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PRIMAL AND SHADOW FUNCTIONS, DUAL AND DUAL-SHADOW FUNCTIONS FOR A CIRCULAR CRACK AND A CIRCULAR 90° V-NOTCH WITH NEUMANN BOUNDARY CONDITIONS

SAMUEL SHANNON, ZOHAR YOSIBASH, MONIQUE DAUGE AND MARTIN COSTABEL

ABSTRACT. This report presents explicit analytical expressions for the primal, primal shadows, dual and dual shadows functions for the Laplace equation in the vicinity of a circular singular edge with Neumann boundary conditions on the faces that intersect at the singular edge. Two configurations are investigated: a penny-shaped crack and a 90° V-notch.

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INTRODUCTION

The edges that we consider are in an axisymmetric configuration. Cylindrical coordinates are denoted by (r, θ, x_3) with the distance r to the axis, the rotation angle $\theta \in [0, 2\pi]$ and the coordinate x_3 along the axis. The circular edges are generated by rotating a point P in the r-x_3 half-plane around the x_3 axis. Let R be $r(P)$. We consider the polar coordinate system (ρ, φ) centered at P such that the following relations hold

$$r = R + \rho \cos\varphi \quad \text{and} \quad x_3 = \rho \sin\varphi \,.$$

In the vicinity of the edge the domain coincides with the set

$$\{(\rho, \varphi, \theta), \quad 0 < \rho < \rho_0, \;\; \varphi_1 < \varphi < \varphi_2, \;\; \theta \in [0, 2\pi]\} \,.$$

In this report we address the cases of

- the penny-shaped crack, for which $\varphi_1 = -\pi$ and $\varphi_2 = \pi$
- the 90° V-notch that corresponds to $\varphi_1 = -\pi$ and $\varphi_2 = \pi/2$.

Solutions τ of the homogeneous Laplace equation $\Delta\tau = 0$ with zero Neumann boundary conditions on the faces abutting the edge can be expressed as follows [1] in coordinates (ρ, φ, θ):

$$\tau(\rho,\varphi,\theta) = \sum_{j\geq 0}\sum_{h=0,2,4\cdots} \partial_\theta^h A_j(\theta)\rho^{\alpha_j} \sum_{f\geq 0}\left(\frac{\rho}{R}\right)^{h+f} \phi_{h,j,f}(\varphi), \tag{1}$$

The functions A_j are coefficients which depend on τ, whereas the functions $\phi_{h,j,f}$ are completely determined by the geometry of the domain around the edge, I.E. by the limiting angles φ_1 and φ_2. For each natural number j, the term $\phi_{0,j,0}$ is an eigenfunction of a 1D Laplace operator and for $h+j>0$, the "shadows" $\phi_{h,j,f}(\varphi)$ are obtained by solving the following recursive set of equations, cf [1, eq. (15-17) and (36)].

For $h = 0$

$$\alpha_j^2\phi_{0,j,0} + \phi''_{0,j,0} = 0 \tag{2}$$

$$(\alpha_j+1)^2\phi_{0,j,1} + \phi''_{0,j,1} = -\left(\alpha_j\cos\varphi\phi_{0,j,0} - \sin\varphi\phi'_{0,j,0}\right) \tag{3}$$

$$(\alpha_j+i)^2\phi_{0,j,f} + \phi''_{0,j,f} = -\left[(\alpha_j+f)(\alpha_j+f-1)\cos\varphi\phi_{0,j,f-1} - \sin\varphi\phi'_{0,j,f-1} + \cos\varphi\phi''_{0,j,f-1}\right] \quad f\geq 2 \tag{4}$$

For $h = 2,4,6\cdots$,

$$\begin{aligned}(\alpha_j+f+h)^2\phi_{h,j,f} + \phi''_{h,j,f} = &-(h+f+\alpha_j-1)\left[2(h+f+\alpha_j)-1\right]\cos\varphi\phi_{h,j,(f-1)}\\ &+\sin\varphi\phi'_{h,j,(f-1)} - 2\cos\varphi\phi''_{h,j,(f-1)}\\ &-(h+\alpha_j+f-2)(h+\alpha_j+f-1)\cos^2\varphi\phi_{h,j,(f-2)}\\ &+\cos\varphi\sin\varphi\phi'_{h,j,(f-2)} - \cos^2\varphi\phi''_{h,j,(f-2)} - \partial_\varphi\phi_{(h-2),j,f}\end{aligned} \tag{5}$$

These equations have to be completed by the Neumann boundary conditions:

$$\partial_\varphi\phi_{h,j,f} = 0, \quad \text{on} \quad \varphi=\varphi_1 \quad \text{and} \quad \varphi=\varphi_2 \tag{6}$$

Equations (2-4) with $h = 0$ are associated with the axisymmetric case, and for $h = 2,4,6,\cdots$ equations (5) are associated with the non-axisymmetric case.

Each eigenfunction and shadow, has a dual and dual shadow counterpart, that can be computed by the following recursive system.

For the dual eigenfunctions and their shadows $\psi_{0,j,f}(\varphi)$ (for $h = 0$ associated with the axisymmetric case), the recursive equation to be solved is [1]:

For $h = 0$

$$\alpha_j^2\psi_{0,j,0} + \psi''_{0,j,0} = 0, \tag{7}$$

$$(-\alpha_j+1)^2\psi_{0,j,1} + \psi''_{0,j,1} = -\left(-\alpha_j\cos\varphi\,\psi_{0,j,0} - \sin\varphi\,\psi'_{0,j,0}\right), \tag{8}$$

$$\begin{aligned}(-\alpha_j+i)^2\psi_{0,j,f} + \psi''_{0,j,f} = &-\left[(-\alpha_j+f)(-\alpha_j+f-1)\cos\varphi\,\psi_{0,j,f-1}\right.\\ &\left.-\sin\varphi\,\psi'_{0,j,f-1} + \cos\varphi\,\psi''_{0,j,f-1}\right], \quad f\geq 2,\end{aligned} \tag{9}$$

For $h = 2,4,6\cdots$,

$$\begin{aligned}&(-\alpha_j+f+h)^2\psi_{h,j,f} + \psi''_{h,j,f} =\\ &\quad -(h+f-\alpha_j-1)\left[2(h+f-\alpha_j)-1\right]\cos\varphi\,\psi_{h,j,(f-1)} + \sin\varphi\,\psi'_{h,j,(f-1)}\\ &\quad -2\cos\varphi\,\psi''_{h,f,(f-1)} - (h-\alpha_j+f-2)(h-\alpha_j+f-1)\cos^2\varphi\,\psi_{h,j,(f-2)}\\ &\quad +\cos\varphi\,\sin\varphi\,\psi'_{h,j,(f-2)} - \cos^2\varphi\,\psi''_{h,j,(f-2)} - \psi_{(h-2),j,f}\end{aligned} \tag{10}$$

with homogeneous Neumann boundary conditions

$$\partial_\varphi\psi_{h,j,f} = 0 \quad \text{on} \quad \varphi=\varphi_1 \quad \text{and} \quad \varphi=\varphi_2. \tag{11}$$

The above systems of equations have been solved using the formal calculus software Mathematica.

REFERENCES

[1] Z. Yosibash, S. Shannon, M. Dauge, and M. Costabel. Circular edge singularities for the Laplace equation and the elasticity system in 3-D domains. *Int. Jour. Fracture*, 168:31–52, 2011. hal.archives-ouvertes.fr/hal-00490971

1. Primal functions and shadows for the crack $-\pi \le \varphi \le \pi$

1.1. First singular exponent ($j = 1$).

$$
\begin{aligned}
\phi_{0,1,0} &= \sin\frac{\varphi}{2}\\
\phi_{0,1,1} &= \frac{1}{4}\sin\frac{\varphi}{2}\\
\phi_{0,1,2} &= \frac{1}{12}\sin\frac{\varphi}{2} - \frac{3}{32}\sin\frac{3\varphi}{2}\\
\phi_{0,1,3} &= \frac{1}{16}\sin\frac{\varphi}{2} - \frac{1}{30}\sin\frac{3\varphi}{2} + \frac{5}{128}\sin\frac{5\varphi}{2}\\
\phi_{0,1,4} &= \frac{3}{80}\sin\frac{\varphi}{2} - \frac{5}{128}\sin\frac{3\varphi}{2} + \frac{1}{70}\sin\frac{5\varphi}{2} - \frac{35}{2048}\sin\frac{7\varphi}{2}\\
\phi_{0,1,5} &= \frac{1}{32}\sin\frac{\varphi}{2} - \frac{3}{140}\sin\frac{3\varphi}{2} + \frac{35}{1536}\sin\frac{5\varphi}{2} - \frac{2}{315}\sin\frac{7\varphi}{2} + \frac{63}{8192}\sin\frac{9\varphi}{2}\\
\phi_{0,1,6} &= \frac{5}{224}\sin\frac{\varphi}{2} - \frac{35}{1536}\sin\frac{3\varphi}{2} + \frac{1}{84}\sin\frac{5\varphi}{2} - \frac{105}{8192}\sin\frac{7\varphi}{2} + \frac{2}{693}\sin\frac{9\varphi}{2} - \frac{231}{65536}\sin\frac{11\varphi}{2}\\
\phi_{0,1,7} &= \frac{5}{256}\sin\frac{\varphi}{2} - \frac{5}{336}\sin\frac{3\varphi}{2} + \frac{63}{4096}\sin\frac{5\varphi}{2} - \frac{1}{154}\sin\frac{7\varphi}{2} + \frac{231}{32768}\sin\frac{9\varphi}{2} - \frac{4}{3003}\sin\frac{11\varphi}{2}\\
&+ \frac{429}{262144}\sin\frac{13\varphi}{2}\\
\phi_{0,1,8} &= \frac{35}{2304}\sin\frac{\varphi}{2} - \frac{63}{4096}\sin\frac{3\varphi}{2} + \frac{5}{528}\sin\frac{5\varphi}{2} - \frac{1617}{163840}\sin\frac{7\varphi}{2} + \frac{1}{286}\sin\frac{9\varphi}{2} - \frac{1001}{262144}\sin\frac{11\varphi}{2}\\
&+ \frac{4}{6435}\sin\frac{13\varphi}{2} - \frac{6435}{8388608}\sin\frac{15\varphi}{2}\\
\phi_{0,1,9} &= \frac{7}{512}\sin\frac{\varphi}{2} - \frac{35}{3168}\sin\frac{3\varphi}{2} + \frac{231}{20480}\sin\frac{5\varphi}{2} - \frac{5}{858}\sin\frac{7\varphi}{2} + \frac{1001}{163840}\sin\frac{9\varphi}{2}\\
&- \frac{4}{2145}\sin\frac{11\varphi}{2} + \frac{2145}{1048576}\sin\frac{13\varphi}{2} - \frac{32}{109395}\sin\frac{15\varphi}{2} + \frac{12155}{33554432}\sin\frac{17\varphi}{2}\\
\phi_{0,1,10} &= \frac{63}{5632}\sin\frac{\varphi}{2} - \frac{231}{20480}\sin\frac{3\varphi}{2} + \frac{35}{4576}\sin\frac{5\varphi}{2} - \frac{1287}{163840}\sin\frac{7\varphi}{2} + \frac{1}{286}\sin\frac{9\varphi}{2}\\
&- \frac{3861}{1048576}\sin\frac{11\varphi}{2} + \frac{12}{12155}\sin\frac{13\varphi}{2} - \frac{36465}{33554432}\sin\frac{15\varphi}{2} + \frac{32}{230945}\sin\frac{17\varphi}{2}\\
&- \frac{46189}{268435456}\sin\frac{19\varphi}{2}
\end{aligned}
$$

$$
\begin{aligned}
\phi_{2,1,0} &= -\frac{1}{6}\sin\frac{\varphi}{2} \\
\phi_{2,1,1} &= -\frac{1}{8}\sin\frac{\varphi}{2} + \frac{7}{60}\sin\frac{3\varphi}{2} \\
\phi_{2,1,2} &= -\frac{31}{240}\sin\frac{\varphi}{2} + \frac{5}{64}\sin\frac{3\varphi}{2} - \frac{19}{280}\sin\frac{5\varphi}{2} \\
\phi_{2,1,3} &= -\frac{41}{384}\sin\frac{\varphi}{2} + \frac{111}{1120}\sin\frac{3\varphi}{2} - \frac{35}{768}\sin\frac{5\varphi}{2} + \frac{187}{5040}\sin\frac{7\varphi}{2} \\
\phi_{2,1,4} &= -\frac{8209}{80640}\sin\frac{\varphi}{2} + \frac{119}{1536}\sin\frac{3\varphi}{2} - \frac{271}{4032}\sin\frac{5\varphi}{2} + \frac{105}{4096}\sin\frac{7\varphi}{2} - \frac{437}{22176}\sin\frac{9\varphi}{2} \\
\phi_{2,1,5} &= -\frac{4069}{46080}\sin\frac{\varphi}{2} + \frac{39899}{483840}\sin\frac{3\varphi}{2} - \frac{427}{8192}\sin\frac{5\varphi}{2} + \frac{1259}{29568}\sin\frac{7\varphi}{2} - \frac{231}{16384}\sin\frac{9\varphi}{2} \\
&+ \frac{1979}{192192}\sin\frac{11\varphi}{2} \\
\phi_{2,1,6} &= -\frac{483047}{5806080}\sin\frac{\varphi}{2} + \frac{5663}{81920}\sin\frac{3\varphi}{2} - \frac{18467}{304128}\sin\frac{5\varphi}{2} + \frac{5467}{163840}\sin\frac{7\varphi}{2} - \frac{2837}{109824}\sin\frac{9\varphi}{2} \\
&+ \frac{1001}{131072}\sin\frac{11\varphi}{2} - \frac{4387}{823680}\sin\frac{13\varphi}{2} \\
\phi_{2,1,7} &= -\frac{153325}{2064384}\sin\frac{\varphi}{2} + \frac{8920073}{127733760}\sin\frac{3\varphi}{2} - \frac{49621}{983040}\sin\frac{5\varphi}{2} + \frac{331871}{7907328}\sin\frac{7\varphi}{2} \\
&- \frac{27027}{1310720}\sin\frac{9\varphi}{2} + \frac{50111}{3294720}\sin\frac{11\varphi}{2} - \frac{2145}{524288}\sin\frac{13\varphi}{2} + \frac{76627}{28005120}\sin\frac{15\varphi}{2} \\
\phi_{2,1,8} &= -\frac{79481719}{1135411200}\sin\frac{\varphi}{2} + \frac{2514853}{41287680}\sin\frac{3\varphi}{2} - \frac{6664667}{123002880}\sin\frac{5\varphi}{2} + \frac{275561}{7864320}\sin\frac{7\varphi}{2} \\
&- \frac{81201}{2928640}\sin\frac{9\varphi}{2} + \frac{13013}{1048576}\sin\frac{11\varphi}{2} - \frac{327121}{37340160}\sin\frac{13\varphi}{2} + \frac{36465}{16777216}\sin\frac{15\varphi}{2} \\
&- \frac{165409}{118243840}\sin\frac{17\varphi}{2}
\end{aligned}
$$

$$
\begin{aligned}
\phi_{4,1,0} &= \frac{1}{120}\sin\frac{\varphi}{2} \\
\phi_{4,1,1} &= \frac{1}{96}\sin\frac{\varphi}{2} - \frac{3}{280}\sin\frac{3\varphi}{2} \\
\phi_{4,1,2} &= \frac{17}{1008}\sin\frac{\varphi}{2} - \frac{7}{768}\sin\frac{3\varphi}{2} + \frac{19}{2016}\sin\frac{5\varphi}{2} \\
\phi_{4,1,3} &= \frac{41}{2304}\sin\frac{\varphi}{2} - \frac{433}{24192}\sin\frac{3\varphi}{2} + \frac{7}{1024}\sin\frac{5\varphi}{2} - \frac{13}{1848}\sin\frac{7\varphi}{2} \\
\phi_{4,1,4} &= \frac{1019}{46080}\sin\frac{\varphi}{2} - \frac{1}{64}\sin\frac{3\varphi}{2} + \frac{595}{38016}\sin\frac{5\varphi}{2} - \frac{77}{16384}\sin\frac{7\varphi}{2} + \frac{263}{54912}\sin\frac{9\varphi}{2} \\
\phi_{4,1,5} &= \frac{1367\sin\frac{\varphi}{2}}{61440} - \frac{5617\sin\frac{3\varphi}{2}}{253440} + \frac{605\sin\frac{5\varphi}{2}}{49152} - \frac{24155\sin\frac{7\varphi}{2}}{1976832} + \frac{1001\sin\frac{9\varphi}{2}}{327680} - \frac{2533\sin\frac{11\varphi}{2}}{823680} \\
\phi_{4,1,6} &= \frac{3211069\sin\frac{\varphi}{2}}{127733760} - \frac{58267\sin\frac{3\varphi}{2}}{2949120} + \frac{514259\sin\frac{5\varphi}{2}}{26357760} - \frac{4433\sin\frac{7\varphi}{2}}{491520} + \frac{81\sin\frac{9\varphi}{2}}{9152} \\
&- \frac{1001\sin\frac{11\varphi}{2}}{524288} + \frac{35401\sin\frac{13\varphi}{2}}{18670080}
\end{aligned}
$$

$$
\begin{aligned}
\phi_{6,1,0} &= -\frac{1}{5040}\sin\frac{\varphi}{2} \\
\phi_{6,1,1} &= -\frac{1}{2880}\sin\frac{\varphi}{2} + \frac{11}{30240}\sin\frac{3\varphi}{2} \\
\phi_{6,1,2} &= -\frac{13}{17280}\sin\frac{\varphi}{2} + \frac{1}{2560}\sin\frac{3\varphi}{2} - \frac{1}{2376}\sin\frac{5\varphi}{2} \\
\phi_{6,1,3} &= -\frac{1}{1024}\sin\frac{\varphi}{2} + \frac{191}{190080}\sin\frac{3\varphi}{2} - \frac{11}{30720}\sin\frac{5\varphi}{2} + \frac{97}{247104}\sin\frac{7\varphi}{2} \\
\phi_{6,1,4} &= -\frac{511}{345600}\sin\frac{\varphi}{2} + \frac{187}{184320}\sin\frac{3\varphi}{2} - \frac{1751}{1647360}\sin\frac{5\varphi}{2} + \frac{143}{491520}\sin\frac{7\varphi}{2} - \frac{59}{183040}\sin\frac{9\varphi}{2}
\end{aligned}
$$

$$
\begin{aligned}
\phi_{8,1,0} &= \frac{1}{362880}\sin\frac{\varphi}{2} \\
\phi_{8,1,1} &= \frac{\sin\frac{\varphi}{2}}{161280} - \frac{13\sin\frac{3\varphi}{2}}{1995840} \\
\phi_{8,1,2} &= \frac{89\sin\frac{\varphi}{2}}{5322240} - \frac{11\sin\frac{3\varphi}{2}}{1290240} + \frac{107\sin\frac{5\varphi}{2}}{11531520}
\end{aligned}
$$

