac{2}{p}}_{p,1}}(1 + \|u_0\|_{B^{-1+\frac{2}{p}}_{p,1}}) \exp\Big\{\frac{C}{\mu^2}\|u_0\|^2_{B^{-1+\frac{2}{p}}_{p,1}}\Big\}. \end{aligned} \tag{3.3}$$

We start the proof of Proposition 3.1 by the following two technical lemmas.

Lemma 3.1. *Let $p \in [1,\infty]$, $w_L \in \widetilde{L}^\infty_t(B^{-1+\frac{2}{p}}_{p,1}) \cap L^1_t(B^{1+\frac{2}{p}}_{p,1})$ and $\bar{w} \in \widetilde{L}^\infty_t(B^0_{2,1}) \cap L^1_t(B^2_{2,1})$ be divergence free vector fields, then for any $\epsilon > 0$, one has*

$$\|\bar{w}\cdot\nabla w_L + w_L\cdot\nabla\bar{w}\|_{L^1_t(B^0_{2,1})} \lesssim \epsilon\|\bar{w}\|_{L^1_t(B^2_{2,1})} + \int_0^t \big(\|w_L\|_{B^{1+\frac{2}{p}}_{p,1}} + \frac{1}{\epsilon}\|w_L\|^2_{B^{\frac{2}{p}}_{p,1}}\big)\|\bar{w}\|_{B^0_{2,1}}\,d\tau.$$

Proof. The proof of this lemma basically follows from that of Lemma 2.5. Note that $\operatorname{div}\bar{w} = \operatorname{div} w_L = 0$, we have

$$\|\bar{w}\cdot\nabla w_L + w_L\cdot\nabla\bar{w}\|_{L^1_t(B^0_{2,1})} \lesssim \|\bar{w}w_L\|_{L^1_t(B^1_{2,1})},$$

and we get by applying Bony's decomposition

$$\bar{w}w_L = T_{w_L}\bar{w} + T_{\bar{w}}w_L + R(\bar{w}, w_L).$$

Applying Lemma 2.1 yields

$$\begin{aligned} \|\Delta_j(T_{\bar{w}}w_L)\|_{L^1_t(L^2)} &\lesssim \int_0^t \sum_{|k-j|\leq 5} \|S_{k-1}\bar{w}\|_{L^2}\|\Delta_k w_L\|_{L^\infty}\,d\tau \\ &\lesssim d_j 2^{-j}\int_0^t \|w_L\|_{B^{1+\frac{2}{p}}_{p,1}}\|\bar{w}\|_{B^0_{2,1}}\,d\tau, \end{aligned}$$

and

$$
\begin{aligned}
\|\Delta_j(R(\bar{w},w_L))\|_{L^1_t(L^2)} &\lesssim \int_0^t 2^{j\frac{2}{p}} \sum_{k\geq j-N_0} \|\widetilde{\Delta}_k\bar{w}\|_{L^2}\|\Delta_k w_L\|_{L^p} d\tau\\
&\lesssim \int_0^t 2^{j\frac{2}{p}} \sum_{k\geq j-N_0} d_k(\tau)2^{-k(1+\frac{2}{p})}\|w_L\|_{B^{1+\frac{2}{p}}_{p,1}}\|\bar{w}\|_{B^0_{2,1}} d\tau\\
&\lesssim d_j 2^{-j}\int_0^t \|w_L\|_{B^{1+\frac{2}{p}}_{p,1}}\|\bar{w}\|_{B^0_{2,1}} d\tau.
\end{aligned}
$$

Along the same line, one has

$$
\begin{aligned}
\|\Delta_j(T_{w_L}\bar{w})\|_{L^1_t(L^2)} &\lesssim \int_0^t \sum_{|k-j|\leq 5} \|S_{k-1}w_L\|_{L^\infty}\|\Delta_k\bar{w}\|_{L^2} d\tau \lesssim \sum_{|k-j|\leq 5}\int_0^t \|w_L\|_{B^{\frac{2}{p}}_{p,1}}\|\Delta_k\bar{w}\|_{L^2}\, d\tau\\
&\lesssim \Big(\sum_{|k-j|\leq 5}\int_0^t 2^{-k}\|w_L\|^2_{B^{\frac{2}{p}}_{p,1}}\|\Delta_k\bar{w}\|_{L^2}\, d\tau\Big)^{\frac12}\Big(\sum_{|k-j|\leq 5}\int_0^t 2^{k}\|\Delta_k\bar{w}\|_{L^2}\, d\tau\Big)^{\frac12}\\
&\lesssim d_j 2^{-j}\big(\frac{1}{\epsilon}\int_0^t \|w\|^2_{B^{\frac{2}{p}}_{p,1}}\|\bar{w}\|_{B^0_{2,1}}\, d\tau + \epsilon\int_0^t \|\bar{w}\|_{B^2_{2,1}} d\tau\big).
\end{aligned}
$$

By summing up the above estimates, we finish the proof of Lemma 3.1. □

Lemma 3.2. *Let $p\in(1,4)$ and w_L be given at the beginning of this section, one has*

$$
\|w_L\cdot\nabla w_L\|_{L^1_t(B^0_{2,1})} \lesssim \frac{1}{\mu}\|u_0\|^2_{B^{-1+\frac{2}{p}}_{p,1}}. \tag{3.4}
$$

Proof. Indeed due to $\mathrm{div} w_L=0$, one has

$$
\|w_L\cdot\nabla w_L\|_{L^1_t(B^0_{2,1})} \lesssim \|w_L\otimes w_L\|_{L^1_t(B^1_{2,1})},
$$

and thanks to Bony's decomposition, we get

$$
w_L\otimes w_L = 2T_{w_L}w_L + R(w_L,w_L).
$$

We first deal with (3.4) for the case when $2\leq p<4$. In this case, we have $p<\frac{2p}{p-2}\leq\infty$, so that applying Lemma 2.1 gives

$$
\begin{aligned}
\|S_{k-1}w_L\|_{L^\infty_t(L^{\frac{2p}{p-2}})} &\lesssim \sum_{\ell\leq k-2} 2^{2\ell(\frac{1}{p}-\frac{p-2}{2p})}\|\Delta_\ell w_L\|_{L^\infty_t(L^p)}\\
&\lesssim d_k 2^{\frac{2k}{p}}\|w_L\|_{\widetilde{L}^\infty_t(B^{-1+\frac{2}{p}}_{p,1})} \lesssim d_k 2^{\frac{2k}{p}}\|u_0\|_{B^{-1+\frac{2}{p}}_{p,1}},
\end{aligned}
$$

where we used (3.1) in the last step. Then applying Lemma 2.1 once again leads to

$$
\begin{aligned}
\|\Delta_j(T_{w_L}w_L)\|_{L^1_t(L^2)} &\lesssim \sum_{|k-j|\leq 5}\|S_{k-1}w_L\|_{L^\infty_t(L^{\frac{2p}{p-2}})}\|\Delta_k w_L\|_{L^1_t(L^p)}\\
&\lesssim \frac{1}{\mu}d_j 2^{-j}\|u_0\|^2_{B^{-1+\frac{2}{p}}_{p,1}}.
\end{aligned} \tag{3.5}
$$

Similarly as $2 \le p < 4$, we get by applying Lemma 2.1 that

$$\begin{aligned}\|\Delta_j(R(w_L,w_L))\|_{L^1_t(L^2)} \lesssim & 2^{2j(\frac{2}{p}-\frac{1}{2})}\sum_{k\ge j-N_0}\|\widetilde{\Delta}_k w_L\|_{L^\infty_t(L^p)}\|\Delta_k w_L\|_{L^1_t(L^p)}\\ \lesssim & \frac{1}{\mu}2^{j(\frac{4}{p}-1)}\sum_{k\ge j-N_0} d_k 2^{-k\frac{4}{p}}\|u_0\|^2_{B^{-1+\frac{2}{p}}_{p,1}}\\ \lesssim & \frac{1}{\mu}d_j 2^{-j}\|u_0\|^2_{B^{-1+\frac{2}{p}}_{p,1}}.\end{aligned}$$

This together with (3.5) proves (3.4) for $p \in [2,4)$.

On the other hand, when $p \in (1,2)$, let p' be determined by $\frac{1}{p'} = 1 - \frac{1}{p}$, we deduce from Lemma 2.1 that

$$\begin{aligned}\|S_{k-1}w_L\|_{L^\infty_t(L^{p'})} \lesssim & \sum_{\ell\le k-2} 2^{2\ell(\frac{2}{p}-1)}\|\Delta_\ell w_L\|_{L^\infty_t(L^p)}\\ \lesssim & d_k 2^{k(\frac{2}{p}-1)}\|w_L\|_{\widetilde{L}^\infty_t(B^{-1+\frac{2}{p}}_{p,1})} \lesssim d_k 2^{k(\frac{2}{p}-1)}\|u_0\|_{B^{-1+\frac{2}{p}}_{p,1}},\end{aligned}$$

and

$$\begin{aligned}\|\Delta_j(T_{w_L}w_L)\|_{L^1_t(L^2)} \lesssim & 2^j\sum_{|k-j|\le 5}\|S_{k-1}w_L\|_{L^\infty_t(L^{p'})}\|\Delta_k w_L\|_{L^1_t(L^p)}\\ \lesssim & \frac{1}{\mu}d_j 2^{-j}\|u_0\|^2_{B^{-1+\frac{2}{p}}_{p,1}}.\end{aligned} \tag{3.6}$$

Along the same line, one has

$$\begin{aligned}\|\Delta_j(R(w_L,w_L))\|_{L^1_t(L^2)} \lesssim & 2^j\sum_{k\ge j-N_0}\|\widetilde{\Delta}_k w_L\|_{L^\infty_t(L^{p'})}\|\Delta_k w_L\|_{L^1_t(L^p)}\\ \lesssim & \frac{1}{\mu}2^j\sum_{k\ge j-N_0} d_k 2^{-2k}\|u_0\|^2_{B^{-1+\frac{2}{p}}_{p,1}}\\ \lesssim & \frac{1}{\mu}d_j 2^{-j}\|u_0\|^2_{B^{-1+\frac{2}{p}}_{p,1}}.\end{aligned}$$

This together with (3.6) ensures (3.4) for $p \in (1,2)$. □

With the above two technical lemmas, we now present the proof of Proposition 3.1.

