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Kleber Carrapatoso. Propagation of chaos for the spatially homogeneous Landau equation for maxwellian molecules. 2012. hal-00765621v1

HAL Id: hal-00765621

<https://hal.science/hal-00765621v1>

Preprint submitted on 15 Dec 2012 (v1), last revised 14 Jul 2014 (v3)

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PROPAGATION OF CHAOS FOR THE SPATIALLY HOMOGENEOUS LANDAU EQUATION FOR MAXWELLIAN MOLECULES

KLEBER CARRAPATOSO

ABSTRACT. We prove a quantitative propagation of chaos, uniformly in time, for the spatially homogeneous Landau equation in the case of Maxwellian molecules. We improve the results of Fontbona, Guérin and Méléard [7] and Fournier [8] where the propagation of chaos is proved for finite time.

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1. INTRODUCTION

An important open problem in kinetic theory is to derive Boltzmann equation from a many-particle system undergoing Newton's laws of dynamics. The correct scaling limit for this is the so-called Boltzmann-Grad or low density limit, see Grad [10]. The best result is Lanford

Date: December 15, 2012.

2000 Mathematics Subject Classification. 82C40, 76P05, 54C70.

Key words and phrases. Chaos; entropic chaos; Landau equation, grazing collisions; Boltzmann equation; maxwellian molecules.

[15] who proved the limit for short times (see also Illner and Pulvirenti [12] and Gallagher, Saint-Raymond and Texier [9]).

Kac [13] proposed to derive the spatially homogeneous Boltzmann equation from a many-particle Markov process, performing a *mean-field limit*. The program set by Kac in [13] was then to investigate the behavior of solutions of the mean-field equation in terms of the behaviour of the solutions of the *master equation*, i.e. the equation for the law of the many-particle process. We refer to Mischler and Mouhot [18] for a detailed introduction on Kac's program and for recent important results.

In the same way, we would like to derive rigorously another equation from kinetic theory, the Landau equation, from a many-particle system described by Newton's laws. It is an open problem, but the correct scalling to this is also known, the weak-coupling limit, and we refer to Bobylev, Pulvirenti and Saffirio [2] and the references therein for more information on this topic and partial results. We do not pursue this problem here.

Instead, in this work, we shall use the approach described above introduced by Kac [13]. Hence, we shall introduce a N -particle Markov process (see section 2.3) from which we derive the spatially homogeneous Landau equation in the mean-field limit. The N -particle process used here is obtained by means of the grazing collisions limit applied to the N -particle master equation for the Boltzmann model. We should mention that the N -particle master equation introduced here was, in fact, originally proposed by Balescu and Prigogine in the 1950's (see [14] and references therein); and it is also studied by Kiessling and Lancellotti [14] and Miot, Pulvirenti and Saffirio [17] (both in the Coulomb case).

Let us briefly explain how we can prove the *mean-field limit* with the approach proposed by Kac. Consider the probability density F_t^N associated to the Landau N -particle system and its evolution equation, i.e. the master equation (section 2.3). Integrating this equation over all variables but the first, we obtain an evolution equation for the first marginal $\Pi_1(F_t^N)$ that depends on the second marginal $\Pi_2(F_t^N)$. If the second marginal of the probability density was the 2-fold tensorization of a one-particle probability f_t , then f_t would satisfy the Landau equation (section 2.2). However, even if at initial time we start with an N -fold tensor probability $F^N(0) = f(0)^{\otimes N}$, this property can not be satisfied at later time because there are interactions between the particles. Kac suggested then that the chaos property (see definition below (1.1)), which is weaker than tensorization, could be propagated in time, which in turns would prove the mean-field limit.

1.1. Known results. Before giving our main results let us present known results concerning the propagation of chaos for the Landau equation for maxwellian molecules.

The work of Fontbona, Guérin and Méléard [7] consider nonlinear diffusion processes driven by a white noise that have an interpretation in terms of PDEs corresponding to the Landau equation. They construct an N -particle system that converges, in the limit $N \rightarrow \infty$ and in finite time, to the nonlinear process and, moreover, obtain quantitative convergence rates in Wasserstein distance W_2 . Then Fournier [8], with the same probabilistic interpretation, improves the rate of convergence of [7].

We should mention that the Landau master equation introduced in this work (section 2.3) differs from the master equation associated to the N -particle process in [7, 8], see remark 2.1.

1.2. Main results. Consider a Polish space E , we shall denote by $\mathcal{P}(E)$ the space of probability measures on E . We denote by $\mathcal{P}_{\text{sym}}(E^N)$ the space of symmetric probabilities on $\mathcal{P}(E^N)$. We say that a symmetric probability $F^N \in \mathcal{P}_{\text{sym}}(E^N)$ is f -chaotic (or Kac chaotic), for some probability $f \in \mathcal{P}(E)$, if for all $\ell \in \mathbb{N}^*$ we have

$$(1.1) \quad F_t^N \rightharpoonup f^{\otimes \ell} \quad \text{when } N \rightarrow \infty,$$

where $F_\ell^N = \Pi_\ell(F^N)$ is the ℓ -th marginal of F^N and the convergence has to be understood in weak sense on $\mathcal{P}(E^\ell)$, i.e. the convergence of integral against continuous and bounded functions $\varphi \in C_b(E)^\otimes \ell$. In this paper we are interested in quantitative rates of convergence, more precisely we shall investigate estimates of the type, for any $\varphi \in \mathcal{F}^\otimes \ell$ with $\mathcal{F} \subset C_b(E)$ and $\|\varphi\|_{\mathcal{F}^\otimes \ell} \leq 1$,

$$|\langle F_\ell^N - f^{\otimes \ell}, \varphi \rangle| \leq C(\ell) \varepsilon(N),$$

with a constant $C(\ell)$ possibly depending on ℓ and a function $\varepsilon(N) \rightarrow 0$ when $N \rightarrow \infty$. Another possibility is to replace the left-hand side of the last equation by some distance on the space of probabilities, as for example the Wasserstein distance, $W_1(F_\ell^N, f^{\otimes \ell})$.

The many-particle process can be considered in \mathbb{R}^{dN} and then its law F^N is a symmetric probability measure on \mathbb{R}^{dN} , however, thanks to the conservation laws, the process can be restricted to some submanifold of \mathbb{R}^{dN} . In our case, the dynamics of the many-particle process conserves momentum and energy (see section 2 for details), which implies that the process can be restricted to

$$(1.2) \quad \mathcal{S}^N(\mathcal{E}, \mathcal{M}) := \left\{ V = (v_1, \dots, v_N) \in \mathbb{R}^{dN} ; \frac{1}{N} \sum_{i=1}^N |v_i - \mathcal{M}|^2 = \mathcal{E}, \frac{1}{N} \sum_{i=1}^N v_i = \mathcal{M} \right\}$$

where $\mathcal{E} \geq 0$ and $\mathcal{M} \in \mathbb{R}^d$. We consider through the paper, without loss of generality, the case $\mathcal{M} = 0$, we denote $\mathcal{S}^N(\mathcal{E}) := \mathcal{S}^N(\mathcal{E}, 0)$ and call these submanifolds Boltzmann's spheres.

Initial data. Considering the process in $\mathcal{S}^N(\mathcal{E})$, we shall need an initial data $F_0^N \in \mathcal{P}_{\text{sym}}(\mathcal{S}^N(\mathcal{E}))$ that is f_0 -chaotic for some $f_0 \in \mathcal{P}(\mathbb{R}^d)$. This problem was studied in [4], where it is proven that for some (regular enough) probability measure $f \in \mathcal{P}(\mathbb{R}^d)$, with zero momentum $\mathcal{M} = \int v f = 0$ and energy $\mathcal{E} = \int |v|^2 f$, we can construct a probability measure $F^N \in \mathcal{P}_{\text{sym}}(\mathcal{S}^N(\mathcal{E}))$ that is f -chaotic (and also entropically f -chaotic, see section 5 for the definition), by tensorization of f and restriction to the Boltzmann's sphere $\mathcal{S}^N(\mathcal{E})$. We shall denote this probability measure by

$$(1.3) \quad F^N = [f^{\otimes N}]_{\mathcal{S}^N(\mathcal{E})} := \frac{f^{\otimes N}}{\int_{\mathcal{S}^N(\mathcal{E})} f^{\otimes N} d\gamma^N} \gamma^N,$$

where γ^N is the uniform probability measure on $\mathcal{S}^N(\mathcal{E})$.

We can now state our main results in a simplified version.

Theorem 1.1. *Consider $f_0 \in \mathcal{P}(\mathbb{R}^d)$, with zero momentum and energy \mathcal{E} , and also $F_0^N = [f_0^{\otimes N}]_{\mathcal{S}^N(\mathcal{E})} \in \mathcal{P}_{\text{sym}}(\mathcal{S}^N(\mathcal{E}))$. Let f_t be the solution of the Landau equation (see (2.10)) with initial data f_0 and F_t^N the solution of the Landau master equation (see (2.29)) with initial data F_0^N .*

(1) (Theorems 4.1 and 4.2) *Then, for all $N \in \mathbb{N}^*$ we have*

$$\sup_{t \geq 0} \frac{W_1(F_t^N, f_t^{\otimes N})}{N} \leq \varepsilon_1(N),$$

where ε_1 is a polynomial function and $\varepsilon_1(N) \rightarrow 0$ as $N \rightarrow \infty$.

(2) (Theorem 5.3) *Then for all $N \in \mathbb{N}^*$ we have*

$$\sup_{t \geq 0} \left| \frac{1}{N} H(F_t^N | \gamma^N) - H(f_t | \gamma) \right| \leq \varepsilon_2(N),$$

where ε_2 is a polynomial function $\varepsilon_2(N) \rightarrow 0$ as $N \rightarrow \infty$, $H(f | \gamma)$ denotes the relative entropy of f_t with respect to γ , the centered Gaussian probability measure in \mathbb{R}^d with energy \mathcal{E} , and $H(F_t^N | \gamma^N)$ denotes the relative entropy of F_t^N with respect to γ^N (see Section 5).

1.3. Strategy. The main idea is to use the consistency-stability method developed by Mischler, Mouhot and Wennberg in [18, 19]. Consider the semigroups associated to the evolution of the N -particle system and the limit mean-field equation, this method reduces the problem of propagation of chaos to proving consistency and stability estimates for these semigroups. First of all, we need to introduce the Landau master equation, which is derived by the asymptotics of grazing collisions from the Boltzmann master equation. Then, with the Landau master equation and the limit Landau equation at hand, we can investigate the estimates needed to apply the consistency-stability method and prove the propagation of chaos.

1.4. Organization of the paper. Section 2 is devoted to deduce a N -particle stochastic process modeling the Landau dynamics and to present the limit Landau equation. In Section 3 we present the consistency-stability method, with some adaptations, developed by Mischler, Mouhot and Wennberg in [18, 19]. Finally, in Section 4 we apply the method presented before to the Landau model in order to prove the propagation of chaos with quantitative rate and uniformly in time.

2. THE LANDAU MODEL

Our aim in this section is to present the N -particle system and the limit mean-field equation corresponding to Landau model. The limit Landau equation is well known and we shall present it in the Subsection 2.2. Furthermore, in Subsection 2.3 we deduce a master equation for the N -particle system corresponding to Landau.

First of all, we present the Boltzmann model, with its master equation and limit equation, which will be very useful in the sequel since Boltzmann and Landau equations are linked by the asymptotics of grazing collision that we shall explain in details later.

2.1. The Boltzmann model. We present here the Boltzmann model, with the limit mean field equation and the master equation. The spatially homogeneous Boltzmann equation [27, 18] is given by, for $f = f(t, v)$,

$$(2.1) \quad \partial_t f = Q(f, f)$$

with the collision operator given by

$$(2.2) \quad Q(g, f) = \int_{\mathbb{R}^d \times \mathbb{S}^{d-1}} B(|v - v_*|, \cos \theta) \left(g(v'_*) f(v') - g(v_*) f(v) \right) dv_* d\sigma,$$

and where the post-collisional velocities v' and v'_* are given by

$$(2.3) \quad v' = \frac{v + v_*}{2} + \frac{|v - v_*|}{2} \sigma, \quad v'_* = \frac{v + v_*}{2} - \frac{|v - v_*|}{2} \sigma$$

and $\cos \theta = \sigma \cdot (v - v_*) / |v - v_*|$.

The collision kernel B satisfies $B(|z|, \cos \theta) = \Gamma(|z|) b(\cos \theta)$ (for more information on the collision kernel we refer to [27]) for some nonnegative functions Γ and b . When the interaction potential is proportional to r^{-s} , where r denotes the distance between particles, then we have

$$\Gamma(|z|) = |z|^\gamma, \quad \sin^{d-2} \theta b(\cos \theta) \sim C_b \theta^{-1-\nu} \text{ when } \theta \sim 0,$$

with $\gamma = (s - 2d + 2)/s$, for some constant $C_b > 0$ and $\nu \in (0, 2)$ (for example in the 3-dimensional case we have $\nu = 2/s$). In this work we are concerned with the case of *true Maxwellian molecules* $\gamma = 0$ (which corresponds to $s = 2d - 2$), more precisely

$$(2.4) \quad B(|v - w|, \cos \theta) = b(\cos \theta) \sim_{\theta \sim 0} C_b \theta^{1-\nu-d}.$$

We remark that in this case we have $\int_{\mathbb{S}^{d-1}} b(\cos \theta) d\sigma = \infty$ but $\int_{\mathbb{S}^{d-1}} b(\cos \theta)(1 - \cos \theta) d\sigma < \infty$.

Another possible way to describe the pre and post-collisional velocities is the ω -representation

$$(2.5) \quad v' = v - (v - v_*, \omega)\omega, \quad v'_* = v_* + (v - v_*, \omega)\omega, \quad \omega \in \mathbb{S}^{d-1},$$

which gives us

$$Q_B(f, f) = \int_{\mathbb{R}^d \times \mathbb{S}^{d-1}} \tilde{B}(|v - v_*|, \omega) \left(f(v'_*)f(v') - f(v_*)f(v) \right) dv_* d\omega,$$

with $\tilde{B}(z, \omega) = |z|^\gamma b_\omega(\alpha)$ and α the angle formed by z and ω , and the following relation holds

$$(2.6) \quad \sigma = \frac{v - v_*}{|v - v_*|} - 2 \left(\omega, \frac{v - v_*}{|v - v_*|} \right) \omega.$$

Let us now present the many-particle model [16, 13, 18, 3, 19]. Given a pre-collisional system of velocities $V = (v_1, \dots, v_N) \in \mathbb{R}^{dN}$ and a collision kernel $B(|z|, \cos \theta) = \Gamma(|z|)b(\cos \theta)$, the process is: for any $i' \neq j'$, pick a random time $T(\Gamma(|v_{i'} - v_{j'}|))$ of collision accordingly to an exponential law of parameter $\Gamma(|v_{i'} - v_{j'}|)$ and choose the minimum time T_1 and the colliding pair (v_i, v_j) ; draw $\sigma \in \mathbb{S}^{d-1} \subset \mathbb{R}^d$ according to the law $b(\cos \theta_{ij})$, with $\cos \theta_{ij} = \sigma \cdot (v_i - v_j)/|v_i - v_j|$; after collision the new velocities become $V'_{ij} = (v_1, \dots, v'_i, \dots, v'_j, \dots, v_N)$ with

$$(2.7) \quad v'_i = \frac{v_i + v_j}{2} + \frac{|v_i - v_j|}{2} \sigma, \quad v'_j = \frac{v_i + v_j}{2} - \frac{|v_i - v_j|}{2} \sigma.$$

Iterating this construction we built then the associated Markov process $(\mathcal{V}_t)_{t \geq 0}$ on \mathbb{R}^{dN} . As explained in the introduction, we can also consider this process on $\mathcal{S}^N(\mathcal{E})$. Then, after a rescaling of time, the master equation is given in dual form by [18, 19],

$$(2.8) \quad \partial_t \langle F_t^N, \varphi \rangle = \langle F_t^N, G_B^N \varphi \rangle$$

where

$$(2.9) \quad G_B^N \varphi = \frac{1}{2N} \sum_{i,j=1}^N \Gamma(|v_i - v_j|) \int_{\mathbb{S}^{d-1}} b(\cos \theta_{ij}) (\varphi'_{ij} - \varphi) d\sigma$$

with the shorthand notation $\varphi'_{ij} = \varphi(V'_{ij})$ and $\varphi = \varphi(V) \in C_b(\mathbb{R}^{dN})$. We shall consider the case of Maxwellian molecules, i.e. $\Gamma(|z|) = 1$ and $b(\cos \theta)$ satisfying (2.4).

2.2. Limit equation. We present here the limit spatially homogeneous Landau equation for maxwellian molecules and some of its basic properties, for more information we refer to [27, 25, 24].

The Landau equation is a kinetic model in plasma physics that describes the evolution of the density f of a gas in the phase space of all positions and velocities of particles. Assuming that the density function does not depend on the position, we obtain the *spatially homogeneous Landau equation* in the form

$$(2.10) \quad \partial_t f = Q_L(f, f)$$

where $f = f(t, v)$ is the density of particles with velocity v at time t , $v \in \mathbb{R}^d$ and $t \in \mathbb{R}^+$. The Landau operator is given by

$$(2.11) \quad Q_L(g, f) = \partial_i \left\{ \int_{\mathbb{R}^d} a_{ij}(v - v_*) (g(v_*) \partial_j f(v) - \partial_j g(v_*) f(v)) dv_* \right\},$$

where here and below we shall use the convention of implicit summation over indices. Moreover, the matrix a is nonnegative, symmetric and depends on the interaction between particles. If two particles interact with a potential proportional to $1/r^s$, where r denotes their distance, then we have

$$a_{ij}(z) = \Lambda |z|^{\gamma+2} \Pi_{ij}(z), \quad \Pi_{ij}(z) = \delta_{ij} - \frac{z_i z_j}{|z|^2},$$

with $\gamma = (s - 2d + 2)/s$ and some constant $\Lambda > 0$. As for the Boltzmann equation, we consider the case of Maxwellian molecules $\gamma = 0$, i.e.

$$(2.12) \quad a_{ij}(z) = \Lambda |z|^2 \Pi_{ij}(z).$$

We also define

$$(2.13) \quad b_i(z) = \partial_j a_{ij} = -\Lambda(d-1)z_i, \quad c(z) = \partial_{ij} \bar{a}_{ij} = -3\Lambda(d-1),$$

and we denote

$$\bar{a}_{ij} = a_{ij} * f, \quad \bar{b}_i = b_i * f, \quad \bar{c} = c * f.$$

Hence, we can write the Landau equation in another form

$$(2.14) \quad \partial_t f = \bar{a}_{ij} \partial_{ij} f - \bar{c} f.$$

Moreover, let $\varphi(v)$ be a test function, then we have the following weak forms

$$(2.15) \quad \int Q_L(f, f) \varphi = -\frac{1}{2} \int dv dv_* f f_* a(v - v_*) \left(\frac{\nabla f}{f} - \frac{\nabla_* f_*}{f_*} \right) (\nabla \varphi - \nabla_* \varphi_*)$$

or

$$(2.16) \quad \begin{aligned} \int Q_L(f, f) \varphi &= \frac{1}{2} \int dv dv_* f f_* a_{ij}(v - v_*) (\partial_{ij} \varphi + (\partial_{ij} \varphi)_*) \\ &+ \int dv dv_* f f_* b_i(v - v_*) (\partial_i \varphi - (\partial_i \varphi)_*). \end{aligned}$$

This equation satisfies the conservation of mass, momentum and energy. Moreover, the entropy $H(f) = \int f \log f$ is nonincreasing, indeed taking $\varphi = \log f$ we obtain

$$(2.17) \quad \frac{d}{dt} H(f) = -\frac{1}{2} \int f f_* a(v - v_*) \left(\frac{\nabla f}{f} - \frac{\nabla_* f_*}{f_*} \right) \cdot \left(\frac{\nabla f}{f} - \frac{\nabla_* f_*}{f_*} \right) dv dv_* \leq 0,$$

since a is nonnegative, which is the Landau version of the H -theorem of Boltzmann. For more information we refer to [25].

The Landau equation was introduced by Landau (citer livre Landau-Lifschitz). Later it was proven that the Landau equation can be obtained as a limit of the Boltzmann equation when grazing collisions prevail (see [5, 24] and the references therein for more details).

2.3. Master equation. We derive a master equation for the Landau model. It is based on [24] where they use the grazing collisions limit to pass from Boltzmann to Landau limit equations (see also [5]). Since we know the master equation for the Boltzmann model (2.8), we shall perform the grazing collisions limit to obtain a master equation for Landau model. As explained in the introduction, the master equation we derive here (see (2.29)) is the same introduced by Balescu and Prigorine in the 1950's, and it is also studied in the works [14, 17].

Grazing collisions. We present here the grazing collision limit as in [24]. Consider the true Maxwellian molecules collision kernel b satisfying (2.4). We say that b_ε concentrates in grazing collision if:

$$(2.18) \quad \left\{ \begin{array}{l} \forall \theta_0 > 0, \quad \sup_{\theta > \theta_0} b_\varepsilon(\cos \theta) \xrightarrow{\varepsilon \rightarrow 0} 0 \\ \Lambda_\varepsilon = \int_{\mathbb{S}^{d-1}} b_\varepsilon(\cos \theta) (1 - \cos \theta) d\sigma \xrightarrow{\varepsilon \rightarrow 0} \Lambda > 0. \end{array} \right.$$

For the sake of simplicity, to derive the Landau master equation in this section, we suppose the dimension $d = 3$ to perform the computations, the other cases being the same.

From (2.6), using a spherical coordinate system (in dimension $d = 3$) with axis $v - v_*$, we have

$$\sigma = \frac{v - v_*}{|v - v_*|} \cos \theta + (\cos \phi \vec{h} + \sin \phi \vec{i}) \sin \theta.$$

Moreover we have $|(v - v_*, \omega)| = |v - v_*| \sin(\theta/2)$. Finally we can write the operator in the following way (see [24])

$$(2.19) \quad Q_B(f, f) = \int_0^{2\pi} d\phi \int_0^{\pi/2} d\theta \int_{\mathbb{R}^d} dv_* \zeta(\theta)(f' f'_* - f f_*),$$

with $\zeta(\theta) = \sin^{d-2} \theta b(\cos \theta)$. In this case, we can rewrite (2.18) and say that ζ_ε concentrates in grazing collisions if for all $\theta_0 \geq 0$

$$(2.20) \quad \left\{ \begin{array}{l} \sup_{\theta \geq \theta_0} \zeta_\varepsilon(\theta) \rightarrow 0 \quad \text{when } \varepsilon \rightarrow 0 \\ \Lambda_\varepsilon := \frac{\pi}{2} \int_0^{\pi/2} \sin^2 \frac{\theta}{2} \zeta_\varepsilon(\theta) d\theta \rightarrow \Lambda < \infty \quad \text{when } \varepsilon \rightarrow 0. \end{array} \right.$$

Let us consider then the Boltzmann master equation (2.8)-(2.9), using the form of (2.19), that is

$$(G_B^N \varphi)(V) = \frac{1}{2N} \sum_{i,j=1}^N \int_0^{2\pi} d\phi \int_0^{\pi/2} d\theta \zeta(\theta)(\varphi'_{ij} - \varphi).$$

In this equation, we take a second order Taylor expansion of φ'_{ij} and obtain

$$(2.21) \quad \begin{aligned} \varphi(V'_{ij}) - \varphi(V) &= D\varphi[V](V'_{ij} - V) + \frac{1}{2}(V'_{ij} - V)^T D^2\varphi[V](V'_{ij} - V) \\ &\quad + O(|V'_{ij} - V|^3). \end{aligned}$$

With the incoming and outgoing velocities V and V'_{ij} (see (2.7)), we have

$$V'_{ij} - V = (0, \dots, 0, v'_i - v_i, 0, \dots, 0, v'_j - v_j, 0, \dots, 0).$$

In (2.21), $D\varphi[V]$ and $D^2\varphi[V]$ are given by

$$D\varphi[V] = (\nabla_i \varphi(V))_{1 \leq i \leq N}, \quad D^2\varphi[V] = (\nabla_{ij}^2 \varphi(V))_{1 \leq i, j \leq N}$$

where $\nabla_i \varphi = (\partial_{v_{i,\alpha}} \varphi)_{1 \leq \alpha \leq 3}$ and $\nabla_{ij}^2 \varphi = (\partial_{v_{i,\alpha}} \partial_{v_{j,\beta}} \varphi)_{1 \leq \alpha, \beta \leq 3}$. Now we substitute $V'_{ij} - V$ in (2.21) and we get

$$(2.22) \quad \begin{aligned} \varphi(V'_{ij}) - \varphi(V) &= \nabla_i \varphi(V)(v'_i - v_i) + \nabla_j \varphi(V)(v'_j - v_j) \\ &\quad + \frac{1}{2} \{ \nabla_{ii}^2 \varphi(V)(v'_i - v_i)^2 + \nabla_{jj}^2 \varphi(V)(v'_j - v_j)^2 \\ &\quad + \nabla_{ij}^2 \varphi(V)(v'_i - v_i)(v'_j - v_j) + \nabla_{ji}^2 \varphi(V)(v'_i - v_i)(v'_j - v_j) \} \\ &\quad + O(|V'_{ij} - V|^3). \end{aligned}$$

Finally, using (2.7) and (2.5) with v_i and v_j , one obtains

$$(2.23) \quad \begin{aligned} \varphi(V'_{ij}) - \varphi(V) &= - (v_i - v_j, \omega)(\nabla_i \varphi - \nabla_j \varphi, \omega) \quad (= T_1) \\ &\quad + \frac{1}{2} (v_i - v_j, \omega)^2 \{ \nabla_{ii}^2 \varphi + \nabla_{jj}^2 \varphi - \nabla_{ij}^2 \varphi - \nabla_{ji}^2 \varphi \} (\omega, \omega) \quad (= T_2) \\ &\quad + O\left(|v_i - v_j|^3 \sin^3 \frac{\theta}{2}\right). \end{aligned}$$

For each pair of particles i et j , in the orthonormal basis $\{\frac{v_i - v_j}{|v_i - v_j|}, \vec{h}, \vec{i}\}$, one has

$$\omega = \frac{v_i - v_j}{|v_i - v_j|} \sin \frac{\theta}{2} + (\cos \phi \vec{h} + \sin \phi \vec{i}) \cos \frac{\theta}{2}$$

and then, using the fact that linear combinations of $\cos \phi$ and $\sin \phi$ vanish when integrated over ϕ , we can compute the contribution of T_1 integrated over ϕ

$$- \int_0^{2\pi} d\phi (v_i - v_j, \omega) (\nabla_i \varphi - \nabla_j \varphi, \omega) = -2\pi \sin^2 \frac{\theta}{2} (v_i - v_j, \nabla_i \varphi - \nabla_j \varphi).$$

Now we have to compute the integral of T_2 over ϕ , we denote

$$\lambda_{\alpha\beta} = \{\partial_{v_i, \alpha} \partial_{v_i, \beta} \varphi + \partial_{v_j, \alpha} \partial_{v_j, \beta} \varphi - \partial_{v_i, \alpha} \partial_{v_j, \beta} \varphi - \partial_{v_j, \alpha} \partial_{v_i, \beta} \varphi\} / 2$$

and in the same orthonormal basis, we compute

$$A = |v_i - v_j|^2 \sin^2 \frac{\theta}{2} \int_0^{2\pi} d\phi \lambda_{\alpha\beta} \omega_\alpha \omega_\beta.$$

Again, linear combinations of $\cos \phi$ and $\sin \phi$ vanish, which implies

$$(2.24) \quad \begin{aligned} A &= |v_i - v_j|^2 \sin^2 \frac{\theta}{2} \int_0^{2\pi} d\phi \left(\lambda_{11} \sin^2 \frac{\theta}{2} + \lambda_{22} \cos^2 \phi \cos^2 \frac{\theta}{2} + \lambda_{33} \sin^2 \phi \cos^2 \frac{\theta}{2} \right) \\ &= 2\pi |v_i - v_j|^2 \left(\lambda_{11} \sin^4 \frac{\theta}{2} + \frac{\lambda_{22}}{2} \cos^2 \frac{\theta}{2} \sin^2 \frac{\theta}{2} + \frac{\lambda_{33}}{2} \cos^2 \frac{\theta}{2} \sin^2 \frac{\theta}{2} \right) \end{aligned}$$

and we remark that the first coefficient is of order greater than 2 in θ .