$$
\phi_{10,1,0} = -\frac{1}{39916800}\sin\frac{\varphi}{2}
$$

1.2. **Second singular exponent ($j = 3$).**

$$
\begin{aligned}
\phi_{0,3,0} &= \sin\frac{3\varphi}{2}\\
\phi_{0,3,1} &= -\frac{1}{4}\sin\frac{\varphi}{2}\\
\phi_{0,3,2} &= -\frac{3}{32}\sin\frac{\varphi}{2}+\frac{1}{10}\sin\frac{3\varphi}{2}\\
\phi_{0,3,3} &= -\frac{3}{40}\sin\frac{\varphi}{2}+\frac{5}{128}\sin\frac{3\varphi}{2}-\frac{3}{70}\sin\frac{5\varphi}{2}\\
\phi_{0,3,4} &= -\frac{3}{64}\sin\frac{\varphi}{2}+\frac{27}{560}\sin\frac{3\varphi}{2}-\frac{35}{2048}\sin\frac{5\varphi}{2}+\frac{2}{105}\sin\frac{7\varphi}{2}\\
\phi_{0,3,5} &= -\frac{9}{224}\sin\frac{\varphi}{2}+\frac{7}{256}\sin\frac{3\varphi}{2}-\frac{1}{35}\sin\frac{5\varphi}{2}+\frac{63}{8192}\sin\frac{7\varphi}{2}-\frac{2}{231}\sin\frac{9\varphi}{2}\\
\phi_{0,3,6} &= -\frac{15}{512}\sin\frac{\varphi}{2}+\frac{5}{168}\sin\frac{3\varphi}{2}-\frac{63}{4096}\sin\frac{5\varphi}{2}+\frac{5}{308}\sin\frac{7\varphi}{2}-\frac{231}{65536}\sin\frac{9\varphi}{2}\\
&+ \frac{4}{1001}\sin\frac{11\varphi}{2}\\
\phi_{0,3,7} &= -\frac{5}{192}\sin\frac{\varphi}{2}+\frac{81}{4096}\sin\frac{3\varphi}{2}-\frac{25}{1232}\sin\frac{5\varphi}{2}+\frac{693}{81920}\sin\frac{7\varphi}{2}-\frac{9}{1001}\sin\frac{9\varphi}{2}\\
&+ \frac{429}{262144}\sin\frac{11\varphi}{2}-\frac{4}{2145}\sin\frac{13\varphi}{2}\\
\phi_{0,3,8} &= -\frac{21}{1024}\sin\frac{\varphi}{2}+\frac{175}{8448}\sin\frac{3\varphi}{2}-\frac{2079}{163840}\sin\frac{5\varphi}{2}+\frac{15}{1144}\sin\frac{7\varphi}{2}-\frac{3003}{655360}\sin\frac{9\varphi}{2}\\
&+ \frac{7}{1430}\sin\frac{11\varphi}{2}-\frac{6435}{8388608}\sin\frac{13\varphi}{2}+\frac{32}{36465}\sin\frac{15\varphi}{2}\\
\phi_{0,3,9} &= -\frac{105}{5632}\sin\frac{\varphi}{2}+\frac{77}{5120}\sin\frac{3\varphi}{2}-\frac{35}{2288}\sin\frac{5\varphi}{2}+\frac{1287}{163840}\sin\frac{7\varphi}{2}-\frac{7}{858}\sin\frac{9\varphi}{2}\\
&+ \frac{1287}{524288}\sin\frac{11\varphi}{2}-\frac{32}{12155}\sin\frac{13\varphi}{2}+\frac{12155}{33554432}\sin\frac{15\varphi}{2}-\frac{96}{230945}\sin\frac{17\varphi}{2}
\end{aligned}
$$

$$
\begin{aligned}
\phi_{2,3,0} &= -\frac{1}{10}\sin\frac{3\varphi}{2}\\
\phi_{2,3,1} &= \frac{3}{40}\sin\frac{\varphi}{2}+\frac{11}{140}\sin\frac{5\varphi}{2}\\
\phi_{2,3,2} &= \frac{3}{64}\sin\frac{\varphi}{2}-\frac{7}{80}\sin\frac{3\varphi}{2}-\frac{41}{840}\sin\frac{7\varphi}{2}\\
\phi_{2,3,3} &= \frac{65}{896}\sin\frac{\varphi}{2}-\frac{7}{256}\sin\frac{3\varphi}{2}+\frac{145\sin\frac{5\varphi}{2}}{2016}+\frac{103\sin\frac{9\varphi}{2}}{3696}\\
\phi_{2,3,4} &= \frac{27}{512}\sin\frac{\varphi}{2}-\frac{17959\sin\frac{3\varphi}{2}}{241920}+\frac{63\sin\frac{5\varphi}{2}}{4096}-\frac{2263\sin\frac{7\varphi}{2}}{44352}-\frac{489\sin\frac{11\varphi}{2}}{32032}\\
\phi_{2,3,5} &= \frac{8929\sin\frac{\varphi}{2}}{138240}-\frac{291\sin\frac{3\varphi}{2}}{8192}+\frac{2483\sin\frac{5\varphi}{2}}{39424}-\frac{693\sin\frac{7\varphi}{2}}{81920}+\frac{12829\sin\frac{9\varphi}{2}}{384384}+\frac{1121\sin\frac{13\varphi}{2}}{137280}\\
\phi_{2,3,6} &= \frac{4151\sin\frac{\varphi}{2}}{81920}-\frac{1358089\sin\frac{3\varphi}{2}}{21288960}+\frac{3729\sin\frac{5\varphi}{2}}{163840}-\frac{63161\sin\frac{7\varphi}{2}}{1317888}+\frac{3003\sin\frac{9\varphi}{2}}{655360}\\
&- \frac{34169\sin\frac{11\varphi}{2}}{1647360}-\frac{20081\sin\frac{15\varphi}{2}}{4667520}\\
\phi_{2,3,7} &= \frac{2161361\sin\frac{\varphi}{2}}{37847040}-\frac{21857\sin\frac{3\varphi}{2}}{589824}+\frac{10216499\sin\frac{5\varphi}{2}}{184504320}-\frac{18447\sin\frac{7\varphi}{2}}{1310720}+\frac{1342447\sin\frac{9\varphi}{2}}{39536640}\\
&- \frac{1287\sin\frac{11\varphi}{2}}{524288}+\frac{63367\sin\frac{13\varphi}{2}}{5091840}+\frac{132641\sin\frac{17\varphi}{2}}{59121920}
\end{aligned}
$$

$$
\begin{aligned}
\phi_{4,3,0} &= \frac{1}{280}\sin\frac{3\varphi}{2} \\
\phi_{4,3,1} &= -\frac{1}{224}\sin\frac{\varphi}{2} - \frac{13\sin\frac{5\varphi}{2}}{2520} \\
\phi_{4,3,2} &= -\frac{1}{256}\sin\frac{\varphi}{2} + \frac{7}{864}\sin\frac{3\varphi}{2} + \frac{109\sin\frac{7\varphi}{2}}{22176} \\
\phi_{4,3,3} &= -\frac{59\sin\frac{\varphi}{2}}{6912} + \frac{3\sin\frac{3\varphi}{2}}{1024} - \frac{25\sin\frac{5\varphi}{2}}{2688} - \frac{17\sin\frac{9\varphi}{2}}{4368} \\
\phi_{4,3,4} &= -\frac{23\sin\frac{\varphi}{2}}{3072} + \frac{17431\sin\frac{3\varphi}{2}}{1520640} - \frac{33\sin\frac{5\varphi}{2}}{16384} + \frac{355\sin\frac{7\varphi}{2}}{41184} + \frac{457\sin\frac{11\varphi}{2}}{164736} \\
\phi_{4,3,5} &= -\frac{7781\sin\frac{\varphi}{2}}{675840} + \frac{869\sin\frac{3\varphi}{2}}{147456} - \frac{120277\sin\frac{5\varphi}{2}}{9884160} + \frac{429\sin\frac{7\varphi}{2}}{327680} - \frac{13919\sin\frac{9\varphi}{2}}{1976832} \\
&- \frac{25879\sin\frac{13\varphi}{2}}{14002560}
\end{aligned}
$$

$$
\begin{aligned}
\phi_{6,3,0} &= -\frac{\sin\frac{3\varphi}{2}}{15120} \\
\phi_{6,3,1} &= \frac{\sin\frac{\varphi}{2}}{8640} + \frac{\sin\frac{5\varphi}{2}}{7392} \\
\phi_{6,3,2} &= \frac{\sin\frac{\varphi}{2}}{7680} - \frac{53\sin\frac{3\varphi}{2}}{190080} - \frac{7\sin\frac{7\varphi}{2}}{41184} \\
\phi_{6,3,3} &= \frac{61\sin\frac{\varphi}{2}}{168960} - \frac{11\sin\frac{3\varphi}{2}}{92160} + \frac{997\sin\frac{5\varphi}{2}}{2471040} + \frac{19\sin\frac{9\varphi}{2}}{112320}
\end{aligned}
$$

$$
\begin{aligned}
\phi_{8,3,0} &= \frac{\sin\frac{3\varphi}{2}}{1330560} \\
\phi_{8,3,1} &= -\frac{\sin\frac{\varphi}{2}}{591360} - \frac{17\sin\frac{5\varphi}{2}}{8648640}
\end{aligned}
$$

1.3. **Third singular exponent ($j = 5$).**

$$
\begin{aligned}
\phi_{0,5,0} &= \sin\frac{5\varphi}{2}\\
\phi_{0,5,1} &= -\frac{1}{4}\sin\frac{3\varphi}{2}\\
\phi_{0,5,2} &= \frac{3}{32}\sin\frac{\varphi}{2}+\frac{3}{28}\sin\frac{5\varphi}{2}\\
\phi_{0,5,3} &= \frac{5}{128}\sin\frac{\varphi}{2}-\frac{9}{112}\sin\frac{3\varphi}{2}-\frac{1}{21}\sin\frac{7\varphi}{2}\\
\phi_{0,5,4} &= \frac{45}{896}\sin\frac{\varphi}{2}-\frac{35}{2048}\sin\frac{3\varphi}{2}+\frac{3}{56}\sin\frac{5\varphi}{2}+\frac{5}{231}\sin\frac{9\varphi}{2}\\
\phi_{0,5,5} &= \frac{15}{512}\sin\frac{\varphi}{2}-\frac{5}{112}\sin\frac{3\varphi}{2}+\frac{63}{8192}\sin\frac{5\varphi}{2}-\frac{5}{154}\sin\frac{7\varphi}{2}-\frac{10}{1001}\sin\frac{11\varphi}{2}\\
\phi_{0,5,6} &= \frac{25}{768}\sin\frac{\varphi}{2}-\frac{135}{8192}\sin\frac{3\varphi}{2}+\frac{125}{3696}\sin\frac{5\varphi}{2}-\frac{231}{65536}\sin\frac{7\varphi}{2}+\frac{75}{4004}\sin\frac{9\varphi}{2}+\frac{2}{429}\sin\frac{13\varphi}{2}\\
\phi_{0,5,7} &= \frac{45}{2048}\sin\frac{\varphi}{2}-\frac{125}{4224}\sin\frac{3\varphi}{2}+\frac{297}{32768}\sin\frac{5\varphi}{2}-\frac{375}{16016}\sin\frac{7\varphi}{2}+\frac{429}{262144}\sin\frac{9\varphi}{2}\\
&- \frac{3}{286}\sin\frac{11\varphi}{2}-\frac{16}{7293}\sin\frac{15\varphi}{2}\\
\phi_{0,5,8} &= \frac{525}{22528}\sin\frac{\varphi}{2}-\frac{231}{16384}\sin\frac{3\varphi}{2}+\frac{875}{36608}\sin\frac{5\varphi}{2}-\frac{1287}{262144}\sin\frac{7\varphi}{2}+\frac{35}{2288}\sin\frac{9\varphi}{2}\\
&- \frac{6435}{8388608}\sin\frac{11\varphi}{2}+\frac{14}{2431}\sin\frac{13\varphi}{2}+\frac{48}{46189}\sin\frac{17\varphi}{2}
\end{aligned}
$$

$$
\begin{aligned}
\phi_{2,5,0} &= -\frac{1}{14}\sin\frac{5\varphi}{2}\\
\phi_{2,5,1} &= \frac{3}{56}\sin\frac{3\varphi}{2}+\frac{5}{84}\sin\frac{7\varphi}{2}\\
\phi_{2,5,2} &= -\frac{15}{448}\sin\frac{\varphi}{2}-\frac{67\sin\frac{5\varphi}{2}}{1008}-\frac{71\sin\frac{9\varphi}{2}}{1848}\\
\phi_{2,5,3} &= -\frac{5}{256}\sin\frac{\varphi}{2}+\frac{445\sin\frac{3\varphi}{2}}{8064}+\frac{1261\sin\frac{7\varphi}{2}}{22176}+\frac{361\sin\frac{11\varphi}{2}}{16016}\\
\phi_{2,5,4} &= -\frac{185\sin\frac{\varphi}{2}}{4608}+\frac{45\sin\frac{3\varphi}{2}}{4096}-\frac{31309\sin\frac{5\varphi}{2}}{532224}-\frac{725\sin\frac{9\varphi}{2}}{17472}-\frac{173\sin\frac{13\varphi}{2}}{13728}\\
\phi_{2,5,5} &= -\frac{665\sin\frac{\varphi}{2}}{24576}+\frac{15589\sin\frac{3\varphi}{2}}{304128}-\frac{99\sin\frac{5\varphi}{2}}{16384}+\frac{33853\sin\frac{7\varphi}{2}}{658944}+\frac{4567\sin\frac{11\varphi}{2}}{164736}+\frac{3197\sin\frac{15\varphi}{2}}{466752}\\
\phi_{2,5,6} &= -\frac{21763\sin\frac{\varphi}{2}}{540672}+\frac{1705\sin\frac{3\varphi}{2}}{98304}-\frac{961937\sin\frac{5\varphi}{2}}{18450432}+\frac{429\sin\frac{7\varphi}{2}}{131072}-\frac{263249\sin\frac{9\varphi}{2}}{6589440}\\
&- \frac{5771\sin\frac{13\varphi}{2}}{329472}-\frac{21613\sin\frac{17\varphi}{2}}{5912192}
\end{aligned}
$$

$$
\begin{aligned}
\phi_{4,5,0} &= \frac{1}{504}\sin\frac{5\varphi}{2}\\
\phi_{4,5,1} &= -\frac{5\sin\frac{3\varphi}{2}}{2016}-\frac{17\sin\frac{7\varphi}{2}}{5544}\\
\phi_{4,5,2} &= \frac{5\sin\frac{\varphi}{2}}{2304}+\frac{10\sin\frac{5\varphi}{2}}{2079}+\frac{295\sin\frac{9\varphi}{2}}{96096}\\
\phi_{4,5,3} &= \frac{5\sin\frac{\varphi}{2}}{3072}-\frac{35\sin\frac{3\varphi}{2}}{6912}-\frac{955\sin\frac{7\varphi}{2}}{164736}-\frac{1}{396}\sin\frac{11\varphi}{2}\\
\phi_{4,5,4} &= \frac{25\sin\frac{\varphi}{2}}{5632}-\frac{55\sin\frac{3\varphi}{2}}{49152}+\frac{28211\sin\frac{5\varphi}{2}}{3953664}+\frac{919\sin\frac{9\varphi}{2}}{164736}+\frac{5197\sin\frac{13\varphi}{2}}{2800512}
\end{aligned}
$$

$$\begin{aligned}
\phi_{6,5,0} &= -\frac{\sin\frac{5\varphi}{2}}{33264}\\
\phi_{6,5,1} &= \frac{\sin\frac{3\varphi}{2}}{19008}+\frac{19\sin\frac{7\varphi}{2}}{288288}\\
\phi_{6,5,2} &= -\frac{\sin\frac{\varphi}{2}}{16896}-\frac{67\sin\frac{5\varphi}{2}}{494208}-\frac{\sin\frac{9\varphi}{2}}{11440}
\end{aligned}$$

$$\phi_{8,5,0} = \frac{\sin\frac{5\varphi}{2}}{3459456}$$

1.4. **Fourth singular exponent ($j=7$).**

$$\begin{aligned}
\phi_{0,7,0} &= \sin\frac{7\varphi}{2}\\
\phi_{0,7,1} &= -\frac{1}{4}\sin\frac{5\varphi}{2}\\
\phi_{0,7,2} &= \frac{3}{32}\sin\frac{3\varphi}{2}+\frac{1}{9}\sin\frac{7\varphi}{2}\\
\phi_{0,7,3} &= -\frac{5}{128}\sin\frac{\varphi}{2}-\frac{1}{12}\sin\frac{5\varphi}{2}-\frac{5}{99}\sin\frac{9\varphi}{2}\\
\phi_{0,7,4} &= -\frac{35\sin\frac{\varphi}{2}}{2048}+\frac{5}{96}\sin\frac{3\varphi}{2}+\frac{5}{88}\sin\frac{7\varphi}{2}+\frac{10}{429}\sin\frac{11\varphi}{2}\\
\phi_{0,7,5} &= -\frac{35\sin\frac{\varphi}{2}}{1152}+\frac{63\sin\frac{3\varphi}{2}}{8192}-\frac{25}{528}\sin\frac{5\varphi}{2}-\frac{5}{143}\sin\frac{9\varphi}{2}-\frac{14\sin\frac{13\varphi}{2}}{1287}\\
\phi_{0,7,6} &= -\frac{35\sin\frac{\varphi}{2}}{2048}+\frac{875\sin\frac{3\varphi}{2}}{25344}-\frac{231\sin\frac{5\varphi}{2}}{65536}+\frac{125\sin\frac{7\varphi}{2}}{3432}+\frac{35\sin\frac{11\varphi}{2}}{1716}+\frac{112\sin\frac{15\varphi}{2}}{21879}\\
\phi_{0,7,7} &= -\frac{525\sin\frac{\varphi}{2}}{22528}+\frac{77\sin\frac{3\varphi}{2}}{8192}-\frac{875\sin\frac{5\varphi}{2}}{27456}+\frac{429\sin\frac{7\varphi}{2}}{262144}-\frac{175\sin\frac{9\varphi}{2}}{6864}-\frac{28\sin\frac{13\varphi}{2}}{2431}-\frac{112\sin\frac{17\varphi}{2}}{46189}
\end{aligned}$$

$$\begin{aligned}
\phi_{2,7,0} &= -\frac{1}{18}\sin\frac{7\varphi}{2}\\
\phi_{2,7,1} &= \frac{1}{24}\sin\frac{5\varphi}{2}+\frac{19}{396}\sin\frac{9\varphi}{2}\\
\phi_{2,7,2} &= -\frac{5}{192}\sin\frac{3\varphi}{2}-\frac{85\sin\frac{7\varphi}{2}}{1584}-\frac{109\sin\frac{11\varphi}{2}}{3432}\\
\phi_{2,7,3} &= \frac{35\sin\frac{\varphi}{2}}{2304}+\frac{565\sin\frac{5\varphi}{2}}{12672}+\frac{647\sin\frac{9\varphi}{2}}{13728}+\frac{391\sin\frac{13\varphi}{2}}{20592}\\
\phi_{2,7,4} &= \frac{35\sin\frac{\varphi}{2}}{4096}-\frac{1645\sin\frac{3\varphi}{2}}{50688}-\frac{48259\sin\frac{7\varphi}{2}}{988416}-\frac{321\sin\frac{11\varphi}{2}}{9152}-\frac{7543\sin\frac{15\varphi}{2}}{700128}\\
\phi_{2,7,5} &= \frac{5915\sin\frac{\varphi}{2}}{270336}-\frac{77\sin\frac{3\varphi}{2}}{16384}+\frac{168343\sin\frac{5\varphi}{2}}{3953664}+\frac{430007\sin\frac{9\varphi}{2}}{9884160}+\frac{66637\sin\frac{13\varphi}{2}}{2800512}+\frac{17517\sin\frac{17\varphi}{2}}{2956096}
\end{aligned}$$

$$\begin{aligned}
\phi_{4,7,0} &= \frac{1}{792}\sin\frac{7\varphi}{2}\\
\phi_{4,7,1} &= -\frac{5\sin\frac{5\varphi}{2}}{3168}-\frac{7\sin\frac{9\varphi}{2}}{3432}\\
\phi_{4,7,2} &= \frac{35\sin\frac{3\varphi}{2}}{25344}+\frac{395\sin\frac{7\varphi}{2}}{123552}+\frac{29\sin\frac{11\varphi}{2}}{13728}\\
\phi_{4,7,3} &= -\frac{35\sin\frac{\varphi}{2}}{33792}-\frac{3325\sin\frac{5\varphi}{2}}{988416}-\frac{179\sin\frac{9\varphi}{2}}{44928}-\frac{625\sin\frac{13\varphi}{2}}{350064}
\end{aligned}$$

$$\phi_{6,7,0} = -\frac{\sin\frac{7\varphi}{2}}{61776}$$
$$\phi_{6,7,1} = \frac{7\sin\frac{5\varphi}{2}}{247104} + \frac{23\sin\frac{9\varphi}{2}}{617760}$$