Proof of Proposition 3.1. As the existence part of Proposition 3.1 essentially follows from (3.3). Again for simplicity, we just present the detailed proof to (3.3) for smooth enough solutions of (1.7). We first get by taking the L^2 inner product of (3.1) with $\bar{w}$ that

$$\frac{1}{2}\frac{d}{dt}\|\bar{w}(t)\|^2_{L^2} + \mu\|\nabla\bar{w}(t)\|^2_{L^2} \le \|\bar{w}\|^2_{L^2}\|\nabla w_L\|_{L^\infty} + \|\bar{w}\|_{L^2}\|w_L\cdot\nabla w_L\|_{L^2}.$$

Applying Gronwall's inequality and then using (3.1), Lemma 3.2, we get that

$$\begin{aligned}\|\bar{w}\|_{L^\infty_t(L^2)} \le & \|w_L\cdot\nabla w_L\|_{L^1_t(L^2)}\exp\bigl\{\|\nabla w_L\|_{L^1_t(L^\infty)}\bigr\}\\ \le & \frac{C}{\mu}\|u_0\|^2_{B^{-1+\frac{2}{p}}_{p,1}}\exp\Bigl\{\frac{C}{\mu}\|u_0\|_{B^{-1+\frac{2}{p}}_{p,1}}\Bigr\} \le C\|u_0\|_{B^{-1+\frac{2}{p}}_{p,1}}\exp\Bigl\{\frac{C}{\mu}\|u_0\|_{B^{-1+\frac{2}{p}}_{p,1}}\Bigr\},\end{aligned} \tag{3.7}$$

and

$$
\begin{aligned}
\mu\|\nabla\bar{w}\|_{L^2_t(L^2)}^2 &\leq \|\bar{w}\|_{L^\infty_t(L^2)}^2\|\nabla w_L\|_{L^1_t(L^\infty)} + \|\bar{w}\|_{L^\infty_t(L^2)}\|w_L\cdot\nabla w_L\|_{L^1_t(L^2)}\\
&\leq \frac{C}{\mu}\|u_0\|^3_{B^{-1+\frac{2}{p}}_{p,1}}\exp\Big\{\frac{C}{\mu}\|u_0\|_{B^{-1+\frac{2}{p}}_{p,1}}\Big\}\\
&\leq C\|u_0\|^2_{B^{-1+\frac{2}{p}}_{p,1}}\exp\Big\{\frac{C}{\mu}\|u_0\|_{B^{-1+\frac{2}{p}}_{p,1}}\Big\}.
\end{aligned}\tag{3.8}
$$

On the other hand, we notice that

$$
\begin{aligned}
\|\bar{w}\cdot\nabla\bar{w}\|_{L^1_t(B^0_{2,1})} \leq& C\int_0^t\|\bar{w}\cdot\nabla\bar{w}\|_{B^0_{2,1}}\,d\tau \leq C\int_0^t\|\bar{w}\|_{\dot{H}^{\frac{1}{2}}}\|\nabla\bar{w}\|_{\dot{H}^{\frac{1}{2}}}\,d\tau\\
\leq& C\int_0^t\|\bar{w}\|^{\frac{1}{2}}_{L^2}\|\nabla\bar{w}\|_{L^2}\|\bar{w}\|^{\frac{1}{2}}_{B^2_{2,1}}\,d\tau.
\end{aligned}
$$

So that it follows from (3.2), Proposition 2.1 and Lemma 3.1 that

$$
\begin{aligned}
&\|\bar{w}\|_{\widetilde{L}^\infty_t(B^0_{2,1})} + \mu\|\bar{w}\|_{L^1_t(B^2_{2,1})} + \|\nabla p\|_{L^1_t(B^0_{2,1})}\\
&\leq C\Big\{\|\bar{w}\cdot\nabla\bar{w}\|_{L^1_t(B^0_{2,1})} + \|\bar{w}\cdot\nabla w_L + w_L\cdot\nabla\bar{w}\|_{L^1_t(B^0_{2,1})} + \|w_L\cdot\nabla w_L\|_{L^1_t(B^0_{2,1})}\Big\}\\
&\leq C\Big\{\epsilon\|\bar{w}\|_{L^1_t(B^2_{2,1})} + \|\bar{w}\|_{L^\infty_t(L^2)}\|\nabla\bar{w}\|^2_{L^2_t(L^2)}\\
&\qquad + \int_0^t\big(\|w_L\|_{B^{1+\frac{2}{p}}_{p,1}} + \frac{1}{\epsilon}\|w_L\|^2_{B^{\frac{2}{p}}_{p,1}}\big)\|\bar{w}\|_{B^0_{2,1}}\,d\tau + \frac{C}{\mu}\|u_0\|^2_{B^{-1+\frac{2}{p}}_{p,1}}\Big\}.
\end{aligned}
$$

Taking $\epsilon = \frac{\mu}{2C}$ in the above inequality and using (3.1), (3.8), we infer

$$
\begin{aligned}
&\|\bar{w}\|_{\widetilde{L}^\infty_t(B^0_{2,1})} + \mu\|\bar{w}\|_{L^1_t(B^2_{2,1})} + \|\nabla p\|_{L^1_t(B^0_{2,1})}\\
&\leq C\exp\Big\{C\int_0^t(\|w_L\|_{B^{1+\frac{2}{p}}_{p,1}} + \frac{1}{\mu}\|w_L\|^2_{B^{\frac{2}{p}}_{p,1}})\,d\tau\Big\}\\
&\qquad\times\Big(\frac{C}{\mu}\|u_0\|^3_{B^{-1+\frac{2}{p}}_{p,1}}\exp\{\frac{C}{\mu}\|u_0\|_{B^{-1+\frac{2}{p}}_{p,1}}\} + \frac{C}{\mu}\|u_0\|^2_{B^{-1+\frac{2}{p}}_{p,1}}\Big)\\
&\leq C\|u_0\|_{B^{-1+\frac{2}{p}}_{p,1}}(1+\|u_0\|_{B^{-1+\frac{2}{p}}_{p,1}})\exp\Big\{\frac{C}{\mu^2}\|u_0\|^2_{B^{-1+\frac{2}{p}}_{p,1}}\Big\}.
\end{aligned}\tag{3.9}
$$

Therefore, summing up (3.1) and (3.9) results in

$$
\begin{aligned}
\|w\|_{\widetilde{L}^\infty_t(B^{-1+\frac{2}{p}}_{p,1})} + \mu\|w\|_{L^1_t(B^{1+\frac{2}{p}}_{p,1})} + \|\nabla p\|_{L^1_t(B^{-1+\frac{2}{p}}_{p,1})}
\leq&\big(\|w_L\|_{\widetilde{L}^\infty_t(B^{-1+\frac{2}{p}}_{p,1})} + \mu\|w_L\|_{L^1_t(B^{1+\frac{2}{p}}_{p,1})}\big)\\
&+\big(\|\bar{w}\|_{\widetilde{L}^\infty_t(B^{-1+\frac{2}{p}}_{p,1})} + \mu\|\bar{w}\|_{L^1_t(B^{1+\frac{2}{p}}_{p,1})} + \|\nabla p\|_{L^1_t(B^{-1+\frac{2}{p}}_{p,1})}\big)\\
\leq& C\|u_0\|_{B^{-1+\frac{2}{p}}_{p,1}}(1+\|u_0\|_{B^{-1+\frac{2}{p}}_{p,1}})\exp\Big\{\frac{C}{\mu^2}\|u_0\|^2_{B^{-1+\frac{2}{p}}_{p,1}}\Big\},
\end{aligned}
$$

which gives rise to (3.3). The uniqueness part of Proposition 3.1 has been proved in [14, 15]. This completes the proof of the proposition. ∎

4. The proof of Theorem 1.1

The goal of this section is to present the proof of Theorem 1.1. In fact, given $a_0 \in B^{\frac{2}{q}}_{q,1}(\mathbb{R}^2)$, $u_0 \in B^{-1+\frac{2}{p}}_{p,1}(\mathbb{R}^2)$ with $\|a_0\|_{B^{\frac{2}{q}}_{q,1}}$ being sufficiently small and p, q satisfying the conditions listed in Theorem 1.1, it follows by a similar argument as that in [2] that there exists a positive time T so that (1.2) has a unique solution $(a, u, \nabla\Pi)$ with

$$
\begin{aligned}
&a \in \mathcal{C}([0,T]; B^{\frac{2}{q}}_{q,1}(\mathbb{R}^2)), \qquad u \in \mathcal{C}([0,T]; B^{-1+\frac{2}{p}}_{p,1}(\mathbb{R}^2)) \cap L^1((0,T); B^{1+\frac{2}{p}}_{p,1}(\mathbb{R}^2)),\\
&\nabla\Pi \in L^1((0,T); B^{-1+\frac{2}{p}}_{p,1}(\mathbb{R}^2)).
\end{aligned}
\tag{4.1}
$$

Moreover, if $\frac{1}{p} + \frac{1}{q} \geq 1$, this solution is unique. We denote T^* to be the largest possible time so that there holds (4.1). Hence the proof of Theorem 1.1 is reduced to show that $T^* = \infty$ under the assumption of (1.4). Toward this, we split the velocity u as $w + v$, with $(w, p), (a, v, \Pi_1)$ solving (1.7) and (1.8) respectively. Then thanks to Proposition 3.1, it remains to solve (1.8) globally.