We introduce $\Pi_{\alpha\beta}(v_i - v_j)$ the projection over the orthogonal space of $\frac{v_i - v_j}{|v_i - v_j|}$, and the dominant term of (2.24) when $\theta \rightarrow 0$ is

$$\pi |v_i - v_j|^2 \sin^2 \frac{\theta}{2} \Pi_{\alpha\beta}(v_i - v_j) \lambda_{\alpha\beta}$$

or in matricial notation

$$\pi |v_i - v_j|^2 \sin^2 \frac{\theta}{2} \Pi(v_i - v_j) : \frac{(\nabla_{ii}^2 \varphi + \nabla_{jj}^2 \varphi - \nabla_{ij}^2 \varphi - \nabla_{ji}^2 \varphi)}{2}.$$

Finally, we obtain

$$(2.25) \quad \begin{aligned} &\frac{1}{2} \int_0^{2\pi} d\phi (\varphi(V'_{ij}) - \varphi(V)) = -\frac{\pi}{2} \sin^2 \frac{\theta}{2} (2(v_i - v_j), \nabla_i \varphi - \nabla_j \varphi) \\ &+ \frac{\pi}{4} |v_i - v_j|^2 \sin^2 \frac{\theta}{2} \Pi(v_i - v_j) : (\nabla_{ii}^2 \varphi + \nabla_{jj}^2 \varphi - \nabla_{ij}^2 \varphi - \nabla_{ji}^2 \varphi) \\ &+ O(|v_i - v_j|^2 \theta^4 \wedge 1) + O(|v_i - v_j|^3 \theta^3 \wedge 1). \end{aligned}$$

Consider now the Boltzmann master equation with kernel ζ_ε satisfying the grazing collisions (2.20) and plug (2.25) in it, we obtain then

$$(2.26) \quad \begin{aligned} G_B^N \varphi &= \frac{1}{N} \sum_{i,j=1}^N \int_0^{\pi/2} d\theta \zeta_\varepsilon(\theta) \frac{1}{2} \int_0^{2\pi} d\phi (\varphi'_{ij} - \varphi) \\ &= \frac{1}{N} \sum_{i,j=1}^N \int_0^{\pi/2} d\theta \zeta_\varepsilon(\theta) \left\{ -\frac{\pi}{2} \sin^2 \frac{\theta}{2} (2(v_i - v_j), \nabla_i \varphi - \nabla_j \varphi) \right. \\ &+ \frac{\pi}{4} |v_i - v_j|^2 \sin^2 \frac{\theta}{2} \Pi(v_i - v_j) : (\nabla_{ii}^2 \varphi + \nabla_{jj}^2 \varphi - \nabla_{ij}^2 \varphi - \nabla_{ji}^2 \varphi) \\ &\left. + O(|v_i - v_j|^2 \theta^4 \wedge 1) + O(|v_i - v_j|^3 \theta^3 \wedge 1) \right\}. \end{aligned}$$

This can be written in the following way

$$\begin{aligned}
(2.27) \quad G_B^N \varphi &= \frac{1}{N} \sum_{i,j=1}^N \frac{\pi}{2} \int_0^{\pi/2} d\theta \sin^2 \frac{\theta}{2} \zeta_\varepsilon(\theta) (-2|v_i - v_j|^2 \frac{(v_i - v_j)}{|v_i - v_j|^2}) \cdot (\nabla_i \varphi - \nabla_j \varphi) \\
&+ \frac{1}{2N} \sum_{i,j=1}^N \frac{\pi}{2} \int_0^{\pi/2} d\theta \sin^2 \frac{\theta}{2} \zeta_\varepsilon(\theta) |v_i - v_j|^2 \Pi(v_i - v_j) : (\nabla_{ii}^2 \varphi + \nabla_{jj}^2 \varphi - \nabla_{ij}^2 \varphi - \nabla_{ji}^2 \varphi) \\
&+ \frac{1}{N} \sum_{i,j=1}^N \int_0^{\pi/2} d\theta \zeta_\varepsilon(\theta) (O(|v_i - v_j|^2 \theta^4 \wedge 1) + O(|v_i - v_j|^3 \theta^3 \wedge 1)).
\end{aligned}$$

As in [24], the last term converges to 0 when $\varepsilon \rightarrow 0$. Then we have, using (2.20) and the definition of the functions a (2.12) and b (2.13), when $\varepsilon \rightarrow 0$

$$\begin{aligned}
(2.28) \quad G_B^N \varphi &\longrightarrow \frac{1}{N} \sum_{i,j=1}^N -2\Lambda |v_i - v_j|^2 \frac{(v_i - v_j)}{|v_i - v_j|^2} \cdot (\nabla_i \varphi - \nabla_j \varphi) \\
&+ \frac{1}{2N} \sum_{i,j=1}^N \Lambda |v_i - v_j|^2 \Pi(v_i - v_j) : (\nabla_{ii}^2 \varphi + \nabla_{jj}^2 \varphi - \nabla_{ij}^2 \varphi - \nabla_{ji}^2 \varphi) \\
&= \frac{1}{N} \sum_{i,j=1}^N b(v_i - v_j) \cdot (\nabla_i \varphi - \nabla_j \varphi) \\
&+ \frac{1}{2N} \sum_{i,j=1}^N a(v_i - v_j) : (\nabla_{ii}^2 \varphi + \nabla_{jj}^2 \varphi - \nabla_{ij}^2 \varphi - \nabla_{ji}^2 \varphi) =: G_L^N \varphi
\end{aligned}$$

and that defines the Landau generator G_L^N . Finally, we derive the following Landau master equation

$$\begin{aligned}
(2.29) \quad \partial_t \langle f_t^N, \varphi \rangle &= \langle f_t^N, G_L^N \varphi \rangle = \frac{1}{N} \int \sum_{i,j=1}^N b(v_i - v_j) \cdot (\nabla_i \varphi - \nabla_j \varphi) f_t^N(dV) \\
&+ \frac{1}{2N} \int \sum_{i,j=1}^N a(v_i - v_j) : (\nabla_{ii}^2 \varphi + \nabla_{jj}^2 \varphi - \nabla_{ij}^2 \varphi - \nabla_{ji}^2 \varphi) f_t^N(dV).
\end{aligned}$$

Remark 2.1. In the paper [7], the Landau equation is studied with a probabilistic approach. In particular they prove that the following process associated to a N -particle system, for $i = 1, \dots, N$,

$$(2.30) \quad dX_t^i = \frac{\sqrt{2}}{\sqrt{N}} \sum_{k=1}^N \sigma(X_t^i - X_t^k) dB_t^{i,k} + \frac{2}{N} \sum_{k=1}^N b(X_t^i - X_t^k) dt,$$

where $B^{i,k}$ are N^2 independant \mathbb{R}^d -valued Brownian motions, converges to the process

$$(2.31) \quad X_t = X_0 + \sqrt{2} \int_0^t \int_{\mathbb{R}^d} \sigma(X_s - y) W(dy, ds) + 2 \int_0^t \int_{\mathbb{R}^d} b(X_s - y) P_s(dy) ds$$

where P_t is the law of V_t and W is a white noise in space-time. Moreover the process (2.31) is associated to the spatially homogeneous Landau equation with the coefficients $a_{\alpha\beta}(z) := (\sigma\sigma^*)_{\alpha\beta}(z)$ and $b_\alpha(z) := \partial_\beta a_{\alpha\beta}(z)$. Then, the Kolmogorov equation of (2.30) is, for a test function

$\varphi : \mathbb{R}^{dN} \rightarrow \mathbb{R}$ and where f_t^N represents the law of X_t ,

$$\begin{aligned}
\partial_t \langle f_t^N, \varphi \rangle &= \langle f_t^N, G_2^N \varphi \rangle \\
&= \frac{1}{N} \sum_{i,j=1}^N \int_{\mathbb{R}^{dN}} b(v_i - v_j) \cdot (\nabla_i \varphi(V) - \nabla_j \varphi(V)) f_t^N(dV) \\
&+ \frac{1}{2N} \sum_{i,j=1}^N \int_{\mathbb{R}^{dN}} a(v_i - v_j) : (\nabla_{ii}^2 \varphi(V) + \nabla_{jj}^2 \varphi(V)) f_t^N(dV).
\end{aligned}
\tag{2.32}$$

In [8], instead of the process (2.31) it was used a similar process with only N independent Brownian motion, i.e. replacing $B^{i,k}$ by B^i in (2.31), and it gives the same master equation (2.32). We remark that this equation differs from (2.29) by the terms $\sum_{i,j} a(v_i - v_j) : (-\nabla_{ij}^2 \varphi - \nabla_{ji}^2 \varphi)$.

Now, in order to obtain a N -particle SDE associated to (2.28)-(2.29), we shall modify (2.31). Consider then, for $i = 1, \dots, N$, \mathbb{R}^d -valued random variables $(X_t^i)_{t \geq 0}$ satisfying the following equation

$$\forall i = 1, \dots, N \quad dX_t^i = \frac{\sqrt{2}}{\sqrt{N}} \sum_{\substack{k=1 \\ k \neq i}}^N \sigma(X_t^i - X_t^k) dZ_t^{i,k} + \frac{2}{N} \sum_{\substack{k=1 \\ k \neq i}}^N b(X_t^i - X_t^k) dt
\tag{2.33}$$

where, for all $1 \leq i \leq N$ and $i < k$, $Z_t^{i,k} = B_t^{i,k}$ are $N(N-1)/2$ independent \mathbb{R}^d -valued Brownian motions and the other terms are anti-symmetric $Z_t^{k,i} = -B_t^{i,k}$. As in (2.31), we have $a(z) = \sigma(z)\sigma^*(z)$ and σ is symmetric (recall that a also is), i.e. $\sigma(-z) = \sigma(z)$. We can rewrite (2.33), let $X = (X^1, \dots, X^N)$ we have the following equation

$$dX_t = \mu(X_t) dt + \Sigma(X_t) dW_t
\tag{2.34}$$

where $\mu = (\mu_1, \dots, \mu_N)$ is given by

$$\forall i = 1, \dots, N, \forall X = (X^1, \dots, X^N) \in \mathbb{R}^{dN}, \quad \mu_i(X) = \frac{2}{N} \sum_{k=1}^N b(X^i - X^k),$$

moreover, W_t is constituted of the $N(N-1)/2$ independent Brownian motion, more precisely

$$W_t = (W_t^1, \dots, W_t^{N(N-1)/2}) = (B_t^{1,2}, \dots, B_t^{1,N}, B_t^{2,3}, \dots, B_t^{2,N}, \dots, B_t^{(N-1),N}),$$

i.e. for $i < k$, $B_t^{i,k} = W_t^\alpha$ with $\alpha = (i-1)(N-i/2) + k - i$. Finally, $\Sigma : \mathbb{R}^{dN} \rightarrow \mathcal{M}_{N, N(N-1)/2}(\mathcal{M}_{d,d}(\mathbb{R}^d))$ is given by, for all $1 \leq i \leq N$, $1 \leq \beta \leq N(N-1)/2$ and $X = (X^1, \dots, X^N) \in \mathbb{R}^{dN}$,

$$(\Sigma(X))_{i\beta} = \begin{cases} -\frac{\sqrt{2}}{\sqrt{N}} \sigma(X^i - X^k); & \beta = Nk - k\frac{(k+1)}{2}, 1 \leq k \leq i-1, \\ \frac{\sqrt{2}}{\sqrt{N}} \sigma(X^i - X^k); & (i-1)(N - \frac{i}{2}) + 1 \leq \beta \leq i(N - \frac{i+1}{2}), k = \beta - (i-1)(N - \frac{i}{2}) + i, \\ 0; & \text{otherwise.} \end{cases}$$

Indeed, multiplying the i th-row of $\Sigma(X)$ by dW_t we obtain

$$-\frac{\sqrt{2}}{\sqrt{N}} \sum_{k=1}^{i-1} \sigma(X^i - X^k) dB_t^{i,k} + \frac{\sqrt{2}}{\sqrt{N}} \sum_{k=i+1}^N \sigma(X^i - X^k) dB_t^{i,k} = \frac{\sqrt{2}}{\sqrt{N}} \sum_{\substack{k=1 \\ k \neq i}}^N \sigma(X^i - X^k) dZ_t^{i,k},$$

which corresponds to the first term on the right-hand side of (2.33).

Consider a test function $\phi : \mathbb{R}^{dN} \rightarrow \mathbb{R}$ and f_t^N the law of X_t , then the Kolmogorov equation of (2.34) is

$$\begin{aligned} \partial_t \langle f_t^N, \phi \rangle &= \frac{2}{N} \sum_{i,j=1}^N \int_{\mathbb{R}^{dN}} b(v_i - v_j) \cdot (\nabla_i \phi) f_t^N(dV) \\ &\quad + \frac{1}{N} \sum_{i,j=1}^N \int_{\mathbb{R}^{dN}} a(v_i - v_j) : (\nabla_{ii}^2 \phi - \nabla_{ij}^2 \phi) f_t^N(dV), \end{aligned}$$

and using the symmetry property of a and anti-symmetry of b we obtain (2.29).

3. THE CONSISTENCY-STABILITY METHOD FOR THE LANDAU EQUATION

In this section we present the method developed in [18, 19] with some modifications, in order to be able to apply it later to the Landau equation in section 4.

3.1. Abstract framework. Consider a polish space E and we shall denote by $\mathcal{P}(E)$ the space of probability measures on E . Consider also E^N and the space of probability measures $\mathcal{P}(E^N)$, over which, through this paper, we will only consider symmetric measures, more precisely, we say that $F^N \in \mathcal{P}(E^N)$ is symmetric if for all $\varphi \in C_b(E^N)$ we have that

$$\int_{E^N} \varphi dF^N = \int_{E^N} \varphi_\sigma dF^N,$$

for any permutation σ of $\{1, \dots, N\}$, and where

$$\varphi_\sigma := \varphi(V_\sigma) = \varphi(v_{\sigma(1)}, \dots, v_{\sigma(N)}),$$

for $V = (v_1, \dots, v_N) \in E^N$.

We then consider a N -particle system with initial probability density $F_0^N \in \mathcal{P}_{\text{sym}}(E^N)$ and its evolution equation in dual form, for all $\varphi \in C_b(E^N)$,

$$(3.1) \quad \partial_t \langle F_t^N, \varphi \rangle = \langle F_t^N, G^N \varphi \rangle.$$

It generates a linear semigroup denoted by $S_t^N : F_0^N \mapsto F_t^N$. Moreover, we define the dual semigroup T_t^N by

$$(3.2) \quad \forall \phi \in C_b(E^N), \forall F^N \in \mathcal{P}(E^N), \quad \langle T_t^N(\phi), F^N \rangle = \langle \phi, S_t^N(F^N) \rangle$$

and its generator G^N by.

$$(3.3) \quad \forall \phi \in C_b(E^N), \quad \partial_t \phi = G^N \phi.$$

At the level of the limit (mean field) equation, we consider a initial probability density $f_0 \in \mathcal{P}(E)$ and the equation

$$(3.4) \quad \partial_t f_t = Q(f_t).$$

We also define its semigroup $S_t^\infty : f_0 \mapsto f_t$. Then, we define the pullback semigroup T_t^∞ by

$$(3.5) \quad \forall \Phi \in C_b(\mathcal{P}(E)), \forall f \in \mathcal{P}(E), \quad T_t^\infty[\Phi](f) := \Phi(S_t^\infty(f))$$

and its generator G^∞ by

$$(3.6) \quad \forall \Phi \in C_b(\mathcal{P}(E)), \quad \partial_t \Phi = G^\infty \Phi.$$

We define some applications relating this objects. The function $\pi_E^N : E^N/\mathfrak{S}_N \rightarrow \mathcal{P}(E)$ is defined by, where \mathfrak{S}_N denotes the group of permutations of $\{1, \dots, N\}$,

$$(3.7) \quad \pi_E^N(V) := \mu_V^N = \frac{1}{N} \sum_{i=1}^N \delta_{v_i}$$

and μ_V^N is called the empirical measure associated to V . The application $\pi_C^N : C_b(\mathcal{P}(E)) \rightarrow C_b(E^N)$ is given by

$$(3.8) \quad \forall V \in E^N, \forall \Phi \in C_b(\mathcal{P}(E)), \quad \pi_C^N[\Phi](V) := \Phi(\mu_V^N).$$

The application $\pi_P^N : \mathcal{P}(E^N) \rightarrow \mathcal{P}(\mathcal{P}(E))$ is

$$(3.9) \quad \begin{aligned} & \forall F^N \in \mathcal{P}(E^N), \forall \Phi \in C_b(\mathcal{P}(E)), \\ & \langle \pi_P^N(F^N), \Phi \rangle := \langle F^N, \pi_C^N(\Phi) \rangle = \int_{E^N} \Phi(\mu_V^N) F^N(dV), \end{aligned}$$

where the first bracket is the duality $\mathcal{P}(\mathcal{P}(E)) \leftrightarrow C_b(\mathcal{P}(E))$ and the second one is the duality $\mathcal{P}(E^N) \leftrightarrow C_b(E^N)$. Finally, the application $R^N : C_b(E^N) \rightarrow C_b(\mathcal{P}(E))$ is defined by

$$(3.10) \quad \begin{aligned} & \forall \varphi \in C_b(E^N), \forall f \in \mathcal{P}(E), \\ & R^N[\varphi](f) := \langle \varphi, f^{\otimes N} \rangle = \int_{E^N} \varphi(V) f(dv_1) \cdots f(dv_N), \end{aligned}$$

in the sequel we will denote $R_\varphi^\ell := R^\ell[\varphi]$ for $\varphi \in C_b(E^\ell)$. The functions R_φ^ℓ are the "polynomials" on the space $\mathcal{P}(E)$, we will see later (exemple 3.6) that they are continuous in the sense of Definitions 3.3 and 3.4, where we develop a differential calculus on $\mathcal{P}(E)$.

For a given weight function $m : E \rightarrow \mathbb{R}_+$ we define the N -particle weight function

$$(3.11) \quad \forall V = (v_1, \dots, v_N) \in E^N, \quad M_m^N(V) := \frac{1}{N} \sum_{i=1}^N m(v_i) = \langle \mu_V^N, m \rangle = M_m(\mu_V^N).$$

Definition 3.1. For a given weight function $m_G : E \rightarrow \mathbb{R}_+$ we define the subspaces of probabilities

$$\mathcal{P}_G := \{f \in \mathcal{P}(E); \langle f, m_G \rangle < \infty\}$$

and the corresponding bounded sets, for $a \in (0, \infty)$,

$$\mathcal{BP}_{G,a} := \{f \in \mathcal{P}_G; \langle f, m_G \rangle \leq a\}.$$

For a given constraint function $\mathbf{m}_G : E \rightarrow \mathbb{R}^D$ such that $\langle f, \mathbf{m}_G \rangle$ is well defined for any $f \in \mathcal{P}_G$ and a given space of constraints $\mathbf{R}_G \subset \mathbb{R}^D$, we define, for any $\mathbf{r} \in \mathbf{R}_G$, the constrained subsets

$$\mathcal{P}_{G,\mathbf{r}} := \{f \in \mathcal{P}_G; \langle f, \mathbf{m}_G \rangle = \mathbf{r}\},$$

and the corresponding bounded constrained subsets

$$\mathcal{BP}_{G,a,\mathbf{r}} := \{f \in \mathcal{BP}_{G,a}; \langle f, \mathbf{m}_G \rangle = \mathbf{r}\}.$$

and the corresponding space of increments

$$\mathcal{IP}_G := \{g - f; \exists \mathbf{r} \in \mathbf{R}_G \text{ s.t. } g, f \in \mathcal{P}_{G,\mathbf{r}}\}.$$

We shall consider a distance dist_G defined on the whole space \mathcal{P}_G or such that there is a Banach space $\mathcal{G} \supset \mathcal{IP}_G$ endowed with a norm $\|\cdot\|_{\mathcal{G}}$ such that dist_G is defined for any $\mathbf{r} \in \mathbf{R}_G$ on $\mathcal{P}_{G,\mathbf{r}}$, by for any $f, g \in \mathcal{P}_{G,\mathbf{r}}$

$$\text{dist}_G(g, f) = \|g - f\|_{\mathcal{G}}.$$

Definition 3.2. We say that two spaces \mathcal{F} and \mathcal{P}_G , endowed with the norm $\|\cdot\|_{\mathcal{F}}$ and the distance dist_G inherited from the norm $\|\cdot\|_{\mathcal{G}}$, are in duality if

$$(3.12) \quad \forall f, g \in \mathcal{G}, \forall \varphi \in \mathcal{F} \quad |\langle g - f, \varphi \rangle| \leq \text{dist}_G(g, f) \|\varphi\|_{\mathcal{F}}.$$

Definition 3.3. Consider two metric spaces $\tilde{\mathcal{G}}_1$ and $\tilde{\mathcal{G}}_2$, some weight function $\Lambda : \tilde{\mathcal{G}}_1 \rightarrow \mathbb{R}_+^*$ and $\eta \in (0, 1]$. We denote by $C_\Lambda^{0,\eta}(\tilde{\mathcal{G}}_1; \tilde{\mathcal{G}}_2)$ the (weighted) space of functions with η -Hölder regularity, that is functions $\mathcal{S} : \tilde{\mathcal{G}}_1 \rightarrow \tilde{\mathcal{G}}_2$ such that there exists a constant $C > 0$

$$(3.13) \quad \forall f, g \in \tilde{\mathcal{G}}_1, \quad \text{dist}_{\mathcal{G}_2}(\mathcal{S}(f), \mathcal{S}(g)) \leq C \Lambda(g, f) \text{dist}_{\mathcal{G}_1}(f, g)^\eta.$$

where $\Lambda(g, f) = \max\{\Lambda(g), \Lambda(f)\}$.

We define then a higher order differential calculus.

Definition 3.4. Consider two normed spaces \mathcal{G}_1 and \mathcal{G}_2 , two metric spaces $\tilde{\mathcal{G}}_1$ and $\tilde{\mathcal{G}}_2$ such that $\tilde{\mathcal{G}}_i - \tilde{\mathcal{G}}_i \subset \mathcal{G}_i$, some weight function $\Lambda : \tilde{\mathcal{G}}_1 \rightarrow [1, \infty)$ and $\eta \in (0, 1]$. We denote by $C_\Lambda^{2,\eta}(\tilde{\mathcal{G}}_1; \tilde{\mathcal{G}}_2)$ the (weighted) space of functions two times continuously differentiable from $\tilde{\mathcal{G}}_1$ to $\tilde{\mathcal{G}}_2$, and such that the 2th derivative satisfies some weighted η -Hölder regularity (in the sense of Definition 3.3).

More precisely, these are functions $\mathcal{S} : \tilde{\mathcal{G}}_1 \rightarrow \tilde{\mathcal{G}}_2$ continuous, such that there exists maps (for $j = 1, 2$) $D^j \mathcal{S} : \tilde{\mathcal{G}}_1 \rightarrow \mathcal{L}^j(\mathcal{G}_1, \mathcal{G}_2)$, where $\mathcal{L}^j(\mathcal{G}_1, \mathcal{G}_2)$ is the space of j -multilinear applications from \mathcal{G}_1 to \mathcal{G}_2 , and there exists some constants $C_j > 0$, so that we have for all $f, g \in \tilde{\mathcal{G}}_1$,

$$(3.14) \quad \begin{aligned} \|\mathcal{S}(g) - \mathcal{S}(f)\|_{\mathcal{G}_2} &\leq C_0 \Lambda(g, f) \|g - f\|_{\mathcal{G}_1}^{\eta_0}, \\ \|\langle D\mathcal{S}[f], g - f \rangle\|_{\mathcal{G}_2} &\leq C_1 \Lambda(g, f) \|g - f\|_{\mathcal{G}_1}^{\eta_0}, \\ \|\mathcal{S}(g) - \mathcal{S}(f) - \langle D\mathcal{S}[f], g - f \rangle\|_{\mathcal{G}_2} &\leq C_2 \Lambda(g, f) \|g - f\|_{\mathcal{G}_1}^{1+\eta_1}, \\ \|\langle D^2\mathcal{S}[f], (g - f)^{\otimes 2} \rangle\|_{\mathcal{G}_2} &\leq C_3 \Lambda(g, f) \|g - f\|_{\mathcal{G}_1}^{1+\eta_1}, \\ \left\| \mathcal{S}(g) - \mathcal{S}(f) - \sum_{i=1}^2 \langle D^i \mathcal{S}[f], (g - f)^{\otimes i} \rangle \right\|_{\mathcal{G}_2} &\leq C_4 \Lambda(g, f) \|g - f\|_{\mathcal{G}_1}^{2+\eta}, \end{aligned}$$

where $\eta_0, \eta_1 \in [\eta, 1]$.