1.5. **Fifth singular exponent ($j = 9$).**

$$\phi_{0,9,0} = \sin\frac{9\varphi}{2}$$
$$\phi_{0,9,1} = -\frac{1}{4}\sin\frac{7\varphi}{2}$$
$$\phi_{0,9,2} = \frac{3}{32}\sin\frac{5\varphi}{2} + \frac{5}{44}\sin\frac{9\varphi}{2}$$
$$\phi_{0,9,3} = -\frac{5}{128}\sin\frac{3\varphi}{2} - \frac{15}{176}\sin\frac{7\varphi}{2} - \frac{15}{286}\sin\frac{11\varphi}{2}$$
$$\phi_{0,9,4} = \frac{35\sin\frac{\varphi}{2}}{2048} + \frac{75\sin\frac{5\varphi}{2}}{1408} + \frac{135\sin\frac{9\varphi}{2}}{2288} + \frac{7}{286}\sin\frac{13\varphi}{2}$$
$$\phi_{0,9,5} = \frac{63\sin\frac{\varphi}{2}}{8192} - \frac{175\sin\frac{3\varphi}{2}}{5632} - \frac{225\sin\frac{7\varphi}{2}}{4576} - \frac{21}{572}\sin\frac{11\varphi}{2} - \frac{28\sin\frac{15\varphi}{2}}{2431}$$
$$\phi_{0,9,6} = \frac{1575\sin\frac{\varphi}{2}}{90112} - \frac{231\sin\frac{3\varphi}{2}}{65536} + \frac{2625\sin\frac{5\varphi}{2}}{73216} + \frac{175\sin\frac{9\varphi}{2}}{4576} + \frac{105\sin\frac{13\varphi}{2}}{4862} + \frac{252\sin\frac{17\varphi}{2}}{46189}$$

$$\phi_{2,9,0} = -\frac{1}{22}\sin\frac{9\varphi}{2}$$
$$\phi_{2,9,1} = \frac{3}{88}\sin\frac{7\varphi}{2} + \frac{23}{572}\sin\frac{11\varphi}{2}$$
$$\phi_{2,9,2} = -\frac{15}{704}\sin\frac{5\varphi}{2} - \frac{103\sin\frac{9\varphi}{2}}{2288} - \frac{31\sin\frac{13\varphi}{2}}{1144}$$
$$\phi_{2,9,3} = \frac{35\sin\frac{3\varphi}{2}}{2816} + \frac{685\sin\frac{7\varphi}{2}}{18304} + \frac{553\sin\frac{11\varphi}{2}}{13728} + \frac{639\sin\frac{15\varphi}{2}}{38896}$$
$$\phi_{2,9,4} = -\frac{315\sin\frac{\varphi}{2}}{45056} - \frac{1995\sin\frac{5\varphi}{2}}{73216} - \frac{5293\sin\frac{9\varphi}{2}}{126720} - \frac{14197\sin\frac{13\varphi}{2}}{466752} - \frac{13933\sin\frac{17\varphi}{2}}{1478048}$$

$$\phi_{4,9,0} = \frac{\sin\frac{9\varphi}{2}}{1144}$$
$$\phi_{4,9,1} = -\frac{5\sin\frac{7\varphi}{2}}{4576} - \frac{5\sin\frac{11\varphi}{2}}{3432}$$
$$\phi_{4,9,2} = \frac{35\sin\frac{5\varphi}{2}}{36608} + \frac{47\sin\frac{9\varphi}{2}}{20592} + \frac{361\sin\frac{13\varphi}{2}}{233376}$$

$$\phi_{6,9,0} = -\frac{\sin\frac{9\varphi}{2}}{102960}$$

1.6. Sixth singular exponent ($j = 11$).

$$
\begin{aligned}
\phi_{0,11,0} &= \sin\frac{11\varphi}{2}\\
\phi_{0,11,1} &= -\frac{1}{4}\sin\frac{9\varphi}{2}\\
\phi_{0,11,2} &= \frac{3}{32}\sin\frac{7\varphi}{2}+\frac{3}{26}\sin\frac{11\varphi}{2}\\
\phi_{0,11,3} &= -\frac{5}{128}\sin\frac{5\varphi}{2}-\frac{9}{104}\sin\frac{9\varphi}{2}-\frac{7}{130}\sin\frac{13\varphi}{2}\\
\phi_{0,11,4} &= \frac{35\sin\frac{3\varphi}{2}}{2048}+\frac{45}{832}\sin\frac{7\varphi}{2}+\frac{63\sin\frac{11\varphi}{2}}{1040}+\frac{28\sin\frac{15\varphi}{2}}{1105}\\
\phi_{0,11,5} &= -\frac{63\sin\frac{\varphi}{2}}{8192}-\frac{105\sin\frac{5\varphi}{2}}{3328}-\frac{21}{416}\sin\frac{9\varphi}{2}-\frac{42\sin\frac{13\varphi}{2}}{1105}-\frac{252\sin\frac{17\varphi}{2}}{20995}
\end{aligned}
$$

$$
\begin{aligned}
\phi_{2,11,0} &= -\frac{1}{26}\sin\frac{11\varphi}{2}\\
\phi_{2,11,1} &= \frac{3}{104}\sin\frac{9\varphi}{2}+\frac{9}{260}\sin\frac{13\varphi}{2}\\
\phi_{2,11,2} &= -\frac{15}{832}\sin\frac{7\varphi}{2}-\frac{121\sin\frac{11\varphi}{2}}{3120}-\frac{209\sin\frac{15\varphi}{2}}{8840}\\
\phi_{2,11,3} &= \frac{35\sin\frac{5\varphi}{2}}{3328}+\frac{161\sin\frac{9\varphi}{2}}{4992}+\frac{3733\sin\frac{13\varphi}{2}}{106080}+\frac{4867\sin\frac{17\varphi}{2}}{335920}
\end{aligned}
$$

$$
\begin{aligned}
\phi_{4,11,0} &= \frac{\sin\frac{11\varphi}{2}}{1560}\\
\phi_{4,11,1} &= -\frac{\sin\frac{9\varphi}{2}}{1248}-\frac{29\sin\frac{13\varphi}{2}}{26520}
\end{aligned}
$$

1.7. Seventh singular exponent ($j = 13$).

$$
\begin{aligned}
\phi_{0,13,0} &= \sin\frac{13\varphi}{2}\\
\phi_{0,13,1} &= -\frac{1}{4}\sin\frac{11\varphi}{2}\\
\phi_{0,13,2} &= \frac{3}{32}\sin\frac{9\varphi}{2}+\frac{7}{60}\sin\frac{13\varphi}{2}\\
\phi_{0,13,3} &= -\frac{5}{128}\sin\frac{7\varphi}{2}-\frac{7}{80}\sin\frac{11\varphi}{2}-\frac{14}{255}\sin\frac{15\varphi}{2}\\
\phi_{0,13,4} &= \frac{35\sin\frac{5\varphi}{2}}{2048}+\frac{7}{128}\sin\frac{9\varphi}{2}+\frac{21}{340}\sin\frac{13\varphi}{2}+\frac{42\sin\frac{17\varphi}{2}}{1615}
\end{aligned}
$$

$$
\begin{aligned}
\phi_{2,13,0} &= -\frac{1}{30}\sin\frac{13\varphi}{2}\\
\phi_{2,13,1} &= \frac{1}{40}\sin\frac{11\varphi}{2}+\frac{31\sin\frac{15\varphi}{2}}{1020}\\
\phi_{2,13,2} &= -\frac{1}{64}\sin\frac{9\varphi}{2}-\frac{139\sin\frac{13\varphi}{2}}{4080}-\frac{271\sin\frac{17\varphi}{2}}{12920}
\end{aligned}
$$

$$
\phi_{4,13,0} = \frac{\sin\frac{13\varphi}{2}}{2040}
$$

1.8. **Higher order singular exponent ($j = 15, 17, 19, 21$).**

$$\begin{aligned}
\phi_{0,15,0} &= \sin\frac{15\varphi}{2} \\
\phi_{0,15,1} &= -\frac{1}{4}\sin\frac{13\varphi}{2} \\
\phi_{0,15,2} &= \frac{3}{32}\sin\frac{11\varphi}{2} + \frac{2}{17}\sin\frac{15\varphi}{2} \\
\phi_{0,15,3} &= -\frac{5}{128}\sin\frac{9\varphi}{2} - \frac{3}{34}\sin\frac{13\varphi}{2} - \frac{18}{323}\sin\frac{17\varphi}{2}
\end{aligned}$$

$$\begin{aligned}
\phi_{2,15,0} &= -\frac{1}{34}\sin\frac{15\varphi}{2} \\
\phi_{2,15,1} &= \frac{3}{136}\sin\frac{13\varphi}{2} + \frac{35\sin\frac{17\varphi}{2}}{1292}
\end{aligned}$$

$$\begin{aligned}
\phi_{0,17,0} &= \sin\frac{17\varphi}{2} \\
\phi_{0,17,1} &= -\frac{1}{4}\sin\frac{15\varphi}{2} \\
\phi_{0,17,2} &= \frac{3}{32}\sin\frac{13\varphi}{2} + \frac{9}{76}\sin\frac{17\varphi}{2}
\end{aligned}$$

$$\phi_{2,17,0} = \frac{1}{38}\sin\frac{17\varphi}{2}$$

$$\begin{aligned}
\phi_{0,19,0} &= \sin\frac{19\varphi}{2} \\
\phi_{0,19,1} &= -\frac{1}{4}\sin\frac{17\varphi}{2}
\end{aligned}$$

$$\phi_{0,21,0} = \sin\frac{21\varphi}{2}$$

2. Dual singular functions and Dual-shadows for the crack $-\pi \le \varphi \le \pi$

2.1. First singular exponent ($j = 1$).

$$\begin{aligned}
\psi_{0,1,0} &= \sin\frac{\varphi}{2}\\
\psi_{0,1,1} &= -\frac{1}{4}\sin\frac{3\varphi}{2}\\
\psi_{0,1,2} &= \frac{3}{32}\sin\frac{5\varphi}{2}\\
\psi_{0,1,3} &= -\frac{5}{128}\sin\frac{7\varphi}{2}\\
\psi_{0,1,4} &= \frac{35}{2048}\sin\frac{9\varphi}{2}
\end{aligned}$$

$$\begin{aligned}
\psi_{2,1,0} &= -\frac{1}{2}\sin\frac{\varphi}{2}\\
\psi_{2,1,1} &= -\frac{1}{4}\sin\frac{\varphi}{2}+\frac{3}{8}\sin\frac{3\varphi}{2}\\
\psi_{2,1,2} &= -\frac{13}{48}\sin\frac{\varphi}{2}+\frac{1}{8}\sin\frac{3\varphi}{2}-\frac{15}{64}\sin\frac{5\varphi}{2}\\
\psi_{2,1,3} &= -\frac{17}{96}\sin\frac{\varphi}{2}+\frac{85}{384}\sin\frac{3\varphi}{2}-\frac{1}{16}\sin\frac{5\varphi}{2}+\frac{35}{256}\sin\frac{7\varphi}{2}\\
\psi_{2,1,4} &= -\frac{2059\sin\frac{\varphi}{2}}{11520}+\frac{7}{64}\sin\frac{3\varphi}{2}-\frac{245\sin\frac{5\varphi}{2}}{1536}+\frac{1}{32}\sin\frac{7\varphi}{2}-\frac{315\sin\frac{9\varphi}{2}}{4096}
\end{aligned}$$

$$\begin{aligned}
\psi_{4,1,0} &= \frac{1}{24}\sin\frac{\varphi}{2}\\
\psi_{4,1,1} &= \frac{1}{24}\sin\frac{\varphi}{2}-\frac{5}{96}\sin\frac{3\varphi}{2}\\
\psi_{4,1,2} &= \frac{19}{288}\sin\frac{\varphi}{2}-\frac{1}{32}\sin\frac{3\varphi}{2}+\frac{35}{768}\sin\frac{5\varphi}{2}\\
\psi_{4,1,3} &= \frac{23}{384}\sin\frac{\varphi}{2}-\frac{161\sin\frac{3\varphi}{2}}{2304}+\frac{1}{48}\sin\frac{5\varphi}{2}-\frac{35\sin\frac{7\varphi}{2}}{1024}\\
\psi_{4,1,4} &= \frac{3431\sin\frac{\varphi}{2}}{46080}-\frac{3}{64}\sin\frac{3\varphi}{2}+\frac{63\sin\frac{5\varphi}{2}}{1024}-\frac{5}{384}\sin\frac{7\varphi}{2}+\frac{385\sin\frac{9\varphi}{2}}{16384}
\end{aligned}$$

2.2. **Second singular exponent ($j = 3$).**

$$\begin{aligned}
\psi_{0,3,0} &= \sin\left(\frac{3\varphi}{2}\right) \\
\psi_{0,3,1} &= -\frac{1}{4}\sin\left(\frac{5\varphi}{2}\right) \\
\psi_{0,3,2} &= \frac{1}{4}\sin\left(\frac{3\varphi}{2}\right) + \frac{3}{32}\sin\left(\frac{7\varphi}{2}\right) \\
\psi_{0,3,3} &= -\frac{3}{16}\sin\left(\frac{5\varphi}{2}\right) - \frac{5}{128}\sin\left(\frac{9\varphi}{2}\right) \\
\psi_{0,3,4} &= \frac{15}{128}\sin\left(\frac{7\varphi}{2}\right) + \frac{35}{2048}\sin\left(\frac{11\varphi}{2}\right)
\end{aligned}$$

$$\begin{aligned}
\psi_{2,3,0} &= \frac{1}{2}\sin\left(\frac{3\varphi}{2}\right) \\
\psi_{2,3,1} &= \frac{1}{4}\sin\left(\frac{\varphi}{2}\right) - \frac{3}{8}\sin\left(\frac{5\varphi}{2}\right) \\
\psi_{2,3,2} &= \frac{1}{8}\sin\left(\frac{\varphi}{2}\right) - \frac{5}{16}\sin\left(\frac{3\varphi}{2}\right) + \frac{15}{64}\sin\left(\frac{7\varphi}{2}\right) \\
\psi_{2,3,3} &= \frac{19}{96}\sin\left(\frac{\varphi}{2}\right) - \frac{1}{16}\sin\left(\frac{3\varphi}{2}\right) + \frac{35}{128}\sin\left(\frac{5\varphi}{2}\right) - \frac{35}{256}\sin\left(\frac{9\varphi}{2}\right) \\
\psi_{2,3,4} &= \frac{23}{192}\sin\left(\frac{\varphi}{2}\right) - \frac{491\sin\left(\frac{3\varphi}{2}\right)}{2304} + \frac{1}{32}\sin\left(\frac{5\varphi}{2}\right) - \frac{105}{512}\sin\left(\frac{7\varphi}{2}\right) + \frac{315\sin\left(\frac{11\varphi}{2}\right)}{4096}
\end{aligned}$$

$$\begin{aligned}
\psi_{4,3,0} &= -\frac{1}{8}\sin\left(\frac{3\varphi}{2}\right) \\
\psi_{4,3,1} &= \frac{1}{24}\sin\left(\frac{\varphi}{2}\right) + \frac{5}{32}\sin\left(\frac{5\varphi}{2}\right) \\
\psi_{4,3,2} &= \frac{1}{96}\sin\left(\frac{\varphi}{2}\right) - \frac{5}{72}\sin\left(\frac{3\varphi}{2}\right) - \frac{35}{256}\sin\left(\frac{7\varphi}{2}\right) \\
\psi_{4,3,3} &= \frac{3}{128}\sin\left(\frac{\varphi}{2}\right) + \frac{175\sin\left(\frac{5\varphi}{2}\right)}{2304} + \frac{105\sin\left(\frac{9\varphi}{2}\right)}{1024} \\
\psi_{4,3,4} &= \frac{1}{384}\sin\left(\frac{\varphi}{2}\right) - \frac{1477\sin\left(\frac{3\varphi}{2}\right)}{46080} - \frac{1}{384}\sin\left(\frac{5\varphi}{2}\right) - \frac{35}{512}\sin\left(\frac{7\varphi}{2}\right) - \frac{1155\sin\left(\frac{11\varphi}{2}\right)}{16384}
\end{aligned}$$

2.3. Third singular exponent ($j = 5$).

$$
\begin{aligned}
\psi_{0,5,0} &= \sin\left(\frac{5\varphi}{2}\right)\\
\psi_{0,5,1} &= -\frac{1}{4}\sin\left(\frac{7\varphi}{2}\right)\\
\psi_{0,5,2} &= \frac{1}{6}\sin\left(\frac{5\varphi}{2}\right)+\frac{3}{32}\sin\left(\frac{9\varphi}{2}\right)\\
\psi_{0,5,3} &= -\frac{1}{6}\sin\left(\frac{3\varphi}{2}\right)-\frac{1}{8}\sin\left(\frac{7\varphi}{2}\right)-\frac{5}{128}\sin\left(\frac{11\varphi}{2}\right)\\
\psi_{0,5,4} &= \frac{3}{16}\sin\left(\frac{5\varphi}{2}\right)+\frac{5}{64}\sin\left(\frac{9\varphi}{2}\right)+\frac{35}{2048}\sin\left(\frac{13\varphi}{2}\right)
\end{aligned}
$$

$$
\begin{aligned}
\psi_{2,5,0} &= \frac{1}{6}\sin\left(\frac{5\varphi}{2}\right)\\
\psi_{2,5,1} &= -\frac{5}{12}\sin\left(\frac{3\varphi}{2}\right)-\frac{1}{8}\sin\left(\frac{7\varphi}{2}\right)\\
\psi_{2,5,2} &= -\frac{1}{8}\sin\left(\frac{\varphi}{2}\right)+\frac{23}{48}\sin\left(\frac{5\varphi}{2}\right)+\frac{5}{64}\sin\left(\frac{9\varphi}{2}\right)\\
\psi_{2,5,3} &= -\frac{1}{16}\sin\left(\frac{\varphi}{2}\right)+\frac{7}{32}\sin\left(\frac{3\varphi}{2}\right)-\frac{155}{384}\sin\left(\frac{7\varphi}{2}\right)-\frac{35}{768}\sin\left(\frac{11\varphi}{2}\right)\\
\psi_{2,5,4} &= -\frac{25}{192}\sin\left(\frac{\varphi}{2}\right)+\frac{1}{32}\sin\left(\frac{3\varphi}{2}\right)-\frac{559\sin\left(\frac{5\varphi}{2}\right)}{2304}+\frac{455\sin\left(\frac{9\varphi}{2}\right)}{1536}+\frac{105\sin\left(\frac{13\varphi}{2}\right)}{4096}
\end{aligned}
$$

$$
\begin{aligned}
\psi_{4,5,0} &= \frac{1}{24}\sin\left(\frac{5\varphi}{2}\right)\\
\psi_{4,5,1} &= \frac{1}{8}\sin\left(\frac{3\varphi}{2}\right)-\frac{5}{96}\sin\left(\frac{7\varphi}{2}\right)\\
\psi_{4,5,2} &= -\frac{5}{96}\sin\left(\frac{\varphi}{2}\right)-\frac{55}{288}\sin\left(\frac{5\varphi}{2}\right)+\frac{35}{768}\sin\left(\frac{9\varphi}{2}\right)\\
\psi_{4,5,3} &= -\frac{1}{48}\sin\left(\frac{\varphi}{2}\right)+\frac{115\sin\left(\frac{3\varphi}{2}\right)}{1152}+\frac{455\sin\left(\frac{7\varphi}{2}\right)}{2304}-\frac{35\sin\left(\frac{11\varphi}{2}\right)}{1024}\\
\psi_{4,5,4} &= -\frac{7}{144}\sin\left(\frac{\varphi}{2}\right)+\frac{1}{128}\sin\left(\frac{3\varphi}{2}\right)-\frac{1141\sin\left(\frac{5\varphi}{2}\right)}{9216}-\frac{175\sin\left(\frac{9\varphi}{2}\right)}{1024}+\frac{385\sin\left(\frac{13\varphi}{2}\right)}{16384}
\end{aligned}
$$