4.1. **The estimate of v.** First we reformulate the v equation of (1.8) to be

$$
\begin{aligned}
\partial_t v - \mu\Delta v =& F - (1+a)\nabla\Pi_1 + \mu a\Delta v + (1+a)\,\mathrm{div}[(\widetilde{\mu}(a) - \mu)\mathcal{M}(v)]\\
&- (v\cdot\nabla v + v\cdot\nabla w + w\cdot\nabla v).
\end{aligned}
\tag{4.2}
$$

Let $f_1(t)$, $f_2(t)$, $a_{\bar\lambda}$, $v_{\bar\lambda}$, $\nabla\Pi_{\bar\lambda}$ be given by (2.8), and $a_{\lambda_1} \stackrel{\text{def}}{=} a\exp\{-\lambda_1\int_0^t f_1(\tau)d\tau\}$. Then it follows from (4.2) that

$$
\begin{aligned}
\partial_t v_{\bar\lambda} + (\lambda_1 f_1(t) + \lambda_2 f_2(t))v_{\bar\lambda} - \mu\Delta v_{\bar\lambda} = F_{\bar\lambda} - (1+a)\nabla\Pi_{\bar\lambda} + \mu a\Delta v_{\bar\lambda}\\
+ (1+a)\,\mathrm{div}[(\widetilde{\mu}(a) - \mu)\mathcal{M}(v_{\bar\lambda})] - (v\cdot\nabla v_{\bar\lambda} + v_{\bar\lambda}\cdot\nabla w + w\cdot\nabla v_{\bar\lambda}).
\end{aligned}
$$

Applying Δ_j to the above equation and taking the L^2 inner product of the resulting equation with $|\Delta_j v_{\bar\lambda}|^{p-2}\Delta_j v_{\bar\lambda}$ (in the case when $p \in (1,2)$, we need a small modification to make this argument rigorous, which we omit here), we obtain

$$
\begin{aligned}
&\frac{1}{p}\frac{d}{dt}\|\Delta_j v_{\bar\lambda}\|^p_{L^p} + (\lambda_1 f_1(t) + \lambda_2 f_2(t))\|\Delta_j v_{\bar\lambda}\|^p_{L^p} - \mu\big(\Delta\Delta_j v_{\bar\lambda} \mid |\Delta_j v_{\bar\lambda}|^{p-2}\Delta_j v_{\bar\lambda}\big)\\
&\quad\leq\Big\{\|\Delta_j F_{\bar\lambda}\|_{L^p} + \|\Delta_j((1+a)\nabla\Pi_{\bar\lambda})\|_{L^p} + \|\Delta_j((1+a)\,\mathrm{div}[(\widetilde{\mu}(a)-\mu)\mathcal{M}(v_{\bar\lambda})])\|_{L^p}\\
&\qquad + \mu\|\Delta_j(a\Delta v_{\bar\lambda})\|_{L^p} + \|\Delta_j(v\cdot\nabla v_{\bar\lambda} + v_{\bar\lambda}\cdot\nabla w + w\cdot\nabla v_{\bar\lambda})\|_{L^p}\Big\}\|\Delta_j v_{\bar\lambda}\|^{p-1}_{L^p}.
\end{aligned}
\tag{4.3}
$$

While applying Lemma A.5 of [7] that

$$
-\big(\Delta\Delta_j v_{\bar\lambda} \mid |\Delta_j v_{\bar\lambda}|^{p-2}\Delta_j v_{\bar\lambda}\big) \geq \bar{c}2^{2j}\|\Delta_j v_{\bar\lambda}\|^p_{L^p}
$$

for some positive constant $\bar{c}$, from which and (4.3), we deduce that

$$
\begin{aligned}
&\|\Delta_j v_{\bar\lambda}\|_{L^\infty_t(L^p)} + \int_0^t (\lambda_1 f_1(t') + \lambda_2 f_2(t'))\|\Delta_j v_{\bar\lambda}\|_{L^p}\,dt' + \bar{c}\mu\|\Delta_j v_{\bar\lambda}\|_{L^1_t(L^p)}\\
&\leq\Big\{\|\Delta_j F_{\bar\lambda}\|_{L^1_t(L^p)} + \|\Delta_j((1+a)\nabla\Pi_{\bar\lambda})\|_{L^1_t(L^p)} + \|\Delta_j((1+a)\,\mathrm{div}[(\widetilde{\mu}(a)-\mu)\mathcal{M}(v_{\bar\lambda})])\|_{L^1_t(L^p)}\\
&\quad + \mu\|\Delta_j(a\Delta v_{\bar\lambda})\|_{L^1_t(L^p)} + \|\Delta_j(v\cdot\nabla v_{\bar\lambda} + v_{\bar\lambda}\cdot\nabla w + w\cdot\nabla v_{\bar\lambda})\|_{L^1_t(L^p)}\Big\}.
\end{aligned}
$$

This gives rise to

$$
\begin{aligned}
&\|v_{\bar\lambda}\|_{\widetilde{L}^\infty_t(B^{-1+\frac{2}{p}}_{p,1})}+\lambda_1\|v_{\bar\lambda}\|_{L^1_{t,f_1}(B^{-1+\frac{2}{p}}_{p,1})}+\lambda_2\|v_{\bar\lambda}\|_{L^1_{t,f_2}(B^{-1+\frac{2}{p}}_{p,1})}+\bar c\mu\|v_{\bar\lambda}\|_{L^1_t(B^{1+\frac{2}{p}}_{p,1})}\\
&\leq C\Big\{\|F_{\bar\lambda}\|_{L^1_t(B^{-1+\frac{2}{p}}_{p,1})}+\|(1+a)\nabla\Pi_{\bar\lambda}\|_{L^1_t(B^{-1+\frac{2}{p}}_{p,1})}+\|v\|_{\widetilde{L}^\infty_t(B^{-1+\frac{2}{p}}_{p,1})}\|v_{\bar\lambda}\|_{L^1_t(B^{1+\frac{2}{p}}_{p,1})}\\
&\quad+\big(\mu+\mathfrak{C}(1+\|a\|_{\widetilde{L}^\infty_t(B^{\frac{2}{q}}_{q,1})})\big)\|a\|_{\widetilde{L}^\infty_t(B^{\frac{2}{q}}_{q,1})}\|v_{\bar\lambda}\|_{L^1_t(B^{1+\frac{2}{p}}_{p,1})}+\|v_{\bar\lambda}\cdot\nabla w+w\cdot\nabla v_{\bar\lambda}\|_{L^1_t(B^{-1+\frac{2}{p}}_{p,1})}\Big\},
\end{aligned}
\tag{4.4}
$$

where the norm $\|v_{\bar\lambda}\|_{L^1_{t,f}(B^{-1+\frac{2}{p}}_{p,1})}$ is given by Definition 2.3 and $\mathfrak{C}$ is a positive constant depending on $\|\widetilde{\mu}'\|_{L^\infty(-1,1)}$ as long as $\|a\|_{L^\infty}\leq 1$.