We define then the seminorms on $C_\Lambda^{2,\eta}(\tilde{\mathcal{G}}_1; \tilde{\mathcal{G}}_2)$

$$[\mathcal{S}]_{C_\Lambda^{1,0}} := \sup_{f \in \tilde{\mathcal{G}}_1, h \in \mathcal{G}_1} \frac{\|\langle D\mathcal{S}[f], h \rangle\|_{\mathcal{G}_2}}{\Lambda(f) \|h\|_{\mathcal{G}_1}^{\eta_0}}, \quad [\mathcal{S}]_{C_\Lambda^{2,0}} := \sup_{f \in \tilde{\mathcal{G}}_1, h \in \mathcal{G}_1} \frac{\|\langle D^2\mathcal{S}[f], (h, h) \rangle\|_{\mathcal{G}_2}}{\Lambda(f) \|h\|_{\mathcal{G}_1}^{1+\eta_1}}$$

and

$$\begin{aligned} [\mathcal{S}]_{C_\Lambda^{0,\eta_0}} &:= \sup_{f, g \in \tilde{\mathcal{G}}_1} \frac{\|\mathcal{S}(g) - \mathcal{S}(f)\|_{\mathcal{G}_2}}{\Lambda(g, f) \|g - f\|_{\mathcal{G}_1}^{\eta_0}}, \\ [\mathcal{S}]_{C_\Lambda^{1,\eta_1}} &:= \sup_{f, g \in \tilde{\mathcal{G}}_1} \frac{\|\mathcal{S}(g) - \mathcal{S}(f) - \langle D\mathcal{S}[f], g - f \rangle\|_{\mathcal{G}_2}}{\Lambda(g, f) \|g - f\|_{\mathcal{G}_1}^{1+\eta_1}}, \\ [\mathcal{S}]_{C_\Lambda^{2,\eta}} &:= \sup_{f, g \in \tilde{\mathcal{G}}_1} \frac{\left\| \mathcal{S}(g) - \mathcal{S}(f) - \sum_{i=1}^2 \langle D^i \mathcal{S}[f], (g - f)^{\otimes i} \rangle \right\|_{\mathcal{G}_2}}{\Lambda(g, f) \|g - f\|_{\mathcal{G}_1}^{2+\eta}}. \end{aligned}$$

Finally we combine these seminorms into

$$\|\mathcal{S}\|_{C_\Lambda^{2,\eta}} := [\mathcal{S}]_{C_\Lambda^{0,\eta_0}} + [\mathcal{S}]_{C_\Lambda^{1,\eta_1}} + [\mathcal{S}]_{C_\Lambda^{2,\eta}} + [\mathcal{S}]_{C_\Lambda^{1,0}} + [\mathcal{S}]_{C_\Lambda^{2,0}}.$$

This differential calculus holds for composition, more precisely for $\mathcal{U} \in C_{\Lambda_U}^{k,\eta}(\tilde{\mathcal{G}}_1; \tilde{\mathcal{G}}_2)$ and $\mathcal{V} \in C_{\Lambda_V}^{k,\eta}(\tilde{\mathcal{G}}_2; \tilde{\mathcal{G}}_3)$ we have $\mathcal{S} = \mathcal{V} \circ \mathcal{U} \in C_{\Lambda_S}^{k,\eta_S}(\tilde{\mathcal{G}}_1; \tilde{\mathcal{G}}_3)$ for some appropriate weight function Λ_S and exponent η_S . We now state the following lemma

Lemma 3.5. *Let \mathcal{G}_i be normed spaces and $\tilde{\mathcal{G}}_i$ be metric spaces for $i = 1, 2, 3$, such that $\tilde{\mathcal{G}}_i - \tilde{\mathcal{G}}_i \subset \mathcal{G}_i$. Consider $\mathcal{U} \in C_\Lambda^{2,\eta} \cap C_\Lambda^{1,(1+2\eta)/3} \cap C_\Lambda^{0,(2+\eta)/3}(\tilde{\mathcal{G}}_1; \tilde{\mathcal{G}}_2)$, with $\eta \in (0, 1]$, and $\mathcal{V} \in C^{2,1}(\tilde{\mathcal{G}}_2; \tilde{\mathcal{G}}_3)$. Then the composition function $\mathcal{S} = \mathcal{V} \circ \mathcal{U} \in C_{\Lambda^3}^{2,\eta} \cap C_{\Lambda^3}^{1,(1+2\eta)/3} \cap C_{\Lambda^3}^{0,(2+\eta)/3}(\tilde{\mathcal{G}}_1; \tilde{\mathcal{G}}_3)$ and we have*

$$\begin{aligned} D\mathcal{S}[f] &= D\mathcal{V}[\mathcal{U}(f)] \circ D\mathcal{U}[f], \\ D^2\mathcal{S}[f] &= D^2\mathcal{V}[\mathcal{U}(f)] \circ (D\mathcal{U}[f] \otimes D\mathcal{U}[f]) + D\mathcal{V}[\mathcal{U}(f)] \circ D^2\mathcal{U}[f]. \end{aligned}$$

More precisely, the following estimates hold

$$\begin{aligned} [\mathcal{S}]_{C_\Lambda^{0,(2+\eta)/3}} &\leq [\mathcal{V}]_{C^{0,1}} [\mathcal{U}]_{C_\Lambda^{0,(2+\eta)/3}}, \\ [\mathcal{S}]_{C_\Lambda^{1,0}} &\leq [\mathcal{V}]_{C^{1,0}} [\mathcal{U}]_{C_\Lambda^{1,0}}, \\ [\mathcal{S}]_{C_{\Lambda^2}^{1,(1+2\eta)/3}} &\leq [\mathcal{V}]_{C^{1,0}} [\mathcal{U}]_{C_\Lambda^{1,(1+2\eta)/3}} + [\mathcal{V}]_{C^{1,1}} [\mathcal{U}]_{C_\Lambda^{0,(2+\eta)/3}}^2, \\ [\mathcal{S}]_{C_{\Lambda^2}^{2,0}} &\leq [\mathcal{V}]_{C^{1,0}} [\mathcal{U}]_{C_\Lambda^{2,0}} + [\mathcal{V}]_{C^{2,0}} [\mathcal{U}]_{C_\Lambda^{1,0}}^2, \\ [\mathcal{S}]_{C_{\Lambda^3}^{2,\eta}} &\leq [\mathcal{V}]_{C^{1,0}} [\mathcal{U}]_{C_\Lambda^{2,\eta}} + [\mathcal{V}]_{C^{2,0}} [\mathcal{U}]_{C_\Lambda^{1,(1+2\eta)/3}}^2 \\ &\quad + 2[\mathcal{V}]_{C^{2,0}} [\mathcal{U}]_{C_\Lambda^{1,0}} [\mathcal{U}]_{C_\Lambda^{1,(1+2\eta)/3}} + [\mathcal{V}]_{C^{2,1}} [\mathcal{U}]_{C_\Lambda^{0,(2+\eta)/3}}^3. \end{aligned}$$

Proof of Lemma 3.5. Let $f, g \in \tilde{\mathcal{G}}_1$ and $\bar{f}, \bar{g} \in \tilde{\mathcal{G}}_2$.

By Definition 3.4 with $\mathcal{U} \in C_\Lambda^{2,\eta} \cap C_\Lambda^{1,(1+2\eta)/3} \cap C_\Lambda^{0,(2+\eta)/3}(\tilde{\mathcal{G}}_1; \tilde{\mathcal{G}}_2)$ and $\mathcal{V} \in C^{2,1}(\tilde{\mathcal{G}}_2; \tilde{\mathcal{G}}_3)$ we have

$$\begin{aligned} (3.15) \quad \mathcal{U}(g) - \mathcal{U}(f) &= \langle D\mathcal{U}[f], g - f \rangle + R_{\mathcal{U}}^1(g, f) \\ \mathcal{U}(g) - \mathcal{U}(f) &= \langle D\mathcal{U}[f], g - f \rangle + \langle D^2\mathcal{U}[f], (g - f)^{\otimes 2} \rangle + R_{\mathcal{U}}^2(g, f) \end{aligned}$$

with

$$(3.16) \quad \|\mathcal{U}(g) - \mathcal{U}(f)\|_{\mathcal{G}_2} \leq [\mathcal{U}]_{C_\Lambda^{0,\eta_0}} \Lambda(g, f) \|g - f\|_{\mathcal{G}_1}^{\eta_0},$$

$$(3.17) \quad \|\langle D\mathcal{U}[f], g - f \rangle\|_{\mathcal{G}_2} \leq [\mathcal{U}]_{C_\Lambda^{1,0}} \Lambda(g, f) \|g - f\|_{\mathcal{G}_1}^{\eta_0}$$

$$(3.18) \quad \|R_{\mathcal{U}}^1(g, f)\|_{\mathcal{G}_2} \leq [\mathcal{U}]_{C_\Lambda^{1,\eta_1}} \Lambda(g, f) \|g - f\|_{\mathcal{G}_1}^{1+\eta_1},$$

$$(3.19) \quad \|\langle D^2\mathcal{U}[f], (g - f)^{\otimes 2} \rangle\|_{\mathcal{G}_2} \leq [\mathcal{U}]_{C_\Lambda^{2,0}} \Lambda(g, f) \|g - f\|_{\mathcal{G}_1}^{1+\eta_1}$$

$$(3.20) \quad \|R_{\mathcal{U}}^2(g, f)\|_{\mathcal{G}_2} \leq [\mathcal{U}]_{C_\Lambda^{2,\eta}} \Lambda(g, f) \|g - f\|_{\mathcal{G}_1}^{2+\eta},$$

where, for simplicity, we denote $\eta_0 = (2 + \eta)/3$ and $\eta_1 = (1 + 2\eta)/3$.

Similarly we have for \mathcal{V} ,

$$\begin{aligned} (3.21) \quad \mathcal{V}(\bar{g}) - \mathcal{V}(\bar{f}) &= \langle D\mathcal{V}[\bar{f}], \bar{g} - \bar{f} \rangle + R_{\mathcal{V}}^1(\bar{g}, \bar{f}) \\ \mathcal{V}(\bar{g}) - \mathcal{V}(\bar{f}) &= \langle D\mathcal{V}[\bar{f}], \bar{g} - \bar{f} \rangle + \langle D^2\mathcal{V}[\bar{f}], (\bar{g} - \bar{f})^{\otimes 2} \rangle + R_{\mathcal{V}}^2(\bar{g}, \bar{f}) \end{aligned}$$

with

$$(3.22) \quad \|\mathcal{V}(\bar{g}) - \mathcal{V}(\bar{f})\|_{\mathcal{G}_3} \leq [\mathcal{V}]_{C^{0,1}} \|\bar{g} - \bar{f}\|_{\mathcal{G}_2}$$

$$(3.23) \quad \|\langle D\mathcal{V}[\bar{f}], \bar{g} - \bar{f} \rangle\|_{\mathcal{G}_3} \leq [\mathcal{V}]_{C^{1,0}} \|\bar{g} - \bar{f}\|_{\mathcal{G}_2}$$

$$(3.24) \quad \|R_{\mathcal{V}}^1(\bar{g}, \bar{f})\|_{\mathcal{G}_3} \leq [\mathcal{V}]_{C^{1,1}} \|\bar{g} - \bar{f}\|_{\mathcal{G}_2}^2$$

$$(3.25) \quad \|\langle D^2\mathcal{V}[\bar{f}], (\bar{g} - \bar{f})^{\otimes 2} \rangle\|_{\mathcal{G}_3} \leq [\mathcal{V}]_{C^{2,0}} \|\bar{g} - \bar{f}\|_{\mathcal{G}_2}^2$$

$$(3.26) \quad \|R_{\mathcal{V}}^2(\bar{g}, \bar{f})\|_{\mathcal{G}_3} \leq [\mathcal{V}]_{C^{2,1}} \|\bar{g} - \bar{f}\|_{\mathcal{G}_2}^3.$$

Using these estimates we can compute for $\mathcal{S} = \mathcal{V} \circ \mathcal{U}$, then we obtain first thanks to (3.22) and (3.16)

$$\begin{aligned} \|\mathcal{S}(g) - \mathcal{S}(f)\|_{\mathcal{G}_3} &= \|\mathcal{V}(\mathcal{U}(g)) - \mathcal{V}(\mathcal{U}(f))\|_{\mathcal{G}_3} \\ &\leq [\mathcal{V}]_{C^{0,1}} \|\mathcal{U}(g) - \mathcal{U}(f)\|_{\mathcal{G}_2} \\ &\leq [\mathcal{V}]_{C^{0,1}} [\mathcal{U}]_{C_\Lambda^{0,\eta_0}} \Lambda(g, f) \|g - f\|_{\mathcal{G}_1}^{\eta_0} \end{aligned}$$

which implies $[\mathcal{S}]_{C_\Lambda^{0,(2+\eta)/3}} \leq [\mathcal{V}]_{C^{0,1}} [\mathcal{U}]_{C_\Lambda^{0,(2+\eta)/3}}$.

We also have, using (3.21) and (3.15),

$$\begin{aligned} \mathcal{S}(g) - \mathcal{S}(f) &= \mathcal{V}(\mathcal{U}(g)) - \mathcal{V}(\mathcal{U}(f)) \\ &= \langle D\mathcal{V}[\mathcal{U}(f)], \mathcal{U}(g) - \mathcal{U}(f) \rangle + R_{\mathcal{V}}^1(\mathcal{U}(g), \mathcal{U}(f)) \\ &= \langle D\mathcal{V}[\mathcal{U}(f)], \{ \langle DU[f], g - f \rangle + R_{\mathcal{U}}^1(g, f) \} \rangle \\ &\quad + R_{\mathcal{V}}^1(\mathcal{U}(g), \mathcal{U}(f)), \end{aligned}$$

from which we deduce $\langle D\mathcal{S}[f], g - f \rangle = \langle D\mathcal{V}[\mathcal{U}(f)], (\langle DU[f], g - f \rangle) \rangle$ and, by (3.23) and (3.17),

$$\begin{aligned} \|\langle D\mathcal{S}[f], g - f \rangle\|_{\mathcal{G}_3} &\leq [\mathcal{V}]_{C^{1,0}} \|\langle DU[f], g - f \rangle\|_{\mathcal{G}_2} \\ &\leq [\mathcal{V}]_{C^{1,0}} [\mathcal{U}]_{C_\Lambda^{1,0}} \Lambda(f) \|g - f\|_{\mathcal{G}_1}^{\eta_0}, \end{aligned}$$

which yields $[\mathcal{S}]_{C_\Lambda^{1,0}} \leq [\mathcal{V}]_{C^{1,0}} [\mathcal{U}]_{C_\Lambda^{1,0}}$.

Therefore, we obtain using (3.23), (3.24), (3.18) and (3.16),

$$\begin{aligned} \|\mathcal{S}(g) - \mathcal{S}(f) - \langle D\mathcal{S}[f], g - f \rangle\|_{\mathcal{G}_3} &\leq \|\langle D\mathcal{V}[\mathcal{U}(f)], R_{\mathcal{U}}^1(g, f) \rangle\|_{\mathcal{G}_3} + \|R_{\mathcal{V}}^1(\mathcal{U}(g), \mathcal{U}(f))\|_{\mathcal{G}_3} \\ &\leq [\mathcal{V}]_{C^{1,0}} \|R_{\mathcal{U}}^1(g, f)\|_{\mathcal{G}_2} + [\mathcal{V}]_{C^{1,1}} \|\mathcal{U}(g) - \mathcal{U}(f)\|_{\mathcal{G}_2}^2 \\ &\leq [\mathcal{V}]_{C^{1,0}} [\mathcal{U}]_{C_\Lambda^{1,\eta_1}} \Lambda(g, f) \|g - f\|_{\mathcal{G}_1}^{1+\eta_1} + [\mathcal{V}]_{C^{1,1}} [\mathcal{U}]_{C_\Lambda^{0,\eta_0}}^2 \Lambda(g, f)^2 \|g - f\|_{\mathcal{G}_1}^{2\eta_0}. \end{aligned}$$

Since $1 + \eta_1 = 2\eta_0 = 1 + (1 + 2\eta)/3$ and $\Lambda \geq 1$, the last inequality implies

$$[\mathcal{S}]_{C_\Lambda^{1,(1+2\eta)/3}} \leq [\mathcal{V}]_{C^{1,0}} [\mathcal{U}]_{C_\Lambda^{1,(1+2\eta)/3}} + [\mathcal{V}]_{C^{1,1}} [\mathcal{U}]_{C_\Lambda^{0,(2+\eta)/3}}^2.$$

Finally, from (3.21) and (3.15), we have

$$\begin{aligned} \mathcal{S}(g) - \mathcal{S}(f) &= \mathcal{V}(\mathcal{U}(g)) - \mathcal{V}(\mathcal{U}(f)) \\ &= \langle D\mathcal{V}[\mathcal{U}(f)], \mathcal{U}(g) - \mathcal{U}(f) \rangle + \left\langle D^2\mathcal{V}[\mathcal{U}(f)], (\mathcal{U}(g) - \mathcal{U}(f))^{\otimes 2} \right\rangle + R_{\mathcal{V}}^2(\mathcal{U}(g), \mathcal{U}(f)) \\ &= \left\langle D\mathcal{V}[\mathcal{U}(f)], \left(\langle DU[f], g - f \rangle + \langle D^2\mathcal{U}[f], (g - f)^{\otimes 2} \rangle + R_{\mathcal{U}}^2(g, f) \right) \right\rangle \\ &\quad + \left\langle D^2\mathcal{V}[\mathcal{U}(f)], \left(\langle DU[f], g - f \rangle + R_{\mathcal{U}}^1(g, f) \right)^{\otimes 2} \right\rangle \\ &\quad + R_{\mathcal{V}}^2(\mathcal{U}(g), \mathcal{U}(f)), \end{aligned}$$

which yields

$$\begin{aligned} \langle D^2\mathcal{S}[f], (g - f)^{\otimes 2} \rangle &= \langle D\mathcal{V}[\mathcal{U}(f)], (\langle D^2\mathcal{U}[f], (g - f)^{\otimes 2} \rangle) \rangle \\ &\quad + \left\langle D^2\mathcal{V}[\mathcal{U}(f)], (\langle DU[f], g - f \rangle)^{\otimes 2} \right\rangle. \end{aligned}$$

Hence we obtain, with (3.23), (3.25), (3.19) and (3.17),

$$\begin{aligned} \|\langle D^2\mathcal{S}[f], (g - f)^{\otimes 2} \rangle\|_{\mathcal{G}_3} &\leq [\mathcal{V}]_{C^{1,0}} \|\langle D^2\mathcal{U}[f], (g - f)^{\otimes 2} \rangle\|_{\mathcal{G}_2} + [\mathcal{V}]_{C^{2,0}} \|\langle DU[f], g - f \rangle\|_{\mathcal{G}_2}^2 \\ &\leq [\mathcal{V}]_{C^{1,0}} [\mathcal{U}]_{C_\Lambda^{2,0}} \Lambda(f) \|g - f\|_{\mathcal{G}_1}^{1+\eta_1} + [\mathcal{V}]_{C^{2,0}} \left([\mathcal{U}]_{C_\Lambda^{1,0}} \Lambda(f) \|g - f\|_{\mathcal{G}_1}^{\eta_0} \right)^2 \\ &\leq \left([\mathcal{V}]_{C^{1,0}} [\mathcal{U}]_{C_\Lambda^{2,0}} + [\mathcal{V}]_{C^{2,0}} [\mathcal{U}]_{C_\Lambda^{1,0}}^2 \right) \Lambda(f)^2 \|g - f\|_{\mathcal{G}_1}^{1+(1+2\eta)/3}, \end{aligned}$$

which gives $[\mathcal{S}]_{C_{\Lambda^2}^{2,0}} \leq [\mathcal{V}]_{C^{1,0}} [\mathcal{U}]_{C_{\Lambda}^{2,0}} + [\mathcal{V}]_{C^{2,0}} [\mathcal{U}]_{C_{\Lambda}^{1,0}}^2$.

Now, for the last estimate we obtain

$$\begin{aligned} & \|\mathcal{S}(g) - \mathcal{S}(f) - \langle D\mathcal{S}[f], g - f \rangle - \langle D^2\mathcal{S}[f], (g - f)^{\otimes 2} \rangle\|_{\mathcal{G}_3} \\ & \leq \|\langle D\mathcal{V}[\mathcal{U}(f)], R_{\mathcal{U}}^2(g, f) \rangle\|_{\mathcal{G}_3} + \left\| \left\langle D^2\mathcal{V}[\mathcal{U}(f)], (R_{\mathcal{U}}^1(g, f))^{\otimes 2} \right\rangle \right\|_{\mathcal{G}_3} \\ & \quad + 2 \|\langle D^2\mathcal{V}[\mathcal{U}(f)], (\langle D\mathcal{U}[f], g - f \rangle \otimes R_{\mathcal{U}}^1(g, f)) \rangle\|_{\mathcal{G}_3} \\ & \quad + \|R_{\mathcal{V}}^2(\mathcal{U}(g), \mathcal{U}(f))\|_{\mathcal{G}_3} \end{aligned}$$

and using the equations (3.22) to (3.26) and (3.16) to (3.20), it gives

$$\begin{aligned} & \|\mathcal{S}(g) - \mathcal{S}(f) - \langle D\mathcal{S}[f], g - f \rangle - \langle D^2\mathcal{S}[f], (g - f)^{\otimes 2} \rangle\|_{\mathcal{G}_3} \\ & \leq [\mathcal{V}]_{C^{1,0}} \|R_{\mathcal{U}}^2(g, f)\|_{\mathcal{G}_2} + [\mathcal{V}]_{C^{2,0}} \|R_{\mathcal{U}}^1(g, f)\|_{\mathcal{G}_2}^2 \\ & \quad + 2 [\mathcal{V}]_{C^{2,0}} \|\langle D\mathcal{U}[f], g - f \rangle\|_{\mathcal{G}_2} \|R_{\mathcal{U}}^1(g, f)\|_{\mathcal{G}_2} + [\mathcal{V}]_{C^{2,1}} \|\mathcal{U}(g) - \mathcal{U}(f)\|_{\mathcal{G}_2}^3 \\ & \leq [\mathcal{V}]_{C^{1,0}} [\mathcal{U}]_{C_{\Lambda}^{2,\eta}} \Lambda(g, f) \|g - f\|_{\mathcal{G}_1}^{2+\eta} \\ & \quad + [\mathcal{V}]_{C^{2,0}} [\mathcal{U}]_{C_{\Lambda}^{1,\eta_1}}^2 \Lambda(g, f)^2 \|g - f\|_{\mathcal{G}_1}^{2+2\eta_1} \\ & \quad + 2 [\mathcal{V}]_{C^{2,0}} [\mathcal{U}]_{C_{\Lambda}^{1,0}} [\mathcal{U}]_{C_{\Lambda}^{1,\eta_1}} \Lambda(g, f)^2 \|g - f\|_{\mathcal{G}_1}^{1+\eta_1+\eta_0} \\ & \quad + [\mathcal{V}]_{C^{2,1}} [\mathcal{U}]_{C_{\Lambda}^{0,\eta_0}}^3 \Lambda(g, f)^3 \|g - f\|_{\mathcal{G}_1}^{3\eta_0}. \end{aligned}$$

Since $1 + \eta_1 + \eta_0 = 3\eta_0 = 2 + \eta < 2 + 2\eta_1$, we deduce

$$\begin{aligned} [\mathcal{S}]_{C_{\Lambda^3}^{2,\eta}} & \leq [\mathcal{V}]_{C^{1,0}} [\mathcal{U}]_{C_{\Lambda}^{2,\eta}} + [\mathcal{V}]_{C^{2,0}} [\mathcal{U}]_{C_{\Lambda}^{1,(1+2\eta)/3}}^2 \\ & \quad + 2 [\mathcal{V}]_{C^{2,0}} [\mathcal{U}]_{C_{\Lambda}^{1,0}} [\mathcal{U}]_{C_{\Lambda}^{1,(1+2\eta)/3}} + [\mathcal{V}]_{C^{2,1}} [\mathcal{U}]_{C_{\Lambda}^{0,(2+\eta)/3}}^3. \end{aligned}$$

□

Exemple 3.6. Consider the pair \mathcal{F} and $\mathcal{P}_{\mathcal{G}}$ in duality (Definition 3.2) where $\mathcal{F} \subset C_b(E)$, and consider $\varphi = \varphi_1 \times \cdots \times \varphi_{\ell} \in \mathcal{F}^{\otimes \ell}$. Then the application R_{φ}^{ℓ} defined in (3.10) is $C^{2,1}(\mathcal{P}_{\mathcal{G}}; \mathbb{R})$. Consider $f, g \in \mathcal{P}_{\mathcal{G}}$, then we have thanks to the multilinearity of R_{φ}^{ℓ} [18, 19] that

$$\begin{aligned} |R_{\varphi}^{\ell}(g) - R_{\varphi}^{\ell}(f)| & \leq \ell \|\varphi\|_{\mathcal{F} \otimes (L^{\infty})^{\ell-1}} \|g - f\|_{\mathcal{G}}, \\ |DR_{\varphi}^{\ell}[f](g - f)| & \leq \ell \|\varphi\|_{\mathcal{F} \otimes (L^{\infty})^{\ell-1}} \|g - f\|_{\mathcal{G}}, \\ |R_{\varphi}^{\ell}(g) - R_{\varphi}^{\ell}(f) - DR_{\varphi}^{\ell}[f](g - f)| & \leq \frac{\ell(\ell-1)}{2} \|\varphi\|_{\mathcal{F} \otimes^2 \otimes (L^{\infty})^{\ell-2}} \|g - f\|_{\mathcal{G}}^2 \\ |D^2R_{\varphi}^{\ell}[f](g - f)^{\otimes 2}| & \leq \frac{\ell(\ell-1)}{2} \|\varphi\|_{\mathcal{F} \otimes^2 \otimes (L^{\infty})^{\ell-2}} \|g - f\|_{\mathcal{G}}^2 \\ \left| R_{\varphi}^{\ell}(g) - R_{\varphi}^{\ell}(f) - DR_{\varphi}^{\ell}[f](g - f) - D^2R_{\varphi}^{\ell}[f](g - f)^{\otimes 2} \right| \\ & \leq \frac{\ell(\ell-1)(\ell-2)}{6} \|\varphi\|_{\mathcal{F} \otimes^3 \otimes (L^{\infty})^{\ell-3}} \|g - f\|_{\mathcal{G}}^3, \end{aligned}$$

where we define

$$\|\varphi\|_{\mathcal{F} \otimes^j \otimes (L^{\infty})^{\ell-j}} := \max_{i_1, \dots, i_j \text{ distincts in } [1, \ell]} \left(\|\varphi_{i_1}\|_{\mathcal{F}} \cdots \|\varphi_{i_j}\|_{\mathcal{F}} \prod_{k \neq i_1, \dots, i_j} \|\varphi_k\|_{L^{\infty}} \right).$$

Since we shall need endow the subspaces of probability measures in Definition 3.1 with metrics, let us give useful examples. We denote by $\mathcal{P}_p(\mathbb{R}^d)$ the space of probabilities with finite moments up to order p , more precisely $\mathcal{P}_p(\mathbb{R}^d) := \{f \in \mathcal{P}(E) ; \langle |v|^p, f \rangle < \infty\}$.