3. Primal functions and shadows for 90° V-notch $-\pi \le \varphi \le \pi/2$

3.1. First singular exponent ($j = 1$).

$$
\begin{aligned}
\phi_{0,1,0} &= \sin\frac{2\varphi}{3} + \frac{1}{\sqrt{3}}\cos\frac{2\varphi}{3}\\
\phi_{0,1,1} &= \frac{1}{4}\sin\frac{\varphi}{3} - \frac{1}{4\sqrt{3}}\cos\frac{\varphi}{3} + \frac{1}{20}\sin\frac{5\varphi}{3} + \frac{1}{20\sqrt{3}}\cos\frac{5\varphi}{3}\\
\phi_{0,1,2} &= \frac{\sqrt{3}}{40}\cos\frac{2\varphi}{3} + \frac{3}{40}\sin\frac{2\varphi}{3} + \frac{\sqrt{3}}{32}\cos\frac{4\varphi}{3} - \frac{3}{32}\sin\frac{4\varphi}{3}\\
\phi_{0,1,3} &= -\frac{13\sqrt{3}}{640}\cos\frac{\varphi}{3} + \frac{39}{640}\sin\frac{\varphi}{3} - \frac{13\sqrt{3}}{1280}\cos\frac{5\varphi}{3} - \frac{39}{1280}\sin\frac{5\varphi}{3} - \frac{5}{128\sqrt{3}}\cos\frac{7\varphi}{3}\\
&+ \frac{5}{128}\sin\frac{7\varphi}{3} + \frac{7}{1280}\sin\frac{11\varphi}{3} + \frac{7}{1280\sqrt{3}}\cos\frac{11\varphi}{3}\\
\phi_{0,1,4} &= \frac{247\sqrt{3}}{20480}\cos\frac{2\varphi}{3} + \frac{741}{20480}\sin\frac{2\varphi}{3} + \frac{5}{128\sqrt{3}}\cos\frac{4\varphi}{3} - \frac{5}{128}\sin\frac{4\varphi}{3} + \frac{83}{7040\sqrt{3}}\cos\frac{8\varphi}{3}\\
&+ \frac{83}{7040}\sin\frac{8\varphi}{3} + \frac{35}{2048\sqrt{3}}\cos\frac{10\varphi}{3} - \frac{35}{2048}\sin\frac{10\varphi}{3}\\
\phi_{0,1,5} &= -\frac{169\sqrt{3}}{16384}\cos\frac{\varphi}{3} + \frac{507}{16384}\sin\frac{\varphi}{3} - \frac{6021\sqrt{3}}{901120}\cos\frac{5\varphi}{3} - \frac{18063}{901120}\sin\frac{5\varphi}{3} - \frac{63\sqrt{3}}{8192}\cos\frac{7\varphi}{3}\\
&+ \frac{189}{8192}\sin\frac{7\varphi}{3} - \frac{415}{78848\sqrt{3}}\cos\frac{11\varphi}{3} - \frac{415}{78848}\sin\frac{11\varphi}{3} - \frac{21\sqrt{3}}{8192}\cos\frac{13\varphi}{3} + \frac{63}{8192}\sin\frac{13\varphi}{3}\\
&+ \frac{109}{157696\sqrt{3}}\cos\frac{17\varphi}{3} + \frac{109}{157696}\sin\frac{17\varphi}{3}\\
\phi_{0,1,6} &= \frac{2609\sqrt{3}}{360448}\cos\frac{2\varphi}{3} + \frac{1001\sqrt{3}}{131072}\cos\frac{4\varphi}{3} + \frac{36745\sqrt{3}}{10092544}\cos\frac{8\varphi}{3} + \frac{357\sqrt{3}}{81920}\cos\frac{10\varphi}{3}\\
&+ \frac{1137\sqrt{3}}{1531904}\cos\frac{14\varphi}{3} + \frac{77\sqrt{3}}{65536}\cos\frac{16\varphi}{3} + \frac{7827}{360448}\sin\frac{2\varphi}{3} - \frac{3003}{131072}\sin\frac{4\varphi}{3}\\
&+ \frac{110235}{10092544}\sin\frac{8\varphi}{3} - \frac{1071}{81920}\sin\frac{10\varphi}{3} + \frac{3411}{1531904}\sin\frac{14\varphi}{3} - \frac{231}{65536}\sin\frac{16\varphi}{3}\\
\phi_{0,1,7} &= -\frac{18641\sqrt{3}}{2883584}\cos\frac{\varphi}{3} - \frac{575075}{40370176\sqrt{3}}\cos\frac{5\varphi}{3} - \frac{1365\sqrt{3}}{262144}\cos\frac{7\varphi}{3} - \frac{1330111\sqrt{3}}{686292992}\cos\frac{11\varphi}{3}\\
&- \frac{3157\sqrt{3}}{1310720}\cos\frac{13\varphi}{3} - \frac{42069\sqrt{3}}{122552320}\cos\frac{17\varphi}{3} - \frac{143\sqrt{3}}{262144}\cos\frac{19\varphi}{3} + \frac{55923}{2883584}\sin\frac{\varphi}{3}\\
&- \frac{575075}{40370176}\sin\frac{5\varphi}{3} + \frac{4095}{262144}\sin\frac{7\varphi}{3} - \frac{3990333}{686292992}\sin\frac{11\varphi}{3} + \frac{9471}{1310720}\sin\frac{13\varphi}{3}\\
&- \frac{126207}{122552320}\sin\frac{17\varphi}{3} + \frac{429}{262144}\sin\frac{19\varphi}{3} + \frac{87217}{857866240}\sin\frac{23\varphi}{3} + \frac{87217}{857866240\sqrt{3}}\cos\frac{23\varphi}{3}\\
\phi_{0,1,8} &= \frac{1373975}{92274688\sqrt{3}}\cos\frac{2\varphi}{3} + \frac{74361\sqrt{3}}{14417920}\cos\frac{4\varphi}{3} + \frac{437105}{49020928\sqrt{3}}\cos\frac{8\varphi}{3} + \frac{141141\sqrt{3}}{41943040}\cos\frac{10\varphi}{3}\\
&+ \frac{56281699\sqrt{3}}{54903439360}\cos\frac{14\varphi}{3} + \frac{429\sqrt{3}}{327680}\cos\frac{16\varphi}{3} + \frac{8995579}{19730923520\sqrt{3}}\cos\frac{20\varphi}{3} + \frac{2145\sqrt{3}}{8388608}\cos\frac{22\varphi}{3}\\
&+ \frac{1373975}{92274688}\sin\frac{2\varphi}{3} - \frac{223083}{14417920}\sin\frac{4\varphi}{3} + \frac{437105}{49020928}\sin\frac{8\varphi}{3} - \frac{423423}{41943040}\sin\frac{10\varphi}{3}\\
&+ \frac{168845097}{54903439360}\sin\frac{14\varphi}{3} - \frac{1287}{327680}\sin\frac{16\varphi}{3} + \frac{8995579}{19730923520}\sin\frac{20\varphi}{3} - \frac{6435}{8388608}\sin\frac{22\varphi}{3}\\
\phi_{0,1,9} &= -\frac{1674225\sqrt{3}}{369098752}\cos\frac{\varphi}{3} - \frac{22357035\sqrt{3}}{6274678784}\cos\frac{5\varphi}{3} - \frac{640381\sqrt{3}}{167772160}\cos\frac{7\varphi}{3} - \frac{79139337\sqrt{3}}{43922751488}\cos\frac{11\varphi}{3}\\
&- \frac{351351\sqrt{3}}{167772160}\cos\frac{13\varphi}{3} - \frac{2140689\sqrt{3}}{3992977408}\cos\frac{17\varphi}{3} - \frac{23595\sqrt{3}}{33554432}\cos\frac{19\varphi}{3}\\
&- \frac{62969053}{293145149440\sqrt{3}}\cos\frac{23\varphi}{3} - \frac{12155}{33554432\sqrt{3}}\cos\frac{25\varphi}{3} + \frac{5022675}{369098752}\sin\frac{\varphi}{3} - \frac{67071105}{6274678784}\sin\frac{5\varphi}{3}\\
&+ \frac{1921143}{167772160}\sin\frac{7\varphi}{3} - \frac{237418011}{43922751488}\sin\frac{11\varphi}{3} + \frac{1054053}{167772160}\sin\frac{13\varphi}{3} - \frac{6422067}{3992977408}\sin\frac{17\varphi}{3}\\
&+ \frac{70785}{33554432}\sin\frac{19\varphi}{3} - \frac{62969053}{293145149440}\sin\frac{23\varphi}{3} + \frac{12155}{33554432}\sin\frac{25\varphi}{3} + \frac{187981}{11152261120}\sin\frac{29\varphi}{3} + \frac{187981}{11152261120\sqrt{3}}\cos\frac{29\varphi}{3}\\
\phi_{0,1,10} &= \frac{2710107\sqrt{3}}{738197504}\cos\frac{2\varphi}{3} + \frac{507745\sqrt{3}}{134217728}\cos\frac{4\varphi}{3} + \frac{61145349\sqrt{3}}{25098715136}\cos\frac{8\varphi}{3} + \frac{898079\sqrt{3}}{335544320}\cos\frac{10\varphi}{3}\\
&+ \frac{2152824343\sqrt{3}}{2020446568448}\cos\frac{14\varphi}{3} + \frac{340197\sqrt{3}}{268435456}\cos\frac{16\varphi}{3} + \frac{20970810791\sqrt{3}}{75045158256640}\cos\frac{20\varphi}{3} + \frac{75361}{67108864\sqrt{3}}\cos\frac{22\varphi}{3}\\
&+ \frac{1787693753}{18310297026560\sqrt{3}}\cos\frac{26\varphi}{3} + \frac{46189}{268435456\sqrt{3}}\cos\frac{28\varphi}{3} + \frac{8130321}{738197504}\sin\frac{2\varphi}{3} - \frac{1523235}{134217728}\sin\frac{4\varphi}{3} + \frac{183436047}{25098715136}\sin\frac{8\varphi}{3}\\
&- \frac{2694237}{335544320}\sin\frac{10\varphi}{3} + \frac{6458473029}{2020446568448}\sin\frac{14\varphi}{3} - \frac{1020591}{268435456}\sin\frac{16\varphi}{3} + \frac{62912432373}{75045158256640}\sin\frac{20\varphi}{3} - \frac{75361}{67108864}\sin\frac{22\varphi}{3}\\
&+ \frac{1787693753}{18310297026560}\sin\frac{26\varphi}{3} - \frac{46189}{268435456}\sin\frac{28\varphi}{3}
\end{aligned}
$$

$$
\begin{aligned}
\phi_{2,1,0} &= \frac{1}{20}\left(-\sqrt{3}\cos\frac{2\varphi}{3} - 3\sin\frac{2\varphi}{3}\right)\\
\phi_{2,1,1} &= \frac{3}{80}\sqrt{3}\cos\frac{\varphi}{3} + \frac{11}{320}\sqrt{3}\cos\frac{5\varphi}{3} + \frac{67\sqrt{3}\cos\frac{11\varphi}{3}}{3520} - \frac{9}{80}\sin\frac{\varphi}{3} + \frac{33}{320}\sin\frac{5\varphi}{3} + \frac{201\sin\frac{11\varphi}{3}}{3520}\\
\phi_{2,1,2} &= -\frac{99\sqrt{3}\cos\frac{2\varphi}{3}}{2560} - \frac{3}{128}\sqrt{3}\cos\frac{4\varphi}{3} - \frac{87\sqrt{3}\cos\frac{8\varphi}{3}}{3520} - \frac{297\sin\frac{2\varphi}{3}}{2560} + \frac{9}{128}\sin\frac{4\varphi}{3} - \frac{261\sin\frac{8\varphi}{3}}{3520}\\
\phi_{2,1,3} &= \frac{33\sqrt{3}\cos\frac{\varphi}{3}}{1024} + \frac{177\sqrt{3}\cos\frac{5\varphi}{3}}{5632} + \frac{7}{512}\sqrt{3}\cos\frac{7\varphi}{3} + \frac{10207\sqrt{3}\cos\frac{11\varphi}{3}}{788480} + \frac{38427\sqrt{3}\cos\frac{17\varphi}{3}}{13404160}\\
&- \frac{99\sin\frac{\varphi}{3}}{1024} + \frac{531\sin\frac{5\varphi}{3}}{5632} - \frac{21}{512}\sin\frac{7\varphi}{3} + \frac{30621\sin\frac{11\varphi}{3}}{788480} + \frac{115281\sin\frac{17\varphi}{3}}{13404160}\\
\phi_{2,1,4} &= -\frac{28387\sqrt{3}\cos\frac{2\varphi}{3}}{901120} - \frac{385\sqrt{3}\cos\frac{4\varphi}{3}}{16384} - \frac{136329\sqrt{3}\cos\frac{8\varphi}{3}}{6307840} - \frac{63\sqrt{3}\cos\frac{10\varphi}{3}}{8192} - \frac{709\sqrt{3}\cos\frac{14\varphi}{3}}{95744}\\
&- \frac{85161\sin\frac{2\varphi}{3}}{901120} + \frac{1155\sin\frac{4\varphi}{3}}{16384} - \frac{408987\sin\frac{8\varphi}{3}}{6307840} + \frac{189\sin\frac{10\varphi}{3}}{8192} - \frac{2127\sin\frac{14\varphi}{3}}{95744}\\
\phi_{2,1,5} &= \frac{195993\sqrt{3}\cos\frac{\varphi}{3}}{7208960} + \frac{163131\sqrt{3}\cos\frac{5\varphi}{3}}{6307840} + \frac{2079\sqrt{3}\cos\frac{7\varphi}{3}}{131072} + \frac{1200027\sqrt{3}\cos\frac{11\varphi}{3}}{85786624} + \frac{693\sqrt{3}\cos\frac{13\varphi}{3}}{163840} + \frac{4022021\sqrt{3}\cos\frac{17\varphi}{3}}{1072332800}\\
&+ \frac{11088127\sqrt{3}\cos\frac{23\varphi}{3}}{24663654400} - \frac{587979\sin\frac{\varphi}{3}}{7208960} + \frac{489393\sin\frac{5\varphi}{3}}{6307840} - \frac{6237\sin\frac{7\varphi}{3}}{131072} + \frac{3600081\sin\frac{11\varphi}{3}}{85786624} - \frac{2079\sin\frac{13\varphi}{3}}{163840} + \frac{12066063\sin\frac{17\varphi}{3}}{1072332800}\\
&+ \frac{33264381\sin\frac{23\varphi}{3}}{24663654400}\\
\phi_{2,1,6} &= -\frac{2611551\sqrt{3}\cos\frac{2\varphi}{3}}{100925440} - \frac{613557\sqrt{3}\cos\frac{4\varphi}{3}}{28835840} - \frac{9448749\sqrt{3}\cos\frac{8\varphi}{3}}{490209280} - \frac{53361\sqrt{3}\cos\frac{10\varphi}{3}}{5242880} - \frac{293357739\sqrt{3}\cos\frac{14\varphi}{3}}{34314649600} - \frac{3003\sqrt{3}\cos\frac{16\varphi}{3}}{1310720}\\
&- \frac{1164771\sqrt{3}\cos\frac{20\varphi}{3}}{580321280} - \frac{7834653\sin\frac{2\varphi}{3}}{100925440} + \frac{1840671\sin\frac{4\varphi}{3}}{28835840} - \frac{28346247\sin\frac{8\varphi}{3}}{490209280} + \frac{160083\sin\frac{10\varphi}{3}}{5242880} - \frac{880073217\sin\frac{14\varphi}{3}}{34314649600} + \frac{9009\sin\frac{16\varphi}{3}}{1310720}\\
&- \frac{3494313\sin\frac{20\varphi}{3}}{580321280}\\
\phi_{2,1,7} &= \frac{37211697\sqrt{3}\cos\frac{\varphi}{3}}{1614807040} + \frac{120244237\sqrt{3}\cos\frac{5\varphi}{3}}{5490343936} + \frac{81543\sqrt{3}\cos\frac{7\varphi}{3}}{5242880} + \frac{1843285679\sqrt{3}\cos\frac{11\varphi}{3}}{137258598400} + \frac{33033\sqrt{3}\cos\frac{13\varphi}{3}}{5242880}\\
&+ \frac{16043532597\sqrt{3}\cos\frac{17\varphi}{3}}{3156947763200} + \frac{1287\sqrt{3}\cos\frac{19\varphi}{3}}{1048576} + \frac{2067082063\sqrt{3}\cos\frac{23\varphi}{3}}{2052016046080} + \frac{22910746871\sqrt{3}\cos\frac{29\varphi}{3}}{297542326681600} - \frac{111635091\sin\frac{\varphi}{3}}{1614807040}\\
&+ \frac{360732711\sin\frac{5\varphi}{3}}{5490343936} - \frac{244629\sin\frac{7\varphi}{3}}{5242880} + \frac{5529857037\sin\frac{11\varphi}{3}}{137258598400} - \frac{99099\sin\frac{13\varphi}{3}}{5242880} + \frac{48130597791\sin\frac{17\varphi}{3}}{3156947763200}\\
&- \frac{3861\sin\frac{19\varphi}{3}}{1048576} + \frac{6201246189\sin\frac{23\varphi}{3}}{2052016046080} + \frac{68732240613\sin\frac{29\varphi}{3}}{297542326681600}\\
\phi_{2,1,8} &= -\frac{23999498037\sqrt{3}\cos\frac{2\varphi}{3}}{1098068787200} - \frac{22191111\sqrt{3}\cos\frac{4\varphi}{3}}{1174405120} - \frac{18777125183\sqrt{3}\cos\frac{8\varphi}{3}}{1098068787200} - \frac{1813851\sqrt{3}\cos\frac{10\varphi}{3}}{167772160} - \frac{225549913639\sqrt{3}\cos\frac{14\varphi}{3}}{25255582105600}\\
&- \frac{127413\sqrt{3}\cos\frac{16\varphi}{3}}{33554432} - \frac{964379391513\sqrt{3}\cos\frac{20\varphi}{3}}{328322567372800} - \frac{21879\sqrt{3}\cos\frac{22\varphi}{3}}{33554432} - \frac{260981373\sqrt{3}\cos\frac{26\varphi}{3}}{497562419200} - \frac{71998494111\sin\frac{2\varphi}{3}}{1098068787200}\\
&+ \frac{66573333\sin\frac{4\varphi}{3}}{1174405120} - \frac{56331375549\sin\frac{8\varphi}{3}}{1098068787200} + \frac{5441553\sin\frac{10\varphi}{3}}{167772160} - \frac{676649740917\sin\frac{14\varphi}{3}}{25255582105600} + \frac{382239\sin\frac{16\varphi}{3}}{33554432}\\
&- \frac{2893138174539\sin\frac{20\varphi}{3}}{328322567372800} + \frac{65637\sin\frac{22\varphi}{3}}{33554432} - \frac{782944119\sin\frac{26\varphi}{3}}{497562419200}
\end{aligned}
$$

$$\phi_{4,1,0} = \frac{3\sqrt{3}\cos\frac{2\varphi}{3}}{1280} + \frac{9\sin\frac{2\varphi}{3}}{1280}$$

$$\phi_{4,1,1} = -\frac{3\sqrt{3}\cos\frac{\varphi}{3}}{1024} - \frac{171\sqrt{3}\cos\frac{5\varphi}{3}}{56320} - \frac{201\sqrt{3}\cos\frac{11\varphi}{3}}{197120} + \frac{1359\sqrt{3}\cos\frac{17\varphi}{3}}{3351040} + \frac{9\sin\frac{\varphi}{3}}{1024} - \frac{513\sin\frac{5\varphi}{3}}{56320} - \frac{603\sin\frac{11\varphi}{3}}{197120} + \frac{4077\sin\frac{17\varphi}{3}}{3351040}$$

$$\phi_{4,1,2} = \frac{27\sqrt{3}\cos\frac{2\varphi}{3}}{5632} + \frac{21\sqrt{3}\cos\frac{4\varphi}{3}}{8192} + \frac{10881\sqrt{3}\cos\frac{8\varphi}{3}}{3153920} + \frac{75\sqrt{3}\cos\frac{14\varphi}{3}}{95744} + \frac{81\sin\frac{2\varphi}{3}}{5632} - \frac{63\sin\frac{4\varphi}{3}}{8192} + \frac{32643\sin\frac{8\varphi}{3}}{3153920} + \frac{225\sin\frac{14\varphi}{3}}{95744}$$

$$\begin{aligned}\phi_{4,1,3} = & -\frac{1827\sqrt{3}\cos\frac{\varphi}{3}}{360448} - \frac{14127\sqrt{3}\cos\frac{5\varphi}{3}}{2523136} - \frac{63\sqrt{3}\cos\frac{7\varphi}{3}}{32768} - \frac{127053\sqrt{3}\cos\frac{11\varphi}{3}}{42893312} - \frac{695133\sqrt{3}\cos\frac{17\varphi}{3}}{1072332800} + \frac{2098839\sqrt{3}\cos\frac{23\varphi}{3}}{24663654400} + \frac{5481\sin\frac{\varphi}{3}}{360448} \\ & - \frac{42381\sin\frac{5\varphi}{3}}{2523136} + \frac{189\sin\frac{7\varphi}{3}}{32768} - \frac{381159\sin\frac{11\varphi}{3}}{42893312} - \frac{2085399\sin\frac{17\varphi}{3}}{1072332800} + \frac{6296517\sin\frac{23\varphi}{3}}{24663654400}\end{aligned}$$