Let

$$
\bar T\stackrel{\rm def}{=}\sup\Big\{\ t<T^*,\quad \|a\|_{\widetilde{L}^\infty_t(B^{\frac{2}{q}}_{q,1})}\leq c_1\ \Big\}
\tag{4.5}
$$

for some c_1 sufficiently small. Then (2.9) ensures that for $t\leq\bar T$

$$
\begin{aligned}
&\|(1+a)\nabla\Pi_{\bar\lambda}\|_{L^1_t(B^{-1+\frac{2}{p}}_{p,1})}\\
&\quad\leq\mathfrak{C}\Big\{\epsilon\|v_{\bar\lambda}\|_{L^1_t(B^{1+\frac{2}{p}}_{p,1})}+\|v\|_{\widetilde{L}^\infty_t(B^{-1+\frac{2}{p}}_{p,1})}\|v_{\bar\lambda}\|_{L^1_t(B^{1+\frac{2}{p}}_{p,1})}\\
&\qquad+\|v_{\bar\lambda}\|_{L^1_{t,f_1}(B^{-1+\frac{2}{p}}_{p,1})}+\frac{1}{\epsilon}\|v_{\bar\lambda}\|_{L^1_{t,f_2}(B^{-1+\frac{2}{p}}_{p,1})}\\
&\qquad+(1+\mu+\|a\|_{\widetilde{L}^\infty_t(B^{\frac{2}{q}}_{q,1})})\big(\|a_{\bar\lambda}\|_{L^1_{t,f_1}(B^{\frac{2}{q}}_{q,1})}+\|a\|_{\widetilde{L}^\infty_t(B^{\frac{2}{q}}_{q,1})}\|v_{\bar\lambda}\|_{L^1_t(B^{1+\frac{2}{p}}_{p,1})}\big)\Big\},
\end{aligned}
$$

from which, Lemma 2.5, we infer from (4.4) that

$$
\begin{aligned}
&\|v_{\bar\lambda}\|_{\widetilde{L}^\infty_t(B^{-1+\frac{2}{p}}_{p,1})}+\lambda_1\|v_{\bar\lambda}\|_{L^1_{t,f_1}(B^{-1+\frac{2}{p}}_{p,1})}+\lambda_2\|v_{\bar\lambda}\|_{L^1_{t,f_2}(B^{-1+\frac{2}{p}}_{p,1})}+\bar c\mu\|v_{\bar\lambda}\|_{L^1_t(B^{1+\frac{2}{p}}_{p,1})}\\
&\leq\mathfrak{C}\Big\{\epsilon\|v_{\bar\lambda}\|_{L^1_t(B^{1+\frac{2}{p}}_{p,1})}+\|v\|_{\widetilde{L}^\infty_t(B^{-1+\frac{2}{p}}_{p,1})}\|v_{\bar\lambda}\|_{L^1_t(B^{1+\frac{2}{p}}_{p,1})}+\|v_{\bar\lambda}\|_{L^1_{t,f_1}(B^{-1+\frac{2}{p}}_{p,1})}+\frac{1}{\epsilon}\|v_{\bar\lambda}\|_{L^1_{t,f_2}(B^{-1+\frac{2}{p}}_{p,1})}\\
&\quad+(1+\mu+\|a\|_{\widetilde{L}^\infty_t(B^{\frac{2}{q}}_{q,1})})\big(\|a_{\bar\lambda}\|_{L^1_{t,f_1}(B^{\frac{2}{q}}_{q,1})}+\|a\|_{\widetilde{L}^\infty_t(B^{\frac{2}{q}}_{q,1})}\|v_{\bar\lambda}\|_{L^1_t(B^{1+\frac{2}{p}}_{p,1})}\big)\Big\}
\end{aligned}
\tag{4.6}
$$

for $t\leq\bar T$ and some positive constant $\mathfrak{C}$ depending on $\|\widetilde{\mu}'\|_{L^\infty(-1,1)}$.

4.2. **The proof of Theorem 1.1.** We get by taking $\lambda=\lambda_1$ in Proposition 2.2 that

$$
\|a_{\lambda_1}\|_{\widetilde{L}^\infty_t(B^{\frac{2}{q}}_{q,1})}+\frac{\lambda_1}{2}\|a_{\lambda_1}\|_{L^1_{t,f_1}(B^{\frac{2}{q}}_{q,1})}\leq\|a_0\|_{B^{\frac{2}{q}}_{q,1}}+C\|v\|_{L^1_t(B^{1+\frac{2}{p}}_{p,1})}\|a_{\lambda_1}\|_{\widetilde{L}^\infty_t(B^{\frac{2}{q}}_{q,1})}.
\tag{4.7}
$$

Note that

$$
\|a_{\bar\lambda}\|_{L^1_{t,f_1}(B^{\frac{2}{q}}_{q,1})}\leq\|a_{\lambda_1}\|_{L^1_{t,f_1}(B^{\frac{2}{q}}_{q,1})},
$$

By summing up (4.6) and (4.7)×(1+μ) and choosing $\varepsilon, \lambda_1, \lambda_2$ satisfying $\mathfrak{C}\epsilon = \frac{\bar{c}}{2}\mu$, $\lambda_1 = 8\mathfrak{C}$, $\lambda_2 = \frac{2\mathfrak{C}^2}{\bar{c}\mu}$, we obtain

$$
\begin{aligned}
&(1+\mu)\|a_{\lambda_1}\|_{\widetilde{L}^\infty_t(B^{\frac{2}{q}}_{q,1})} + \|v_{\bar{\lambda}}\|_{\widetilde{L}^\infty_t(B^{-1+\frac{2}{p}}_{p,1})} + \frac{\lambda_1}{2}\Big(\frac{1+\mu}{2}\|a_{\lambda_1}\|_{L^1_{t,f_1}(B^{\frac{2}{q}}_{q,1})} + \|v_{\bar{\lambda}}\|_{L^1_{t,f_1}(B^{-1+\frac{2}{p}}_{p,1})}\Big)\\
&\quad + \frac{\lambda_2}{2}\|v_{\bar{\lambda}}\|_{L^1_{t,f_2}(B^{-1+\frac{2}{p}}_{p,1})} + \frac{\bar{c}\mu}{2}\|v_{\bar{\lambda}}\|_{L^1_t(B^{1+\frac{2}{p}}_{p,1})}\\
&\leq (1+\mu)\|a_0\|_{B^{\frac{2}{q}}_{q,1}} + C_1\Big\{(1+\mu)\|v\|_{L^1_t(B^{1+\frac{2}{p}}_{p,1})}\|a_{\lambda_1}\|_{\widetilde{L}^\infty_t(B^{\frac{2}{q}}_{q,1})} + \|v\|_{\widetilde{L}^\infty_t(B^{-1+\frac{2}{p}}_{p,1})}\|v_{\bar{\lambda}}\|_{L^1_t(B^{1+\frac{2}{p}}_{p,1})}\\
&\quad + (1+\mu+\|a\|_{\widetilde{L}^\infty_t(B^{\frac{2}{q}}_{q,1})})\|a\|_{\widetilde{L}^\infty_t(B^{\frac{2}{q}}_{q,1})}\|v_{\bar{\lambda}}\|_{L^1_t(B^{1+\frac{2}{p}}_{p,1})}\Big\}
\end{aligned}
\tag{4.8}
$$

for $t \leq \bar{T}$ and some positive constant C_1 depending on $\|\widetilde{\mu}'\|_{L^\infty(-1,1)}$.

Now let c_2 be a small enough positive constant, which will be determined later on. We define Υ by

$$
\Upsilon \stackrel{def}{=} sup\Big\{t < T^* : (1+\mu)\|a\|_{\widetilde{L}^\infty_t(B^{\frac{2}{q}}_{q,1})} + \|v\|_{\widetilde{L}^\infty_t(B^{-1+\frac{2}{p}}_{p,1})} + \mu\|v\|_{L^1_t(B^{1+\frac{2}{p}}_{p,1})} \leq c_2\mu\Big\}.
\tag{4.9}
$$

(4.9) together with (4.5) implies that $\Upsilon \leq \bar{T}$, if we take $c_2 \leq c_1$. We shall prove that $\Upsilon = \infty$ under the assumption of (1.4). Otherwise, taking $c_2 \leq \min\big(\frac{\bar{c}}{12C_1}, \frac{1}{2C_1}\big)$, we deduce form (4.8) that

$$
\|v_{\bar{\lambda}}\|_{\widetilde{L}^\infty_t(B^{-1+\frac{2}{p}}_{p,1})} + \frac{1+\mu}{2}\|a_{\lambda_1}\|_{\widetilde{L}^\infty_t(B^{\frac{2}{q}}_{q,1})} + \frac{\bar{c}\mu}{4}\|v_{\bar{\lambda}}\|_{L^1_t(B^{1+\frac{2}{p}}_{p,1})} \leq (1+\mu)\|a_0\|_{B^{\frac{2}{q}}_{q,1}},
$$

for $t \leq \Upsilon$. This together with (2.8) gives rise to

$$
\begin{aligned}
&\|v\|_{\widetilde{L}^\infty_t(B^{-1+\frac{2}{p}}_{p,1})} + \frac{1+\mu}{2}\|a\|_{\widetilde{L}^\infty_t(B^{\frac{2}{q}}_{q,1})} + \frac{\bar{c}\mu}{4}\|v\|_{L^1_t(B^{1+\frac{2}{p}}_{p,1})}\\
&\leq (1+\mu)\|a_0\|_{B^{\frac{2}{q}}_{q,1}} \exp\Big\{C_2\int_0^t \big(\|w(\tau)\|_{B^{1+\frac{2}{p}}_{p,1}} + \frac{1}{\mu}\|\nabla p(\tau)\|_{B^{-1+\frac{2}{p}}_{p,1}} + \frac{1}{\mu}\|w(\tau)\|^2_{B^{\frac{2}{p}}_{p,1}}\big)\,d\tau\Big\},
\end{aligned}
\tag{4.10}
$$

for $t \leq \Upsilon$.