Definition 3.7 (Monge-Kantorovich-Wasserstein distance). For $f, g \in \mathcal{P}_p(\mathbb{R}^d)$ we define the distance

$$W_p^p(f, g) := \inf_{\pi \in \Pi(f, g)} \int_{\mathbb{R}^d \times \mathbb{R}^d} \text{dist}(x, y)^p \pi(dx, dy)$$

where $\Pi(f, g)$ is the set of probability measures on $\mathbb{R}^d \times \mathbb{R}^d$ with marginals f and g respectively.

In an analogous way, we also define, for $\mu, \nu \in \mathcal{P}(\mathcal{P}(\mathbb{R}^d))$ and a distance D over $\mathcal{P}(E)$, the distance

$$\mathcal{W}_{1,D}(\mu, \nu) := \inf_{\pi \in \Pi(\mu, \nu)} \int_{\mathcal{P}(\mathbb{R}^d) \times \mathcal{P}(\mathbb{R}^d)} D(f, g) \pi(df, dg)$$

where $\Pi(\mu, \nu)$ denotes the set of probability measures on $\mathcal{P}(\mathbb{R}^d) \times \mathcal{P}(\mathbb{R}^d)$ with marginals μ and ν .

Definition 3.8 (Fourier based distances). For $f, g \in \mathcal{P}_s(\mathbb{R}^d)$ we define the distance

$$(3.27) \quad |f - g|_s := \sup_{\xi \in \mathbb{R}^d} \frac{|\widehat{f}(\xi) - \widehat{g}(\xi)|}{|\xi|^s}$$

which is well defined if f and g have equal moments up to order $s - 1$ if s is a integer or $\lfloor s \rfloor$ if not. We denote by \mathcal{H}^{-s} the space associated to this norm.

Definition 3.9 (General Fourier based distances). Let $k \in \mathbb{N}^*$ and set

$$m_{\mathcal{G}} := |v|^k, \quad \mathbf{m}_{\mathcal{G}} := (v^\alpha)_{\alpha \in \mathbb{N}^d}, \quad |\alpha| \leq k - 1$$

and

$$v^\alpha = (v_1^{\alpha_1}, \dots, v_d^{\alpha_d}), \quad \alpha = (\alpha_1, \dots, \alpha_d)$$

and we define

$$\forall f \in \mathcal{IP}_{\mathcal{G}}, \quad \|f\|_{\mathcal{G}} = |f|_s := \sup_{\xi \in \mathbb{R}^d} \frac{|\widehat{f}(\xi)|}{|\xi|^s}, \quad s \in (0, k].$$

We extend the above norm to $M_k^1(\mathbb{R}^d)$ in the following way. First we define for $f \in M_{k-1}^1(\mathbb{R}^d)$ and $\alpha \in \mathbb{N}^d$, $|\alpha| \leq k - 1$ the moment

$$\mathcal{M}_\alpha[f] := \int_{\mathbb{R}^d} v^\alpha f(dv).$$

For a fixed function χ in the Schwartz space such that $\chi = 1$ over $\{v \in \mathbb{R}^d, |v| \leq 1\}$, we define then

$$\hat{\mathcal{M}}_k[f](\xi) := \chi(\xi) \left(\sum_{|\alpha| \leq k-1} \mathcal{M}_\alpha[f] \frac{\xi^\alpha}{\alpha!} \right).$$

We may define then the semi-norms

$$|f|_k := \sup_{\xi \in \mathbb{R}^d} \frac{|\widehat{f}(\xi) - \hat{\mathcal{M}}_k[f](\xi)|}{|\xi|^k}$$

and the norm

$$\|f\|_k := |f|_k + \sum_{|\alpha| \leq k-1} |\mathcal{M}_\alpha[f]|.$$

3.2. Abstract theorem. We state the assumptions of our abstract theorem 3.11.

Assumption A1 (N-particle system). The semigroup T_t^N and its generator G^N are well defined on $C_b(E^N)$ and are invariant under permutation so that F_t^N is well defined. Moreover, we assume that the following conditions hold:

- (i) *Conservation constraint:* There exists a constraint function $\mathbf{m}_{\mathcal{G}_1} : E \rightarrow \mathbb{R}^D$ and a subset $\mathbf{R}_{\mathcal{G}_1} \subset \mathbb{R}^D$ such that defining the set

$$\mathbb{E}_N := \{V \in E^N; \langle \mu_V^N, \mathbf{m}_{\mathcal{G}_1} \rangle \in \mathbf{R}_{\mathcal{G}_1}\}$$

there holds

$$\forall t \geq 0, \quad \text{supp } F_t^N \subset \mathbb{E}_N.$$

- (ii) *Propagation of integral moment bound:* There exists a weight function $m_{\mathcal{G}_1}$, a time $T \in (0, \infty]$ and a constant $C_{m_{\mathcal{G}_1}}^T > 0$, possibly depending on $m_{\mathcal{G}_1}$, but not on N , such that

$$(3.28) \quad \forall N \geq 1 \quad \sup_{t \geq 0} \langle F_t^N, M_{m_{\mathcal{G}_1}}^N \rangle \leq C_{m_{\mathcal{G}_1}}.$$

Assumption A2 (Nonlinear semigroup). Consider a probability space $\mathcal{P}_{\mathcal{G}_1}$ as in Definition 3.1, associated to some function $m_{\mathcal{G}_1}$ and a constraint function \mathbf{m}_1 , and endowed with the metric induced from \mathcal{G}_1 . For some $\delta \in (0, 1]$ and some $\bar{a} \in (0, \infty)$, we have for any $a \in (\bar{a}, \infty)$:

- (i) The nonlinear semigroup is well defined $S_t^\infty : \mathcal{BP}_{\mathcal{G}_1, a} \rightarrow \mathcal{BP}_{\mathcal{G}_1, a}$, which is δ -Hölder continuous locally uniformly in time, in the sense that for any $\tau \in (0, \infty)$ there exists $C_\tau > 0$ such that

$$\forall f, g \in \mathcal{BP}_{\mathcal{G}_1, a} \quad \sup_{0 \leq t \leq \tau} \|S_t^\infty g - S_t^\infty f\|_{\mathcal{G}_1} \leq C_\tau \|g - f\|_{\mathcal{G}_1}^\delta.$$

- (ii) The application Q is bounded and δ -Hölder continuous from $\mathcal{BP}_{\mathcal{G}_1, a}$ into \mathcal{G}_1 .

With **(A2)** we have the following lemma [18, Lemma 2.11] (see also [19, Lemma 2.9])

Lemma 3.10. *Assume **(A2)**. For any $a \in (\bar{a}, \infty)$ the pullback semigroup T_t^∞ defined by*

$$\forall f \in \mathcal{BP}_{\mathcal{G}, a}, \quad \Phi \in C_b(\mathcal{BP}_{\mathcal{G}, a}), \quad T_t^\infty[\Phi](f) := \Phi(S_t^{NL}(f))$$

is a C_0 -semigroup of contractions on the Banach space $C_b(\mathcal{BP}_{\mathcal{G}, a}(E))$.

Its generator G^∞ is an unbounded linear operator on $C_b(\mathcal{BP}_{\mathcal{G}, a})$ with domain $\text{Dom}(G^\infty)$ containing $C_b^{1, \eta}(\mathcal{BP}_{\mathcal{G}, a})$. On the latter space, it is defined by the formula

$$(3.29) \quad \forall \Phi \in C_b^{1, \eta}(\mathcal{BP}_{\mathcal{G}, a}), \quad \forall f \in \mathcal{BP}_{\mathcal{G}, a}, \quad (G^\infty \Phi)(f) := \langle D\Phi[f], Q(f) \rangle.$$

Assumption A3 (Convergence of the generators). Let $\mathcal{P}_{\mathcal{G}_1}, m_{\mathcal{G}_1}, \mathbf{R}_{\mathcal{G}_1}$ be such as introduced in **(A2)**. Define a weight function $1 \leq m'_{\mathcal{G}_1} \leq C m_{\mathcal{G}_1}$ and then the corresponding weight $\Lambda_1(f) := \langle f, m'_{\mathcal{G}_1} \rangle$.

We assume that there exist a function $\varepsilon(N)$ going to 0 as $N \rightarrow \infty$ and $\eta \in (0, 1]$ such that for all $\Phi \in C_{\Lambda_1}^{2, \eta}(\mathcal{P}_{\mathcal{G}_1, \mathbf{r}}; \mathbb{R})$ we have

$$(3.30) \quad \left\| \left(M_{m_{\mathcal{G}_1}}^N \right)^{-1} (G^N \pi_N - \pi_N G^\infty) \Phi \right\|_{L^\infty(\mathbb{E}_N)} \leq \varepsilon(N) \sup_{\mathbf{r} \in \mathbf{R}_{\mathcal{G}_1}} \left([\Phi]_{C_{\Lambda_1}^{1, \eta}(\mathcal{P}_{\mathcal{G}_1, \mathbf{r}}; \mathbb{R})} + [\Phi]_{C_{\Lambda_1}^{2, 0}(\mathcal{P}_{\mathcal{G}_1, \mathbf{r}}; \mathbb{R})} \right).$$

Assumption A4 (Differential stability). We assume that the flow

$$S_t^\infty \in C_{\Lambda_2}^{2, \eta} \cap C_{\Lambda_2}^{1, (1+2\eta)/3} \cap C_{\Lambda_2}^{0, (2+\eta)/3}(\mathcal{P}_{\mathcal{G}_1, \mathbf{r}}; \mathcal{P}_{\mathcal{G}_2}),$$

for any $\mathbf{r} \in \mathbf{R}_{\mathcal{G}_1}$, and that there exists $C_4^\infty > 0$ such that

$$(3.31) \quad \sup_{\mathbf{r} \in \mathbf{R}_{\mathcal{G}_1}} \int_0^\infty \left([S_t^\infty]_{C_{\Lambda_2}^{1,(1+2\eta)/3}} + [S_t^\infty]_{C_{\Lambda_2}^{0,(2+\eta)/3}}^2 + [S_t^\infty]_{C_{\Lambda_2}^{2,0}} + [S_t^\infty]_{C_{\Lambda_2}^{1,0}}^2 \right) dt \leq C_4^\infty,$$

where $\mathcal{P}_{\mathcal{G}_2}$ is the same probability space as $\mathcal{P}_{\mathcal{G}_1}$ but endowed with the norm associated to some Banach space $\mathcal{G}_2 \supset \mathcal{G}_1$, with $\Lambda_2 = \Lambda_1^{\frac{1}{3}}$ and η is the same as in **(A3)**.

Assumption A5 (Weak stability). We assume that, for some probability space $\mathcal{P}_{\mathcal{G}_3}$ associated to a weight function $m_{\mathcal{G}_3}$, a constraint $\mathbf{m}_{\mathcal{G}_3}$, a set of constraints $\mathbf{R}_{\mathcal{G}_3}$ and a distance $\text{dist}_{\mathcal{G}_3}$, for any $a > 0$ there exists a constant $C_5^\infty > 0$ such that for any $\mathbf{r} \in \mathbf{R}_{\mathcal{G}_3}$,

$$(3.32) \quad \forall f, g \in \mathcal{BP}_{\mathcal{G}_3, a, \mathbf{r}}, \quad \sup_{t \geq 0} \text{dist}_{\mathcal{G}_3} (S_t^\infty(f), S_t^\infty(g)) \leq C_5^\infty \text{dist}_{\mathcal{G}_3} (f, g).$$

Theorem 3.11 (Abstract theorem). *Let us consider a family of N -particle initial conditions $F_0^N \in \mathcal{P}_{\text{sym}}(E^N)$ and the solution associated $F_t^N = S_t^N(F_0^N)$. Consider also a one-particle initial condition $f_0 \in \mathcal{P}(E)$ and the solution associated $f_t = S_t^\infty(f_0)$. Assume that **(A1)**-**(A2)**-**(A3)**-**(A4)**-**(A5)** hold for some spaces $\mathcal{P}_{\mathcal{G}_i}$, \mathcal{G}_i and \mathcal{F}_i , $i = 1, 2, 3$, and where \mathcal{G}_i and \mathcal{F}_i are in duality.*

Then there exists a constant $C \in (0, \infty)$ such that for any $N, \ell \in \mathbb{N}$, with $N \geq 2\ell$, and for any $\varphi = \varphi_1 \otimes \cdots \otimes \varphi_\ell \in \mathcal{F}^{\otimes \ell}$, $\mathcal{F} := \mathcal{F}_1 \cap \mathcal{F}_2 \cap \mathcal{F}_3$ we have

$$(3.33) \quad \sup_{t \geq 0} \left| \left\langle S_t^N(F_0^N) - (S_t^\infty(f_0))^{\otimes N}, \varphi \otimes \mathbf{1}^{N-\ell} \right\rangle \right| \\ \leq C \left[\frac{\ell^2}{N} \|\varphi\|_{L^\infty} + C_{m_{\mathcal{G}_1}} C_4^\infty \varepsilon(N) \ell^3 \|\varphi\|_{\mathcal{F}_2^{\otimes 3} \otimes (L^\infty)^{\ell-3}} + C_5^\infty \ell \|\varphi\|_{\mathcal{F}_3 \otimes (L^\infty)^{\ell-1}} \mathcal{W}_{1, \mathcal{G}_3}(\pi_P^N F_0^N, \delta_{f_0}) \right].$$

As a consequence, if F_0^N is f_0 -chaotic the propagation of chaos holds.

Proof of Theorem 3.11. We split the term (3.33) in three parts :

$$\begin{aligned} & \left| \left\langle S_t^N(F_0^N) - (S_t^\infty(f_0))^{\otimes N}, \varphi \otimes \mathbf{1}^{N-\ell} \right\rangle \right| \\ & \leq \left| \left\langle S_t^N(F_0^N), \varphi \otimes \mathbf{1}^{N-\ell} \right\rangle - \left\langle S_t^N(F_0^N), R_\varphi^\ell \circ \mu_V^N \right\rangle \right| \quad (= T_1) \\ & + \left| \left\langle F_0^N, T_t^N(R_\varphi^\ell \circ \mu_V^N) \right\rangle - \left\langle F_0^N, (T_t^\infty R_\varphi^\ell) \circ \mu_V^N \right\rangle \right| \quad (= T_2) \\ & + \left| \left\langle F_0^N, (T_t^\infty R_\varphi^\ell) \circ \mu_V^N \right\rangle - \left\langle (S_t^\infty(f_0))^{\otimes \ell}, \varphi \right\rangle \right| \quad (= T_3) \end{aligned}$$

and we evaluate each of them.

Step 1. For the first term T_1 , a classical combinatorial trick (see [21] and [19, Lemma 2.14]) implies

$$T_1 \leq \frac{2\ell^2 \|\varphi\|_{L^\infty(E^\ell)}}{N}.$$

Step 2. For the term T_2 , we have the following identity

$$\begin{aligned} T_t^N \pi_N - \pi_N T_t^\infty &= - \int_0^t \frac{d}{ds} [T_{t-s}^N \pi_N T_s^\infty] ds \\ &= \int_0^t T_{t-s}^N (G^N \pi_N - \pi_N G^\infty) T_s^\infty ds \end{aligned}$$

and then for any $t \geq 0$

$$\begin{aligned}
(3.34) \quad T_2 &\leq \int_0^\infty \left| \left\langle (M_{m_{\mathcal{G}_1}}^N) S_{t-s}^N(F_0^N), (M_{m_{\mathcal{G}_1}}^N)^{-1} [G^N \pi_N - \pi_N G^\infty] T_s^\infty R_\varphi^\ell \right\rangle \right| ds \\
&\leq \sup_{t \geq 0} \left\langle M_{m_{\mathcal{G}_1}}^N, S_t^N(F_0^N) \right\rangle \int_0^\infty \left\| (M_{m_{\mathcal{G}_1}}^N)^{-1} [G^N \pi_N - \pi_N G^\infty] T_s^\infty R_\varphi^\ell \right\|_{L^\infty(\mathbb{E}_N)} ds \\
&\leq C_{m_{\mathcal{G}_1}} \varepsilon(N) \sup_{\mathbf{r} \in \mathbf{R}_{\mathcal{G}_1}} \int_0^\infty \left([T_s^\infty R_\varphi^\ell]_{C_{\Lambda_1}^{1,\eta}(\mathcal{P}_{\mathcal{G}_1,\mathbf{r}};\mathbb{R})} + [T_s^\infty R_\varphi^\ell]_{C_{\Lambda_1}^{2,0}(\mathcal{P}_{\mathcal{G}_1,\mathbf{r}};\mathbb{R})} \right) ds,
\end{aligned}$$

thanks to **(A1)-(i)** and **(A3)**.

Furthermore we know that the application $T_s^\infty(R_\varphi^\ell) = R_\varphi^\ell \circ S_s^\infty$ and by assumption **(A4)** the semigroup

$$S_s^\infty \in C_{\Lambda_2}^{2,\eta} \cap C_{\Lambda_2}^{1,(1+2\eta)/3} \cap C_{\Lambda_2}^{0,(2+\eta)/3}(\mathcal{P}_{\mathcal{G}_1,\mathbf{r}}; \mathcal{P}_{\mathcal{G}_2}).$$

Moreover, with $\varphi \in \mathcal{F}_2^{\otimes \ell}$ we have $R_\varphi^\ell \in C^{2,1}(\mathcal{P}_{\mathcal{G}_2}; \mathbb{R})$ (see exemple 3.6). Finally, by Lemma 3.5 we obtain that

$$T_s^\infty(R_\varphi^\ell) \in C_{\Lambda_2}^{2,\eta} \cap C_{\Lambda_2}^{1,(1+2\eta)/3} \cap C_{\Lambda_2}^{0,(2+\eta)/3}(\mathcal{P}_{\mathcal{G}_1,\mathbf{r}}; \mathbb{R})$$

with

$$\begin{aligned}
[T_s^\infty R_\varphi^\ell]_{C_{\Lambda_2}^{1,(1+2\eta)}(\mathcal{P}_{\mathcal{G}_1,\mathbf{r}};\mathbb{R})} &\leq \|R_\varphi^\ell\|_{C^{2,1}(\mathcal{P}_{\mathcal{G}_2};\mathbb{R})} \left([S_t^\infty]_{C_{\Lambda_2}^{1,(1+2\eta)/3}(\mathcal{P}_{\mathcal{G}_1,\mathbf{r}};\mathcal{P}_{\mathcal{G}_2})} + [S_t^\infty]_{C_{\Lambda_2}^{0,(2+\eta)/3}(\mathcal{P}_{\mathcal{G}_1,\mathbf{r}};\mathcal{P}_{\mathcal{G}_2})}^2 \right), \\
[T_s^\infty R_\varphi^\ell]_{C_{\Lambda_2}^{2,0}(\mathcal{P}_{\mathcal{G}_1,\mathbf{r}};\mathbb{R})} &\leq \|R_\varphi^\ell\|_{C^{2,1}(\mathcal{P}_{\mathcal{G}_2};\mathbb{R})} \left([S_t^\infty]_{C_{\Lambda_2}^{2,0}(\mathcal{P}_{\mathcal{G}_1,\mathbf{r}};\mathcal{P}_{\mathcal{G}_2})} + [S_t^\infty]_{C_{\Lambda_2}^{1,0}(\mathcal{P}_{\mathcal{G}_1,\mathbf{r}};\mathcal{P}_{\mathcal{G}_2})}^2 \right).
\end{aligned}$$

From assumption **(A4)**, $\Lambda_2 = \Lambda_1^{1/(3)}$ and the estimate of $\|R_\varphi^\ell\|_{C^{2,1}}$ in exemple 3.6, we can deduce, plugging the last estimate on (3.34),

$$(3.35) \quad T_2 \leq C_{m_{\mathcal{G}_1}} C_4^\infty \varepsilon(N) \ell^3 \|\varphi\|_{\mathcal{F}_2^{\otimes 3} \otimes (L^\infty)^{\ell-3}}.$$

Step 3. For the third term T_3 we deduce, thanks to the assumption **(A5)** and the fact that $R_\varphi^\ell \in C^{0,1}(\mathcal{P}_{\mathcal{G}_3}; \mathbb{R})$ for $\varphi \in \mathcal{F}_3^{\otimes \ell}$,

$$\begin{aligned}
(3.36) \quad T_3 &= \left| \left\langle F_0^N, R_\varphi^\ell(S_t^\infty(\mu_V^N)) - R_\varphi^\ell(S_t^\infty(f_0)) \right\rangle \right| \\
&\leq \|R_\varphi^\ell\|_{C^{0,1}(\mathcal{P}_{\mathcal{G}_3};\mathbb{R})} \left\langle F_0^N, \text{dist}_{\mathcal{G}_3}(S_t^\infty(\mu_V^N), S_t^\infty(f_0)) \right\rangle \\
&\leq \ell \|\varphi\|_{\mathcal{F}_3 \otimes (L^\infty)^{\ell-1}} C_5^\infty \left\langle F_0^N, \text{dist}_{\mathcal{G}_3}(\mu_V^N, f_0) \right\rangle.
\end{aligned}$$

By the definition of the Wasserstein distance

$$\forall \alpha, \beta \in \mathcal{P}(\mathcal{P}_{\mathcal{G}_3}) \quad \mathcal{W}_{1,\mathcal{G}_3}(\alpha, \beta) := \inf_{\pi \in \Pi(\alpha, \beta)} \int_{\mathcal{P}_{\mathcal{G}_3} \times \mathcal{P}_{\mathcal{G}_3}} \text{dist}_{\mathcal{G}_3}(f, g) \pi(df, dg)$$

with $\Pi(\alpha, \beta)$ the set of probability measures over $\mathcal{P}_{\mathcal{G}_3} \times \mathcal{P}_{\mathcal{G}_3}$ with marginals α and β respectively. Taking $\beta = \delta_{f_0}$, the set $\Pi(\alpha, \delta_{f_0}) = \{\alpha \otimes \delta_{f_0}\}$ has only one element and we obtain, with $\alpha = \pi_P^N F_0^N$,

$$\begin{aligned}
(3.37) \quad \mathcal{W}_{1,\mathcal{G}_3}(\pi_P^N F_0^N, \delta_{f_0}) &:= \inf_{\pi \in \Pi(\pi_P^N F_0^N, \delta_{f_0})} \iint_{\mathcal{P}_{\mathcal{G}_3} \times \mathcal{P}_{\mathcal{G}_3}} \text{dist}_{\mathcal{G}_3}(f, g) \pi(df, dg) \\
&= \iint_{\mathcal{P}_{\mathcal{G}_3} \times \mathcal{P}_{\mathcal{G}_3}} \text{dist}_{\mathcal{G}_3}(f, g) \pi_P^N F_0^N(df) \delta_{f_0}(dg) \\
&= \int_{\mathcal{P}_{\mathcal{G}_3}} \text{dist}_{\mathcal{G}_3}(f, f_0) \pi_P^N F_0^N(df) \\
&= \int_{E^N} \text{dist}_{\mathcal{G}_3}(\mu_V^N, f_0) F_0^N(dV).
\end{aligned}$$

where we have used the definition of π_P^N (see subsection 3.1) in the last equality. We conclude then plugging this estimate on (3.36). \square

4. APPLICATION TO THE LANDAU EQUATION

In this section we will use the consistency-stability method presented in the Section 3 to show the propagation of chaos for the Landau equation for maxwellian molecules. To prove some estimates for the Landau equation we will prove first the same estimates for the Boltzmann equation (as in [18]) with a collisional kernel satisfying the grazing collisions (2.18). Then passing to the limit we will recover the same results for Landau.

Our main theorems are:

Theorem 4.1. *Consider a N -particle initial condition $F_0^N \in \mathcal{P}_{\text{sym}}(\mathbb{R}^{dN})$ and, for all $t \geq 0$, the associated solution of the N -particle Landau dynamics $F_t^N = S_t^N(F_0^N)$. Consider also a one-particle initial condition $f_0 \in \mathcal{P}_2(\mathbb{R}^d)$, with zero momentum $\int v f_0 = 0$ and energy $\int |v|^2 f_0 =: \mathcal{E} \in (0, \infty)$, and the associated solution of the limit (mean-field) Landau equation $f_t = S_t^\infty(f_0)$. Suppose further that there exists $\mathcal{E}_0 \in (0, \infty)$ such that*

$$(4.1) \quad \text{supp } F_0^N \subset \left\{ V \in \mathbb{R}^{dN}, \frac{1}{N} \sum_{i=1}^N |v_i|^2 \leq \mathcal{E}_0 \right\}.$$

Then, for $\ell \in \mathbb{N}^*$, for all

$$\varphi = \varphi_1 \otimes \cdots \otimes \varphi_\ell \in \mathcal{F}^{\otimes \ell}, \quad \mathcal{F} := \left\{ \varphi : \mathbb{R}^d \rightarrow \mathbb{R}^d, \|\varphi\|_{\mathcal{F}} = \int_{\mathbb{R}^d} (1 + |\xi|^6) |\hat{\varphi}(\xi)| d\xi < \infty \right\}$$

we have, for any $N \geq 2\ell$,

$$\begin{aligned} & \sup_{t \geq 0} |\langle S_t^N(F_0^N) - (S_t^\infty(f_0))^{\otimes N}, \varphi \rangle| \\ & \leq C \left[\frac{\ell^2}{N} \|\varphi\|_{L^\infty} + C_4^\infty \frac{\ell^3}{N} \|\varphi\|_{\mathcal{F}^3 \otimes (L^\infty)^{\ell-3}} + C_5^\infty \ell \|\varphi\|_{W^{1,\infty} \otimes (L^\infty)^{\ell-1}} \mathcal{W}_{1,W_2}(\pi_P^N F_0^N, \delta_{f_0}) \right]. \end{aligned}$$

As a consequence, if F_0^N is f_0 -chaotic the third term of the right-hand side goes to 0 when $N \rightarrow \infty$, which implies the propagation of chaos.