$$\begin{aligned}\phi_{4,1,4} = & \frac{47757\sqrt{3}\cos\frac{2\varphi}{3}}{7208960} + \frac{3213\sqrt{3}\cos\frac{4\varphi}{3}}{720896} + \frac{5907\sqrt{3}\cos\frac{8\varphi}{3}}{1114112} + \frac{693\sqrt{3}\cos\frac{10\varphi}{3}}{524288} + \frac{39272811\sqrt{3}\cos\frac{14\varphi}{3}}{17157324800} + \frac{100143\sqrt{3}\cos\frac{20\varphi}{3}}{246636544} + \frac{143271\sin\frac{2\varphi}{3}}{7208960} \\ & - \frac{9639\sin\frac{4\varphi}{3}}{720896} + \frac{17721\sin\frac{8\varphi}{3}}{1114112} - \frac{2079\sin\frac{10\varphi}{3}}{524288} + \frac{117818433\sin\frac{14\varphi}{3}}{17157324800} + \frac{300429\sin\frac{20\varphi}{3}}{246636544}\end{aligned}$$

$$\begin{aligned}\phi_{4,1,5} = & -\frac{34371\sqrt{3}\cos\frac{\varphi}{3}}{5242880} - \frac{6843897\sqrt{3}\cos\frac{5\varphi}{3}}{980418560} - \frac{7371\sqrt{3}\cos\frac{7\varphi}{3}}{2097152} - \frac{305405949\sqrt{3}\cos\frac{11\varphi}{3}}{68629299200} - \frac{9009\sqrt{3}\cos\frac{13\varphi}{3}}{10485760} - \frac{362650311\sqrt{3}\cos\frac{17\varphi}{3}}{225496268800} \\ & - \frac{7307871\sqrt{3}\cos\frac{23\varphi}{3}}{28187033600} + \frac{4039929\sqrt{3}\cos\frac{29\varphi}{3}}{260089446400} + \frac{103113\sin\frac{\varphi}{3}}{5242880} - \frac{20531691\sin\frac{5\varphi}{3}}{980418560} + \frac{22113\sin\frac{7\varphi}{3}}{2097152} - \frac{916217847\sin\frac{11\varphi}{3}}{68629299200} + \frac{27027\sin\frac{13\varphi}{3}}{10485760} \\ & - \frac{1087950933\sin\frac{17\varphi}{3}}{225496268800} - \frac{21923613\sin\frac{23\varphi}{3}}{28187033600} + \frac{12119787\sin\frac{29\varphi}{3}}{260089446400}\end{aligned}$$

$$\begin{aligned}\phi_{4,1,6} = & \frac{10523907\sqrt{3}\cos\frac{2\varphi}{3}}{1372585984} + \frac{484713\sqrt{3}\cos\frac{4\varphi}{3}}{83886080} + \frac{101102037\sqrt{3}\cos\frac{8\varphi}{3}}{15686696960} + \frac{27027\sqrt{3}\cos\frac{10\varphi}{3}}{10485760} + \frac{491571873\sqrt{3}\cos\frac{14\varphi}{3}}{143497625600} + \frac{9009\sqrt{3}\cos\frac{16\varphi}{3}}{16777216} \\ & + \frac{15993341307\sqrt{3}\cos\frac{20\varphi}{3}}{14923753062400} + \frac{3421914141\sqrt{3}\cos\frac{26\varphi}{3}}{22887871283200} + \frac{31571721\sin\frac{2\varphi}{3}}{1372585984} - \frac{1454139\sin\frac{4\varphi}{3}}{83886080} + \frac{303306111\sin\frac{8\varphi}{3}}{15686696960} - \frac{81081\sin\frac{10\varphi}{3}}{10485760} \\ & + \frac{1474715619\sin\frac{14\varphi}{3}}{143497625600} - \frac{27027\sin\frac{16\varphi}{3}}{16777216} + \frac{47980023921\sin\frac{20\varphi}{3}}{14923753062400} + \frac{10265742423\sin\frac{26\varphi}{3}}{22887871283200}\end{aligned}$$

$$\phi_{6,1,0} = -\frac{3\sqrt{3}\cos\frac{2\varphi}{3}}{56320} - \frac{9\sin\frac{2\varphi}{3}}{56320}$$

$$\begin{aligned}\phi_{6,1,1} = & \frac{21\sqrt{3}\cos\frac{\varphi}{3}}{225280} + \frac{39\sqrt{3}\cos\frac{5\varphi}{3}}{394240} + \frac{603\sqrt{3}\cos\frac{11\varphi}{3}}{26808320} \\ & - \frac{4077\sqrt{3}\cos\frac{17\varphi}{3}}{268083200} + \frac{21951\sqrt{3}\cos\frac{23\varphi}{3}}{6165913600} - \frac{63\sin\frac{\varphi}{3}}{225280} + \frac{117\sin\frac{5\varphi}{3}}{394240} + \frac{1809\sin\frac{11\varphi}{3}}{26808320} - \frac{12231\sin\frac{17\varphi}{3}}{268083200} + \frac{65853\sin\frac{23\varphi}{3}}{6165913600}\end{aligned}$$

$$\begin{aligned}\phi_{6,1,2} = & -\frac{369\sqrt{3}\cos\frac{2\varphi}{3}}{1802240} - \frac{189\sqrt{3}\cos\frac{4\varphi}{3}}{1802240} - \frac{4401\sqrt{3}\cos\frac{8\varphi}{3}}{30638080} \\ & - \frac{963\sqrt{3}\cos\frac{14\varphi}{3}}{38297600} + \frac{7911\sqrt{3}\cos\frac{20\varphi}{3}}{616591360} - \frac{1107\sin\frac{2\varphi}{3}}{1802240} + \frac{567\sin\frac{4\varphi}{3}}{1802240} - \frac{13203\sin\frac{8\varphi}{3}}{30638080} - \frac{2889\sin\frac{14\varphi}{3}}{38297600} + \frac{23733\sin\frac{20\varphi}{3}}{616591360}\end{aligned}$$

$$\begin{aligned}\phi_{6,1,3} = & \frac{1917\sqrt{3}\cos\frac{\varphi}{3}}{7208960} + \frac{7371\sqrt{3}\cos\frac{5\varphi}{3}}{24510464} + \frac{63\sqrt{3}\cos\frac{7\varphi}{3}}{655360} + \frac{194433\sqrt{3}\cos\frac{11\varphi}{3}}{1225523200} + \frac{4882743\sqrt{3}\cos\frac{17\varphi}{3}}{197309235200} - \frac{4186251\sqrt{3}\cos\frac{23\varphi}{3}}{366431436800} \\ & - \frac{17914173\sqrt{3}\cos\frac{29\varphi}{3}}{14877116334080} - \frac{5751\sin\frac{\varphi}{3}}{7208960} + \frac{22113\sin\frac{5\varphi}{3}}{24510464} - \frac{189\sin\frac{7\varphi}{3}}{655360} + \frac{583299\sin\frac{11\varphi}{3}}{1225523200} + \frac{14648229\sin\frac{17\varphi}{3}}{197309235200} - \frac{12558753\sin\frac{23\varphi}{3}}{366431436800} \\ & - \frac{53742519\sin\frac{29\varphi}{3}}{14877116334080}\end{aligned}$$

$$\begin{aligned}\phi_{6,1,4} = & -\frac{1041957\sqrt{3}\cos\frac{2\varphi}{3}}{2451046400} - \frac{1449\sqrt{3}\cos\frac{4\varphi}{3}}{5242880} - \frac{1717143\sqrt{3}\cos\frac{8\varphi}{3}}{4902092800} - \frac{819\sqrt{3}\cos\frac{10\varphi}{3}}{10485760} - \frac{238063431\sqrt{3}\cos\frac{14\varphi}{3}}{1578473881600} - \frac{50690457\sqrt{3}\cos\frac{20\varphi}{3}}{2565020057600} \\ & + \frac{12531\sqrt{3}\cos\frac{26\varphi}{3}}{1541478400} - \frac{3125871\sin\frac{2\varphi}{3}}{2451046400} + \frac{4347\sin\frac{4\varphi}{3}}{5242880} - \frac{5151429\sin\frac{8\varphi}{3}}{4902092800} + \frac{2457\sin\frac{10\varphi}{3}}{10485760} - \frac{714190293\sin\frac{14\varphi}{3}}{1578473881600} - \frac{152071371\sin\frac{20\varphi}{3}}{2565020057600} \\ & + \frac{37593\sin\frac{26\varphi}{3}}{1541478400}\end{aligned}$$

$$\phi_{8,1,0} = \frac{9\sqrt{3}\cos\frac{2\varphi}{3}}{12615680} + \frac{27\sin\frac{2\varphi}{3}}{12615680}$$

$$\begin{aligned}\phi_{8,1,1} = & -\frac{81\sqrt{3}\cos\frac{\varphi}{3}}{50462720} - \frac{1467\sqrt{3}\cos\frac{5\varphi}{3}}{857866240} - \frac{603\sqrt{3}\cos\frac{11\varphi}{3}}{2144665600} + \frac{12231\sqrt{3}\cos\frac{17\varphi}{3}}{49327308800} - \frac{65853\sqrt{3}\cos\frac{23\varphi}{3}}{641255014400} + \frac{44469\sqrt{3}\cos\frac{29\varphi}{3}}{2656627916800} + \frac{243\sin\frac{\varphi}{3}}{50462720} \\ & - \frac{4401\sin\frac{5\varphi}{3}}{857866240} - \frac{1809\sin\frac{11\varphi}{3}}{2144665600} + \frac{36693\sin\frac{17\varphi}{3}}{49327308800} - \frac{197559\sin\frac{23\varphi}{3}}{641255014400} + \frac{133407\sin\frac{29\varphi}{3}}{2656627916800}\end{aligned}$$

$$\begin{aligned}\phi_{8,1,2} = & \frac{7533\sqrt{3}\cos\frac{2\varphi}{3}}{1715732480} + \frac{81\sqrt{3}\cos\frac{4\varphi}{3}}{36700160} + \frac{20277\sqrt{3}\cos\frac{8\varphi}{3}}{6862929920} + \frac{2403\sqrt{3}\cos\frac{14\varphi}{3}}{7046758400} - \frac{405\sqrt{3}\cos\frac{20\varphi}{3}}{1165918208} + \frac{129519\sqrt{3}\cos\frac{26\varphi}{3}}{1430491955200} \\ & + \frac{22599\sin\frac{2\varphi}{3}}{1715732480} - \frac{243\sin\frac{4\varphi}{3}}{36700160} + \frac{60831\sin\frac{8\varphi}{3}}{6862929920} + \frac{7209\sin\frac{14\varphi}{3}}{7046758400} - \frac{1215\sin\frac{20\varphi}{3}}{1165918208} + \frac{388557\sin\frac{26\varphi}{3}}{1430491955200}\end{aligned}$$

$$\phi_{10,1,0} = -\frac{27\sqrt{3}\cos\frac{2\varphi}{3}}{4289331200} - \frac{81\sin\frac{2\varphi}{3}}{4289331200}$$

3.2. Second singular exponent ($j = 2$).

$$
\begin{aligned}
\phi_{0,2,0} &= \sin\frac{4\varphi}{3} - \frac{1}{\sqrt{3}}\cos\frac{4\varphi}{3}\\
\phi_{0,2,1} &= \frac{1}{4\sqrt{3}}\cos\frac{\varphi}{3} - \frac{1}{4}\sin\frac{\varphi}{3} - \frac{1}{28}\sin\frac{7\varphi}{3} + \frac{1}{28\sqrt{3}}\cos\frac{7\varphi}{3}\\
\phi_{0,2,2} &= -\frac{\sqrt{3}}{32}\cos\frac{2\varphi}{3} - \frac{\sqrt{3}}{28}\cos\frac{4\varphi}{3} - \frac{3}{32}\sin\frac{2\varphi}{3} + \frac{3}{28}\sin\frac{4\varphi}{3}\\
\phi_{0,2,3} &= \frac{23\sqrt{3}}{896}\cos\frac{\varphi}{3} + \frac{5}{128\sqrt{3}}\cos\frac{5\varphi}{3} + \frac{17\sqrt{3}}{1120}\cos\frac{7\varphi}{3} - \frac{69}{896}\sin\frac{\varphi}{3} + \frac{5}{128}\sin\frac{5\varphi}{3}\\
&- \frac{51}{1120}\sin\frac{7\varphi}{3} - \frac{1}{280}\sin\frac{13\varphi}{3} + \frac{1}{280\sqrt{3}}\cos\frac{13\varphi}{3}\\
\phi_{0,2,4} &= -\frac{85}{1792\sqrt{3}}\cos\frac{2\varphi}{3} - \frac{17\sqrt{3}}{1024}\cos\frac{4\varphi}{3} - \frac{35}{2048\sqrt{3}}\cos\frac{8\varphi}{3} - \frac{35}{1664\sqrt{3}}\cos\frac{10\varphi}{3}\\
&- \frac{85}{1792}\sin\frac{2\varphi}{3} + \frac{51}{1024}\sin\frac{4\varphi}{3} + -\frac{35}{2048}\sin\frac{8\varphi}{3} + \frac{35}{1664}\sin\frac{10\varphi}{3}\\
\phi_{0,2,5} &= \frac{391\sqrt{3}}{28672}\cos\frac{\varphi}{3} + \frac{75\sqrt{3}}{8192}\cos\frac{5\varphi}{3} + \frac{531\sqrt{3}}{53248}\cos\frac{7\varphi}{3} + \frac{21\sqrt{3}}{8192}\cos\frac{11\varphi}{3} + \frac{1015}{106496\sqrt{3}}\cos\frac{13\varphi}{3}\\
&- \frac{1173}{28672}\sin\frac{\varphi}{3} + \frac{225}{8192}\sin\frac{5\varphi}{3} - \frac{1593}{53248}\sin\frac{7\varphi}{3} + \frac{63}{8192}\sin\frac{11\varphi}{3} - \frac{1015}{106496}\sin\frac{13\varphi}{3}\\
&- \frac{319}{745472}\sin\frac{19\varphi}{3} + \frac{319}{745472\sqrt{3}}\cos\frac{19\varphi}{3}\\
\phi_{0,2,6} &= -\frac{323\sqrt{3}}{32768}\cos\frac{2\varphi}{3} - \frac{3797\sqrt{3}}{372736}\cos\frac{4\varphi}{3} - \frac{21\sqrt{3}}{4096}\cos\frac{8\varphi}{3} - \frac{19459\sqrt{3}}{3407872}\cos\frac{10\varphi}{3} - \frac{77\sqrt{3}}{65536}\cos\frac{14\varphi}{3}\\
&- \frac{5307\sqrt{3}}{3540992}\cos\frac{16\varphi}{3} - \frac{969}{32768}\sin\frac{2\varphi}{3} + \frac{11391}{372736}\sin\frac{4\varphi}{3} - \frac{63}{4096}\sin\frac{8\varphi}{3} + \frac{58377}{3407872}\sin\frac{10\varphi}{3}\\
&- \frac{231}{65536}\sin\frac{14\varphi}{3} + \frac{15921}{3540992}\sin\frac{16\varphi}{3}\\
\phi_{0,2,7} &= \frac{15023\sqrt{3}}{1703936}\cos\frac{\varphi}{3} + \frac{867\sqrt{3}}{131072}\cos\frac{5\varphi}{3} + \frac{2002421}{95420416\sqrt{3}}\cos\frac{7\varphi}{3}\\
&+ \frac{737\sqrt{3}}{262144}\cos\frac{11\varphi}{3} + \frac{5786399\sqrt{3}}{1812987904}\cos\frac{13\varphi}{3} + \frac{143\sqrt{3}}{262144}\cos\frac{17\varphi}{3} + \frac{217587\sqrt{3}}{311607296}\cos\frac{19\varphi}{3}\\
&- \frac{45069}{1703936}\sin\frac{\varphi}{3} + \frac{2601}{131072}\sin\frac{5\varphi}{3} - \frac{2002421}{95420416}\sin\frac{7\varphi}{3} + \frac{2211}{262144}\sin\frac{11\varphi}{3}\\
&- \frac{17359197}{1812987904}\sin\frac{13\varphi}{3} + \frac{429}{262144}\sin\frac{17\varphi}{3} - \frac{652761}{311607296}\sin\frac{19\varphi}{3} - \frac{4139}{68485120}\sin\frac{25\varphi}{3}\\
&+ \frac{4139}{68485120\sqrt{3}}\cos\frac{25\varphi}{3}\\
\phi_{0,2,8} &= -\frac{7341\sqrt{3}}{1064960}\cos\frac{2\varphi}{3} - \frac{4616423}{218103808\sqrt{3}}\cos\frac{4\varphi}{3} - \frac{44319\sqrt{3}}{10485760}\cos\frac{8\varphi}{3} - \frac{49508365}{3625975808\sqrt{3}}\cos\frac{10\varphi}{3}\\
&- \frac{5577\sqrt{3}}{3670016}\cos\frac{14\varphi}{3} - \frac{279019063\sqrt{3}}{159542935552}\cos\frac{16\varphi}{3} - \frac{2145\sqrt{3}}{8388608}\cos\frac{20\varphi}{3} - \frac{11327483}{11331174400\sqrt{3}}\cos\frac{22\varphi}{3}\\
&- \frac{22023}{1064960}\sin\frac{2\varphi}{3} + \frac{4616423}{218103808}\sin\frac{4\varphi}{3} - \frac{132957}{10485760}\sin\frac{8\varphi}{3} + \frac{49508365}{3625975808}\sin\frac{10\varphi}{3}\\
&- \frac{16731}{3670016}\sin\frac{14\varphi}{3} + \frac{837057189}{159542935552}\sin\frac{16\varphi}{3} - \frac{6435}{8388608}\sin\frac{20\varphi}{3} + \frac{11327483}{11331174400}\sin\frac{22\varphi}{3}\\
\phi_{0,2,9} &= \frac{5494833\sqrt{3}}{872415232}\cos\frac{\varphi}{3} + \frac{42185\sqrt{3}}{8388608}\cos\frac{5\varphi}{3} + \frac{86863845\sqrt{3}}{16575889408}\cos\frac{7\varphi}{3} + \frac{21879\sqrt{3}}{8388608}\cos\frac{11\varphi}{3}\\
&+ \frac{1818260415\sqrt{3}}{638171742208}\cos\frac{13\varphi}{3} + \frac{190905\sqrt{3}}{234881024}\cos\frac{17\varphi}{3} + \frac{15107532969\sqrt{3}}{15954293555200}\cos\frac{19\varphi}{3}\\
&+ \frac{12155}{33554432\sqrt{3}}\cos\frac{23\varphi}{3} + \frac{600356599}{1269091532800\sqrt{3}}\cos\frac{25\varphi}{3} - \frac{16484499}{872415232}\sin\frac{\varphi}{3} + \frac{126555}{8388608}\sin\frac{5\varphi}{3}\\
&- \frac{260591535}{16575889408}\sin\frac{7\varphi}{3} + \frac{65637}{8388608}\sin\frac{11\varphi}{3} - \frac{5454781245}{638171742208}\sin\frac{13\varphi}{3} + \frac{572715}{234881024}\sin\frac{17\varphi}{3}\\
&- \frac{45322598907}{15954293555200}\sin\frac{19\varphi}{3} + \frac{12155}{33554432}\sin\frac{23\varphi}{3} - \frac{600356599}{1269091532800}\sin\frac{25\varphi}{3}\\
&- \frac{134768503}{13960006860800}\sin\frac{31\varphi}{3} + \frac{134768503}{13960006860800\sqrt{3}}\cos\frac{31\varphi}{3}\\
\phi_{0,2,10} &= -\frac{18025733\sqrt{3}\cos\frac{2\varphi}{3}}{3489660928} - \frac{87168519\sqrt{3}\cos\frac{4\varphi}{3}}{16575889408} - \frac{102421\sqrt{3}\cos\frac{8\varphi}{3}}{29360128} - \frac{1344919515\sqrt{3}\cos\frac{10\varphi}{3}}{364669566976}\\
&- \frac{736593\sqrt{3}\cos\frac{14\varphi}{3}}{469762048} - \frac{172871033\sqrt{3}\cos\frac{16\varphi}{3}}{99714334720} - \frac{303875\cos\frac{20\varphi}{3}}{234881024\sqrt{3}} - \frac{906362008781\sqrt{3}\cos\frac{22\varphi}{3}}{1786880878182400}\\
&- \frac{46189\cos\frac{26\varphi}{3}}{268435456\sqrt{3}} - \frac{7032308149\cos\frac{28\varphi}{3}}{30911443763200\sqrt{3}} - \frac{54077199\sin\frac{2\varphi}{3}}{3489660928} + \frac{261505557\sin\frac{4\varphi}{3}}{16575889408} - \frac{307263\sin\frac{8\varphi}{3}}{29360128}\\
&+ \frac{4034758545\sin\frac{10\varphi}{3}}{364669566976} - \frac{2209779\sin\frac{14\varphi}{3}}{469762048} + \frac{518613099\sin\frac{16\varphi}{3}}{99714334720} - \frac{303875\sin\frac{20\varphi}{3}}{234881024}\\
&+ \frac{2719086026343\sin\frac{22\varphi}{3}}{1786880878182400} - \frac{46189\sin\frac{26\varphi}{3}}{268435456} + \frac{7032308149\sin\frac{28\varphi}{3}}{30911443763200}
\end{aligned}
$$