Combining (4.10) with (3.3), we reach

$$
\begin{aligned}
&\|v\|_{\widetilde{L}^\infty_t(B^{-1+\frac{2}{p}}_{p,1})} + \frac{1+\mu}{2}\|a\|_{\widetilde{L}^\infty_t(B^{\frac{2}{q}}_{q,1})} + \frac{\bar{c}\mu}{4}\|v\|_{L^1_t(B^{1+\frac{2}{p}}_{p,1})}\\
&\leq (1+\mu)\|a_0\|_{B^{\frac{2}{q}}_{q,1}} \exp\Big\{\Big[\frac{C_3}{\mu}\|u_0\|_{B^{-1+\frac{2}{p}}_{p,1}}(1+\|u_0\|_{B^{-1+\frac{2}{p}}_{p,1}})\\
&\qquad + \frac{C_3^2}{\mu^2}\|u_0\|^2_{B^{-1+\frac{2}{p}}_{p,1}}(1+\|u_0\|^2_{B^{-1+\frac{2}{p}}_{p,1}})\Big]\exp\Big(\frac{C_3}{\mu^2}\|u_0\|^2_{B^{-1+\frac{2}{p}}_{p,1}}\Big)\Big\}\\
&\leq (1+\mu)\|a_0\|_{B^{\frac{2}{q}}_{q,1}} \exp\Big\{C_4(1+\|u_0\|^2_{B^{-1+\frac{2}{p}}_{p,1}})\exp\Big(\frac{C_4}{\mu^2}\|u_0\|^2_{B^{-1+\frac{2}{p}}_{p,1}}\Big)\Big\}\\
&\leq (1+\mu)\|a_0\|_{B^{\frac{2}{q}}_{q,1}} \exp\Big\{\bar{C}(1+\mu^2)\exp\Big(\frac{\bar{C}}{\mu^2}\|u_0\|^2_{B^{-1+\frac{2}{p}}_{p,1}}\Big)\Big\}
\end{aligned}
$$

for $t \leq \Upsilon$ and some positive constants $\bar{C}$ which depends on $\|\widetilde{\mu}'\|_{L^\infty(-1,1)}$. If we take C_0 large enough and c_0 sufficiently small in (1.4), which depend on $\|\widetilde{\mu}'\|_{L^\infty(-1,1)}$, there holds

$$
(1+\mu)\|a\|_{\widetilde{L}^\infty_t(B^{\frac{2}{q}}_{q,1})} + \|v\|_{\widetilde{L}^\infty_t(B^{-1+\frac{2}{p}}_{p,1})} + \mu\|v\|_{L^1_t(B^{1+\frac{2}{p}}_{p,1})} \leq \frac{c_2}{2}\mu
$$

for $t \leq \Upsilon$, which contradicts with (4.9). Whence we conclude that $\Upsilon = T^* = \infty$. This completes the proof of Theorem 1.1 ■

5. The proof of Theorem 1.3

The proof of Theorem 1.3 basically follows the same line of the proof to Theorem 1.1. More precisely,

5.1. **Estimates of the transport equation.** As we shall not use Lagrange approach in [10], we need first to investigate the following transport equation

$$\partial_t f + u \cdot \nabla f = 0, \qquad f|_{t=0} = f_0 \tag{5.1}$$

with initial data f_0 in multiplier space of $B^s_{p,1}(\mathbb{R}^2)$.

Lemma 5.1. *Let $f \in B^s_{p,1}(\mathbb{R}^d)$ with $-1 < s < 1$, and $u \in L^1((0,T); Lip(\mathbb{R}^d))$. Let X_u be the flow map determined by* (1.6). *Then $f \circ X_u \in \widetilde{L}^\infty((0,T); B^s_{p,1}(\mathbb{R}^d))$, and there holds*

$$\|f \circ X_u\|_{\widetilde{L}^\infty_t(B^s_{p,1})} \leq C\|f\|_{B^s_{p,1}} \exp\Big\{C \int_0^t \|\nabla u(\tau)\|_{L^\infty}\, d\tau\Big\} \quad \text{for} \quad t \leq T. \tag{5.2}$$

Proof. Let $f_\ell \overset{\text{def}}{=} \Delta_\ell f$, we deduce from Lemma 2.2 that

$$\|\Delta_j(f_\ell \circ X_u)\|_{L^\infty_t(L^p)} \leq C d_\ell 2^{-\ell s}\|f\|_{B^s_{p,1}} \min\big(2^{j-\ell}, 2^{\ell-j}\big) \exp\Big\{C \int_0^t \|\nabla u(\tau)\|_{L^\infty} d\tau\Big\},$$

from which and $-1 < s < 1$, we infer for any $j \in \mathbb{Z}$

$$\begin{aligned}\|\Delta_j(f \circ X_u)\|_{L^\infty_t(L^p)} \leq& \Big(\sum_{\ell<j} + \sum_{\ell\geq j}\Big)\|\Delta_j(f_\ell \circ X_u)\|_{L^\infty_t(L^p)}\\ \leq& C\|f\|_{B^s_{p,1}}\Big(\sum_{\ell<j} d_\ell 2^{-\ell s}2^{\ell-j} + \sum_{\ell\geq j} d_\ell 2^{-\ell s}2^{j-\ell}\Big) \exp\Big\{C \int_0^t \|\nabla u(\tau)\|_{L^\infty}\, d\tau\Big\}\\ \leq& C d_j 2^{-js}\|f\|_{B^s_{p,1}} \exp\Big\{C \int_0^t \|\nabla u(\tau)\|_{L^\infty}\, d\tau\Big\},\end{aligned} \tag{5.3}$$

this together with Definition 2.2 implies (5.2), and we complete the proof of the lemma. □

Remark 5.1. *The case when $\frac{d}{p} \geq 1$, $s \in (-1, \frac{d}{p})$, and $u \in L^1((0,T); B^{1+\frac{d}{p}}(\mathbb{R}^d))$, we have a similar version of Lemma 5.1. For simplicity, we just present the case when $\frac{d}{p} = 1$. Instead of* (5.2), *we shall prove*

$$\|f \circ X_u\|_{\widetilde{L}^\infty_t(B^1_{p,1})} \leq C\|f\|_{B^1_{p,1}}(1 + \|u\|_{L^1_t(B^2_{p,1})}) \exp\Big\{C \int_0^t \|\nabla u(\tau)\|_{L^\infty}\, d\tau\Big\}. \tag{5.4}$$

We first deduce from (5.3) *that*

$$\|f \circ X_u\|_{\widetilde{L}^\infty_t(B^1_{p,\infty})} \leq C\|f\|_{B^1_{p,1}} \exp\Big\{C \int_0^t \|\nabla u(\tau)\|_{L^\infty}\, d\tau\Big\}. \tag{5.5}$$

While we get by taking ∇_y to (1.6) *that*

$$\nabla_y X_u(t,y) = Id + \int_0^t \nabla u(\tau, X_u(\tau,y))\nabla_y X_u(t,y)\, d\tau,$$

from which, and the standard product laws in Besov spaces, we infer

$$\begin{aligned}\|\nabla_y X_u - Id\|_{\widetilde{L}^\infty_t(B^1_{p,\infty})} \leq & C\int_0^t \big(\|\nabla u(\tau, X_u(\tau,\cdot))\|_{B^1_{p,\infty}}(1+\|\nabla_y X_u(t,\cdot)\|_{L^\infty}) \\ & + \|\nabla u(\tau,\cdot)\|_{L^\infty}\|\nabla_y X_u - Id\|_{B^1_{p,\infty}}\big)\,d\tau.\end{aligned}$$

Applying Gronwall's inequality and (5.5) *gives*

$$\|\nabla_y X_u - Id\|_{\widetilde{L}^\infty_t(B^1_{p,\infty})} \leq C\|u\|_{L^1_t(B^2_{p,1})}\exp\Big\{C\int_0^t \|\nabla u(\tau)\|_{L^\infty}\,d\tau\Big\}. \tag{5.6}$$

On the other hand, notice that

$$\nabla_y(f\circ X_u) = \nabla f\circ X_u(\nabla_y X_u - Id) + \nabla f\circ X_u,$$

from which and Bony's decomposition, we infer

$$\begin{aligned}\|f\circ X_u\|_{\widetilde{L}^\infty_t(B^1_{p,1})} &= \|\nabla_y(f\circ X_u)\|_{\widetilde{L}^\infty_t(B^0_{p,1})} \\ &\leq C\|\nabla f\circ X_u\|_{\widetilde{L}^\infty_t(B^0_{p,1})}\big(1+\|\nabla_y X_u - Id\|_{L^\infty_t(L^\infty)} + \|\nabla_y X_u - Id\|_{\widetilde{L}^\infty_t(B^1_{p,\infty})}\big).\end{aligned} \tag{5.7}$$

While applying (5.2) *yields*

$$\|\nabla f\circ X_u\|_{\widetilde{L}^\infty_t(B^0_{p,1})} \leq C\|f\|_{B^1_{p,1}}\exp\Big\{C\int_0^t \|\nabla u(\tau)\|_{L^\infty}\,d\tau\Big\}.$$

This together with (5.6) *and* (5.7) *enures* (5.4).

The main result of this subsection is as follows:

Proposition 5.1. *Let* $2<p<4$, $-1<s\leq \frac{2}{p}$ *and* $u\in L^1((0,T),B^{1+\frac{2}{p}}_{p,1}(\mathbb{R}^2))$ *be a divergence free vector fields. Then given* $f_0\in\mathcal{M}(B^s_{p,1}(\mathbb{R}^2))$, (5.1) *has a unique solution* $f\in L^\infty((0,T);\mathcal{M}(B^s_{p,1}(\mathbb{R}^2)))$, *moreover, there holds*

$$\|f\|_{L^\infty_t(\mathcal{M}(B^s_{p,1}))} \leq C\|f_0\|_{\mathcal{M}(B^s_{p,1})}\exp\Big\{C\int_0^t \|\nabla u(\tau)\|_{L^\infty}\,d\tau\Big\} \tag{5.8}$$

for any $t\in(0,T]$.