Theorem 4.2. *Consider the same framework of Theorem 4.1. Assume moreover that $f_0 \in \mathcal{P}_6(\mathbb{R}^d) \cap L^p(\mathbb{R}^d)$ for some $p > 1$ and let $F_0^N := [f_0^{\otimes N}]_{\mathcal{S}^N(\mathcal{E})}$ (observe that (4.1) is satisfied for this choice of initial data with $\mathcal{E}_0 = \mathcal{E}$). Then it holds:*

(1) *For all $0 < \epsilon < 9[(7d+6)^2(d+9)]^{-1}$, there exists a constant $C_\epsilon > 0$ such that*

$$\sup_{t \geq 0} \frac{W_1(F_t^N, f_t^{\otimes N})}{N} \leq C_\epsilon N^{-\epsilon}.$$

(2) *For all $t \geq 0$, for all $N \in \mathbb{N}^*$*

$$\frac{W_1(F_t^N, \gamma^N)}{N} \leq p(t),$$

for a polynomial rate $p(t) \rightarrow 0$ as $t \rightarrow \infty$ and where γ^N is the uniform probability measure on $\mathcal{S}^N(\mathcal{E})$.

Remark 4.3. This theorem also holds (with different quantitative rates) for other choices of initial data F_0^N that are f_0 -chaotic. In particular, if we consider $f_0 \in \mathcal{P}_2(\mathbb{R}^d)$ with compact support and $F_0^N = f_0^{\otimes N} \in \mathcal{P}_{\text{sym}}(\mathbb{R}^{dN})$.

The proof of Theorem 4.1 relies on the proof of assumptions **(A1)**-**(A2)**-**(A3)**-**(A4)**-**(A5)**, with a suitable choice of spaces, and then on the application of Theorem 3.11. Furthermore, we shall prove Theorem 4.2 using Theorem 4.1 and some results from [11, 18, 4].

4.1. Proof of A1. Consider the N -particle SDE (2.33)-(2.34). Since b and σ are Lipschitz, existence and uniqueness hold by standard arguments (see [20, Chapter 5]). Hence it defines a semigroup T_t^N , we can then define its generator $G^N = G_L^N$ (given by (2.28)-(2.29)) and its dual semigroup S_t^N , as explained in section 3.

We have the following lemma.

Lemma 4.4. *The dynamics of the N -particle system (2.29) conserves momentum and energy, more precisely there holds, for all $t \geq 0$,*

$$\int_{\mathbb{R}^{dN}} \varphi \left(\sum_{i=1}^N v_{i,\alpha} \right) F_t^N(dV) = \int_{\mathbb{R}^{dN}} \varphi \left(\sum_{i=1}^N v_{i,\alpha} \right) F_0^N(dV), \quad \alpha \in \{1, \dots, d\}$$

and

$$\int_{\mathbb{R}^{dN}} \varphi(|V|^2) F_t^N(dV) = \int_{\mathbb{R}^{dN}} \varphi(|V|^2) F_0^N(dV).$$

Remark 4.5. We can easily observe during the proof that if we consider the N -particle system of Remark 2.1 with generator G_2^N (2.32), which is different from the system we considered here (2.29), we have conservation of energy

$$\partial_t \left\langle f_t^N, \sum_{i=1}^N |v_i|^2 \right\rangle = \left\langle f_t^N, G_2^N \sum_{i=1}^N |v_i|^2 \right\rangle = 0,$$

however this is not true for all functions $\varphi = \varphi(|V|^2)$. Then, Lemma 4.6, which is a consequence of this lemma, does not hold for the N -particle system of Remark 2.1.

Proof of Lemma 4.4. Let us prove the second equality (energy conservation), the proof of the first one (momentum conservation) being similar. Consider the Landau master equation (2.29) and $\varphi(V) = \varphi(|V|^2)$ smooth enough, we have then

$$\nabla_i(\varphi(|V|^2)) = (\partial_{v_{i,\alpha}} \varphi(|V|^2))_{1 \leq \alpha \leq d} = 2\varphi'(|V|^2)v_i$$

and, for $i \neq j$,

$$\begin{aligned} \partial_{v_{i,\alpha}} \partial_{v_{j,\beta}} \varphi(|V|^2) &= 4\varphi''(|V|^2)v_{i,\alpha}v_{j,\beta} \\ \partial_{v_{i,\alpha}} \partial_{v_{i,\beta}} \varphi(|V|^2) &= 4\varphi''(|V|^2)v_{i,\alpha}v_{i,\beta} + 2\varphi'(|V|^2)\delta_{\alpha\beta}. \end{aligned}$$

Denoting $\varphi' = \varphi'(|V|^2)$ and $\varphi'' = \varphi''(|V|^2)$ for simplicity, we obtain

$$\begin{aligned} (\nabla_{ii}^2 \varphi + \nabla_{jj}^2 \varphi - \nabla_{ij}^2 \varphi - \nabla_{ji}^2 \varphi)_{\alpha\beta} &= 4\varphi' \delta_{\alpha\beta} + 4\varphi''(v_{i,\alpha}v_{i,\beta} + v_{j,\alpha}v_{j,\beta} - v_{i,\alpha}v_{j,\beta} - v_{j,\alpha}v_{i,\beta}) \\ &= 4\varphi' \delta_{\alpha\beta} + 4\varphi''(v_i - v_j)_\alpha(v_i - v_j)_\beta. \end{aligned}$$

Therefore we have

$$\begin{aligned} (4.2) \quad b(v_i - v_j)(\nabla_i \varphi(|V|^2) - \nabla_j \varphi(|V|^2)) &= -2|v_i - v_j|^\gamma(v_i - v_j) \cdot 2\varphi'(|V|^2)(v_i - v_j) \\ &= -4\varphi'(|V|^2)|v_i - v_j|^{\gamma+2} \end{aligned}$$

and

$$\begin{aligned}
& a(v_i - v_j) : (\nabla_{ii}^2 \varphi + \nabla_{jj}^2 \varphi - \nabla_{ij}^2 \varphi - \nabla_{ji}^2 \varphi) = \\
& = |v_i - v_j|^\gamma \sum_{\alpha, \beta=1}^d \left\{ [|v_i - v_j|^2 \delta_{\alpha\beta} - (v_i - v_j)_\alpha (v_i - v_j)_\beta] 4\varphi' \delta_{\alpha\beta} \right. \\
& \quad \left. + [|v_i - v_j|^2 \delta_{\alpha\beta} - (v_i - v_j)_\alpha (v_i - v_j)_\beta] 4\varphi'' (v_i - v_j)_\alpha (v_i - v_j)_\beta \right\} \\
& =: |v_i - v_j|^\gamma \{T_1 + T_2\}.
\end{aligned}$$

Computing T_1 we have

$$\begin{aligned}
(4.3) \quad T_1 &= 4\varphi' \sum_{\alpha, \beta=1}^d [|v_i - v_j|^2 \delta_{\alpha\beta} - (v_i - v_j)_\alpha (v_i - v_j)_\beta] \delta_{\alpha\beta} \\
&= 8\varphi' |v_i - v_j|^2,
\end{aligned}$$

and computing T_2

$$\begin{aligned}
(4.4) \quad T_2 &= 4\varphi'' \sum_{\alpha, \beta=1}^d [|v_i - v_j|^2 (v_i - v_j)_\alpha (v_i - v_j)_\beta \delta_{\alpha\beta} - (v_i - v_j)_\alpha^2 (v_i - v_j)_\beta^2] \\
&= 4\varphi'' \left\{ |v_i - v_j|^4 - \left[\sum_{\alpha=1}^d (v_i - v_j)_\alpha^2 \right]^2 \right\} \\
&= 0.
\end{aligned}$$

Gathering (4.2), (4.3) and (4.4) we obtain

$$\begin{aligned}
(4.5) \quad & b(v_i - v_j) (\nabla_i \varphi - \nabla_j \varphi) + \frac{1}{2} a(v_i - v_j) : (\nabla_{ii}^2 \varphi + \nabla_{jj}^2 \varphi - \nabla_{ij}^2 \varphi - \nabla_{ji}^2 \varphi) \\
&= -4\varphi' |v_i - v_j|^{\gamma+2} + \frac{1}{2} 8\varphi' |v_i - v_j|^{\gamma+2} \\
&= 0,
\end{aligned}$$

which implies, for all $t \geq 0$,

$$(4.6) \quad \int \varphi(|V|^2) F_t^N(dV) = \int \varphi(|V|^2) F_0^N(dV).$$

□

Lemma 4.6. *Consider F_0^N such that*

$$\text{supp } F_0^N \subset \{V \in \mathbb{R}^{dN}; M_2^N(V) = \frac{1}{N} \sum_{i=1}^N |v_i|^2 \leq \mathcal{E}_0\}.$$

Then there holds

$$\forall t > 0, \quad \text{supp } F_t^N \subset \{V \in \mathbb{R}^{dN}; M_2^N(V) \leq \mathcal{E}_0\}.$$

Proof of Lemma 4.6. It is a consequence of Lemma 4.4, with $\varphi(|V|^2) = \mathbf{1}_{|V|^2 > N\mathcal{E}_0}$. Consider a mollifier ρ_η for $\eta > 0$, i.e. $\rho_\eta(x) = \eta^{-1} \rho(\eta^{-1}x)$, with $\rho \in C_c^\infty(\mathbb{R})$, $\rho \geq 0$ and $\text{supp } \rho \subset B_1$, and define $\varphi_\eta = \rho_\eta * \varphi$. Using Lemma 4.4 we have, for all η and for all $t \geq 0$,

$$\int_{\mathbb{R}^{dN}} \varphi_\eta F_t^N(dV) = \int_{\mathbb{R}^{dN}} \varphi_\eta F_0^N(dV).$$

Passing to the limit $\eta \rightarrow 0$ we obtain

$$\int_{\mathbb{R}^{dN}} \mathbf{1}_{|V|^2 > N\mathcal{E}_0} F_t^N(dV) = \int_{\mathbb{R}^{dN}} \mathbf{1}_{|V|^2 > N\mathcal{E}_0} F_0^N(dV) = 0.$$

□

Lemma 4.7. *Consider F_0^N such that $\langle F_0^N, M_k^N \rangle \leq C_k$ for $k > 2$. Then there holds*

$$\sup_{t \geq 0} \langle F_t^N, M_k^N \rangle \leq C_k.$$

Proof of Lemma 4.7. Consider $F_t^{N,\varepsilon}$ the solution of the Boltzmann N -particle system (2.8)-(2.9) with grazing collisions (2.18). Then from [18, Lemma 5.3], we obtain the desired result for $F_t^{N,\varepsilon}$ with a constant independent of ε . We conclude passing to the grazing collisions limit $\varepsilon \rightarrow 0$. □

Consider the constraint function $\mathbf{m}_{\mathcal{G}_1} : \mathbb{R}^d \rightarrow \mathbb{R}_+ \times \mathbb{R}^d$, $\mathbf{m}_{\mathcal{G}_1}(v) = (|v|^2, v)$ with the set of constraints $\mathbf{R}_{\mathcal{G}_1} : \{(r, \bar{r}) \in \mathbb{R}_+ \times \mathbb{R}^d; |\bar{r}|^2 \leq r \leq \mathcal{E}_0\}$. Then Lemma 4.6 proves **(A1i)**. Moreover, Lemma 4.7 proves **(A1ii)** with the weight function $m_{\mathcal{G}_1}(v) := \langle v \rangle^6 = (1 + |v|^2)^3$ for all $v \in \mathbb{R}^d$.

4.2. Proof of A2. Let us define the spaces of probabilities (and the corresponding bounded, constrained and increments subsets, see Definition 3.1)

$$\mathcal{P}_{\mathcal{G}_1} := \{f \in \mathcal{P}(\mathbb{R}^d); M_6(f) < \infty\},$$

and, for $\mathbf{r} \in \mathbf{R}_{\mathcal{G}_1}$, more precisely $\mathbf{r} = (r, \bar{r}) = (r, r_1, \dots, r_d)$, the constrained space

$$\mathcal{P}_{\mathcal{G}_1, \mathbf{r}} := \{f \in \mathcal{P}_{\mathcal{G}_1}; \langle f, |v|^2 \rangle = r, \langle f, v_i \rangle = r_i \text{ for } i = 1, \dots, d\}.$$

We define then the for some $a \in (0, \infty)$ the bounded set

$$\mathcal{BP}_{\mathcal{G}_1, a} := \{f \in \mathcal{P}_{\mathcal{G}_1}; M_6(f) \leq a\},$$

and, for any $\mathbf{r} \in \mathbf{R}_{\mathcal{G}_1}$, the bounded constrained set

$$\mathcal{BP}_{\mathcal{G}_1, a, \mathbf{r}} := \{f \in \mathcal{BP}_{\mathcal{G}_1, a}; \langle f, |v|^2 \rangle = r, \langle f, v_i \rangle = r_i \text{ for } i = 1, \dots, d\}.$$

We endow these spaces with the distance $\text{dist}_{\mathcal{G}_1}$ associated with the norm $\|\cdot\|_{\mathcal{G}_1} = |\cdot|_2$ (see definitions 3.8 and 3.9).

We have the following lemma.

Lemma 4.8. *Let $f_0, g_0 \in \mathcal{P}_2(\mathbb{R}^d)$ with same momentum, i.e. $\langle f_0, v \rangle = \langle g_0, v \rangle$, and consider the solutions f_t, g_t of Landau equation (2.10)-(2.11). Then*

$$(4.7) \quad \sup_{t \geq 0} |f_t - g_t|_2 \leq |f_0 - g_0|_2.$$

Remark 4.9. Let us mention that this result can be found in [25] proving uniqueness for the Landau equation for maxwellian molecules. There the author indicates that we can prove it using the known result for the Boltzmann equation for maxwellian molecules from [23] and then passing to the limit of grazing collisions.

Proof of Lemma 4.8. Let us split the prove in two steps. First we prove the lemma for the Boltzmann equation then we recover the result for Landau equation passing to the limit of grazing collisions.

Step 1. We shall prove the deasired result for the Boltzmann equation with true Maxwellian molecules. This result is prove in [23, 18], but we write it for completeness.

Consider the solutions f_t^ε and g_t^ε of Boltzmann equation (2.1)-(2.2) with initial data f_0 and g_0 respectively. Denote $d^\varepsilon := g^\varepsilon - f^\varepsilon$ and $s^\varepsilon := g^\varepsilon + f^\varepsilon$, then the equation satisfied for d is

$$\partial_t d^\varepsilon = \frac{1}{2} \left[Q_{B,\varepsilon}(s^\varepsilon, d^\varepsilon) + Q_{B,\varepsilon}(d^\varepsilon, s^\varepsilon) \right].$$

Performing the Fourier transform ([1]), we have

$$\partial_t \widehat{d^\varepsilon}(\xi) = \int_{\mathbb{S}^{d-1}} b_\varepsilon(\sigma \cdot \widehat{\xi}) \left[\frac{\widehat{d^\varepsilon}(\xi^+) \widehat{s^\varepsilon}(\xi^-)}{2} + \frac{\widehat{d^\varepsilon}(\xi^-) \widehat{s^\varepsilon}(\xi^+)}{2} - \widehat{d^\varepsilon}(\xi) \right] d\sigma$$

where $\xi^+ = \frac{\xi + |\xi|\sigma}{2}$, $\xi^- = \frac{\xi - |\xi|\sigma}{2}$ and $\widehat{\xi} = \xi/|\xi|$.

We recall that b_ε is not integrable so we perform the following cut-off, which will be relaxed in the end,

$$(4.8) \quad \int_{\mathbb{S}^{d-1}} b_\varepsilon^K(\sigma \cdot \widehat{\xi}) d\sigma = K, \quad b_\varepsilon^K = b_\varepsilon \mathbf{1}_{|\theta| \geq \delta(K)},$$

for some function δ , so that $b_\varepsilon = b_\varepsilon^K + b_\varepsilon^C$. In [23, 18], we observe that the remainder term

$$R_\varepsilon^K(\xi) := \int_{\mathbb{S}^{d-1}} b_\varepsilon^C(\sigma \cdot \widehat{\xi}) \left[\frac{\widehat{d^\varepsilon}(\xi^+) \widehat{s^\varepsilon}(\xi^-)}{2} + \frac{\widehat{d^\varepsilon}(\xi^-) \widehat{s^\varepsilon}(\xi^+)}{2} - \widehat{d^\varepsilon}(\xi) \right] d\sigma$$

verifies, for any $\xi \in \mathbb{R}^d$, $|R_\varepsilon^K(\xi)| \leq r_\varepsilon^K |\xi|$, where $r_\varepsilon^K \rightarrow 0$ as $K \rightarrow \infty$, and r_ε^K is bounded by the second order moment.

Using that $\|\widehat{s}\|_\infty \leq 2$, we have

$$\frac{d}{dt} \frac{|\widehat{d^\varepsilon}(\xi)|}{|\xi|^2} + K \frac{|\widehat{d^\varepsilon}(\xi)|}{|\xi|^2} \leq \sup_{\xi \in \mathbb{R}^d} \frac{|\widehat{d^\varepsilon}(\xi)|}{|\xi|^2} \left(\sup_{\xi \in \mathbb{R}^d} \int_{\mathbb{S}^{d-1}} b_\varepsilon^K(\sigma \cdot \widehat{\xi}) (|\widehat{\xi}^+|^2 + |\widehat{\xi}^-|^2) d\sigma \right) + R_\varepsilon^K$$

with

$$|\widehat{\xi}^+|^2 = \frac{1}{2} (1 + \sigma \cdot \widehat{\xi}), \quad |\widehat{\xi}^-|^2 = \frac{1}{2} (1 - \sigma \cdot \widehat{\xi}).$$

One obtains

$$\frac{d}{dt} \frac{|\widehat{d^\varepsilon}(\xi)|}{|\xi|^2} + K \frac{|\widehat{d^\varepsilon}(\xi)|}{|\xi|^2} \leq K \sup_{\xi \in \mathbb{R}^d} \frac{|\widehat{d^\varepsilon}(\xi)|}{|\xi|^2} + R_\varepsilon^K,$$

and by a Gronwall's lemma one deduces

$$\sup_{\xi \in \mathbb{R}^d} \frac{|\widehat{d^\varepsilon}(\xi)|}{|\xi|^2} \leq \sup_{\xi \in \mathbb{R}^d} \frac{|\widehat{d_0}(\xi)|}{|\xi|^2} + t R_\varepsilon^K.$$

Relaxing the cut-off $K \rightarrow \infty$ one proves

$$(4.9) \quad \|f_t^\varepsilon - g_t^\varepsilon\|_2 \leq \|f_0 - g_0\|_2.$$

Step 2. Since (4.9) does not depend on ε and the solution of the Boltzmann equation f_t^ε converges towards the solution of the Landau equation f_t (see [24]) when $\varepsilon \rightarrow 0$, we obtain the desired result. \square

Therefore we have that the Landau semigroup S_t^∞ is $C^{0,1}(\mathcal{P}_{G_1}; \mathcal{P}_{G_1})$ and **(A2)-i** is proved.

To prove **(A2)-ii** we use [18, Lemma 5.5], valid for the Boltzmann operator with grazing collisions $Q_{B,\varepsilon}$, which says that there exists $C > 0$ and $\delta \in (0, 1]$ such that for any $f, g \in \mathcal{BP}_{G_1, a, r}$ we have

$$|Q_{B,\varepsilon}(f, f) - Q_{B,\varepsilon}(g, g)|_2 \leq C \|f - g\|_2^\delta,$$

with a constant C that does not depend on ε . Finally, passing to the limit of grazing collisions $\varepsilon \rightarrow 0$, we have that $Q_{B,\varepsilon} \rightarrow Q_L$. We prove then **(A2)-ii** also for the Landau equation.

4.3. Proof of A3. Let $\Lambda_1(f) := \langle f, m'_{G_1} \rangle$ with the weight $m'_{G_1}(v) := \langle v \rangle^4$, where we recall that $m_{G_1} = \langle v \rangle^6$, and then consider the generator G^N of the Landau master equation (2.29).

Then we have the following lemma, which proves **(A3)**.

Lemma 4.10. *For all $\Phi \in \bigcap_{\mathbf{r} \in \mathbf{R}_{G_1}} C_{\Lambda_1}^{2,\eta}(\mathcal{P}_{G_1,\mathbf{r}}; \mathbb{R})$ there exists $C > 0$ such that*

$$(4.10) \quad \left\| \left(M_{m_{G_1}}^N \right)^{-1} \left(G^N \pi_N - \pi_N G^\infty \right) \Phi \right\|_{L^\infty(\mathbb{E}_N)} \leq \frac{C}{N} \sup_{\mathbf{r} \in \mathbf{R}_{G_1}} [\Phi]_{C_{\Lambda_1}^{2,0}(\mathcal{P}_{G_1,\mathbf{r}}; \mathbb{R})}.$$

Proof of Lemma 4.10. The application $\mathbb{R}^{dN} \rightarrow \mathcal{P}_{G_1}$, $V \mapsto \mu_V^N$ is of class $C^{2,1}$ with (see [19, Lemma 5.4])

$$(4.11) \quad \begin{aligned} \partial_{i\alpha}(\mu_V^N) &= \frac{1}{N} \partial_\alpha \delta_{V_i} \\ \partial_{i\alpha, i\beta}^2(\mu_V^N) &= \frac{1}{N} \partial_{\alpha\beta}^2 \delta_{V_i} \end{aligned}$$

and for $i \neq j$, $\partial_{i\alpha, j\beta}^2(\mu_V^N) = 0$.

Let $\Phi \in C_{\Lambda_1}^{2,\eta}(\mathcal{P}_{G_1}; \mathbb{R})$, so $\mathbb{R}^{dN} \rightarrow \mathbb{R}$, $V \mapsto \Phi(\mu_V^N)$ is also $C^{2,\eta}$. Indeed, let $\phi = D\Phi[\mu_V^N]$ and we have

$$\begin{aligned} \partial_{v_i, \alpha}(\Phi(\mu_V^N)) &= \langle D\Phi[\mu_V^N], \partial_{v_i, \alpha} \mu_V^N \rangle = \langle D\Phi[\mu_V^N], \frac{1}{N} \partial_{v_i, \alpha} \delta_{v_i} \rangle = \frac{1}{N} \partial_\alpha \phi(v_i) \\ \partial_{v_i, \alpha, v_i, \beta}^2 \Phi(\mu_V^N) &= \langle D\Phi[\mu_V^N], \frac{1}{N} \partial_{v_i, \alpha, v_i, \beta}^2 \delta_{v_i} \rangle + D^2\Phi[\mu_V^N] \left(\frac{1}{N} \partial_{v_i, \alpha} \delta_{v_i}, \frac{1}{N} \partial_{v_i, \beta} \delta_{v_i} \right) \\ &= \frac{1}{N} \partial_{\alpha, \beta}^2 \phi(v_i) + \frac{1}{N^2} D^2\Phi[\mu_V^N] (\partial_{v_i, \alpha} \delta_{v_i}, \partial_{v_i, \beta} \delta_{v_i}). \end{aligned}$$

We compute

$$\begin{aligned} (G^N \pi^N \Phi)(V) &= G^N \Phi(\mu_V^N) \\ &= \frac{1}{N} \sum_{i,j=1}^N \sum_{\alpha=1}^d b_\alpha(v_i - v_j) [\partial_{v_i, \alpha}(\Phi(\mu_V^N)) - \partial_{v_j, \alpha}(\Phi(\mu_V^N))] \\ &\quad + \frac{1}{2N} \sum_{i,j=1}^N \sum_{\alpha, \beta=1}^d a_{\alpha\beta}(v_i - v_j) \left[\partial_{v_i, \alpha, v_i, \beta}^2 \Phi(\mu_V^N) + \partial_{v_j, \alpha, v_j, \beta}^2 \Phi(\mu_V^N) \right. \\ &\quad \left. - \partial_{v_i, \alpha, v_j, \beta}^2 \Phi(\mu_V^N) - \partial_{v_j, \alpha, v_i, \beta}^2 \Phi(\mu_V^N) \right] \\ &= \frac{1}{N} \sum_{i,j=1}^N \sum_{\alpha=1}^d b_\alpha(v_i - v_j) \left[\frac{1}{N} \partial_\alpha \phi(v_i) - \frac{1}{N} \partial_\alpha \phi(v_j) \right] \\ &\quad + \frac{1}{2N} \sum_{i,j=1}^N \sum_{\alpha, \beta=1}^d a_{\alpha\beta}(v_i - v_j) \left[\frac{1}{N} \partial_{\alpha, \beta}^2 \phi(v_i) + \frac{1}{N} \partial_{\alpha, \beta}^2 \phi(v_j) \right] \quad (= I_1) \\ &\quad + \frac{1}{2N} \sum_{i,j=1}^N \sum_{\alpha, \beta=1}^d a_{\alpha\beta}(v_i - v_j) \left[\frac{1}{N^2} D^2\Phi[\mu_V^N] (\partial_{v_i, \alpha} \delta_{v_i}, \partial_{v_i, \beta} \delta_{v_i}) \right. \\ &\quad \left. + \frac{1}{N^2} D^2\Phi[\mu_V^N] (\partial_{v_j, \alpha} \delta_{v_j}, \partial_{v_j, \beta} \delta_{v_j}) - 2 \frac{1}{N^2} D^2\Phi[\mu_V^N] (\partial_{v_i, \alpha} \delta_{v_i}, \partial_{v_j, \beta} \delta_{v_j}) \right] \quad (= I_2). \end{aligned}$$

For the first term we have

$$\begin{aligned} I_1 &= \iint \sum_{\alpha=1}^d b_\alpha(v - v_*) [\partial_\alpha \phi(v) - \partial_\alpha \phi(v_*)] \mu_V^N(dv) \mu_V^N(dv_*) \\ &\quad + \frac{1}{2} \iint \sum_{\alpha, \beta=1}^d a_{\alpha\beta}(v - v_*) [\partial_{\alpha, \beta}^2 \phi(v) + \partial_{\alpha, \beta}^2 \phi(v_*)] \mu_V^N(dv) \mu_V^N(dv_*) \\ &= \langle Q_L(\mu_V^N, \mu_V^N), \phi \rangle = \langle Q_L(\mu_V^N, \mu_V^N), D\Phi[\mu_V^N] \rangle = (\pi^N G^\infty \Phi)(V), \end{aligned}$$

thanks to Lemma 3.10. For the second one, we deduce that there exists $C > 0$ depending on d such that

$$|I_2| \leq \frac{C}{N} |D^2 \Phi[\mu_V^N](\partial_{v_{i, \alpha}} \delta_{v_i}, \partial_{v_{i, \beta}} \delta_{v_i})| \frac{1}{N^2} \sum_{i, j=1}^N |v_i - v_j|^2,$$

since $|a_{\alpha\beta}(v_i - v_j)| \leq |v_i - v_j|^2$. We conclude then

$$\begin{aligned} |I_2| &\leq \frac{C}{N} [\Phi]_{C_{\Lambda_1}^{2,0}(\mathcal{P}_{\mathcal{G}_1, \mathbf{r}; \mathbb{R}})} \Lambda_1(\mu_V^N) \frac{1}{N^2} \sum_{i, j=1}^N |v_i - v_j|^2 \\ &\leq \frac{C\mathcal{E}}{N} [\Phi]_{C_{\Lambda_1}^{2,0}(\mathcal{P}_{\mathcal{G}_1, \mathbf{r}; \mathbb{R}})} M_{m_{\mathcal{G}_1}}^N(V) \end{aligned}$$

and therefore we prove (4.10). \square

4.4. Proof of A4. In the same way of the subsection 4.2, we will use here the Boltzmann equation and then perform the asymptotics of grazing collisions to prove the results for the Landau equation.