$$\phi_{2,2,0} = \frac{1}{28}\sqrt{3}\cos\frac{4\varphi}{3} - \frac{3}{28}\sin\frac{4\varphi}{3}$$

$$\phi_{2,2,1} = -\frac{3}{112}\sqrt{3}\cos\frac{\varphi}{3} - \frac{1}{35}\sqrt{3}\cos\frac{7\varphi}{3} - \frac{97\sqrt{3}\cos\frac{13\varphi}{3}}{7280} + \frac{9}{112}\sin\frac{\varphi}{3} + \frac{3}{35}\sin\frac{7\varphi}{3} + \frac{291\sin\frac{13\varphi}{3}}{7280}$$

$$\phi_{2,2,2} = \frac{15}{896}\sqrt{3}\cos\frac{2\varphi}{3} + \frac{141\sqrt{3}\cos\frac{4\varphi}{3}}{4480} + \frac{123\sqrt{3}\cos\frac{10\varphi}{3}}{5824} + \frac{45}{896}\sin\frac{2\varphi}{3} - \frac{423\sin\frac{4\varphi}{3}}{4480} - \frac{369\sin\frac{10\varphi}{3}}{5824}$$

$$\begin{aligned}\phi_{2,2,3} = {} & -\frac{93\sqrt{3}\cos\frac{\varphi}{3}}{3584} - \frac{5}{512}\sqrt{3}\cos\frac{5\varphi}{3} - \frac{633\sqrt{3}\cos\frac{7\varphi}{3}}{23296} \\ & - \frac{21839\sqrt{3}\cos\frac{13\varphi}{3}}{1863680} - \frac{68787\sqrt{3}\cos\frac{19\varphi}{3}}{35409920} + \frac{279\sin\frac{\varphi}{3}}{3584} - \frac{15}{512}\sin\frac{5\varphi}{3} + \frac{1899\sin\frac{7\varphi}{3}}{23296} + \frac{65517\sin\frac{13\varphi}{3}}{1863680} + \frac{206361\sin\frac{19\varphi}{3}}{35409920}\end{aligned}$$

$$\begin{aligned}\phi_{2,2,4} = {} & \frac{77\sqrt{3}\cos\frac{2\varphi}{3}}{4096} + \frac{50497\sqrt{3}\cos\frac{4\varphi}{3}}{1863680} + \frac{45\sqrt{3}\cos\frac{8\varphi}{3}}{8192} \\ & + \frac{3207\sqrt{3}\cos\frac{10\varphi}{3}}{163840} + \frac{751\sqrt{3}\cos\frac{16\varphi}{3}}{110656} + \frac{231\sin\frac{2\varphi}{3}}{4096} - \frac{151491\sin\frac{4\varphi}{3}}{1863680} + \frac{135\sin\frac{8\varphi}{3}}{8192} - \frac{9621\sin\frac{10\varphi}{3}}{163840} - \frac{2253\sin\frac{16\varphi}{3}}{110656}\end{aligned}$$

$$\begin{aligned}\phi_{2,2,5} = {} & -\frac{49509\sqrt{3}\cos\frac{\varphi}{3}}{2129920} - \frac{207\sqrt{3}\cos\frac{5\varphi}{3}}{16384} - \frac{21369\sqrt{3}\cos\frac{7\varphi}{3}}{917504} - \frac{99\sqrt{3}\cos\frac{11\varphi}{3}}{32768} - \frac{1137957\sqrt{3}\cos\frac{13\varphi}{3}}{87162880} - \frac{22170977\sqrt{3}\cos\frac{19\varphi}{3}}{6232145920} \\ & - \frac{9624067\sqrt{3}\cos\frac{25\varphi}{3}}{31160729600} + \frac{148527\sin\frac{\varphi}{3}}{2129920} - \frac{621\sin\frac{5\varphi}{3}}{16384} + \frac{64107\sin\frac{7\varphi}{3}}{917504} - \frac{297\sin\frac{11\varphi}{3}}{32768} + \frac{3413871\sin\frac{13\varphi}{3}}{87162880} + \frac{66512931\sin\frac{19\varphi}{3}}{6232145920} \\ & + \frac{28872201\sin\frac{25\varphi}{3}}{31160729600}\end{aligned}$$

$$\begin{aligned}\phi_{2,2,6} = {} & \frac{5931\sqrt{3}\cos\frac{2\varphi}{3}}{327680} + \frac{2762703\sqrt{3}\cos\frac{4\varphi}{3}}{119275520} + \frac{10593\sqrt{3}\cos\frac{8\varphi}{3}}{1310720} + \frac{32529369\sqrt{3}\cos\frac{10\varphi}{3}}{1812987904} + \frac{429\sqrt{3}\cos\frac{14\varphi}{3}}{262144} \\ & + \frac{816384879\sqrt{3}\cos\frac{16\varphi}{3}}{99714334720} + \frac{54482031\sqrt{3}\cos\frac{22\varphi}{3}}{28327936000} + \frac{17793\sin\frac{2\varphi}{3}}{327680} - \frac{8288109\sin\frac{4\varphi}{3}}{119275520} + \frac{31779\sin\frac{8\varphi}{3}}{1310720} - \frac{97588107\sin\frac{10\varphi}{3}}{1812987904} + \frac{1287\sin\frac{14\varphi}{3}}{262144} \\ & - \frac{2449154637\sin\frac{16\varphi}{3}}{99714334720} - \frac{163446093\sin\frac{22\varphi}{3}}{28327936000}\end{aligned}$$

$$\begin{aligned}\phi_{2,2,7} = {} & -\frac{78308991\sqrt{3}\cos\frac{\varphi}{3}}{3816816640} - \frac{449031\sqrt{3}\cos\frac{5\varphi}{3}}{34078720} - \frac{1470654989\sqrt{3}\cos\frac{7\varphi}{3}}{72519516160} - \frac{26169\sqrt{3}\cos\frac{11\varphi}{3}}{5242880} - \frac{1279456777\sqrt{3}\cos\frac{13\varphi}{3}}{99714334720} - \frac{6435\sqrt{3}\cos\frac{17\varphi}{3}}{7340032} \\ & - \frac{4498216551\sqrt{3}\cos\frac{19\varphi}{3}}{906493952000} - \frac{34493067223\sqrt{3}\cos\frac{25\varphi}{3}}{34900017152000} - \frac{58410950531\sqrt{3}\cos\frac{31\varphi}{3}}{1081900531712000} + \frac{234926973\sin\frac{\varphi}{3}}{3816816640} - \frac{1347093\sin\frac{5\varphi}{3}}{34078720} \\ & + \frac{4411964967\sin\frac{7\varphi}{3}}{72519516160} - \frac{78507\sin\frac{11\varphi}{3}}{5242880} + \frac{3838370331\sin\frac{13\varphi}{3}}{99714334720} - \frac{19305\sin\frac{17\varphi}{3}}{7340032} + \frac{13494649653\sin\frac{19\varphi}{3}}{906493952000} + \frac{103479201669\sin\frac{25\varphi}{3}}{34900017152000} \\ & + \frac{175232851593\sin\frac{31\varphi}{3}}{1081900531712000}\end{aligned}$$

$$\begin{aligned}\phi_{2,2,8} = {} & \frac{102271257\sqrt{3}\cos\frac{2\varphi}{3}}{6106906624} + \frac{641736027\sqrt{3}\cos\frac{4\varphi}{3}}{31876710400} + \frac{2680227\sqrt{3}\cos\frac{8\varphi}{3}}{293601280} + \frac{10376443733\sqrt{3}\cos\frac{10\varphi}{3}}{638171742208} + \frac{176319\sqrt{3}\cos\frac{14\varphi}{3}}{58720256} \\ & + \frac{36582771103\sqrt{3}\cos\frac{16\varphi}{3}}{4198498304000} + \frac{109395\sqrt{3}\cos\frac{20\varphi}{3}}{234881024} + \frac{3259134726429\sqrt{3}\cos\frac{22\varphi}{3}}{1116800548864000} + \frac{10028501487\sqrt{3}\cos\frac{28\varphi}{3}}{19319652352000} + \frac{306813771\sin\frac{2\varphi}{3}}{6106906624} \\ & - \frac{1925208081\sin\frac{4\varphi}{3}}{31876710400} + \frac{8040681\sin\frac{8\varphi}{3}}{293601280} - \frac{31129331199\sin\frac{10\varphi}{3}}{638171742208} + \frac{528957\sin\frac{14\varphi}{3}}{58720256} - \frac{109748313309\sin\frac{16\varphi}{3}}{4198498304000} + \frac{328185\sin\frac{20\varphi}{3}}{234881024} \\ & - \frac{9777404179287\sin\frac{22\varphi}{3}}{1116800548864000} - \frac{30085504461\sin\frac{28\varphi}{3}}{19319652352000}\end{aligned}$$

$$
\begin{aligned}
\phi_{4,2,0} &= -\frac{3\sqrt{3}\cos\frac{4\varphi}{3}}{2240} + \frac{9\sin\frac{4\varphi}{3}}{2240} \\
\phi_{4,2,1} &= \frac{3\sqrt{3}\cos\frac{\varphi}{3}}{1792} + \frac{45\sqrt{3}\cos\frac{7\varphi}{3}}{23296} + \frac{291\sqrt{3}\cos\frac{13\varphi}{3}}{465920} - \frac{1737\sqrt{3}\cos\frac{19\varphi}{3}}{8852480} - \frac{9\sin\frac{\varphi}{3}}{1792} - \frac{135\sin\frac{7\varphi}{3}}{23296} - \frac{873\sin\frac{13\varphi}{3}}{465920} + \frac{5211\sin\frac{19\varphi}{3}}{8852480} \\
\phi_{4,2,2} &= -\frac{3\sqrt{3}\cos\frac{2\varphi}{3}}{2048} - \frac{141\sqrt{3}\cos\frac{4\varphi}{3}}{46592} - \frac{17181\sqrt{3}\cos\frac{10\varphi}{3}}{7454720} - \frac{111\sqrt{3}\cos\frac{16\varphi}{3}}{221312} - \frac{9\sin\frac{2\varphi}{3}}{2048} + \frac{423\sin\frac{4\varphi}{3}}{46592} + \frac{51543\sin\frac{10\varphi}{3}}{7454720} + \frac{333\sin\frac{16\varphi}{3}}{221312} \\
\phi_{4,2,3} &= \frac{339\sqrt{3}\cos\frac{\varphi}{3}}{106496} + \frac{9\sqrt{3}\cos\frac{5\varphi}{3}}{8192} + \frac{22401\sqrt{3}\cos\frac{7\varphi}{3}}{5963776} + \frac{1161297\sqrt{3}\cos\frac{13\varphi}{3}}{566558720} + \frac{1288377\sqrt{3}\cos\frac{19\varphi}{3}}{3116072960} - \frac{731631\sqrt{3}\cos\frac{25\varphi}{3}}{15580364800} - \frac{1017\sin\frac{\varphi}{3}}{106496} \\
&+ \frac{27\sin\frac{5\varphi}{3}}{8192} - \frac{67203\sin\frac{7\varphi}{3}}{5963776} - \frac{3483891\sin\frac{13\varphi}{3}}{566558720} - \frac{3865131\sin\frac{19\varphi}{3}}{3116072960} + \frac{2194893\sin\frac{25\varphi}{3}}{15580364800} \\
\phi_{4,2,4} &= -\frac{297\sqrt{3}\cos\frac{2\varphi}{3}}{106496} - \frac{75477\sqrt{3}\cos\frac{4\varphi}{3}}{17039360} - \frac{99\sqrt{3}\cos\frac{8\varphi}{3}}{131072} - \frac{210081\sqrt{3}\cos\frac{10\varphi}{3}}{56655872} - \frac{11511609\sqrt{3}\cos\frac{16\varphi}{3}}{7122452480} - \frac{48771\sqrt{3}\cos\frac{22\varphi}{3}}{186368000} - \frac{891\sin\frac{2\varphi}{3}}{106496} \\
&+ \frac{226431\sin\frac{4\varphi}{3}}{17039360} - \frac{297\sin\frac{8\varphi}{3}}{131072} + \frac{630243\sin\frac{10\varphi}{3}}{56655872} + \frac{34534827\sin\frac{16\varphi}{3}}{7122452480} + \frac{146313\sin\frac{22\varphi}{3}}{186368000} \\
\phi_{4,2,5} &= \frac{595161\sqrt{3}\cos\frac{\varphi}{3}}{136314880} + \frac{14949\sqrt{3}\cos\frac{5\varphi}{3}}{6815744} + \frac{12637593\sqrt{3}\cos\frac{7\varphi}{3}}{2589982720} + \frac{1287\sqrt{3}\cos\frac{11\varphi}{3}}{2621440} + \frac{638879919\sqrt{3}\cos\frac{13\varphi}{3}}{199428669440} + \frac{88487757\sqrt{3}\cos\frac{19\varphi}{3}}{76703334400} \\
&+ \frac{2921545173\sqrt{3}\cos\frac{25\varphi}{3}}{17450008576000} - \frac{1085706519\sqrt{3}\cos\frac{31\varphi}{3}}{108190053171200} - \frac{1785483\sin\frac{\varphi}{3}}{136314880} + \frac{44847\sin\frac{5\varphi}{3}}{6815744} - \frac{37912779\sin\frac{7\varphi}{3}}{2589982720} + \frac{3861\sin\frac{11\varphi}{3}}{2621440} \\
&- \frac{1916639757\sin\frac{13\varphi}{3}}{199428669440} - \frac{265463271\sin\frac{19\varphi}{3}}{76703334400} - \frac{8764635519\sin\frac{25\varphi}{3}}{17450008576000} + \frac{3257119557\sin\frac{31\varphi}{3}}{108190053171200} \\
\phi_{4,2,6} &= -\frac{8363223\sqrt{3}\cos\frac{2\varphi}{3}}{2181038080} - \frac{48435399\sqrt{3}\cos\frac{4\varphi}{3}}{9064939520} - \frac{1683\sqrt{3}\cos\frac{8\varphi}{3}}{1048576} - \frac{212164605\sqrt{3}\cos\frac{10\varphi}{3}}{45583695872} - \frac{1287\sqrt{3}\cos\frac{14\varphi}{3}}{4194304} - \frac{25127548689\sqrt{3}\cos\frac{16\varphi}{3}}{9971433472000} \\
&- \frac{652815981\sqrt{3}\cos\frac{22\varphi}{3}}{839699660800} - \frac{1069062327\sqrt{3}\cos\frac{28\varphi}{3}}{11039801344000} - \frac{25089669\sin\frac{2\varphi}{3}}{2181038080} + \frac{145306197\sin\frac{4\varphi}{3}}{9064939520} - \frac{5049\sin\frac{8\varphi}{3}}{1048576} + \frac{636493815\sin\frac{10\varphi}{3}}{45583695872} \\
&- \frac{3861\sin\frac{14\varphi}{3}}{4194304} + \frac{75382646067\sin\frac{16\varphi}{3}}{9971433472000} + \frac{1958447943\sin\frac{22\varphi}{3}}{839699660800} + \frac{3207186981\sin\frac{28\varphi}{3}}{11039801344000}
\end{aligned}
$$

$$
\begin{aligned}
\phi_{6,2,0} &= \frac{3\sqrt{3}\cos\frac{4\varphi}{3}}{116480} - \frac{9\sin\frac{4\varphi}{3}}{116480} \\
\phi_{6,2,1} &= -\frac{3\sqrt{3}\cos\frac{\varphi}{3}}{66560} - \frac{3\sqrt{3}\cos\frac{7\varphi}{3}}{57344} - \frac{873\sqrt{3}\cos\frac{13\varphi}{3}}{70819840} + \frac{5211\sqrt{3}\cos\frac{19\varphi}{3}}{779018240} + \frac{9\sin\frac{\varphi}{3}}{66560} + \frac{9\sin\frac{7\varphi}{3}}{57344} + \frac{2619\sin\frac{13\varphi}{3}}{70819840} - \frac{15633\sin\frac{19\varphi}{3}}{779018240} \\
\phi_{6,2,2} &= \frac{27\sqrt{3}\cos\frac{2\varphi}{3}}{532480} + \frac{459\sqrt{3}\cos\frac{4\varphi}{3}}{4259840} + \frac{9153\sqrt{3}\cos\frac{10\varphi}{3}}{113311744} + \frac{1503\sqrt{3}\cos\frac{16\varphi}{3}}{97377280} - \frac{5211\sqrt{3}\cos\frac{22\varphi}{3}}{856064000} + \frac{81\sin\frac{2\varphi}{3}}{532480} - \frac{1377\sin\frac{4\varphi}{3}}{4259840} - \frac{27459\sin\frac{10\varphi}{3}}{113311744} \\
&- \frac{4509\sin\frac{16\varphi}{3}}{97377280} + \frac{15633\sin\frac{22\varphi}{3}}{856064000} \\
\phi_{6,2,3} &= -\frac{297\sqrt{3}\cos\frac{\varphi}{3}}{2129920} - \frac{99\sqrt{3}\cos\frac{5\varphi}{3}}{2129920} - \frac{2871\sqrt{3}\cos\frac{7\varphi}{3}}{17039360} - \frac{1167711\sqrt{3}\cos\frac{13\varphi}{3}}{12464291840} - \frac{1231299\sqrt{3}\cos\frac{19\varphi}{3}}{77901824000} + \frac{23785659\sqrt{3}\cos\frac{25\varphi}{3}}{4362502144000} \\
&+ \frac{57409293\sqrt{3}\cos\frac{31\varphi}{3}}{67618783232000} + \frac{891\sin\frac{\varphi}{3}}{2129920} - \frac{297\sin\frac{5\varphi}{3}}{2129920} + \frac{8613\sin\frac{7\varphi}{3}}{17039360} + \frac{3503133\sin\frac{13\varphi}{3}}{12464291840} + \frac{3693897\sin\frac{19\varphi}{3}}{77901824000} - \frac{71356977\sin\frac{25\varphi}{3}}{4362502144000} \\
&- \frac{172227879\sin\frac{31\varphi}{3}}{67618783232000} \\
\phi_{6,2,4} &= \frac{19701\sqrt{3}\cos\frac{2\varphi}{3}}{136314880} + \frac{767421\sqrt{3}\cos\frac{4\varphi}{3}}{3237478400} + \frac{99\sqrt{3}\cos\frac{8\varphi}{3}}{2621440} + \frac{293115\sqrt{3}\cos\frac{10\varphi}{3}}{1424490496} + \frac{35369211\sqrt{3}\cos\frac{16\varphi}{3}}{383516672000} + \frac{65479527\sqrt{3}\cos\frac{22\varphi}{3}}{4985716736000} \\
&- \frac{306636441\sqrt{3}\cos\frac{28\varphi}{3}}{77278609408000} + \frac{59103\sin\frac{2\varphi}{3}}{136314880} - \frac{2302263\sin\frac{4\varphi}{3}}{3237478400} + \frac{297\sin\frac{8\varphi}{3}}{2621440} - \frac{879345\sin\frac{10\varphi}{3}}{1424490496} - \frac{106107633\sin\frac{16\varphi}{3}}{383516672000} - \frac{196438581\sin\frac{22\varphi}{3}}{4985716736000} \\
&+ \frac{919909323\sin\frac{28\varphi}{3}}{77278609408000}
\end{aligned}
$$

$$
\begin{aligned}
\phi_{8,2,0} &= -\frac{9\sqrt{3}\cos\frac{4\varphi}{3}}{29818880} + \frac{27\sin\frac{4\varphi}{3}}{29818880} \\
\phi_{8,2,1} &= \frac{81\sqrt{3}\cos\frac{\varphi}{3}}{119275520} + \frac{1773\sqrt{3}\cos\frac{7\varphi}{3}}{2266234880} + \frac{873\sqrt{3}\cos\frac{13\varphi}{3}}{6232145920} - \frac{15633\sqrt{3}\cos\frac{19\varphi}{3}}{155803648000} - \frac{243\sin\frac{\varphi}{3}}{119275520} - \frac{5319\sin\frac{7\varphi}{3}}{2266234880} - \frac{2619\sin\frac{13\varphi}{3}}{6232145920} + \frac{46899\sin\frac{19\varphi}{3}}{155803648000} \\
\phi_{8,2,2} &= -\frac{891\sqrt{3}\cos\frac{2\varphi}{3}}{954204160} - \frac{81\sqrt{3}\cos\frac{4\varphi}{3}}{40468480} - \frac{2619\sqrt{3}\cos\frac{10\varphi}{3}}{1812987904} - \frac{4023\sqrt{3}\cos\frac{16\varphi}{3}}{19475456000} + \frac{1516401\sqrt{3}\cos\frac{22\varphi}{3}}{8725004288000} - \frac{2673\sin\frac{2\varphi}{3}}{954204160} + \frac{243\sin\frac{4\varphi}{3}}{40468480} \\
&+ \frac{7857\sin\frac{10\varphi}{3}}{1812987904} + \frac{12069\sin\frac{16\varphi}{3}}{19475456000} - \frac{4549203\sin\frac{22\varphi}{3}}{8725004288000}
\end{aligned}
$$

$$
\phi_{10,2,0} = \frac{27\sqrt{3}\cos\frac{4\varphi}{3}}{11331174400} - \frac{81\sin\frac{4\varphi}{3}}{11331174400}
$$