Proof. Both the existence and uniqueness part of Proposition 5.1 follows from (5.8). Indeed let X_u be the flow map determined by (1.6). Then we infer from (5.1) that $f(t,x)=f_0(X_u^{-1}(t,x))$, from which, Definition 1.1 and Lemma 5.1, we infer

$$\begin{aligned}\|f(t)\|_{\mathcal{M}(B^s_{p,1})} &= \sup_{\|\psi\|_{B^s_{p,1}}=1}\|\psi f(t)\|_{B^s_{p,1}} \\ &= \sup_{\|\psi\|_{B^s_{p,1}}=1}\|(\psi\circ X_u(t)f_0)\circ X_u^{-1}(t)\|_{B^s_{p,1}} \\ &\leq C\sup_{\|\psi\|_{B^s_{p,1}}=1}\|\psi\circ X_u(t)f_0\|_{B^s_{p,1}}\exp\Big\{C\int_0^t \|\nabla u(\tau)\|_{L^\infty}\,d\tau\Big\},\end{aligned}$$

applying Lemma 5.1 once again leads to

$$\begin{aligned}\|f(t)\|_{\mathcal{M}(B^s_{p,1})} &\leq C\|f_0\|_{\mathcal{M}(B^s_{p,1})}\exp\Big\{C\int_0^t \|\nabla u(\tau)\|_{L^\infty}\Big\}\sup_{\|\psi\|_{B^s_{p,1}}=1}\|\psi\circ X_u(t)\|_{B^s_{p,1}} \\ &\leq C\|f_0\|_{\mathcal{M}(B^s_{p,1})}\exp\Big\{C\int_0^t \|\nabla u(\tau)\|_{L^\infty}\Big\}\sup_{\|\psi\|_{B^s_{p,1}}=1}\|\psi\|_{B^s_{p,1}} \\ &\leq C\|f_0\|_{\mathcal{M}(B^s_{p,1})}\exp\Big\{C\int_0^t \|\nabla u(\tau)\|_{L^\infty}\Big\} \quad \text{for any} \quad t\leq T.\end{aligned}$$

This completes the proof of Proposition 5.1. $\square$

5.2. **Estimates of the pressure.** In this subsection, we aim at providing similar version of Proposition 2.3 in the case when $a \in L^\infty((0,T);\mathcal{M}(B_{p,1}^{-1+\frac{2}{p}}(\mathbb{R}^2)))$ and $\widetilde{\mu}(a)-\mu \in L^\infty((0,T);\mathcal{M}(B_{p,1}^{\frac{2}{p}}(\mathbb{R}^2)))$.

Proposition 5.2. *Let $p \in [2,4)$, $a \in L^\infty((0,T);\mathcal{M}(B_{p,1}^{-1+\frac{2}{p}}(\mathbb{R}^2)))$ and $\widetilde{\mu}(a)-\mu \in L^\infty((0,T);$ $\mathcal{M}(B_{p,1}^{\frac{2}{p}}(\mathbb{R}^2)))$. Let $w, v \in \widetilde{L}_T^\infty(B_{p,1}^{-1+\frac{2}{p}}) \cap L_T^1(B_{p,1}^{1+\frac{2}{p}})$ and $\nabla p \in L_T^1(B_{p,1}^{-1+\frac{2}{p}})$. Then (2.7) has a unique solution with $\nabla \Pi_1 \in L_T^1(B_{p,1}^{-1+\frac{2}{p}})$, and for any $\epsilon>0$, $t \le T$, there holds*

$$
\begin{aligned}
\|\nabla \Pi_{\bar\lambda}\|_{L_t^1(B_{p,1}^{-1+\frac{2}{p}})} \le \frac{C}{1-C\|a\|_{L_T^\infty(\mathcal{M}(B_{p,1}^{-1+\frac{2}{p}}))}} \Big\{ & \epsilon \|v_{\bar\lambda}\|_{L_t^1(B_{p,1}^{1+\frac{2}{p}})} + \|v\|_{\widetilde{L}_t^\infty(B_{p,1}^{-1+\frac{2}{p}})} \|v_{\bar\lambda}\|_{L_t^1(B_{p,1}^{1+\frac{2}{p}})} \\
& + \|v_{\bar\lambda}\|_{L^1_{t,f_1}(B_{p,1}^{-1+\frac{2}{p}})} + \frac{1}{\epsilon}\|v_{\bar\lambda}\|_{L^1_{t,f_2}(B_{p,1}^{-1+\frac{2}{p}})} \\
& + \big(\|v_{\bar\lambda}\|_{L_t^1(B_{p,1}^{1+\frac{2}{p}})} + \int_0^t f_1(\tau)d\tau\big)\big[\mu\|a\|_{L_t^\infty(\mathcal{M}(B_{p,1}^{-1+\frac{2}{p}}))} \\
& + (1+\|a\|_{L_t^\infty(\mathcal{M}(B_{p,1}^{-1+\frac{2}{p}}))})\|\widetilde{\mu}(a)-\mu\|_{L_t^\infty(\mathcal{M}(B_{p,1}^{\frac{2}{p}}))}\big]\Big\}
\end{aligned} \tag{5.9}
$$

provided that $C\|a\|_{L_T^\infty(\mathcal{M}(B_{p,1}^{-1+\frac{2}{p}}))} \le \frac12$, where $f_1(t), f_2(t)$ and $\Pi_{\bar\lambda}, v_{\bar\lambda}$ are defined by (2.8).

Proof. Similar to the proof of Proposition 2.3, we just present the proof of (5.9) for smooth enough solutions of (2.7). Indeed along the same line to the proof of Proposition 2.3, we have (2.10). While applying Definition 1.1 and standard product laws in Besov spaces leads to

$$
\begin{aligned}
&\|v\cdot\nabla v_{\bar\lambda}\|_{L_t^1(B_{p,1}^{-1+\frac{2}{p}})} \lesssim \|v\|_{\widetilde{L}_t^\infty(B_{p,1}^{-1+\frac{2}{p}})}\|v_{\bar\lambda}\|_{L_t^1(B_{p,1}^{1+\frac{2}{p}})},\\
&\|a\nabla\Pi_{\bar\lambda}\|_{L_t^1(B_{p,1}^{-1+\frac{2}{p}})} \lesssim \|a\|_{L_t^\infty(\mathcal{M}(B_{p,1}^{-1+\frac{2}{p}}))}\|\nabla\Pi_{\bar\lambda}\|_{L_t^1(B_{p,1}^{-1+\frac{2}{p}})},\\
&\|a\Delta v_{\bar\lambda}\|_{L_t^1(B_{p,1}^{-1+\frac{2}{p}})} \lesssim \|a\|_{L_t^\infty(\mathcal{M}(B_{p,1}^{-1+\frac{2}{p}}))}\|v_{\bar\lambda}\|_{L_t^1(B_{p,1}^{1+\frac{2}{p}})},\\
&\|a(\mu\Delta w-\nabla p)\|_{L_t^1(B_{p,1}^{-1+\frac{2}{p}})} \lesssim \mu\|a\|_{L_t^\infty(\mathcal{M}(B_{p,1}^{-1+\frac{2}{p}}))}\int_0^t f_1(\tau)\,d\tau,
\end{aligned}
$$

and

$$
\begin{aligned}
&\|(1+a)\,\mathrm{div}((\widetilde{\mu}(a)-\mu)\mathcal{M}(v_{\bar\lambda}))\|_{L_t^1(B_{p,1}^{-1+\frac{2}{p}})}\\
&\qquad\lesssim (1+\|a\|_{L_t^\infty(\mathcal{M}(B_{p,1}^{-1+\frac{2}{p}}))})\|\widetilde{\mu}(a)-\mu\|_{L_t^\infty(\mathcal{M}(B_{p,1}^{\frac{2}{p}}))}\|v_{\bar\lambda}\|_{L_t^1(B_{p,1}^{1+\frac{2}{p}})},\\
&\|(1+a)\,\mathrm{div}((\widetilde{\mu}(a)-\mu)\mathcal{M}(w))\|_{L_t^1(B_{p,1}^{-1+\frac{2}{p}})}\\
&\qquad\lesssim (1+\|a\|_{L_t^\infty(\mathcal{M}(B_{p,1}^{-1+\frac{2}{p}}))})\|\widetilde{\mu}(a)-\mu\|_{L_t^\infty(\mathcal{M}(B_{p,1}^{\frac{2}{p}}))}\int_0^t f_1(\tau)\,d\tau,
\end{aligned}
$$

so that

$$
\begin{aligned}
\|F_{\bar\lambda}\|_{L_t^1(B_{p,1}^{-1+\frac{2}{p}})} &\le \|a(\mu\Delta w-\nabla p)\|_{L_t^1(B_{p,1}^{-1+\frac{2}{p}})} + \|(1+a)\,\mathrm{div}((\widetilde{\mu}(a)-\mu)\mathcal{M}(w))\|_{L_t^1(B_{p,1}^{-1+\frac{2}{p}})}\\
&\lesssim \big[\mu\|a\|_{L_t^\infty(\mathcal{M}(B_{p,1}^{-1+\frac{2}{p}}))} + (1+\|a\|_{L_t^\infty(\mathcal{M}(B_{p,1}^{-1+\frac{2}{p}}))})\|\widetilde{\mu}(a)-\mu\|_{L_t^\infty(\mathcal{M}(B_{p,1}^{\frac{2}{p}}))}\big]\int_0^t f_1(\tau)\,d\tau.
\end{aligned} \tag{5.10}
$$