We define the following equations, denoting by Q the symmetrized version of the Landau operator Q_L (2.11), i.e. $Q(f, g) = [Q_L(f, g) + Q_L(g, f)]/2$,

$$(4.12) \quad \begin{cases} \partial_t f = Q(f, f), & f|_{t=0} = f_0, \\ \partial_t g = Q(g, g), & g|_{t=0} = g_0, \\ \partial_t h = 2Q(f, h), & h|_{t=0} = g_0 - f_0, \\ \partial_t r = 2Q(f, r) + Q(h, h), & r|_{t=0} = 0, \end{cases}$$

and the new variables

$$d := g - f, \quad s := g + f, \quad \omega := g - f - h, \quad \psi := g - f - h - r,$$

which satisfy

$$(4.13) \quad \begin{cases} \partial_t d = Q(s, d), & d|_{t=0} = g_0 - f_0, \\ \partial_t \omega = Q(s, \omega) + Q(h, d), & \omega|_{t=0} = 0, \\ \partial_t \psi = Q(s, \psi) + Q(h, \omega) + Q(r, d), & \psi|_{t=0} = 0. \end{cases}$$

Lemma 4.11. *Consider $f_0, g_0 \in \mathcal{P}_{\mathcal{G}_1, \mathbf{r}}$, $\mathbf{r} \in \mathbf{R}_{\mathcal{G}_1}$, and the solutions f_t, g_t, h_t of (4.12)-(4.13). There exists $\lambda_1 \in (0, \infty)$ that for any $\eta \in [2/3, 1]$, there exists C_η such that we have*

$$(4.14) \quad \begin{aligned} |g_t - f_t|_2 &\leq C_\eta e^{-(1-\eta)\lambda_1 t} M_4(f_0 + g_0)^{1/3} |g_0 - f_0|_2^\eta, \\ |h_t|_2 &\leq C_\eta e^{-(1-\eta)\lambda_1 t} M_4(f_0 + g_0)^{1/3} |g_0 - f_0|_2^\eta. \end{aligned}$$

Proof of Lemma 4.11. We split the proof into two steps. Again, we shall first prove the lemma for Boltzmann equation with a kernel satisfying the grazing collisions, which is proved in [18], and then passing to the limit of grazing collisions we prove the same result for the Landau equation.

Step 1. Let us denote by Q_ε the symmetrized version of the Boltzmann operator $Q_{B,\varepsilon}$ (2.2) with kernel b_ε satisfying (2.18), i.e. $Q_\varepsilon(f, g) = [Q_{B,\varepsilon}(f, g) + Q_{B,\varepsilon}(g, f)]/2$.

Consider the solutions f_t^ε , g_t^ε and h_t^ε of

$$(4.15) \quad \begin{cases} \partial_t f^\varepsilon = Q_\varepsilon(f^\varepsilon, f^\varepsilon), & f^\varepsilon|_{t=0} = f_0, \\ \partial_t g^\varepsilon = Q_\varepsilon(g^\varepsilon, g^\varepsilon), & g^\varepsilon|_{t=0} = g_0, \\ \partial_t h^\varepsilon = 2Q_\varepsilon(f^\varepsilon, h^\varepsilon), & h^\varepsilon|_{t=0} = g_0 - f_0, \end{cases}$$

and define $d^\varepsilon := g^\varepsilon - f^\varepsilon$ which satisfies (where $s^\varepsilon := g^\varepsilon + f^\varepsilon$)

$$\partial_t d^\varepsilon = Q_\varepsilon(s^\varepsilon, d^\varepsilon), \quad d^\varepsilon|_{t=0} = g_0 - f_0.$$

Define $D = \widehat{d}^\varepsilon - \widehat{\mathcal{M}}_4[d^\varepsilon]$. Then the equation satisfied for D is

$$\partial_t D = \widehat{Q}_\varepsilon(\widehat{d}^\varepsilon, \widehat{s}^\varepsilon) - \partial_t \widehat{\mathcal{M}}_4[d^\varepsilon] = \widehat{Q}_\varepsilon(D, \widehat{s}^\varepsilon) + \widehat{Q}_\varepsilon(\widehat{\mathcal{M}}_4[d^\varepsilon], \widehat{s}^\varepsilon) - \widehat{\mathcal{M}}_4[Q_\varepsilon(d^\varepsilon, s^\varepsilon)].$$

From [18, Lemma 5.6] we know that, for any $\xi \in \mathbb{R}^d$,

$$\left| \widehat{Q}_\varepsilon(\widehat{\mathcal{M}}_4[d^\varepsilon], \widehat{s}^\varepsilon) - \widehat{\mathcal{M}}_4[Q_\varepsilon(d^\varepsilon, s^\varepsilon)] \right| \leq C|\xi|^4 \sum_{|\alpha| \leq 3} |M_\alpha[f^\varepsilon - g^\varepsilon]|,$$

and also, from [22, Theorem 8.1], that there are constants $C, \delta > 0$ such that for all $t \geq 0$

$$\sum_{|\alpha| \leq 3} |M_\alpha[f_t^\varepsilon - g_t^\varepsilon]| \leq C e^{-\delta t} \sum_{|\alpha| \leq 3} |M_\alpha[f_0 - g_0]|.$$

Then, following [18] and performing the same cut-off as in the proof of lemma 4.8, we have that

$$(4.16) \quad \begin{aligned} \frac{d}{dt} \frac{|D|}{|\xi|^4} + K \frac{|D|}{|\xi|^4} &\leq \left(\sup_{\xi \in \mathbb{R}^d} \frac{|D|}{|\xi|^4} \right) \left(\sup_{\xi \in \mathbb{R}^d} \int_{\mathbb{S}^{d-1}} b_\varepsilon^K(\sigma \cdot \widehat{\xi}) \left(|\widehat{\xi}^+|^4 + |\widehat{\xi}^-|^4 \right) d\sigma \right) \\ &+ C e^{-\delta t} \left(\sum_{|\alpha| \leq 3} |M_\alpha[f_0 - g_0]| \right) + R_\varepsilon^K. \end{aligned}$$

where

$$R_\varepsilon^K(\xi) := \int_{\mathbb{S}^{d-1}} b_\varepsilon^K(\sigma \cdot \widehat{\xi}) \left[\frac{1}{2} D(\xi^+) \widehat{s}^\varepsilon(\xi^-) + \frac{1}{2} D(\xi^-) \widehat{s}^\varepsilon(\xi^+) - D(\xi) \right] d\sigma$$

satisfies, for any $\xi \in \mathbb{R}^d$, $|R_\varepsilon^K(\xi)| \leq r_\varepsilon^K |\xi|^4$, with $r_\varepsilon^K \rightarrow 0$ as $K \rightarrow \infty$.

We denote

$$\lambda_K := \int_{\mathbb{S}^{d-1}} b_\varepsilon^K(\sigma \cdot \widehat{\xi}) \left(|\widehat{\xi}^+|^4 + |\widehat{\xi}^-|^4 \right) d\sigma = \int_{\mathbb{S}^{d-1}} b_\varepsilon^K(\sigma \cdot \widehat{\xi}) \frac{1}{2} \left(1 + (\sigma \cdot \widehat{\xi})^2 \right) d\sigma$$

and we compute

$$\begin{aligned} \lambda_K - K &= -\frac{1}{2} \int_{\mathbb{S}^{d-1}} b_\varepsilon^K(\sigma \cdot \widehat{\xi}) \left(1 - (\sigma \cdot \widehat{\xi})^2 \right) d\sigma \\ &\xrightarrow{K \rightarrow \infty} -\frac{1}{2} \int_{\mathbb{S}^{d-1}} b_\varepsilon(\sigma \cdot \widehat{\xi}) \left(1 - (\sigma \cdot \widehat{\xi})^2 \right) d\sigma =: -\bar{\lambda}_\varepsilon \in (-\infty, 0) \\ &\xrightarrow{\varepsilon \rightarrow 0} -\bar{\lambda} \in (-\infty, 0). \end{aligned}$$

One can now apply Gronwall's lemma to obtain

$$\begin{aligned} \sup_{\xi \in \mathbb{R}^d} \frac{|D_t|}{|\xi|^4} &\leq e^{(\lambda_K - K)t} \sup_{\xi \in \mathbb{R}^d} \frac{|\widetilde{D}_0|}{|\xi|^4} + C \left(\sum_{|\alpha| \leq 3} |M_\alpha[f_0 - g_0]| \right) \left(\frac{e^{-\delta t} - e^{(\lambda_K - K)t}}{K - \lambda_K - \delta} \right) \\ &+ R_\varepsilon^K \left(\frac{1 - e^{(\lambda_K - K)t}}{K - \lambda_K} \right). \end{aligned}$$

Then relaxing the cut-off $K \rightarrow \infty$ and choosing $0 < \lambda < \min(\delta, \bar{\lambda}_\varepsilon)$ one has

$$(4.17) \quad \sup_{\xi \in \mathbb{R}^d} \frac{|D_t|}{|\xi|^4} \leq C e^{-\lambda t} \left(\sup_{\xi \in \mathbb{R}^d} \frac{|D_0|}{|\xi|^4} + \sum_{|\alpha| \leq 3} |M_\alpha[f_0 - g_0]| \right).$$

Using a standard interpolation argument [18], one obtains

$$(4.18) \quad \begin{aligned} |g - f|_2 &\leq |g - f - \mathcal{M}_4[g - f]|_2 + C \left(\sum_{|\alpha| \leq 3} |M_\alpha[g - f]| \right) \\ &\leq \|g - f - \mathcal{M}_4[g - f]\|_{M_1}^{1/2} |g - f - \mathcal{M}_4[g - f]|_4^{1/2} + C \left(\sum_{|\alpha| \leq 3} |M_\alpha[g - f]| \right) \\ &\leq C M_4(f_0 + g_0) e^{-(\lambda/2)t}. \end{aligned}$$

Finally one concludes by writing

$$(4.19) \quad \begin{aligned} |g_t^\varepsilon - f_t^\varepsilon|_2 &\leq |g_t^\varepsilon - f_t^\varepsilon|_2^\eta |g_t^\varepsilon - f_t^\varepsilon|_2^{1-\eta} \\ &\leq C_\eta e^{-(1-\eta)\lambda t} M_4(f_0 + g_0)^{1/3} |g_0 - f_0|_2^\eta \end{aligned}$$

where we have used the last estimate (4.18), lemma 4.8 and the fact that $M_4(f_0 + g_0)^{1-\eta} \leq M_4(f_0 + g_0)^{1/3}$ for $\eta \in [2/3, 1]$. For h_t one has the same computations.

Step 2. Let us now deduce the result for solutions f_t and g_t of the Landau equation. Coming back to (4.17) and choosing $0 < \lambda_1 < \min(\delta, \bar{\lambda})$, where $\bar{\lambda}_\varepsilon \rightarrow \bar{\lambda} \in (0, \infty)$ as $\varepsilon \rightarrow 0$, we recover (4.19) with the exponent λ_1 which does not depend on ε . Hence, passing to the limit $\varepsilon \rightarrow 0$, we have $g^\varepsilon - f^\varepsilon \rightarrow g - f$ and then

$$|g_t - f_t|_2 \leq C_\eta e^{-(1-\eta)\lambda_1 t} M_4(f_0 + g_0)^{1/3} |g_0 - f_0|_2^\eta.$$

Rigorously, we write

$$|g_t - f_t|_2 \leq |g_t - g_t^\varepsilon|_2 + |f_t - f_t^\varepsilon|_2 + |g_t^\varepsilon - f_t^\varepsilon|_2,$$

then for the third term on the right-hand side we use (4.19) with exponent λ_1 that does not depend on ε , and for the other two terms we use that g_t^ε weakly converges towards g_t in L^1 (see Villani [24]), hence $|g_t - g_t^\varepsilon|_2 \rightarrow 0$ when $\varepsilon \rightarrow 0$ and we deduce

$$|g_t - f_t|_2 \leq C_\eta e^{-(1-\eta)\lambda_1 t} M_4(f_0 + g_0)^{1/3} |g_0 - f_0|_2^\eta.$$

□

Lemma 4.12. *Consider $f_0, g_0 \in \mathcal{P}_{\mathcal{G}_1, \mathbf{r}}$, $\mathbf{r} \in \mathbf{R}_{\mathcal{G}_1}$, and the solutions f_t, g_t, h_t, ω_t and r_t of (4.12) and (4.13). There exists $\lambda_1 \in (0, \infty)$ that for any $\eta \in [2/3, 1]$, there exists C_η such that we have*

$$(4.20) \quad \begin{aligned} |g_t - f_t - h_t|_4 &\leq C_\eta e^{-(1-\eta)\lambda_1 t} M_4(f_0 + g_0)^{1/3} |g_0 - f_0|_2^{1+\eta} \\ |r_t|_4 &\leq C_\eta e^{-(1-\eta)\lambda_1 t} M_4(f_0 + g_0)^{1/3} |g_0 - f_0|_2^{1+\eta} \end{aligned}$$

Proof of Lemma 4.12. Let us split the proof into two steps.

Step 1. As in Lemma 4.11, we consider Q_ε the symmetrized version of the Boltzmann operator $Q_{B,\varepsilon}$ and the solutions f_t^ε , g_t^ε and h_t^ε of (4.15).

Consider also r_t^ε solution of

$$(4.21) \quad \partial_t r^\varepsilon = 2Q_\varepsilon(f^\varepsilon, r^\varepsilon) + Q_\varepsilon(h^\varepsilon, h^\varepsilon), \quad r^\varepsilon|_{t=0} = 0,$$

and define $\omega^\varepsilon := g^\varepsilon - f^\varepsilon - h^\varepsilon$ which satisfies

$$\partial_t \omega^\varepsilon = Q_\varepsilon(s^\varepsilon, w^\varepsilon) + Q_\varepsilon(h^\varepsilon, d^\varepsilon), \quad \omega^\varepsilon|_{t=0} = 0.$$

First of all, we remark that ω_t^ε has moments equals to zero up to order 3. Indeed, let us prove that, for $\alpha \in \mathbb{N}^d$,

$$(4.22) \quad \forall |\alpha| \leq 3, \quad M_\alpha(\omega_t^\varepsilon) := \int_{\mathbb{R}^d} v^\alpha \omega_t^\varepsilon(v) dv = 0.$$

Following [18, Lemma 5.8] we know that for maxwellian molecules the α -moment of the Boltzmann operator $Q_{B,\varepsilon}(g, h)$ is a sum of terms given by the product of moments of g and h , then we obtain

$$(4.23) \quad \forall |\alpha| \leq 3, \quad \frac{d}{dt} M_\alpha(\omega_t^\varepsilon) = \sum_{\beta \leq \alpha} a_{\alpha,\beta} M_\beta(\omega_t^\varepsilon) M_{\alpha-\beta}(s_t^\varepsilon) + \sum_{\beta \leq \alpha} a_{\alpha,\beta} M_\beta(h_t^\varepsilon) M_{\alpha-\beta}(d_t^\varepsilon)$$

and we deduce that

$$(4.24) \quad \forall |\alpha| \leq 3, \quad \frac{d}{dt} M_\alpha(\omega_t^\varepsilon) = \sum_{\beta \leq \alpha} a_{\alpha,\beta} M_\beta(\omega_t^\varepsilon) M_{\alpha-\beta}(s_t^\varepsilon)$$

because for all $|\alpha| \leq 1$ we have $M_\alpha(h_t^\varepsilon) = M_\alpha(d_t^\varepsilon) = 0$. We conclude (4.22) by the fact that $\omega_0 = 0$. Therefore $|\omega|_4$ is well defined and we do not need to "take-off the moments of ω ".

We perform then the same cut-off as in Lemmas 4.8 and 4.11 and we have the following equation for ω_t^ε

$$(4.25) \quad \begin{aligned} & \frac{d}{dt} \frac{|\widehat{\omega}(\xi)|}{|\xi|^4} + K \frac{|\widehat{\omega}(\xi)|}{|\xi|^4} \\ & \leq \sup_{\xi \in \mathbb{R}^d} \int_{\mathbb{S}^{d-1}} b_\varepsilon^K(\sigma \cdot \widehat{\xi}) \left(\frac{|\widehat{\omega}(\xi^+)| |\widehat{s}(\xi^-)|}{2|\xi|^4} + \frac{|\widehat{\omega}(\xi^-)| |\widehat{s}(\xi^+)|}{2|\xi|^4} \right) d\sigma \quad (= : T_1) \\ & + \frac{1}{2} \sup_{\xi \in \mathbb{R}^d} \int_{\mathbb{S}^{d-1}} b_\varepsilon^K(\sigma \cdot \widehat{\xi}) \left(\frac{|\widehat{h}(\xi^+)| |\widehat{d}(\xi^-)|}{2|\xi|^4} + \frac{|\widehat{h}(\xi^-)| |\widehat{d}(\xi^+)|}{2|\xi|^4} \right) d\sigma \quad (= : T_2) \\ & + R_\varepsilon^K, \end{aligned}$$

where R_ε^K goes to 0 when $K \rightarrow \infty$.

We compute first T_1 using the fact that $\|\widehat{s}\|_\infty \leq 2$

$$\begin{aligned} T_1 & \leq \left(\sup_{\xi \in \mathbb{R}^d} \frac{|\widehat{\omega}(\xi)|}{|\xi|^4} \right) \sup_{\xi \in \mathbb{R}^d} \int_{\mathbb{S}^{d-1}} b_\varepsilon^K(\sigma \cdot \widehat{\xi}) \left(|\widehat{\xi}^+|^4 + |\widehat{\xi}^-|^4 \right) d\sigma \\ & \leq \lambda_K \sup_{\xi \in \mathbb{R}^d} \frac{|\widehat{\omega}(\xi)|}{|\xi|^4}. \end{aligned}$$

where λ_K is the same that in the proof of lemma 4.11. Next, we compute T_2

$$\begin{aligned} T_2 &\leq \left(\sup_{\xi \in \mathbb{R}^d} \frac{|\widehat{h}(\xi)|}{|\xi|^2} \right) \left(\sup_{\xi \in \mathbb{R}^d} \frac{|\widehat{d}(\xi)|}{|\xi|^2} \right) \sup_{\xi \in \mathbb{R}^d} \int_{\mathbb{S}^{d-1}} b_\varepsilon^K(\sigma \cdot \widehat{\xi}) \left(\frac{|\xi^+|^2 |\xi^-|^2}{|\xi|^4} \right) d\sigma \\ &\leq |h_t^\varepsilon|_2 |d_t^\varepsilon|_2 \sup_{\xi \in \mathbb{R}^d} \int_{\mathbb{S}^{d-1}} b_\varepsilon^K(\sigma \cdot \widehat{\xi}) (1 - \sigma \cdot \widehat{\xi}) d\sigma \\ &\leq \Lambda_\varepsilon e^{-(1-\eta)\lambda t} M_4(f_0 + g_0)^{1-\eta} |h_0|_2 |d_0|_2^\eta \end{aligned}$$

where we have used the estimates of lemmas 4.8 and 4.11, and Λ_ε is defined in (2.18).

After these computations we obtain

$$\frac{d}{dt} \frac{|\widehat{\omega}(\xi)|}{|\xi|^4} + K \frac{|\widehat{\omega}(\xi)|}{|\xi|^4} \leq \lambda_K \sup_{\xi \in \mathbb{R}^d} \frac{|\widehat{\omega}(\xi)|}{|\xi|^4} + \Lambda_\varepsilon e^{-(1-\eta)\lambda t} M_4(f_0 + g_0)^{1-\eta} |d_0|_2^{1+\eta} + R_\varepsilon^K$$

and by Gronwall's lemma

$$(4.26) \quad \sup_{\xi \in \mathbb{R}^d} \frac{|\widehat{\omega}_t(\xi)|}{|\xi|^4} \leq \Lambda_\varepsilon M_4(f_0 + g_0)^{1-\eta} |d_0|_2^{1+\eta} \left(\frac{e^{-(1-\eta)\lambda t} - e^{(\lambda_K - K)t}}{K - \lambda_K - (1-\eta)\lambda} \right) + R_\varepsilon^K \left(\frac{1 - e^{(\lambda_K - K)t}}{K - \lambda_K} \right).$$

Finally, we conclude by relaxing the cut-off parameter $K \rightarrow \infty$ and choosing $(1-\eta)\lambda \in (0, \bar{\lambda}_\varepsilon)$ where $\bar{\lambda}_\varepsilon$ is the same that in lemma 4.11, therefore we have

$$(4.27) \quad |\omega_t^\varepsilon|_4 \leq C_\eta \Lambda_\varepsilon e^{-(1-\eta)\lambda t} M_4(f_0 + g_0)^{1-\eta} |g_0 - f_0|_2^{1+\eta}.$$

We obtain the same estimation for r_t^ε .

Step 2. Consider the solutions f, g and h of (4.12).

Let us choose λ_1 such that $0 < (1-\eta)\lambda_1 < \bar{\lambda}$, where $\bar{\lambda}_\varepsilon \rightarrow \bar{\lambda} \in (0, \infty)$ as $\varepsilon \rightarrow 0$. Then we recover (4.27) with the exponent λ_1 which does not depend on ε . Hence, passing to the limit $\varepsilon \rightarrow 0$, we have $g^\varepsilon - f^\varepsilon - h^\varepsilon \rightarrow g - f - h$ (grazing collisions limit), and in the right-hand side of (4.27) we have $\Lambda_\varepsilon \rightarrow \Lambda > 0$ (see (2.18)). Then

$$|g_t - f_t - h_t|_4 \leq C_\eta \Lambda e^{-(1-\eta)\lambda_1 t} M_4(f_0 + g_0)^{1/3} |g_0 - f_0|_2^{1+\eta}.$$

□

Lemma 4.13. *Consider $f_0, g_0 \in \mathcal{P}_{\mathcal{G}_1, \mathbf{r}}$, $\mathbf{r} \in \mathbf{R}_{\mathcal{G}_1}$, and the solution ψ_t of (4.13). There exists $\lambda_1 \in (0, \infty)$ such that for any $\eta \in [2/3, 1]$, there exists C_η such that we have*

$$(4.28) \quad |g_t - f_t - h_t - r_t|_6 \leq C_\eta e^{-(1-\eta)\lambda_1 t} M_4(f_0 + g_0)^{1/3} |g_0 - f_0|_2^{2+\eta}$$

Proof of Lemma 4.13. We prove the lemma in two steps.

Step 1. Consider the solutions $g_t^\varepsilon, f_t^\varepsilon$ and h_t^ε of (4.15) and r_t^ε solution of (4.21). Define $\psi_t^\varepsilon := g_t^\varepsilon - f_t^\varepsilon - h_t^\varepsilon - r_t^\varepsilon$ that satisfies

$$\partial_t \psi^\varepsilon = Q_\varepsilon(s^\varepsilon, \psi^\varepsilon) + Q_\varepsilon(h^\varepsilon, w^\varepsilon) + Q_\varepsilon(r^\varepsilon, d^\varepsilon), \quad \psi^\varepsilon|_{t=0} = 0.$$

First of all, let us prove that ψ_t^ε has moments equals to zero up to order 5, more precisely, for $\alpha \in \mathbb{N}^d$,

$$(4.29) \quad \forall |\alpha| \leq 5, \quad M_\alpha(\psi_t^\varepsilon) := \int_{\mathbb{R}^d} v^\alpha \psi_t^\varepsilon(v) dv = 0.$$

In fact, as in the proof of Lemma 4.12, we can compute the α -moment of ψ

$$(4.30) \quad \forall |\alpha| \leq 5, \quad \frac{d}{dt} M_\alpha(\psi_t^\varepsilon) = \sum_{\beta \leq \alpha} a_{\alpha, \beta} M_\beta(\psi_t^\varepsilon) M_{\alpha-\beta}(s_t^\varepsilon) + \sum_{\beta \leq \alpha} a_{\alpha, \beta} M_\beta(h_t^\varepsilon) M_{\alpha-\beta}(\omega_t^\varepsilon) \\ + \sum_{\beta \leq \alpha} a_{\alpha, \beta} M_\beta(r_t^\varepsilon) M_{\alpha-\beta}(d_t^\varepsilon).$$

Since

$$(4.31) \quad \forall |\alpha| \leq 2, M_\alpha(h_t^\varepsilon) = M_\alpha(d_t^\varepsilon) = 0, \\ \forall |\alpha| \leq 3, M_\alpha(\omega_t^\varepsilon) = M_\alpha(r_t^\varepsilon) = 0,$$

we deduce that

$$(4.32) \quad \forall |\alpha| \leq 5, \quad \frac{d}{dt} M_\alpha(\psi_t^\varepsilon) = \sum_{\beta \leq \alpha} a_{\alpha, \beta} M_\beta(\psi_t^\varepsilon) M_{\alpha-\beta}(s_t^\varepsilon)$$

and we conclude thanks to $\psi_0 = 0$. Then $|\psi|_6$ is well defined.