3.3. **Higher order exponents ($j = 3, \ldots, 11$).**

$$
\begin{aligned}
\phi_{0,3,0} &= \cos 2\varphi \\
\phi_{0,3,1} &= -\frac{\cos\varphi}{4} - \frac{1}{12}\cos 3\varphi \\
\phi_{0,3,2} &= \frac{3}{32} + \frac{1}{8}\cos 2\varphi \\
\phi_{0,3,3} &= -\frac{\cos\varphi}{8} - \frac{7}{128}\cos 3\varphi - \frac{1}{128}\cos 5\varphi \\
\phi_{0,3,4} &= \frac{5}{96} + \frac{77\cos 2\varphi}{1024} + \frac{17}{640}\cos 4\varphi \\
\phi_{0,3,5} &= -\frac{315\cos\varphi}{4096} - \frac{897\cos 3\varphi}{20480} - \frac{187\cos 5\varphi}{15360} - \frac{7\cos 7\varphi}{7680} \\
\phi_{0,3,6} &= \frac{2205}{65536} + \frac{433\cos 2\varphi}{8192} + \frac{4037\cos 4\varphi}{163840} + \frac{843\cos 6\varphi}{143360} \\
\phi_{0,3,7} &= -\frac{875\cos\varphi}{16384} - \frac{4521\cos 3\varphi}{131072} - \frac{62597\cos 5\varphi}{4587520} - \frac{2529\cos 7\varphi}{917504} - \frac{349\cos 9\varphi}{2752512} \\
\phi_{0,3,8} &= \frac{1575}{65536} + \frac{248941\cos 2\varphi}{6291456} + \frac{59729\cos 4\varphi}{2752512} + \frac{218139\cos 6\varphi}{29360128} + \frac{1835\cos 8\varphi}{1376256} \\
\phi_{0,3,9} &= -\frac{1670823\cos\varphi}{41943040} - \frac{233745\cos 3\varphi}{8388608} - \frac{1558145\cos 5\varphi}{117440512} - \frac{1412281\cos 7\varphi}{352321536} \\
&\quad - \frac{6973\cos 9\varphi}{11010048} - \frac{2201\cos 11\varphi}{110100480}
\end{aligned}
$$

$$
\begin{aligned}
\phi_{2,3,0} &= -\frac{1}{12}\cos 2\varphi \\
\phi_{2,3,1} &= \frac{\cos\varphi}{16} + \frac{7}{96}\cos 3\varphi + \frac{1}{32}\cos 5\varphi \\
\phi_{2,3,2} &= -\frac{5}{128} - \frac{61}{768}\cos 2\varphi - \frac{53}{960}\cos 4\varphi \\
\phi_{2,3,3} &= \frac{45\cos\varphi}{512} + \frac{137\cos 3\varphi}{1920} + \frac{487\cos 5\varphi}{15360} + \frac{71\cos 7\varphi}{15360} \\
\phi_{2,3,4} &= -\frac{385}{8192} - \frac{15437\cos 2\varphi}{184320} - \frac{19523\cos 4\varphi}{368640} - \frac{143\cos 6\varphi}{7680} \\
\phi_{2,3,5} &= \frac{67949\cos\varphi}{737280} + \frac{309119\cos 3\varphi}{4423680} + \frac{372203\cos 5\varphi}{10321920} + \frac{2447\cos 7\varphi}{245760} + \frac{2365\cos 9\varphi}{3096576} \\
\phi_{2,3,6} &= -\frac{61733}{1310720} - \frac{648079\cos 2\varphi}{7864320} - \frac{2200529\cos 4\varphi}{41287680} - \frac{844089\cos 6\varphi}{36700160} - \frac{168421\cos 8\varphi}{30965760} \\
\phi_{2,3,7} &= \frac{2806877\cos\varphi}{31457280} + \frac{6434681\cos 3\varphi}{94371840} + \frac{10053427\cos 5\varphi}{264241152} + \frac{55988167\cos 7\varphi}{3963617280} + \frac{2806673\cos 9\varphi}{990904320} + \frac{301339\cos 11\varphi}{2179989504}
\end{aligned}
$$

$$
\begin{aligned}
\phi_{4,3,0} &= \frac{1}{384}\cos 2\varphi \\
\phi_{4,3,1} &= -\frac{5\cos\varphi}{1536} - \frac{31\cos 3\varphi}{7680} - \frac{1}{768}\cos 5\varphi + \frac{3\cos 7\varphi}{8960} \\
\phi_{4,3,2} &= \frac{35}{12288} + \frac{29\cos 2\varphi}{4608} + \frac{923\cos 4\varphi}{184320} + \frac{29\cos 6\varphi}{26880} \\
\phi_{4,3,3} &= -\frac{161\cos\varphi}{18432} - \frac{3611\cos 3\varphi}{442368} - \frac{23633\cos 5\varphi}{5160960} - \frac{3077\cos 7\varphi}{3440640} + \frac{9103\cos 9\varphi}{92897280} \\
\phi_{4,3,4} &= \frac{189}{32768} + \frac{391699\cos 2\varphi}{35389440} + \frac{17161\cos 4\varphi}{2064384} + \frac{67513\cos 6\varphi}{18350080} + \frac{1325\cos 8\varphi}{2322432} \\
\phi_{4,3,5} &= -\frac{73213\cos\varphi}{5242880} - \frac{1681349\cos 3\varphi}{141557760} - \frac{138697\cos 5\varphi}{18874368} - \frac{5281387\cos 7\varphi}{1981808640} - \frac{683371\cos 9\varphi}{1857945600} + \frac{334729\cos 11\varphi}{13624934400}
\end{aligned}
$$

$$
\begin{aligned}
\phi_{6,3,0} &= -\frac{\cos 2\varphi}{23040} \\
\phi_{6,3,1} &= \frac{7\cos\varphi}{92160} + \frac{13\cos 3\varphi}{138240} + \frac{\cos 5\varphi}{43008} - \frac{3\cos 7\varphi}{286720} + \frac{43\cos 9\varphi}{23224320} \\
\phi_{6,3,2} &= -\frac{7}{81920} - \frac{427\cos 2\varphi}{2211840} - \frac{787\cos 4\varphi}{5160960} - \frac{9\cos 6\varphi}{286720} + \frac{53\cos 8\varphi}{5806080} \\
\phi_{6,3,3} &= \frac{161\cos\varphi}{491520} + \frac{2801\cos 3\varphi}{8847360} + \frac{3035\cos 5\varphi}{16515072} + \frac{2767\cos 7\varphi}{82575360} - \frac{1571\cos 9\varphi}{176947200} - \frac{7361\cos 11\varphi}{4541644800}
\end{aligned}
$$

$$
\begin{aligned}
\phi_{8,3,0} &= \frac{\cos 2\varphi}{2211840} \\
\phi_{8,3,1} &= -\frac{\cos\varphi}{983040} - \frac{11\cos 3\varphi}{8847360} - \frac{\cos 5\varphi}{4128768} + \frac{\cos 7\varphi}{6881280} - \frac{43\cos 9\varphi}{928972800} + \frac{\cos 11\varphi}{162201600}
\end{aligned}
$$

$$
\begin{aligned}
\phi_{0,4,0} &= \sin\frac{8}{3}\varphi + \frac{1}{\sqrt{3}}\cos\frac{8}{3}\varphi \\
\phi_{0,4,1} &= -\frac{1}{4\sqrt{3}}\cos\frac{5\varphi}{3} - \frac{1}{4}\sin\frac{5\varphi}{3} - \frac{5}{44}\sin\frac{11\varphi}{3} - \frac{5}{44\sqrt{3}}\cos\frac{11\varphi}{3} \\
\phi_{0,4,2} &= \frac{\sqrt{3}}{32}\cos\frac{2\varphi}{3} + \frac{3}{32}\sin\frac{2\varphi}{3} + \frac{\sqrt{3}}{22}\cos\frac{8\varphi}{3} + \frac{3}{22}\sin\frac{8\varphi}{3} \\
\phi_{0,4,3} &= -\frac{5}{128\sqrt{3}}\cos\frac{\varphi}{3} + \frac{5}{128}\sin\frac{\varphi}{3} - \frac{43\sqrt{3}}{1408}\cos\frac{5\varphi}{3} - \frac{129}{1408}\sin\frac{5\varphi}{3} \\
&- \frac{25\sqrt{3}}{1232}\cos\frac{11\varphi}{3} - \frac{75}{1232}\sin\frac{11\varphi}{3} - \frac{25}{2464}\sin\frac{17\varphi}{3} - \frac{25}{2464\sqrt{3}}\cos\frac{17\varphi}{3} \\
\phi_{0,4,4} &= \frac{155}{2816\sqrt{3}}\cos\frac{2\varphi}{3} + \frac{155}{2816}\sin\frac{2\varphi}{3} + \frac{35}{2048\sqrt{3}}\cos\frac{4\varphi}{3} - \frac{35}{2048}\sin\frac{4\varphi}{3} + \frac{1675\sqrt{3}}{78848}\cos\frac{8\varphi}{3} \\
&+ \frac{5025}{78848}\sin\frac{8\varphi}{3} + \frac{725}{23936\sqrt{3}}\cos\frac{14\varphi}{3} + \frac{725}{23936}\sin\frac{14\varphi}{3} \\
\phi_{0,4,5} &= -\frac{945\sqrt{3}}{90112}\cos\frac{\varphi}{3} + \frac{2835}{90112}\sin\frac{\varphi}{3} - \frac{5375\sqrt{3}}{315392}\cos\frac{5\varphi}{3} - \frac{16125}{315392}\sin\frac{5\varphi}{3} - \frac{21\sqrt{3}}{8192}\cos\frac{7\varphi}{3} \\
&+ \frac{63}{8192}\sin\frac{7\varphi}{3} - \frac{72225\sqrt{3}}{5361664}\cos\frac{11\varphi}{3} - \frac{216675}{5361664}\sin\frac{11\varphi}{3} - \frac{5365}{382976\sqrt{3}}\cos\frac{17\varphi}{3} \\
&- \frac{5365}{382976}\sin\frac{17\varphi}{3} - \frac{3133}{2680832}\sin\frac{23\varphi}{3} - \frac{3133}{2680832\sqrt{3}}\cos\frac{23\varphi}{3} \\
\phi_{0,4,6} &= \frac{4375\sqrt{3}}{360448}\cos\frac{2\varphi}{3} + \frac{13125}{360448}\sin\frac{2\varphi}{3} + \frac{525\sqrt{3}}{90112}\cos\frac{4\varphi}{3} - \frac{1575}{90112}\sin\frac{4\varphi}{3} + \frac{5125\sqrt{3}}{382976}\cos\frac{8\varphi}{3} \\
&+ \frac{15375}{382976}\sin\frac{8\varphi}{3} + \frac{77\sqrt{3}}{65536}\cos\frac{10\varphi}{3} - \frac{231}{65536}\sin\frac{10\varphi}{3} + \frac{684685\sqrt{3}}{85786624}\cos\frac{14\varphi}{3} \\
&+ \frac{2054055}{85786624}\sin\frac{14\varphi}{3} + \frac{140577\sqrt{3}}{61659136}\cos\frac{20\varphi}{3} + \frac{421731}{61659136}\sin\frac{20\varphi}{3} \\
\phi_{0,4,7} &= -\frac{11625\sqrt{3}\cos\frac{\varphi}{3}}{1441792} - \frac{25375\sqrt{3}\cos\frac{5\varphi}{3}}{2228224} - \frac{833\sqrt{3}\cos\frac{7\varphi}{3}}{262144} - \frac{9794825\cos\frac{11\varphi}{3}}{343146496\sqrt{3}} \\
&- \frac{143\sqrt{3}\cos\frac{13\varphi}{3}}{262144} - \frac{36189151\sqrt{3}\cos\frac{17\varphi}{3}}{7892369408} - \frac{984039\sqrt{3}\cos\frac{23\varphi}{3}}{916078592} - \frac{129905\cos\frac{29\varphi}{3}}{801568768\sqrt{3}} \\
&+ \frac{34875\sin\frac{\varphi}{3}}{1441792} - \frac{76125\sin\frac{5\varphi}{3}}{2228224} + \frac{2499\sin\frac{7\varphi}{3}}{262144} - \frac{9794825\sin\frac{11\varphi}{3}}{343146496} \\
&+ \frac{429\sin\frac{13\varphi}{3}}{262144} - \frac{108567453\sin\frac{17\varphi}{3}}{7892369408} - \frac{2952117\sin\frac{23\varphi}{3}}{916078592} - \frac{129905\sin\frac{29\varphi}{3}}{801568768} \\
\phi_{0,4,8} &= \frac{53925\sqrt{3}\cos\frac{2\varphi}{3}}{6127616} + \frac{10725\sqrt{3}\cos\frac{4\varphi}{3}}{2097152} + \frac{1305175\cos\frac{8\varphi}{3}}{46137344\sqrt{3}} + \frac{897\sqrt{3}\cos\frac{10\varphi}{3}}{524288} \\
&+ \frac{301061615\cos\frac{14\varphi}{3}}{15784738816\sqrt{3}} + \frac{2145\sqrt{3}\cos\frac{16\varphi}{3}}{8388608} + \frac{300557929\sqrt{3}\cos\frac{20\varphi}{3}}{117258059776} + \frac{89746385\cos\frac{26\varphi}{3}}{57219678208\sqrt{3}} \\
&+ \frac{161775\sin\frac{2\varphi}{3}}{6127616} - \frac{32175\sin\frac{4\varphi}{3}}{2097152} + \frac{1305175\sin\frac{8\varphi}{3}}{46137344} - \frac{2691\sin\frac{10\varphi}{3}}{524288} + \frac{301061615\sin\frac{14\varphi}{3}}{15784738816} \\
&- \frac{6435\sin\frac{16\varphi}{3}}{8388608} + \frac{901673787\sin\frac{20\varphi}{3}}{117258059776} + \frac{89746385\sin\frac{26\varphi}{3}}{57219678208}
\end{aligned}
$$

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\begin{aligned}
\phi_{2,4,0} &= -\frac{1}{44}\sqrt{3}\cos\frac{8\varphi}{3} - \frac{3}{44}\sin\frac{8\varphi}{3} \\
\phi_{2,4,1} &= \frac{3}{176}\sqrt{3}\cos\frac{5\varphi}{3} + \frac{13}{616}\sqrt{3}\cos\frac{11\varphi}{3} + \frac{9}{176}\sin\frac{5\varphi}{3} + \frac{39}{616}\sin\frac{11\varphi}{3} \\
\phi_{2,4,2} &= -\frac{15\sqrt{3}\cos\frac{2\varphi}{3}}{1408} - \frac{225\sqrt{3}\cos\frac{8\varphi}{3}}{9856} - \frac{37\sqrt{3}\cos\frac{14\varphi}{3}}{2618} - \frac{45\sin\frac{2\varphi}{3}}{1408} - \frac{675\sin\frac{8\varphi}{3}}{9856} - \frac{111\sin\frac{14\varphi}{3}}{2618} \\
\phi_{2,4,3} &= \frac{35\sqrt{3}\cos\frac{\varphi}{3}}{5632} + \frac{105\sqrt{3}\cos\frac{5\varphi}{3}}{5632} + \frac{13599\sqrt{3}\cos\frac{11\varphi}{3}}{670208} + \frac{5751\sqrt{3}\cos\frac{17\varphi}{3}}{670208} - \frac{105\sin\frac{\varphi}{3}}{5632} + \frac{315\sin\frac{5\varphi}{3}}{5632} + \frac{40797\sin\frac{11\varphi}{3}}{670208} + \frac{17253\sin\frac{17\varphi}{3}}{670208} \\
\phi_{2,4,4} &= -\frac{55\sqrt{3}\cos\frac{2\varphi}{3}}{4096} - \frac{315\sqrt{3}\cos\frac{4\varphi}{3}}{90112} - \frac{110427\sqrt{3}\cos\frac{8\varphi}{3}}{5361664} - \frac{163539\sqrt{3}\cos\frac{14\varphi}{3}}{10723328} - \frac{302101\sqrt{3}\cos\frac{20\varphi}{3}}{61659136} - \frac{165\sin\frac{2\varphi}{3}}{4096} + \frac{945\sin\frac{4\varphi}{3}}{90112} \\
&- \frac{331281\sin\frac{8\varphi}{3}}{5361664} - \frac{490617\sin\frac{14\varphi}{3}}{10723328} - \frac{906303\sin\frac{20\varphi}{3}}{61659136} \\
\phi_{2,4,5} &= \frac{405\sqrt{3}\cos\frac{\varphi}{3}}{45056} + \frac{53\sqrt{3}\cos\frac{5\varphi}{3}}{2992} + \frac{63\sqrt{3}\cos\frac{7\varphi}{3}}{32768} + \frac{990783\sqrt{3}\cos\frac{11\varphi}{3}}{53616640} + \frac{10275141\sqrt{3}\cos\frac{17\varphi}{3}}{986546176} + \frac{155323\sqrt{3}\cos\frac{23\varphi}{3}}{57254912} - \frac{1215\sin\frac{\varphi}{3}}{45056} \\
&+ \frac{159\sin\frac{5\varphi}{3}}{2992} - \frac{189\sin\frac{7\varphi}{3}}{32768} + \frac{2972349\sin\frac{11\varphi}{3}}{53616640} + \frac{30825423\sin\frac{17\varphi}{3}}{986546176} + \frac{465969\sin\frac{23\varphi}{3}}{57254912} \\
\phi_{2,4,6} &= -\frac{337743\sqrt{3}\cos\frac{2\varphi}{3}}{24510464} - \frac{1503\sqrt{3}\cos\frac{4\varphi}{3}}{262144} - \frac{15848761\sqrt{3}\cos\frac{8\varphi}{3}}{857866240} - \frac{273\sqrt{3}\cos\frac{10\varphi}{3}}{262144} - \frac{579169167\sqrt{3}\cos\frac{14\varphi}{3}}{39461847040} - \frac{687402189\sqrt{3}\cos\frac{20\varphi}{3}}{102600802304} \\
&- \frac{38987053\sqrt{3}\cos\frac{26\varphi}{3}}{26566279168} - \frac{1013229\sin\frac{2\varphi}{3}}{24510464} + \frac{4509\sin\frac{4\varphi}{3}}{262144} - \frac{47546283\sin\frac{8\varphi}{3}}{857866240} + \frac{819\sin\frac{10\varphi}{3}}{262144} - \frac{1737507501\sin\frac{14\varphi}{3}}{39461847040} - \frac{2062206567\sin\frac{20\varphi}{3}}{102600802304} \\
&- \frac{116961159\sin\frac{26\varphi}{3}}{26566279168}
\end{aligned}
$$