Substituting the above estimates and Lemma 2.5 into (2.10) ensures (5.9) provided that

$$C\|a\|_{L^\infty_T(\mathcal{M}(B^{-1+\frac{2}{p}}_{p,1}))} \leq \frac{1}{2}$$

This completes the proof of Proposition 5.2. $\square$

5.3. **The proof of Theorem 1.3.** For $p \in (2,4)$, given $a_0 \in \mathcal{M}(B^{-1+\frac{2}{p}}_{p,1}(\mathbb{R}^2))$ with $\widetilde{\mu}(a_0) - \mu \in \mathcal{M}(B^{\frac{2}{p}}_{p,1}(\mathbb{R}^2))$, $u_0 \in B^{-1+\frac{2}{p}}_{p,1}(\mathbb{R}^2)$ with $\|a_0\|_{\mathcal{M}(B^{-1+\frac{2}{p}}_{p,1})} + \|\widetilde{\mu}(a_0) - \mu\|_{\mathcal{M}(B^{\frac{2}{p}}_{p,1})}$ being sufficiently small, it follows from Theorem 2 in [10] and Proposition 5.1 that there exists a positive time T so that (1.2) has a unique solution $(a, u, \nabla\Pi)$ with

$$\begin{aligned}
&a \in L^\infty((0,T);\mathcal{M}(B^{-1+\frac{2}{p}}_{p,1}(\mathbb{R}^2))), \qquad \widetilde{\mu}(a) - \mu \in L^\infty((0,T);\mathcal{M}(B^{\frac{2}{p}}_{p,1}(\mathbb{R}^2))),\\
&u \in \mathcal{C}([0,T];B^{-1+\frac{2}{p}}_{p,1}(\mathbb{R}^2)) \cap L^1((0,T);B^{1+\frac{2}{p}}_{p,1}(\mathbb{R}^2)), \qquad \nabla\Pi \in L^1((0,T);B^{-1+\frac{2}{p}}_{p,1}(\mathbb{R}^2)).
\end{aligned} \tag{5.11}$$

We denote by T^* the largest possible time so that there holds (5.11). Hence the proof of Theorem 1.3 is reduced to show that $T^* = \infty$ provided that there holds (1.5). Toward this, as in the proof of Theorem 1.1, we split the velocity field u as $w+v$, with $w, (a,v)$ solving (1.7) and (1.8) respectively. Then thanks to Proposition 3.1, it remains to solve (1.8) globally. In order to do so, let $f_1(t)$, $f_2(t)$, $v_{\bar\lambda}$, $\nabla\Pi_{\bar\lambda}$ be given by (2.8), along the same line to the proof of Theorem 1.1, we deduce from (4.3) that

$$\begin{aligned}
&\|v_{\bar\lambda}\|_{\widetilde{L}^\infty_t(B^{-1+\frac{2}{p}}_{p,1})} + \lambda_1\|v_{\bar\lambda}\|_{L^1_{t,f_1}(B^{-1+\frac{2}{p}}_{p,1})} + \lambda_2\|v_{\bar\lambda}\|_{L^1_{t,f_2}(B^{-1+\frac{2}{p}}_{p,1})} + \bar{c}\mu\|v_{\bar\lambda}\|_{L^1_t(B^{1+\frac{2}{p}}_{p,1})}\\
&\leq C\Big\{\|F_{\bar\lambda}\|_{L^1_t(B^{-1+\frac{2}{p}}_{p,1})} + \|(1+a)\nabla\Pi_{\bar\lambda}\|_{L^1_t(B^{-1+\frac{2}{p}}_{p,1})} + \|v\|_{\widetilde{L}^\infty_t(B^{-1+\frac{2}{p}}_{p,1})}\|v_{\bar\lambda}\|_{L^1_t(B^{1+\frac{2}{p}}_{p,1})}\\
&\quad + \|v_{\bar\lambda}\cdot\nabla w + w\cdot\nabla v_{\bar\lambda}\|_{L^1_t(B^{-1+\frac{2}{p}}_{p,1})} + \mu\|a\|_{L^\infty_t(\mathcal{M}(B^{-1+\frac{2}{p}}_{p,1}))}\|v_{\bar\lambda}\|_{L^1_t(B^{1+\frac{2}{p}}_{p,1})}\\
&\quad + \|(1+a)\,\mathrm{div}((\widetilde{\mu}(a)-\mu)\mathcal{M}(v_{\bar\lambda}))\|_{L^1_t(B^{-1+\frac{2}{p}}_{p,1})}\Big\},
\end{aligned} \tag{5.12}$$

where the norm $\|v_{\bar\lambda}\|_{L^1_{t,f}(B^{-1+\frac{2}{p}}_{p,1})}$ is given by Definition 2.3.

We denote

$$\bar{T} \stackrel{\text{def}}{=} \sup\Big\{\ t < T^*,\ \ \mu\|a\|_{L^\infty_t(\mathcal{M}(B^{-1+\frac{2}{p}}_{p,1}))} + \|\widetilde{\mu}(a) - \mu\|_{L^\infty_t(\mathcal{M}(B^{\frac{2}{p}}_{p,1}))} \leq c_1\mu\ \Big\} \tag{5.13}$$

for some c_1 being sufficiently small. Then we get by substituting (5.9) and (5.10) into (5.12) that for $t \leq \bar{T}$

$$\begin{aligned}
&\|v_{\bar\lambda}\|_{\widetilde{L}^\infty_t(B^{-1+\frac{2}{p}}_{p,1})} + \lambda_1\|v_{\bar\lambda}\|_{L^1_{t,f_1}(B^{-1+\frac{2}{p}}_{p,1})} + \lambda_2\|v_{\bar\lambda}\|_{L^1_{t,f_2}(B^{-1+\frac{2}{p}}_{p,1})} + \bar{c}\mu\|v_{\bar\lambda}\|_{L^1_t(B^{1+\frac{2}{p}}_{p,1})}\\
&\leq C\Big\{\epsilon\|v_{\bar\lambda}\|_{L^1_t(B^{1+\frac{2}{p}}_{p,1})} + \|v\|_{\widetilde{L}^\infty_t(B^{-1+\frac{2}{p}}_{p,1})}\|v_{\bar\lambda}\|_{L^1_t(B^{1+\frac{2}{p}}_{p,1})} + \|v_{\bar\lambda}\|_{L^1_{t,f_1}(B^{-1+\frac{2}{p}}_{p,1})}\\
&\quad + \frac{1}{\epsilon}\|v_{\bar\lambda}\|_{L^1_{t,f_2}(B^{-1+\frac{2}{p}}_{p,1})} + \big(\mu\|a\|_{L^\infty_t(\mathcal{M}(B^{-1+\frac{2}{p}}_{p,1}))} + \|\widetilde{\mu}(a) - \mu\|_{L^\infty_t(\mathcal{M}(B^{\frac{2}{p}}_{p,1}))}\big)\int_0^t f_1(\tau)\,d\tau\Big\}.
\end{aligned} \tag{5.14}$$

Choosing ε, λ_1 and λ_2 in (5.14) so that $C\epsilon = \frac{\bar{c}\mu}{4}$, $\lambda_1 = 2C$, $\lambda_2 = \frac{8C^2}{\bar{c}\mu}$ and $c_1 \leq \frac{\bar{c}}{4C}$, we obtain

$$
\begin{aligned}
&\|v_{\bar{\lambda}}\|_{\widetilde{L}^\infty_t(B_{p,1}^{-1+\frac{2}{p}})} + \frac{\lambda_1}{2}\|v_{\bar{\lambda}}\|_{L^1_{t,f_1}(B_{p,1}^{-1+\frac{2}{p}})} + \frac{\lambda_2}{2}\|v_{\bar{\lambda}}\|_{L^1_{t,f_2}(B_{p,1}^{-1+\frac{2}{p}})} + \frac{\bar{c}\mu}{2}\|v_{\bar{\lambda}}\|_{L^1_t(B_{p,1}^{1+\frac{2}{p}})} \\
&\leq C_1\Big\{\|v\|_{\widetilde{L}^\infty_t(B_{p,1}^{-1+\frac{2}{p}})}\|v_{\bar{\lambda}}\|_{L^1_t(B_{p,1}^{1+\frac{2}{p}})} + \big(\mu\|a\|_{L^\infty_t(\mathcal{M}(B_{p,1}^{-1+\frac{2}{p}}))} \\
&\qquad + \|\widetilde{\mu}(a) - \mu\|_{L^\infty_t(\mathcal{M}(B_{p,1}^{\frac{2}{p}}))}\big)\int_0^t f_1(\tau)\,d\tau\Big\} \quad \text{for} \quad t \leq \bar{T}.
\end{aligned} \tag{5.15}
$$