We perform the same cut-off as in lemmas 4.8, 4.11 and 4.12, and it gives the following equation for ψ_t

$$(4.33) \quad \frac{d}{dt} \frac{|\widehat{\psi}(\xi)|}{|\xi|^6} + K \frac{|\widehat{\psi}(\xi)|}{|\xi|^4} \\ \leq \sup_{\xi \in \mathbb{R}^d} \int_{\mathbb{S}^{d-1}} b_\varepsilon^K(\sigma \cdot \widehat{\xi}) \left(\frac{|\widehat{\psi}(\xi^+)||\widehat{s}(\xi^-)|}{2|\xi|^6} + \frac{|\widehat{\psi}(\xi^-)||\widehat{s}(\xi^+)|}{2|\xi|^6} \right) d\sigma \quad (=: T_1) \\ + \frac{1}{2} \sup_{\xi \in \mathbb{R}^d} \int_{\mathbb{S}^{d-1}} b_\varepsilon^K(\sigma \cdot \widehat{\xi}) \left(\frac{|\widehat{h}(\xi^+)||\widehat{\omega}(\xi^-)|}{2|\xi|^6} + \frac{|\widehat{h}(\xi^-)||\widehat{\omega}(\xi^+)|}{2|\xi|^6} \right) d\sigma \quad (=: T_2) \\ + \frac{1}{2} \sup_{\xi \in \mathbb{R}^d} \int_{\mathbb{S}^{d-1}} b_\varepsilon^K(\sigma \cdot \widehat{\xi}) \left(\frac{|\widehat{r}(\xi^+)||\widehat{d}(\xi^-)|}{2|\xi|^6} + \frac{|\widehat{r}(\xi^-)||\widehat{d}(\xi^+)|}{2|\xi|^6} \right) d\sigma \quad (=: T_3) \\ + R_\varepsilon^K,$$

with $R_\varepsilon^K \rightarrow 0$ as $K \rightarrow \infty$.

We compute first T_1 using the fact that $\|\widehat{s}\|_\infty \leq 2$

$$T_1 \leq \sup_{\xi \in \mathbb{R}^d} \frac{|\widehat{\psi}(\xi)|}{|\xi|^4} \sup_{\xi \in \mathbb{R}^d} \int_{\mathbb{S}^{d-1}} b_\varepsilon^K(\sigma \cdot \widehat{\xi}) \left(|\widehat{\xi}^+|^6 + |\widehat{\xi}^-|^6 \right) d\sigma \\ \leq \alpha_K \sup_{\xi \in \mathbb{R}^d} \frac{|\widehat{\psi}(\xi)|}{|\xi|^4}.$$

Let us analyze α_K ,

$$\alpha_K = \int_{\mathbb{S}^{d-1}} b_\varepsilon^K(\sigma \cdot \widehat{\xi}) \left(|\widehat{\xi}^+|^6 + |\widehat{\xi}^-|^6 \right) d\sigma = \int_{\mathbb{S}^{d-1}} b_\varepsilon^K(\sigma \cdot \widehat{\xi}) \frac{1}{4} \left(1 + 3(\sigma \cdot \widehat{\xi})^2 \right) d\sigma$$

and we compute

$$\alpha_K - K = - \int_{\mathbb{S}^{d-1}} b_\varepsilon^K(\sigma \cdot \widehat{\xi}) \frac{1}{(4/3)} \left(1 - (\sigma \cdot \widehat{\xi})^2 \right) d\sigma \\ \xrightarrow{K \rightarrow \infty} - \int_{\mathbb{S}^{d-1}} b_\varepsilon(\sigma \cdot \widehat{\xi}) \frac{1}{(4/3)} \left(1 - (\sigma \cdot \widehat{\xi})^2 \right) d\sigma =: -\bar{\alpha}_\varepsilon \in (-\infty, 0) \\ \xrightarrow{\varepsilon \rightarrow 0} -\bar{\alpha} \in (-\infty, 0).$$

Next, we compute T_2

$$\begin{aligned} T_2 &\leq \left(\sup_{\xi \in \mathbb{R}^d} \frac{|\widehat{h}(\xi)|}{|\xi|^2} \right) \left(\sup_{\xi \in \mathbb{R}^d} \frac{|\widehat{\omega}(\xi)|}{|\xi|^4} \right) \sup_{\xi \in \mathbb{R}^d} \int_{\mathbb{S}^{d-1}} b_\varepsilon^K(\sigma \cdot \widehat{\xi}) \frac{1}{2} \left(\frac{|\xi^+|^2 |\xi^-|^4}{|\xi|^2 |\xi|^4} + \frac{|\xi^+|^4 |\xi^-|^2}{|\xi|^4 |\xi|^2} \right) d\sigma \\ &\leq |h_t^\varepsilon|_2 |d_t^\varepsilon|_2 \sup_{\xi \in \mathbb{R}^d} \int_{\mathbb{S}^{d-1}} b_\varepsilon^K(\sigma \cdot \widehat{\xi}) \frac{1}{2} \left(|\widehat{\xi}^-|^4 + |\widehat{\xi}^-|^2 \right) d\sigma \\ &\leq \beta_K e^{-(1-\eta)\lambda t} M_4(f_0 + g_0)^{1-\eta} |h_0|_2 |d_0|_2^{1+\eta} \end{aligned}$$

where we have used the estimates of lemmas 4.8 and 4.12. We compute β_K

$$\begin{aligned} \beta_K &= \int_{\mathbb{S}^{d-1}} b_\varepsilon^K(\sigma \cdot \widehat{\xi}) \frac{1}{2} \left(|\widehat{\xi}^-|^4 + |\widehat{\xi}^-|^2 \right) d\sigma \\ &= \int_{\mathbb{S}^{d-1}} b_\varepsilon^K(\sigma \cdot \widehat{\xi}) \frac{1}{2} \left(1 - \sigma \cdot \widehat{\xi} \right) d\sigma - \int_{\mathbb{S}^{d-1}} b_\varepsilon^K(\sigma \cdot \widehat{\xi}) \frac{1}{8} \left(1 - (\sigma \cdot \widehat{\xi})^2 \right) d\sigma \\ &\xrightarrow{K \rightarrow \infty} \bar{\Lambda}_\varepsilon := \frac{\Lambda_\varepsilon}{2} - \frac{\bar{\lambda}_\varepsilon}{4} \\ &\xrightarrow{\varepsilon \rightarrow 0} \bar{\Lambda} := \frac{\Lambda}{2} - \frac{\bar{\lambda}}{4} \end{aligned}$$

with $\Lambda > 0$ (see (2.18)), $\bar{\lambda} > 0$ and we have the same estimate for T_3 .

After these computations we obtain

$$\frac{d}{dt} \frac{|\widehat{\psi}(\xi)|}{|\xi|^6} + K \frac{|\widehat{\psi}(\xi)|}{|\xi|^6} \leq \alpha_K \sup_{\xi \in \mathbb{R}^d} \frac{|\widehat{\psi}(\xi)|}{|\xi|^6} + \beta_K e^{-(1-\eta)\lambda t} M_4(f_0 + g_0)^{1-\eta} |d_0|_2^{2+\eta} + R_\varepsilon^K$$

and by Gronwall's lemma

$$\sup_{\xi \in \mathbb{R}^d} \frac{|\widehat{\psi}(\xi)|}{|\xi|^6} \leq \beta_K M_4(f_0 + g_0)^{1-\eta} |d_0|_2^{2+\eta} \left(\frac{e^{-(1-\eta)\lambda t} - e^{(\alpha_K - K)t}}{K - \alpha_K - (1-\eta)\lambda} \right) + R_\varepsilon^K \left(\frac{1 - e^{(\alpha_K - K)t}}{K - \alpha_K} \right).$$

We conclude by relaxing the cut-off parameter $K \rightarrow \infty$ and choosing $(1-\eta)\lambda \in (0, \bar{\alpha}_\varepsilon)$, therefore we have

$$(4.34) \quad |\psi_t^\varepsilon|_6 \leq C_\eta \bar{\Lambda}_\varepsilon e^{-(1-\eta)\lambda t} M_4(f_0 + g_0)^{1-\eta} |d_0|_2^{2+\eta}.$$

Step 2. Consider the solutions f, g, h and r of (4.12).

Let us choose λ_1 such that $0 < (1-\eta)\lambda_1 < \bar{\alpha}$, where $\bar{\alpha}_\varepsilon \rightarrow \bar{\alpha} \in (0, \infty)$ as $\varepsilon \rightarrow 0$. Then we recover (4.34) with the exponent λ_1 which does not depend on ε . Hence, passing to the limit $\varepsilon \rightarrow 0$, we have $g^\varepsilon - f^\varepsilon - h^\varepsilon - r^\varepsilon \rightarrow g - f - h - r$ (grazing collisions limit), and in the right-hand side of (4.34) we have $\bar{\Lambda}_\varepsilon \rightarrow \bar{\Lambda}$. Then

$$|g_t - f_t - h_t - r_t|_6 \leq C_\eta \bar{\Lambda} e^{-(1-\eta)\lambda_1 t} M_4(f_0 + g_0)^{1/3} |g_0 - f_0|_2^{1+\eta}.$$

□

Therefore the semigroup of the Landau equation

$$S_t^\infty \in C_{\Lambda_2}^{2,\eta} \cap C_{\Lambda_2}^{1,(1+2\eta)/3} \cap C_{\Lambda_2}^{0,(2+\eta)/3}(\mathcal{P}_{\mathcal{G}_1, \mathbf{r}}; \mathcal{P}_{\mathcal{G}_2}),$$

where $\mathcal{P}_{\mathcal{G}_2}$ is defined as $\mathcal{P}_{\mathcal{G}_1}$ but endowed with the distance associated to the norm $\|\cdot\|_{\mathcal{G}_2} = |\cdot|_6$, with $\Lambda_2(f) := M_4(f)^{1/3} = \Lambda_1(f)^{1/3}$. Moreover there exists a constant $C_4 > 0$ such that one has

$$(4.35) \quad \sup_{\mathbf{r} \in \mathbf{R}_{\mathcal{G}_1}} \int_0^\infty \left([S_t^\infty]_{C_{\Lambda_2}^{2,0}} + [S_t^\infty]_{C_{\Lambda_2}^{1,0}}^2 \right) dt \leq C_4,$$

which proves **(A4)**.

Remark 4.14. In fact, we can deduce that

$$\sup_{\mathbf{r} \in \mathbf{R}_{\mathcal{G}_1}} \int_0^\infty \left([S_t^\infty]_{C_{\Lambda_2}^{1,(1+2\eta)/3}} + [S_t^\infty]_{C_{\Lambda_2}^{0,(2+\eta)/3}}^2 + [S_t^\infty]_{C_{\Lambda_2}^{2,0}} + [S_t^\infty]_{C_{\Lambda_2}^{1,0}}^2 \right) dt \leq C_4.$$

However, coming back to the proof of Theorem 3.11 and from the proof of **(A3)** (4.10), which gives a upper bound of only $[\Phi]_{C^{2,0}}$ instead of $[\Phi]_{C^{1,\eta}} + [\Phi]_{C^{2,0}}$, we remark that (4.35) is sufficient.

4.5. Proof of A5. We define the space of probability measures $\mathcal{P}_{\mathcal{G}_3} := \mathcal{P}_2(\mathbb{R}^d) = \{f \in \mathcal{P}(\mathbb{R}^d); M_2(f) < \infty\}$, the constraint function $\mathbf{m}_{\mathcal{G}_3}(v) = (|v|^2, v)$ and $\mathbf{R}_{\mathcal{G}_3} = \{(r, \bar{r}) \in \mathbb{R}_+ \times \mathbb{R}^d; r \leq \mathcal{E}_0\}$. Then we define for $a \in (0, \infty)$, the bounded set

$$\mathcal{BP}_{\mathcal{G}_3,a} := \{f \in \mathcal{P}_{\mathcal{G}_3}; M_2(f) \leq a\},$$

the constrained bounded set

$$\mathcal{BP}_{\mathcal{G}_3,a,\mathbf{r}} := \{f \in \mathcal{BP}_{\mathcal{G}_3,a}; \langle f, |v|^2 \rangle = r, \langle f, v_i \rangle \text{ for } i = 1, \dots, d\},$$

endowed with the distance $\text{dist}_{\mathcal{G}_3} = W_2$. The following lemma proves **(A5)** with $\mathcal{F}_3 = \text{Lip}(\mathbb{R}^d)$.

Lemma 4.15. *Let f_0, g_0 have the same momentum and energy, and consider $f_t = S_t^\infty(f_0)$, $g_t = S_t^\infty(g_0)$ theirs respective solutions of the Landau Maxwell equation. Then*

$$(4.36) \quad \sup_{t \geq 0} W_2(f_t, g_t) \leq W_2(f_0, g_0).$$

Proof of Lemma 4.15. Consider $f_t^\varepsilon, g_t^\varepsilon$ the solutions of the Boltzmann equation with kernel b_ε satisfying the grazing collisions (2.18) and with initial data f_0 and g_0 , respectively. We know from [22] that

$$\sup_{t \geq 0} W_2(f_t^\varepsilon, g_t^\varepsilon) \leq W_2(f_0, g_0)$$

We know also that [24] f_t^ε converges weakly in L^1 to a weak solution f_t of the Landau equation (grazing collisions limit). Moreover, both equations conserve energy so we have, for all ε

$$\int |v|^2 f_t^\varepsilon(v) dv = \int |v|^2 f_t(v) dv = \int |v|^2 f_0(v) dv.$$

Using the fact that the Wasserstein distance W_2 is equivalent to the weak convergence in $\mathcal{P}(\mathbb{R}^d)$ plus the second order momentum convergence [28], we obtain

$$W_2(f_t, g_t) \leq W_2(f_t, f_t^\varepsilon) + W_2(f_t^\varepsilon, g_t^\varepsilon) + W_2(g_t, g_t^\varepsilon)$$

and then, passing to the limit $\varepsilon \rightarrow 0$, we have $W_2(f_t, f_t^\varepsilon) \rightarrow 0$, $W_2(g_t, g_t^\varepsilon) \rightarrow 0$ and

$$\sup_{t \geq 0} W_2(f_t, g_t) \leq W_2(f_0, g_0).$$

□

4.6. Proof of Theorem 4.2. The proof is a consequence of Theorem 4.1, some results on differents forms of measuring chaos from [11], quantitative estimates on the chaoticity of initial data from [4].

Proof of Theorem 4.2 (1). Thanks to Theorem 4.1, taking $\ell = 2$, we have for all $\phi = \phi_1 \otimes \phi_2 \in \mathcal{F}^{\otimes 2}$ that

$$\sup_{t \geq 0} \frac{|\langle \Pi_2(F_t^N) - f_t^{\otimes 2}, \phi \rangle|}{\|\phi\|_{\mathcal{F}}} \leq C \left(\mathcal{W}_{W_2}(\pi_P^N F_0^N, \delta_{f_0}) + \frac{1}{N} \right),$$

where we recall that $\|\phi\|_{\mathcal{F}} = \int (1 + |\xi|^6) |\widehat{\phi}(\xi)|$. Then we observe that, for $r > 0$, applying Cauchy-Schwarz inequality,

$$\begin{aligned} \|\phi_1\|_{\mathcal{F}} &= \int (1 + |\xi|^6) (1 + |\xi|^2)^{r/2} |\widehat{\phi}_1(\xi)| (1 + |\xi|^2)^{-r/2} d\xi \\ &\leq C \left(\int (1 + |\xi|^2)^{6+r} |\widehat{\phi}_1(\xi)|^2 \right)^{1/2} \left(\int (1 + |\xi|^2)^{-r} \right)^{1/2}. \end{aligned}$$

The first integral in the right-hand side is the norm $\|\phi_1\|_{H^{6+r}}$ and the second one is finite if $2r > d$. We have then $H^s \subset \mathcal{F}$ for $s > 6 + d/2$ which implies

$$(4.37) \quad \sup_{t \geq 0} \|\Pi_2(F_t^N) - f_t^{\otimes 2}\|_{H^{-s}} \leq C \left(\mathcal{W}_{W_2}(\pi_P^N F_0^N, \delta_{f_0}) + \frac{1}{N} \right).$$

Let us denote $M_k = M_k(\Pi_2(F_t^N)) + M_k(f_t^{\otimes 2})$. Thanks to [11], for any $0 < \alpha < k(dk + d + k)^{-1}$ there exists $C := C(\alpha, d, s, M_k)$ such that

$$\frac{W_1(F_t^N, f_t^{\otimes N})}{N} \leq C \left(\|\Pi_2(F_t^N) - f_t^{\otimes 2}\|_{H^{-s}}^{\frac{\alpha k}{d+ks}} + N^{-\frac{\alpha}{2}} \right),$$

which implies with (4.37)

$$(4.38) \quad \frac{W_1(F_t^N, f_t^{\otimes N})}{N} \leq C \left(\mathcal{W}_{1, W_2}(\pi_P^N F_0^N, f_0^{\otimes N})^{\frac{\alpha k}{d+ks}} + N^{-\frac{\alpha}{2}} \right).$$

Now, we have just to estimate the first term of the right-hand side of (4.37).

We have from [4, Proof of Theorem 8] that for any $0 < \beta < (7d + 6)^{-1}$ there exists $C = C(\beta)$ such that

$$\mathcal{W}_{1, W_2}(\pi_P^N F_0^N, \delta_{f_0}) \leq C N^{-\beta}.$$

We assumed that $M_6(f_0)$ is finite, which implies by construction that $M_6(\Pi_2(F_0^N))$ is also finite. Then, for all $t \geq 0$ we have $M_6(f_t)$ finite (see [25]) and $M_6(\Pi_2(F_t^N))$ also finite (see Lemma 4.7). We can conclude gathering the last equation with (4.38), $k = 6$ and $s > 6 + d/2$. \square

Using this result, we can prove the second part of the theorem following [18].

Proof of Theorem 4.2 (2). We split the expression into

$$\frac{W_1(F_t^N, \gamma^N)}{N} = \frac{W_1(F_t^N, f_t^{\otimes N})}{N} + \frac{W_1(\gamma^{\otimes N}, \gamma^N)}{N} + W_1(f_t, \gamma),$$

where γ is the equilibrium Gaussian probability with zero momentum and energy $\mathcal{E} = \int |v|^2 d\gamma$. For the first term we have from point (1) that for all $\epsilon < 9[(7d + 6)^2(d + 9)]^{-1}$ there exists C_ϵ such that

$$\frac{W_1(F_t^N, f_t^{\otimes N})}{N} \leq C_\epsilon N^{-\epsilon}$$

The second term can be estimated by [4, Theorem 18]

$$\frac{W_1(\gamma^{\otimes N}, \gamma^N)}{N} \leq C N^{-\theta},$$

for some $\theta > \epsilon$. For the third term, thanks to [25, Theorem 6] we have

$$W_1(f_t, \gamma) \leq \|(f_t - \gamma)\langle v \rangle\|_{L^1} \leq C e^{-\lambda t}.$$

for constants $C > 0$ and $\lambda > 0$. Finally, putting together these estimates it follows

$$(4.39) \quad \frac{W_1(F_t^N, \gamma^N)}{N} \leq C'_\epsilon (N^{-\epsilon} + e^{-\lambda t}).$$

Moreover, consider h_t^N the Radon-Nicodym derivative of F_t^N with respect to γ^N , i.e. $h_t^N = dF_t^N/d\gamma^N$. Thanks to [14], for all $N \in \mathbb{N}^*$ and $t \geq 0$, it holds

$$\|h_t^N - 1\|_{L^2(\mathcal{S}^N(\mathcal{E}), d\gamma^N)} \leq e^{-\lambda_1 t} \|h_0^N - 1\|_{L^2(\mathcal{S}^N(\mathcal{E}), d\gamma^N)},$$

where $\lambda_1 > 0$. Since $F_0^N = [f_0^{\otimes N}]_{\mathcal{S}^N(\mathcal{E})}$ and $f_0 \in \mathcal{P}_6(\mathbb{R}^d)$, it is possible to bound the right-hand side by

$$\|h_0^N - 1\|_{L^2(\mathcal{S}^N(\mathbb{E}), d\gamma^N)} \leq A^N,$$

with $A > 1$ that depends on f_0 . Hence we deduce, with $\phi : \mathbb{R}^{dN} \rightarrow \mathbb{R}$,

$$\begin{aligned} W_1(F_t^N, \gamma^N) &= \sup_{\|\phi\|_{C^{0,1}} \leq 1} \int_{\mathbb{R}^{dN}} \phi(dF^N - d\gamma^N) \\ &\leq \int_{\mathbb{R}^{dN}} \sum_{j=1}^N |v_j| |dF^N - d\gamma^N| \\ &\leq N \mathcal{E}^{1/2} \|h_t^N - 1\|_{L^1(\mathcal{S}^N(\mathcal{E}), d\gamma^N)} \\ &\leq N \mathcal{E}^{1/2} \|h_t^N - 1\|_{L^2(\mathcal{S}^N(\mathcal{E}), d\gamma^N)}, \end{aligned}$$

which implies

$$(4.40) \quad \frac{W_1(F_t^N, \gamma^N)}{N} \leq A^N e^{-\lambda_1 t}.$$

Define $N(t)$ by $N(t) := \lambda_1 t (2 \log A)^{-1}$ for some $\delta > 0$. Then, choosing (4.39) for $N > N(t)$ and (4.40) for $N \leq N(t)$ it yields, for all $N \in \mathbb{N}^*$ and $t \geq 0$,

$$\frac{W_1(F_t^N, \gamma^N)}{N} \leq p(t) := \min \left\{ C'_\varepsilon (N(t))^{-\varepsilon} + e^{-\lambda t}, e^{-\frac{\lambda_1}{2} t} \right\},$$

with a polynomial function $p(t) \rightarrow 0$ as $t \rightarrow \infty$. □

5. ENTROPIC CHAOS

We can define the master equation (2.29) on \mathbb{R}^{dN} or $\mathcal{S}^N(\mathcal{E})$ thanks to the conservation of momentum and energy, hence for $g^N \in \mathcal{P}_{\text{sym}}(\mathbb{R}^{dN})$ and $f^N \in \mathcal{P}_{\text{sym}}(\mathcal{S}^N(\mathcal{E}))$ we have

$$(5.1) \quad \partial_t \langle g^N, \psi \rangle = \langle g^N, G^N \psi \rangle, \quad \forall \psi \in C_b^2(\mathbb{R}^{dN})$$

$$(5.2) \quad \partial_t \langle f^N, \phi \rangle = \langle f^N, G^N \phi \rangle, \quad \forall \phi \in C_b^2(\mathcal{S}^N(\mathcal{E})),$$

where G^N is given by (2.29).

Suppose that g^N is absolutely continuous with respect to the Lebesgue measure (and we still denote by g^N its Radon-Nicodym derivative). Taking $\psi = \log g^N$ in (5.1), we obtain an equation for the entropy of g^N , i.e. $H(g^N) := \int_{\mathbb{R}^{dN}} g^N \log g^N dV$,

$$(5.3) \quad \frac{d}{dt} \int_{\mathbb{R}^{dN}} g^N \log g^N dV = -\frac{1}{2N} \sum_{i,j} \int_{\mathbb{R}^{dN}} a(v_i - v_j) \left(\frac{\nabla_i g^N}{g^N} - \frac{\nabla_j g^N}{g^N} \right) \cdot \left(\frac{\nabla_i g^N}{g^N} - \frac{\nabla_j g^N}{g^N} \right) g^N dV \leq 0,$$

since a is nonnegative.

Considering now f^N absolutely continuous with respect to γ^N , the uniform probability measure on $\mathcal{S}^N(\mathcal{E})$, and denoting by $h^N := df^N/d\gamma^N$ its derivative, we want to obtain a similar expression for the relative entropy of f^N with respect to γ^N , given by

$$(5.4) \quad H(f^N | \gamma^N) := \int_{\mathcal{S}^N} h^N \log h^N d\gamma^N.$$

For this purpose we could take $\phi = \log h^N$ in (5.2), but we have to give a meaning to $\nabla_i h^N$ for a function h^N defined on \mathcal{S}^N .

Let us consider h a function on $\mathcal{S}^N(\mathcal{E})$ and we define \tilde{h} on \mathbb{R}^{dN} by

$$(5.5) \quad \tilde{h}(V) = \rho(\mathcal{E}(V)) \eta(\mathcal{M}(V)) h \left(\mathcal{E} \frac{V - \mathcal{M}(V)}{\mathcal{E}(V)} \right), \quad \forall V \in \mathbb{R}^{dN}.$$

where $\mathcal{E}(V) = N^{-1} \sum_{i=1}^N |v_i - \mathcal{M}(V)|^2$, $\mathcal{M}(V) = N^{-1} \sum_{i=1}^N v_i$ and the functions ρ and η are smooth.

Denoting by $\nabla_{\mathcal{S}^N}$ the gradient with respect to $\mathcal{S}^N(\mathcal{E})$ and by ∇_{\perp} the gradient with respect to its orthogonal space $(\mathcal{S}^N)^{\perp}$, we can decompose the gradient on \mathbb{R}^{dN}

$$(5.6) \quad \nabla_{\mathbb{R}^{dN}} \tilde{h} = \nabla_{\perp} \tilde{h} + \nabla_{\mathcal{S}^N} \tilde{h} = (\nabla_{\perp} \rho \eta) h + \rho \eta \nabla_{\mathcal{S}^N} h = (\nabla_{\perp} \log(\rho \eta)) \tilde{h} + \rho \eta \nabla_{\mathcal{S}^N} h.$$

For \tilde{h} we can compute $\nabla_i \tilde{h} \in \mathbb{R}^d$, for $1 \leq i \leq N$, as

$$\nabla_i \tilde{h} = \left(\partial_{v_{i,\alpha}} \tilde{h} \right)_{1 \leq \alpha \leq d} = \left(\nabla_{\mathbb{R}^{dN}} \tilde{h} \cdot e_{i,\alpha} \right)_{1 \leq \alpha \leq d},$$

where $(e_{i,\alpha})_{j,\beta} = \delta_{ij} \delta_{\alpha\beta} \in \mathbb{R}^{dN}$. Hence by (5.6), for all $1 \leq i \leq N$ and all $1 \leq \alpha \leq d$,

$$\partial_{v_{i,\alpha}} \tilde{h} = (\nabla_{\perp} \log(\rho \eta) \cdot e_{i,\alpha}) \tilde{h} + \rho \eta (\nabla_{\mathcal{S}^N} h \cdot e_{i,\alpha}).$$

Now, observing that $(\nabla_{\perp} \log(\rho \eta) \cdot (e_{i,\alpha} - e_{j,\alpha}))_{1 \leq \alpha \leq d}$ is proportional to $(v_i - v_j)$ and using that $a(z)z = 0$ for all $z \in \mathbb{R}^d$, we can evaluate the expression

$$a(v_i - v_j) \left(\nabla_i \tilde{h} - \nabla_j \tilde{h} \right) \cdot \left(\nabla_i \tilde{h} - \nabla_j \tilde{h} \right) = (\rho \eta)^2 a(v_i - v_j) (\nabla_{\mathcal{S}_i^N} h - \nabla_{\mathcal{S}_j^N} h) \cdot (\nabla_{\mathcal{S}_i^N} h - \nabla_{\mathcal{S}_j^N} h),$$

where we define

$$(5.7) \quad \nabla_{\mathcal{S}_i^N} h = (\nabla_{\mathcal{S}^N} h \cdot e_{i,\alpha})_{1 \leq \alpha \leq d}.$$

Since we have the following Fubini-like theorem for Boltzmann's spheres (see [6, 4])

$$(5.8) \quad \int_{\mathbb{R}^{dN}} \rho(\mathcal{E}(V)) \eta(\mathcal{M}(V)) A \left(h \left(\mathcal{E} \frac{V - \mathcal{M}(V)}{\mathcal{E}(V)} \right) \right) dV \\ = \left(\int_{\mathbb{R}^+ \times \mathbb{R}^d} B(\rho(\mathcal{E}), \eta(\mathcal{M})) d\mathcal{E} d\mathcal{M} \right) \left(\int_{\mathcal{S}^N(\mathcal{E})} A(h) d\gamma^N \right),$$

for some functions A and B , thanks to (5.3) with $h = h^N$ and $\tilde{h} = g^N$, we obtain the equation for $H(f^N | \gamma^N)$

$$(5.9) \quad \frac{d}{dt} \int_{\mathcal{S}^N(\mathcal{E})} h^N \log h^N d\gamma^N \\ = -\frac{1}{2N} \sum_{i,j} \int_{\mathcal{S}^N(\mathcal{E})} a(v_i - v_j) \left(\frac{\nabla_{\mathcal{S}_i^N} h^N}{h^N} - \frac{\nabla_{\mathcal{S}_j^N} h^N}{h^N} \right) \cdot \left(\frac{\nabla_{\mathcal{S}_i^N} h^N}{h^N} - \frac{\nabla_{\mathcal{S}_j^N} h^N}{h^N} \right) h^N d\gamma^N \\ =: -D^N(F^N) \leq 0,$$

and D^N is called the entropy-production functional. This implies

$$(5.10) \quad \frac{1}{N} H(F_t^N | \gamma^N) + \int_0^t \frac{1}{N} D^N(F_s^N) ds = \frac{1}{N} H(F_0^N | \gamma^N).$$

Moreover for the limit equation we have [25]

$$\frac{d}{dt} H(f) := \frac{d}{dt} \int f \log f dv = -\frac{1}{2} \int f f_* a(v - v_*) \left(\frac{\nabla f}{f} - \frac{\nabla_* f_*}{f_*} \right) \cdot \left(\frac{\nabla f}{f} - \frac{\nabla_* f_*}{f_*} \right) dv dv_*$$

and then for the relative entropy $H(f|\gamma) = \int (f/\gamma) \log(f/\gamma) \gamma(dv)$, we obtain

$$(5.11) \quad H(f_t|\gamma) + \int_0^t D(f_s) ds = H(f_0|\gamma).$$

Lemma 5.1. *If F^N is f -chaotic, then*

$$H(f|\gamma) \leq \liminf_{N \rightarrow \infty} \frac{1}{N} H(F^N|\gamma^N) \quad \text{and} \quad D(f) \leq \liminf_{N \rightarrow \infty} \frac{1}{N} D^N(F^N).$$

Proof of Lemma 5.1. The lower semicontinuity property of the relative entropy is proved in [4, Theorem 21], thus we prove only the second inequality.

Let us denote $\nabla_{12} = \nabla_1 - \nabla_2$, $\nabla_{S_{12}^N} = \nabla_{S_1^N} - \nabla_{S_2^N}$, and for all $x, y, z \in \mathbb{R}^d$ we denote $a(z)xy = (a(z)x) \cdot y$. Since a is nonnegative, considering a function $\varphi : \mathbb{R}^{2d} \rightarrow \mathbb{R}^d$, we have

$$a(v_1 - v_2) \left(\nabla_{12} \log f_1 f_2 - \frac{\varphi}{2} \right) \left(\nabla_{12} \log f_1 f_2 - \frac{\varphi}{2} \right) \geq 0,$$

which gives the following representation for $D(f)$,

$$\begin{aligned} D(f) &= \frac{1}{2} \sup_{\varphi: \mathbb{R}^{2d} \rightarrow \mathbb{R}^d} \iint a(v_1 - v_2) \left[(\nabla_{12} \log f_1 f_2) \varphi - \frac{\varphi \varphi}{4} \right] f_1 f_2 dv_1 dv_2 \\ &= \frac{1}{2} \sup_{\varphi: \mathbb{R}^{2d} \rightarrow \mathbb{R}^d} \iint \left\{ -\nabla_{12} \cdot (a(v_1 - v_2) \varphi) - a(v_1 - v_2) \frac{\varphi \varphi}{4} \right\} f_1 f_2 dv_1 dv_2 \end{aligned}$$

where $f_1 = f(v_1)$ and $f_2 = f(v_2)$. Let $\varepsilon > 0$ and choose $\varphi = \varphi(v_1, v_2) : \mathbb{R}^{2d} \rightarrow \mathbb{R}^d$ such that

$$D(f) - \varepsilon \leq \frac{1}{2} \iint \left\{ -\nabla_{12} (a(v_1 - v_2) \varphi) - a(v_1 - v_2) \frac{\varphi \varphi}{4} \right\} f_1 f_2 dv_1 dv_2.$$

For the N -particle entropy-production D^N defined in (5.9), we have by symmetry

$$\begin{aligned} \frac{1}{N} D^N(F^N) &= \frac{N(N-1)}{N^2} \frac{1}{2} \int_{S^N} a(v_1 - v_2) \left(\frac{\nabla_{S_1^N} h^N}{h^N} - \frac{\nabla_{S_2^N} h^N}{h^N} \right) \cdot \left(\frac{\nabla_{S_1^N} h^N}{h^N} - \frac{\nabla_{S_2^N} h^N}{h^N} \right) h^N d\gamma^N \\ &=: \frac{N(N-1)}{N^2} D_{12}^N(F^N), \end{aligned}$$

and then $\liminf_{N \rightarrow \infty} N^{-1} D^N(F^N) \geq \liminf_{N \rightarrow \infty} D_{12}^N(F^N)$. For $\Phi : \mathbb{R}^{dN} \rightarrow \mathbb{R}^d$, $\Phi \in C_b^1$, we have, with $F^N = h^N \gamma^N$,

$$\begin{aligned} D_{12}^N(F^N) &= \frac{1}{2} \int_{S^N} a(v_1 - v_2) \nabla_{S_{12}^N} \log h^N \cdot \nabla_{12} \log h^N h^N d\gamma^N \\ &= \frac{1}{2} \sup_{\Phi: \mathbb{R}^{dN} \rightarrow \mathbb{R}^d} \int_{S^N} a(v_1 - v_2) \left(\nabla_{S_{12}^N} \log h^N \Phi - \Phi \Phi / 4 \right) h^N d\gamma^N \\ &= \frac{1}{2} \sup_{\Phi} \left\{ \int_{S^N} a(v_1 - v_2) \nabla_{S_{12}^N} h^N \Phi d\gamma^N - \int_{S^N} a(v_1 - v_2) \frac{\Phi \Phi}{4} h^N d\gamma^N \right\}. \end{aligned}$$

Choosing $\Phi(V) = \varphi(v_1, v_2)$ we obtain, using (5.7),

$$(5.12) \quad \begin{aligned} D_{12}^N(F^N) &\geq \frac{1}{2} \int_{S^N} a(v_1 - v_2) \nabla_{S_{12}^N} h^N \varphi d\gamma^N - \frac{1}{2} \int_{S^N} a(v_1 - v_2) \frac{\varphi \varphi}{4} h^N d\gamma^N \\ &\geq \frac{1}{2} \sum_{\alpha, \beta=1}^d \int_{S^N} \nabla_{S^N} h^N \cdot [(e_{1,\alpha} - e_{2,\alpha}) a_{\alpha\beta} (v_1 - v_2) \varphi_\beta] d\gamma^N - \frac{1}{2} \int_{S^N} a(v_1 - v_2) \frac{\varphi \varphi}{4} h^N d\gamma^N. \end{aligned}$$

We need an integration by parts formula for the first term on the right-hand side, thanks to [4, Lemma 22], for a function A and a vector field Ψ , we have

$$\int_{\mathcal{S}^N} \left\{ \nabla_{\mathcal{S}^N} A(V) \cdot \Psi(V) + A(V) \operatorname{div}_{\mathcal{S}^N} \Psi(V) - \frac{d(N-1)-1}{dN} A(V) \Psi(V) \cdot V \right\} d\gamma^N(V) = 0,$$

with

$$(5.13) \quad \operatorname{div}_{\mathcal{S}^N} \Psi(V) = \operatorname{div} \Psi(V) - \frac{1}{N} \sum_{i,j=1}^N \sum_{\beta=1}^d \partial_{v_{i,\beta}} \Psi_{j,\beta}(V) - \sum_{j=1}^N \sum_{\beta=1}^d V \cdot \nabla \Psi_{j,\beta} \frac{v_{j,\beta}}{|V|^2}.$$

Taking $\Psi(V) = (e_{1,\alpha} - e_{2,\alpha}) a_{\alpha\beta} (v_1 - v_2) \varphi_\beta$ we obtain

$$\begin{aligned} & \int_{\mathcal{S}^N} \nabla_{\mathcal{S}^N} h^N \cdot (e_{1,\alpha} - e_{2,\alpha}) a_{\alpha\beta} (v_1 - v_2) \varphi_\beta d\gamma^N \\ &= - \int_{\mathcal{S}^N} h^N \operatorname{div}_{\mathcal{S}^N} [(e_{1,\alpha} - e_{2,\alpha}) a_{\alpha\beta} \varphi_\beta] d\gamma^N + \frac{d(N-1)-1}{dN} \int_{\mathcal{S}^N} h^N a_{\alpha\beta} \varphi_\beta (e_{1,\alpha} - e_{2,\alpha}) \cdot V d\gamma^N. \end{aligned}$$

Since $(e_{1,\alpha} - e_{2,\alpha}) \cdot V = (v_{1,\alpha} - v_{2,\alpha})$, when performing the summation $\alpha, \beta = 1$ to d in the second term of the right-hand side of last equation, we obtain

$$\int_{\mathcal{S}^N} h^N a(v_1 - v_2)(v_1 - v_2) \varphi d\gamma^N = 0.$$

For the first term, thanks to (5.13),

$$\begin{aligned} & \sum_{\alpha,\beta=1}^d \int_{\mathcal{S}^N} h^N \operatorname{div}_{\mathcal{S}^N} [(e_{1,\alpha} - e_{2,\alpha}) a_{\alpha\beta} \varphi_\beta] d\gamma^N \\ &= \int_{\mathcal{S}^N} \nabla_{12} \cdot (a(v_1 - v_2) \varphi) h^N d\gamma^N \\ &\quad - \sum_{\alpha,\beta=1}^d \frac{1}{|V|^2} \int_{\mathcal{S}^N} \{v_1 \cdot \nabla_1(a_{\alpha\beta} \varphi_\beta) + v_2 \cdot \nabla_2(a_{\alpha\beta} \varphi_\beta)\} (v_{1,\alpha} - v_{2,\alpha}) h^N d\gamma^N \end{aligned}$$

Getting back to (5.12) with last expression, we split the integral over (v_1, v_2) and $\mathcal{S}^N(v_1, v_2) := \{(v_3, \dots, v_N) \in \mathbb{R}^{d(N-2)}; V \in \mathcal{S}^N\}$, use that $|V|^2 = \mathcal{E}N$ and $\int_{\mathcal{S}^N(v_1, v_2)} h^N d\gamma^N = F_2^N$, which yields

$$(5.14) \quad \begin{aligned} D_{12}^N(F^N) &\geq -\frac{1}{2} \iint \nabla_{12} \cdot (a(v_1 - v_2) \varphi) F_2^N(v_1, v_2) dv_1 dv_2 - \frac{1}{2} \iint a(v_1 - v_2) \frac{\varphi\varphi}{4} F_2^N(v_1, v_2) dv_1 dv_2 \\ &\quad + O\left(\frac{1}{N}\right) \sum_{\alpha,\beta=1}^d \iint \{v_1 \cdot \nabla_1(a_{\alpha\beta} \varphi_\beta) + v_2 \cdot \nabla_2(a_{\alpha\beta} \varphi_\beta)\} (v_{1,\alpha} - v_{2,\alpha}) F_2^N(v_1, v_2) dv_1 dv_2. \end{aligned}$$

Passing to the limit $N \rightarrow \infty$, since $F_2^N \rightharpoonup f^{\otimes 2}$ we obtain

$$\liminf_{N \rightarrow \infty} D_{12}^N(F^N) \geq \frac{1}{2} \int \left\{ -\nabla_{12} \cdot (a(v_1 - v_2) \varphi) - a(v_1 - v_2) \frac{\varphi\varphi}{4} \right\} f_1 f_2 dv_1 dv_2 \geq D(f) - \varepsilon$$

and we conclude letting ε go to 0. \square

We define the Fisher information of $G \in \mathcal{P}(\mathbb{R}^{dN})$ that is absolute continuous with respect to the Lebesgue measure by

$$I(G) := \int_{\mathbb{R}^{dN}} \frac{|\nabla_{\mathbb{R}^{dN}} G|^2}{G} dV.$$

Moreover, for a $F \in \mathcal{P}(\mathcal{S}^N(\mathcal{E}))$ absolute continuous with respect to γ^N , we define the relative Fisher's information by

$$(5.15) \quad I(F|\gamma^N) := \int_{\mathcal{S}^N(\mathcal{E})} \frac{|\nabla_{\mathcal{S}^N} h|^2}{h} d\gamma^N, \quad h = \frac{dF}{d\gamma^N},$$

where $\nabla_{\mathcal{S}^N}$ stands for the gradient on $\mathcal{S}^N(\mathcal{E})$.

We can now give the following result.

Lemma 5.2. *Let $F_0^N \in \mathcal{P}_{\text{sym}}(\mathcal{S}^N(\mathcal{E}))$ with finite relative Fisher information $I(F_0^N|\gamma^N)$. For all $t > 0$ consider the solution F_t^N of the Landau master equation (2.29). Then we have*

$$I(F_t^N|\gamma^N) \leq I(F_0^N|\gamma^N).$$

Proof of Lemma 5.2. Denote $h_0^N := dF_0^N/d\gamma^N$ and, for all $t \geq 0$, $h_t^N := dF_t^N/d\gamma^N$. Consider \tilde{h}_t^N defined on \mathbb{R}^{dN} given by (5.5) and define then $\tilde{F}_t^N = \tilde{h}_t^N \mathcal{L}$ a solution of (5.1), where \mathcal{L} is the Lebesgue measure on \mathbb{R}^{dN} . Following [18, Lemma 7.4], we claim that is enough to prove that

$$I(\tilde{F}_t^N) := \int_{\mathbb{R}^{dN}} \frac{|\nabla_{\mathbb{R}^{dN}} \tilde{h}_t^N|^2}{\tilde{h}_t^N} dV \leq \int_{\mathbb{R}^{dN}} \frac{|\nabla_{\mathbb{R}^{dN}} \tilde{h}_0^N|^2}{\tilde{h}_0^N} dV =: I(\tilde{F}_0^N).$$

Indeed, this equation, (5.6), (5.8) and the conservation of momentum and energy yield

$$\begin{aligned} I(\tilde{F}_t^N) &= |\nabla_{\perp} \log(\rho\eta)|^2 + \left(\int_{\mathbb{R}^+ \times \mathbb{R}^d} B(\rho(\mathcal{E}), \eta(\mathcal{M})) d\mathcal{E} d\mathcal{M} \right) \left(\int_{\mathcal{S}^N(\mathcal{E})} \frac{|\nabla_{\mathcal{S}^N} h_t^N|^2}{h_t^N} d\gamma^N \right) \\ &\leq |\nabla_{\perp} \log(\rho\eta)|^2 + \left(\int_{\mathbb{R}^+ \times \mathbb{R}^d} B(\rho(\mathcal{E}), \eta(\mathcal{M})) d\mathcal{E} d\mathcal{M} \right) \left(\int_{\mathcal{S}^N(\mathcal{E})} \frac{|\nabla_{\mathcal{S}^N} h_0^N|^2}{h_0^N} d\gamma^N \right) = I(\tilde{F}_0^N), \end{aligned}$$

which implies, dropping the time independent terms,

$$I(F_t^N|\gamma^N) \leq I(F_0^N|\gamma^N).$$

Now, let $F_{t,\varepsilon}^N \in \mathcal{P}_{\text{sym}}(\mathcal{S}^N(\mathcal{E}))$ be the solution of the Boltzmann master equation (2.8)-(2.9) with collision kernel b_{ε} satisfying the grazing collisions assumptions (2.18) and initial datum $F_0^N \in \mathcal{P}_{\text{sym}}(\mathcal{S}^N(\mathcal{E}))$. Then we have from [18, Lemma 7.4], for all $t \geq 0$,

$$I(\tilde{F}_{t,\varepsilon}^N) \leq I(\tilde{F}_0^N),$$

where $\tilde{F}_0^N, \tilde{F}_{t,\varepsilon}^N \in \mathcal{P}_{\text{sym}}(\mathbb{R}^{dN})$ are constructed as before.

Since $\tilde{F}_{t,\varepsilon}^N$ weakly converges towards \tilde{F}_t^N when $\varepsilon \rightarrow 0$ and the Fisher information functional is weakly lower semicontinuous, we obtain

$$I(\tilde{F}_t^N) \leq \liminf_{\varepsilon \rightarrow 0} I(\tilde{F}_{t,\varepsilon}^N) \leq I(\tilde{F}_0^N)$$

and that concludes the proof. \square

Now, with the definitions of relative entropy (5.4), relative Fisher information (5.15) and the notion of entropic chaos, described below, we are able to state our main theorem of this section, concerning the propagation of entropic chaos.

Let F^N be a sequence of probability measures $\mathcal{S}^N(\mathcal{E})$ such that F_1^N weakly converges to f in measure sense, for some $f \in \mathcal{P}(\mathbb{R}^d)$. We say that F^N is entropically f -chaotic if

$$(5.16) \quad \frac{H(F^N|\gamma^N)}{N} \xrightarrow{N \rightarrow \infty} H(f|\gamma).$$

For more information on entropic chaos we refer to [3, 11, 4].

Theorem 5.3. *Let $f_0 \in \mathcal{P}(\mathbb{R}^d)$ and $F_0^N \in \mathcal{P}_{\text{sym}}(\mathcal{S}^N)$ that is f_0 -chaotic. Consider then, for all $t > 0$, the solution F_t^N of the Landau master equation (2.29) with initial condition F_0^N , and the solution f_t of the limit Landau equation (2.10)-(2.11) with initial data f_0 .*

Then we have

- (1) *If F_0^N is entropically f_0 -chaotic, then for all $t > 0$ F_t^N is entropically f_t -chaotic, more precisely*

$$\frac{1}{N} H(F_t^N | \gamma^N) \longrightarrow H(f_t | \gamma) \quad \text{as } N \rightarrow \infty.$$

- (2) *Consider $f_0 \in \mathcal{P}_6(\mathbb{R}^d)$ with $I(f_0 | \gamma) < \infty$ and $F_0^N = [f_0^{\otimes N}]_{\mathcal{S}^N} \in \mathcal{P}_{\text{sym}}(\mathcal{S}^N)$. Then, for all $t > 0$, F_t^N is entropically f_t -chaotic, more precisely, for any $0 < \varepsilon < 18[5(7d+6)^2(d+9)]^{-1}$ there exists a constant $C := C(\varepsilon)$ such that*

$$\sup_{t \geq 0} \left| \frac{1}{N} H(F_t^N | \gamma^N) - H(f_t | \gamma) \right| \leq CN^{-\varepsilon}.$$

- (3) *Consider $f_0 \in \mathcal{P}_6(\mathbb{R}^d)$ with $I(f_0 | \gamma) < \infty$ and $F_0^N = [f_0^{\otimes N}]_{\mathcal{S}^N} \in \mathcal{P}_{\text{sym}}(\mathcal{S}^N)$. Then for all N it holds*

$$\frac{1}{N} H(F_t^N | \gamma^N) \leq p(t),$$

for some polynomial function $p(t) \rightarrow 0$ as $t \rightarrow \infty$.

Proof of Theorem 5.3 (1). The idea is from [18]. Using (5.10), (5.11) and the entropic chaoticity at initial time, one has

$$\begin{aligned} \frac{1}{N} H(f_t^N | \gamma^N) + \int_0^t \frac{1}{N} D^N(F_s^N) ds &= \frac{1}{N} H(f_0^N | \gamma^N) \\ &\xrightarrow{N \rightarrow \infty} H(f_0 | \gamma) = H(f_t | \gamma) + \int_0^t D(f_s) ds. \end{aligned}$$

By Lemma 5.1 one also has

$$\liminf_{N \rightarrow \infty} \left(H(f_t^N | \gamma^N) + \int_0^t \frac{1}{N} D^N(F_s^N) ds \right) \geq H(f_t | \gamma) + \int_0^t D(f_s) ds,$$

and we can conclude with these two last equations together with Lemma 5.1. \square

Proof of Theorem 5.3 (2). From Lemma 5.2 we know that, for all $t \geq 0$, $N^{-1}I(F_t^N | \gamma^N) \leq N^{-1}I(F_0^N | \gamma^N)$ and the later one is bounded by construction, we deduce then that normalized relative Fisher's information $N^{-1}I(F_t^N | \gamma^N)$ is bounded. Since the limit Landau equation propagates moments and the Fisher's information's bound [25, 26], we have, for all $t > 0$, $M_6(f_t)$ and $I(f_t | \gamma)$ bounded.

We can then apply [4, Theorem 31] to F_t^N and we obtain that for any $\beta < (7d+6)^{-1}$ there exists $C' = C'(\beta)$ such that

$$\left| \frac{1}{N} H(F_t^N | \gamma^N) - H(f_t | \gamma) \right| \leq C' \left(\frac{W_2(F_t^N, f_t^{\otimes N})}{\sqrt{N}} + N^{-\beta} \right).$$

We have then to estimate the first term of the right-hand side. From [11], the following estimation holds,

$$(5.17) \quad \frac{W_2(F_t^N, f_t^{\otimes N})}{\sqrt{N}} \leq C \left(\frac{M_6(F_t^N, f_t^{\otimes N})}{N} \right)^{1/10} \left(\frac{W_1(F_t^N, f_t^{\otimes N})}{N} \right)^{2/5}$$

where $M_6(F_t^N, f_t^{\otimes N}) = M_6(F_t^N) + M_6(f_t^{\otimes N})$. We observe that $N^{-1}M_6(F_t^N, f_t^{\otimes N})$ is bounded since $N^{-1}M_6(f_t^{\otimes N}) = M_6(f_t)$, $N^{-1}M_6(F_t^N) \leq C N^{-1}M_6(F_0^N)$ by Lemma 4.7 and $N^{-1}M_6(F_0^N)$ is bounded by construction, thanks to the assumption $M_6(f_0)$ finite.

Finally, Theorem 4.2 and last equation imply that for any $\epsilon < 9[(7d+6)^2(d+9)]^{-1}$ and any $\beta < (7d+6)^{-1}$ there exists a positive constant $C = C(\epsilon, \beta)$ such that

$$(5.18) \quad \left| \frac{1}{N} H(F_t^N | \gamma^N) - H(f_t | \gamma) \right| \leq C \left(N^{-2\epsilon/5} + N^{-\beta} \right),$$

which concludes the proof. □

Proof of Theorem 5.3 (3). By the HWI inequality [28, Theorem 30.21], for all $t \geq 0$, we have

$$\frac{H(F_t^N | \gamma^N)}{N} \leq \frac{\pi}{2} \sqrt{\frac{I(F_t^N | \gamma^N)}{N} \frac{W_2(F_t^N, \gamma^N)}{\sqrt{N}}}.$$

From Lemma 5.2 we have $N^{-1}I(F_t^N | \gamma^N) \leq N^{-1}I(F_0^N | \gamma^N) \leq C$ for some constant $C > 0$ independent of N , by construction. Moreover, thanks to Lemma 4.7 and (5.17) we deduce

$$\frac{W_2(F_t^N, \gamma^N)}{\sqrt{N}} \leq C \left(\frac{W_1(F_t^N, \gamma^N)}{N} \right)^{2/5}.$$

Gathering these two estimates with point (2) in Theorem 4.2 it follows

$$\frac{H(F_t^N | \gamma^N)}{N} \leq C \left(\frac{W_1(F_t^N, \gamma^N)}{N} \right)^{2/5} \leq p(t),$$

for a polynomial function $p(t) \rightarrow 0$ as $t \rightarrow \infty$. □

Acknowledgements. The author would like to thank Stéphane Mischler and Clément Mouhot for suggesting the subject, their encouragement and fruitful discussions.

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