Now let c_2 be a small enough positive constant, which will be determined later on. We define Υ by

$$
\begin{aligned}
\Upsilon \stackrel{def}{=} sup\big\{t < T^* : \|v\|_{\widetilde{L}^\infty_t(B_{p,1}^{-1+\frac{2}{p}})} + \|\widetilde{\mu}(a) - \mu\|_{L^\infty_t(\mathcal{M}(B_{p,1}^{\frac{2}{p}}))} \\
+ \mu(\|a\|_{L^\infty_t(\mathcal{M}(B_{p,1}^{-1+\frac{2}{p}}))} + \|v\|_{L^1_t(B_{p,1}^{1+\frac{2}{p}})}) \leq c_2\mu\big\}.
\end{aligned} \tag{5.16}
$$

(5.13) and (5.16) implies that $\Upsilon \leq \bar{T}$ if we take $c_2 \leq c_1$. We shall prove that $\Upsilon = \infty$ under the assumption (1.5). Otherwise, taking $c_2 \leq \frac{\bar{c}}{8C_1}$, we deduce form (5.15) that

$$
\begin{aligned}
&\|v_{\bar{\lambda}}\|_{\widetilde{L}^\infty_t(B_{p,1}^{-1+\frac{2}{p}})} + \frac{\bar{c}\mu}{4}\|v_{\bar{\lambda}}\|_{L^1_t(B_{p,1}^{1+\frac{2}{p}})} \\
&\leq C(\mu\|a\|_{L^\infty_t(\mathcal{M}(B_{p,1}^{-1+\frac{2}{p}}))} + \|\widetilde{\mu}(a) - \mu\|_{L^\infty_t(\mathcal{M}(B_{p,1}^{\frac{2}{p}}))})\int_0^t f_1(\tau)d\tau \quad \text{for} \ \ t \leq \Upsilon.
\end{aligned} \tag{5.17}
$$

On the other hand, notice from (1.8) that both a and $\widetilde{\mu}(a) - \mu$ satisfy (5.1) so that applying Proposition 5.1 gives rise to

$$
\|a\|_{L^\infty_t(\mathcal{M}(B_{p,1}^{-1+\frac{2}{p}}))} \leq C\|a_0\|_{\mathcal{M}(B_{p,1}^{-1+\frac{2}{p}})} \exp\Big\{C(\|v\|_{L^1_t(B_{p,1}^{1+\frac{2}{p}})} + \|w\|_{L^1_t(B_{p,1}^{1+\frac{2}{p}})})\Big\}, \tag{5.18}
$$

and

$$
\|\widetilde{\mu}(a) - \mu\|_{L^\infty_t(\mathcal{M}(B_{p,1}^{\frac{2}{p}}))} \leq C\|\widetilde{\mu}(a_0) - \mu\|_{\mathcal{M}(B_{p,1}^{\frac{2}{p}})} \exp\Big\{C(\|v\|_{L^1_t(B_{p,1}^{1+\frac{2}{p}})} + \|w\|_{L^1_t(B_{p,1}^{1+\frac{2}{p}})})\Big\}. \tag{5.19}
$$

Then we get by summing up (5.17) with (5.18)$\times\mu$ and (5.19) that

$$
\begin{aligned}
&\|v_{\bar{\lambda}}\|_{\widetilde{L}^\infty_t(B_{p,1}^{-1+\frac{2}{p}})} + \|\widetilde{\mu}(a) - \mu\|_{L^\infty_t(\mathcal{M}(B_{p,1}^{\frac{2}{p}}))} + \frac{\mu}{4}(\|a\|_{L^\infty_t(\mathcal{M}(B_{p,1}^{-1+\frac{2}{p}}))} + \|v_{\bar{\lambda}}\|_{L^1_t(B_{p,1}^{1+\frac{2}{p}})}) \\
&\leq C(\mu\|a_0\|_{\mathcal{M}(B_{p,1}^{-1+\frac{2}{p}})} + \|\widetilde{\mu}(a_0) - \mu\|_{\mathcal{M}(B_{p,1}^{\frac{2}{p}})})\exp\{C\|w\|_{L^1_t(B_{p,1}^{1+\frac{2}{p}})}\}\big(\int_0^t f_1(\tau)d\tau + 1\big),
\end{aligned}
$$

for $t \leq \Upsilon$. This together with (2.8) gives rise to

$$
\begin{aligned}
&\|v\|_{\widetilde{L}^\infty_t(B^{-1+\frac{2}{p}}_{p,1})} + \|\widetilde{\mu}(a)-\mu\|_{L^\infty_t(\mathcal{M}(B^{\frac{2}{p}}_{p,1}))} + \frac{\mu}{4}(\|a\|_{L^\infty_t(\mathcal{M}(B^{-1+\frac{2}{p}}_{p,1}))} + \|v\|_{L^1_t(B^{1+\frac{2}{p}}_{p,1})})\\
&\leq C(\mu\|a_0\|_{\mathcal{M}(B^{-1+\frac{2}{p}}_{p,1})} + \|\widetilde{\mu}(a_0)-\mu\|_{\mathcal{M}(B^{\frac{2}{p}}_{p,1})})\\
&\qquad \times \Big(\int_0^t (\|w(\tau)\|_{B^{1+\frac{2}{p}}_{p,1}} + \frac{1}{\mu}\|\nabla p(\tau)\|_{B^{-1+\frac{2}{p}}_{p,1}})d\tau + 1\Big)\exp\{C\|w\|_{L^1_t(B^{1+\frac{2}{p}}_{p,1})}\}\\
&\qquad \times \exp\Big\{4C\int_0^t (\|w(\tau)\|_{B^{1+\frac{2}{p}}_{p,1}} + \frac{1}{\mu}\|\nabla p(\tau)\|_{B^{-1+\frac{2}{p}}_{p,1}} + \frac{1}{\mu}\|w(\tau)\|^2_{B^{\frac{2}{p}}_{p,1}})\,d\tau\Big\}\\
&\leq C(\mu\|a_0\|_{\mathcal{M}(B^{-1+\frac{2}{p}}_{p,1})} + \|\widetilde{\mu}(a_0)-\mu\|_{\mathcal{M}(B^{\frac{2}{p}}_{p,1})})\\
&\qquad \times \exp\Big\{4C\int_0^t (\|w(\tau)\|_{B^{1+\frac{2}{p}}_{p,1}} + \frac{1}{\mu}\|\nabla p(\tau)\|_{B^{-1+\frac{2}{p}}_{p,1}} + \frac{1}{\mu}\|w(\tau)\|^2_{B^{\frac{2}{p}}_{p,1}})\,d\tau\Big\},
\end{aligned}
$$

from which and (3.3), we infer

$$
\begin{aligned}
&\|v\|_{\widetilde{L}^\infty_t(B^{-1+\frac{2}{p}}_{p,1})} + \|\widetilde{\mu}(a)-\mu\|_{L^\infty_t(\mathcal{M}(B^{\frac{2}{p}}_{p,1}))} + \frac{\mu}{4}(\|a\|_{L^\infty_t(\mathcal{M}(B^{-1+\frac{2}{p}}_{p,1}))} + \|v\|_{L^1_t(B^{1+\frac{2}{p}}_{p,1})})\\
&\leq C(\mu\|a_0\|_{\mathcal{M}(B^{-1+\frac{2}{p}}_{p,1})} + \|\widetilde{\mu}(a_0)-\mu\|_{\mathcal{M}(B^{\frac{2}{p}}_{p,1})})\exp\Big\{\bar{C}(1+\mu^2)\exp\big(\frac{\bar{C}}{\mu^2}\|u_0\|^2_{B^{-1+\frac{2}{p}}_{p,1}}\big)\Big\}
\end{aligned}
\tag{5.20}
$$

for $t \leq \Upsilon$ and some positive constants $\bar{C}$ which depends on $\bar{c}$ and c_2. If we take C_0 large enough and c_0 sufficiently small in (1.5), there holds

$$
\|v\|_{\widetilde{L}^\infty_t(B^{-1+\frac{2}{p}}_{p,1})} + \|\widetilde{\mu}(a)-\mu\|_{L^\infty_t(\mathcal{M}(B^{\frac{2}{p}}_{p,1}))} + \mu(\|a\|_{L^\infty_t(\mathcal{M}(B^{-1+\frac{2}{p}}_{p,1}))} + \|v\|_{L^1_t(B^{1+\frac{2}{p}}_{p,1})}) \leq \frac{c_2}{2}\mu
$$

for $t \leq \Upsilon$, which contradicts with (5.16). Whence we conclude that $\Upsilon = T^* = \infty$. This completes the proof of Theorem 1.3 ∎

Acknowledgments. We would like to thank the anonymous referees for the valuable suggestions concerning the structure of this paper and profitable suggestions. Part of this work was done when M. Paicu was visiting Morningside Center of the Chinese Academy of Sciences in the Fall of 2011. We appreciate the hospitality and the financial support from MCM. P. Zhang is partially supported by NSF of China under Grant 10421101 and 10931007, the one hundred talents' plan from Chinese Academy of Sciences under Grant GJHZ200829 and the National Center for Mathematics and Interdisciplinary Sciences, CAS.

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