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\begin{aligned}
\phi_{4,4,0} &= \frac{3\sqrt{3}\cos\frac{8\varphi}{3}}{4928} + \frac{9\sin\frac{8\varphi}{3}}{4928} \\
\phi_{4,4,1} &= -\frac{15\sqrt{3}\cos\frac{5\varphi}{3}}{19712} - \frac{333\sqrt{3}\cos\frac{11\varphi}{3}}{335104} + \frac{597\sqrt{3}\cos\frac{23\varphi}{3}}{1926848} - \frac{45\sin\frac{5\varphi}{3}}{19712} - \frac{999\sin\frac{11\varphi}{3}}{335104} + \frac{1791\sin\frac{23\varphi}{3}}{1926848} \\
\phi_{4,4,2} &= \frac{15\sqrt{3}\cos\frac{2\varphi}{3}}{22528} + \frac{1035\sqrt{3}\cos\frac{8\varphi}{3}}{670208} + \frac{5517\sqrt{3}\cos\frac{14\varphi}{3}}{5361664} - \frac{597\sqrt{3}\cos\frac{20\varphi}{3}}{7707392} + \frac{45\sin\frac{2\varphi}{3}}{22528} + \frac{3105\sin\frac{8\varphi}{3}}{670208} + \frac{16551\sin\frac{14\varphi}{3}}{5361664} - \frac{1791\sin\frac{20\varphi}{3}}{7707392} \\
\phi_{4,4,3} &= -\frac{45\sqrt{3}\cos\frac{\varphi}{3}}{90112} - \frac{225\sqrt{3}\cos\frac{5\varphi}{3}}{139264} - \frac{41163\sqrt{3}\cos\frac{11\varphi}{3}}{21446656} - \frac{414027\sqrt{3}\cos\frac{17\varphi}{3}}{493273088} + \frac{4179\sqrt{3}\cos\frac{23\varphi}{3}}{114509824} + \frac{375033\sqrt{3}\cos\frac{29\varphi}{3}}{4226453504} + \frac{135\sin\frac{\varphi}{3}}{90112} \\
&- \frac{675\sin\frac{5\varphi}{3}}{139264} - \frac{123489\sin\frac{11\varphi}{3}}{21446656} - \frac{1242081\sin\frac{17\varphi}{3}}{493273088} + \frac{12537\sin\frac{23\varphi}{3}}{114509824} + \frac{1125099\sin\frac{29\varphi}{3}}{4226453504} \\
\phi_{4,4,4} &= \frac{135\sqrt{3}\cos\frac{2\varphi}{3}}{95744} + \frac{45\sqrt{3}\cos\frac{4\varphi}{3}}{131072} + \frac{143487\sqrt{3}\cos\frac{8\varphi}{3}}{61276160} + \frac{926991\sqrt{3}\cos\frac{14\varphi}{3}}{493273088} + \frac{32002167\sqrt{3}\cos\frac{20\varphi}{3}}{51300401152} - \frac{51351\sqrt{3}\cos\frac{26\varphi}{3}}{1300447232} + \frac{405\sin\frac{2\varphi}{3}}{95744} \\
&- \frac{135\sin\frac{4\varphi}{3}}{131072} + \frac{430461\sin\frac{8\varphi}{3}}{61276160} + \frac{2780973\sin\frac{14\varphi}{3}}{493273088} + \frac{96006501\sin\frac{20\varphi}{3}}{51300401152} - \frac{154053\sin\frac{26\varphi}{3}}{1300447232} \\
\phi_{4,4,5} &= -\frac{9855\sqrt{3}\cos\frac{\varphi}{3}}{8912896} - \frac{572103\sqrt{3}\cos\frac{5\varphi}{3}}{245104640} - \frac{117\sqrt{3}\cos\frac{7\varphi}{3}}{524288} - \frac{2968203\sqrt{3}\cos\frac{11\varphi}{3}}{1127481344} - \frac{329537211\sqrt{3}\cos\frac{17\varphi}{3}}{205201604608} - \frac{2526434613\sqrt{3}\cos\frac{23\varphi}{3}}{5950846533632} \\
&+ \frac{3132411\sqrt{3}\cos\frac{29\varphi}{3}}{166457245696} + \frac{81227253\sqrt{3}\cos\frac{35\varphi}{3}}{3769767034880} + \frac{29565\sin\frac{\varphi}{3}}{8912896} - \frac{1716309\sin\frac{5\varphi}{3}}{245104640} + \frac{351\sin\frac{7\varphi}{3}}{524288} - \frac{8904609\sin\frac{11\varphi}{3}}{1127481344} - \frac{988611633\sin\frac{17\varphi}{3}}{205201604608} \\
&- \frac{7579303839\sin\frac{23\varphi}{3}}{5950846533632} + \frac{9397233\sin\frac{29\varphi}{3}}{166457245696} + \frac{243681759\sin\frac{35\varphi}{3}}{3769767034880}
\end{aligned}
$$

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\begin{aligned}
\phi_{6,4,0} &= -\frac{3\sqrt{3}\cos\frac{8\varphi}{3}}{335104} - \frac{9\sin\frac{8\varphi}{3}}{335104} \\
\phi_{6,4,1} &= \frac{3\sqrt{3}\cos\frac{5\varphi}{3}}{191488} + \frac{39\sqrt{3}\cos\frac{11\varphi}{3}}{1914880} - \frac{1791\sqrt{3}\cos\frac{23\varphi}{3}}{200392192} + \frac{9\sin\frac{5\varphi}{3}}{191488} + \frac{117\sin\frac{11\varphi}{3}}{1914880} - \frac{5373\sin\frac{23\varphi}{3}}{200392192} \\
\phi_{6,4,2} &= -\frac{27\sqrt{3}\cos\frac{2\varphi}{3}}{1531904} - \frac{639\sqrt{3}\cos\frac{8\varphi}{3}}{15319040} - \frac{68463\sqrt{3}\cos\frac{14\varphi}{3}}{2466365440} + \frac{5373\sqrt{3}\cos\frac{20\varphi}{3}}{801568768} + \frac{191637\sqrt{3}\cos\frac{26\varphi}{3}}{23245494272} - \frac{81\sin\frac{2\varphi}{3}}{1531904} - \frac{1917\sin\frac{8\varphi}{3}}{15319040} \\
&- \frac{205389\sin\frac{14\varphi}{3}}{2466365440} + \frac{16119\sin\frac{20\varphi}{3}}{801568768} + \frac{574911\sin\frac{26\varphi}{3}}{23245494272} \\
\phi_{6,4,3} &= \frac{9\sqrt{3}\cos\frac{\varphi}{3}}{557056} + \frac{1647\sqrt{3}\cos\frac{5\varphi}{3}}{30638080} + \frac{11493\sqrt{3}\cos\frac{11\varphi}{3}}{176168960} + \frac{1622793\sqrt{3}\cos\frac{17\varphi}{3}}{64125501440} - \frac{26865\sqrt{3}\cos\frac{23\varphi}{3}}{2905686784} - \frac{46559115\sqrt{3}\cos\frac{29\varphi}{3}}{5950846533632} - \frac{27\sin\frac{\varphi}{3}}{557056} \\
&+ \frac{4941\sin\frac{5\varphi}{3}}{30638080} + \frac{34479\sin\frac{11\varphi}{3}}{176168960} + \frac{4868379\sin\frac{17\varphi}{3}}{64125501440} - \frac{80595\sin\frac{23\varphi}{3}}{2905686784} - \frac{139677345\sin\frac{29\varphi}{3}}{5950846533632}
\end{aligned}
$$

$$
\begin{aligned}
\phi_{8,4,0} &= \frac{9\sqrt{3}\cos\frac{8\varphi}{3}}{107233280} + \frac{27\sin\frac{8\varphi}{3}}{107233280} \\
\phi_{8,4,1} &= -\frac{81\sqrt{3}\cos\frac{5\varphi}{3}}{428933120} - \frac{477\sqrt{3}\cos\frac{11\varphi}{3}}{1973092352} + \frac{5373\sqrt{3}\cos\frac{23\varphi}{3}}{46490988544} - \frac{243\sin\frac{5\varphi}{3}}{428933120} - \frac{1431\sin\frac{11\varphi}{3}}{1973092352} + \frac{16119\sin\frac{23\varphi}{3}}{46490988544}
\end{aligned}
$$

$$
\begin{aligned}
\phi_{0,5,0} &= \sin\frac{10\varphi}{3} - \frac{1}{\sqrt{3}}\cos\frac{10\varphi}{3} \\
\phi_{0,5,1} &= \frac{\cos\frac{7\varphi}{3}}{4\sqrt{3}} + \frac{7\cos\frac{13\varphi}{3}}{52\sqrt{3}} - \frac{1}{4}\sin\frac{7\varphi}{3} - \frac{7}{52}\sin\frac{13\varphi}{3} \\
\phi_{0,5,2} &= -\frac{1}{32}\sqrt{3}\cos\frac{4\varphi}{3} - \frac{5}{104}\sqrt{3}\cos\frac{10\varphi}{3} + \frac{3}{32}\sin\frac{4\varphi}{3} + \frac{15}{104}\sin\frac{10\varphi}{3} \\
\phi_{0,5,3} &= \frac{5\cos\frac{\varphi}{3}}{128\sqrt{3}} + \frac{53\sqrt{3}\cos\frac{7\varphi}{3}}{1664} + \frac{145\sqrt{3}\cos\frac{13\varphi}{3}}{6656} + \frac{77\cos\frac{19\varphi}{3}}{6656\sqrt{3}} - \frac{5}{128}\sin\frac{\varphi}{3} - \frac{159\sin\frac{7\varphi}{3}}{1664} \\
&- \frac{435\sin\frac{13\varphi}{3}}{6656} - \frac{77\sin\frac{19\varphi}{3}}{6656} \\
\phi_{0,5,4} &= -\frac{35\cos\frac{2\varphi}{3}}{2048\sqrt{3}} - \frac{95\cos\frac{4\varphi}{3}}{1664\sqrt{3}} - \frac{2407\sqrt{3}\cos\frac{10\varphi}{3}}{106496} - \frac{1043\cos\frac{16\varphi}{3}}{31616\sqrt{3}} - \frac{35\sin\frac{2\varphi}{3}}{2048} + \frac{95\sin\frac{4\varphi}{3}}{1664} \\
&+ \frac{7221\sin\frac{10\varphi}{3}}{106496} + \frac{1043\sin\frac{16\varphi}{3}}{31616} \\
\phi_{0,5,5} &= \frac{1155\sqrt{3}\cos\frac{\varphi}{3}}{106496} + \frac{21\sqrt{3}\cos\frac{5\varphi}{3}}{8192} + \frac{7685\sqrt{3}\cos\frac{7\varphi}{3}}{425984} + \frac{117621\sqrt{3}\cos\frac{13\varphi}{3}}{8093696} + \frac{42763\cos\frac{19\varphi}{3}}{2782208\sqrt{3}} \\
&+ \frac{14749\cos\frac{25\varphi}{3}}{11128832\sqrt{3}} - \frac{3465\sin\frac{\varphi}{3}}{106496} + \frac{63\sin\frac{5\varphi}{3}}{8192} - \frac{23055\sin\frac{7\varphi}{3}}{425984} - \frac{352863\sin\frac{13\varphi}{3}}{8093696} \\
&- \frac{42763\sin\frac{19\varphi}{3}}{2782208} - \frac{14749\sin\frac{25\varphi}{3}}{11128832} \\
\phi_{0,5,6} &= -\frac{1281\sqrt{3}\cos\frac{2\varphi}{3}}{212992} - \frac{43645\sqrt{3}\cos\frac{4\varphi}{3}}{3407872} - \frac{77\sqrt{3}\cos\frac{8\varphi}{3}}{65536} - \frac{17885\sqrt{3}\cos\frac{10\varphi}{3}}{1245184} - \frac{562807\sqrt{3}\cos\frac{16\varphi}{3}}{64749568} \\
&- \frac{254793\sqrt{3}\cos\frac{22\varphi}{3}}{101171200} - \frac{3843\sin\frac{2\varphi}{3}}{212992} + \frac{130935\sin\frac{4\varphi}{3}}{3407872} - \frac{231\sin\frac{8\varphi}{3}}{65536} + \frac{53655\sin\frac{10\varphi}{3}}{1245184} \\
&+ \frac{1688421\sin\frac{16\varphi}{3}}{64749568} + \frac{764379\sin\frac{22\varphi}{3}}{101171200} \\
\phi_{0,5,7} &= \frac{57855\sqrt{3}\cos\frac{\varphi}{3}}{6815744} + \frac{11165\sqrt{3}\cos\frac{5\varphi}{3}}{3407872} + \frac{1577555\sqrt{3}\cos\frac{7\varphi}{3}}{129499136} + \frac{143\sqrt{3}\cos\frac{11\varphi}{3}}{262144} \\
&+ \frac{88150615\cos\frac{13\varphi}{3}}{2848980992\sqrt{3}} + \frac{32566961\sqrt{3}\cos\frac{19\varphi}{3}}{6474956800} + \frac{1929147\sqrt{3}\cos\frac{25\varphi}{3}}{1618739200} + \frac{3279997\cos\frac{31\varphi}{3}}{17806131200\sqrt{3}} \\
&- \frac{173565\sin\frac{\varphi}{3}}{6815744} + \frac{33495\sin\frac{5\varphi}{3}}{3407872} - \frac{4732665\sin\frac{7\varphi}{3}}{129499136} + \frac{429\sin\frac{11\varphi}{3}}{262144} - \frac{88150615\sin\frac{13\varphi}{3}}{2848980992} \\
&- \frac{97700883\sin\frac{19\varphi}{3}}{6474956800} - \frac{5787441\sin\frac{25\varphi}{3}}{1618739200} - \frac{3279997\sin\frac{31\varphi}{3}}{17806131200} \\
\phi_{0,5,8} &= -\frac{1172325\sqrt{3}\cos\frac{2\varphi}{3}}{218103808} - \frac{1216131\sqrt{3}\cos\frac{4\varphi}{3}}{129499136} - \frac{231\sqrt{3}\cos\frac{8\varphi}{3}}{131072} - \frac{107170105\cos\frac{10\varphi}{3}}{3506438144\sqrt{3}} \\
&- \frac{2145\sqrt{3}\cos\frac{14\varphi}{3}}{8388608} - \frac{2318071\cos\frac{16\varphi}{3}}{111288320\sqrt{3}} - \frac{292877947\sqrt{3}\cos\frac{22\varphi}{3}}{103599308800} - \frac{964432279\cos\frac{28\varphi}{3}}{551990067200\sqrt{3}} \\
&- \frac{3516975\sin\frac{2\varphi}{3}}{218103808} + \frac{3648393\sin\frac{4\varphi}{3}}{129499136} - \frac{693\sin\frac{8\varphi}{3}}{131072} + \frac{107170105\sin\frac{10\varphi}{3}}{3506438144} - \frac{6435\sin\frac{14\varphi}{3}}{8388608} \\
&+ \frac{2318071\sin\frac{16\varphi}{3}}{111288320} + \frac{878633841\sin\frac{22\varphi}{3}}{103599308800} + \frac{964432279\sin\frac{28\varphi}{3}}{551990067200}
\end{aligned}
$$

$$
\begin{aligned}
\phi_{0,6,0} &= \cos 4\varphi \\
\phi_{0,6,1} &= -\frac{1}{4}\cos 3\varphi - \frac{3}{20}\cos 5\varphi \\
\phi_{0,6,2} &= \frac{3}{32}\cos 2\varphi + \frac{3}{20}\cos 4\varphi \\
\phi_{0,6,3} &= -\frac{5\cos\varphi}{128} - \frac{63}{640}\cos 3\varphi - \frac{11}{160}\cos 5\varphi - \frac{1}{80}\cos 7\varphi \\
\phi_{0,6,4} &= \frac{35}{2048} + \frac{15}{256}\cos 2\varphi + \frac{363\cos 4\varphi}{5120} + \frac{157\cos 6\varphi}{4480} \\
\phi_{0,6,5} &= -\frac{21\cos\varphi}{512} - \frac{231\cos 3\varphi}{4096} - \frac{6603\cos 5\varphi}{143360} - \frac{471\cos 7\varphi}{28672} - \frac{41\cos 9\varphi}{28672} \\
\phi_{0,6,6} &= \frac{189}{10240} + \frac{2849\cos 2\varphi}{65536} + \frac{1301\cos 4\varphi}{28672} + \frac{25461\cos 6\varphi}{917504} + \frac{349\cos 8\varphi}{43008} \\
\phi_{0,6,7} &= -\frac{47817\cos\varphi}{1310720} - \frac{10495\cos 3\varphi}{262144} - \frac{120495\cos 5\varphi}{3670016} - \frac{59261\cos 7\varphi}{3670016} \\
&- \frac{6631\cos 9\varphi}{1720320} - \frac{229\cos 11\varphi}{1146880}
\end{aligned}
$$

$$
\begin{aligned}
\phi_{0,7,0} &= \sin\frac{14\varphi}{3} + \frac{1}{\sqrt{3}}\cos\frac{14\varphi}{3} \\
\phi_{0,7,1} &= -\frac{\cos\frac{11\varphi}{3}}{4\sqrt{3}} - \frac{11\cos\frac{17\varphi}{3}}{68\sqrt{3}} - \frac{1}{4}\sin\frac{11\varphi}{3} - \frac{11}{68}\sin\frac{17\varphi}{3} \\
\phi_{0,7,2} &= \frac{1}{32}\sqrt{3}\cos\frac{8\varphi}{3} + \frac{7}{136}\sqrt{3}\cos\frac{14\varphi}{3} + \frac{3}{32}\sin\frac{8\varphi}{3} + \frac{21}{136}\sin\frac{14\varphi}{3} \\
\phi_{0,7,3} &= -\frac{5\cos\frac{5\varphi}{3}}{128\sqrt{3}} - \frac{73\sqrt{3}\cos\frac{11\varphi}{3}}{2176} - \frac{259\sqrt{3}\cos\frac{17\varphi}{3}}{10880} - \frac{143\cos\frac{23\varphi}{3}}{10880\sqrt{3}} \\
&- \frac{5}{128}\sin\frac{5\varphi}{3} - \frac{219\sin\frac{11\varphi}{3}}{2176} - \frac{777\sin\frac{17\varphi}{3}}{10880} - \frac{143\sin\frac{23\varphi}{3}}{10880} \\
\phi_{0,7,4} &= \frac{35\cos\frac{2\varphi}{3}}{2048\sqrt{3}} + \frac{65\cos\frac{8\varphi}{3}}{1088\sqrt{3}} + \frac{851\sqrt{3}\cos\frac{14\varphi}{3}}{34816} + \frac{1835\cos\frac{20\varphi}{3}}{50048\sqrt{3}} + \frac{35\sin\frac{2\varphi}{3}}{2048} \\
&+ \frac{65\sin\frac{8\varphi}{3}}{1088} + \frac{2553\sin\frac{14\varphi}{3}}{34816} + \frac{1835\sin\frac{20\varphi}{3}}{50048} \\
\phi_{0,7,5} &= -\frac{21\sqrt{3}\cos\frac{\varphi}{3}}{8192} - \frac{1575\sqrt{3}\cos\frac{5\varphi}{3}}{139264} - \frac{2701\sqrt{3}\cos\frac{11\varphi}{3}}{139264} - \frac{51273\sqrt{3}\cos\frac{17\varphi}{3}}{3203072} \\
&- \frac{89915\cos\frac{23\varphi}{3}}{5204992\sqrt{3}} - \frac{979\cos\frac{29\varphi}{3}}{650624\sqrt{3}} + \frac{63\sin\frac{\varphi}{3}}{8192} - \frac{4725\sin\frac{5\varphi}{3}}{139264} - \frac{8103\sin\frac{11\varphi}{3}}{139264} \\
&- \frac{153819\sin\frac{17\varphi}{3}}{3203072} - \frac{89915\sin\frac{23\varphi}{3}}{5204992} - \frac{979\sin\frac{29\varphi}{3}}{650624} \\
\phi_{0,7,6} &= \frac{1743\sqrt{3}\cos\frac{2\varphi}{3}}{278528} + \frac{77\sqrt{3}\cos\frac{4\varphi}{3}}{65536} + \frac{15281\sqrt{3}\cos\frac{8\varphi}{3}}{1114112} + \frac{100799\sqrt{3}\cos\frac{14\varphi}{3}}{6406144} \\
&+ \frac{3231697\sqrt{3}\cos\frac{20\varphi}{3}}{333119488} + \frac{132627\sqrt{3}\cos\frac{26\varphi}{3}}{46444544} + \frac{5229\sin\frac{2\varphi}{3}}{278528} - \frac{231\sin\frac{4\varphi}{3}}{65536} \\
&+ \frac{45843\sin\frac{8\varphi}{3}}{1114112} + \frac{302397\sin\frac{14\varphi}{3}}{6406144} + \frac{9695091\sin\frac{20\varphi}{3}}{333119488} + \frac{397881\sin\frac{26\varphi}{3}}{46444544}
\end{aligned}
$$

$$
\begin{aligned}
\phi_{0,8,0} &= \sin\frac{16\varphi}{3} - \frac{1}{\sqrt{3}}\cos\frac{16\varphi}{3} \\
\phi_{0,8,1} &= \frac{\cos\frac{13\varphi}{3}}{4\sqrt{3}} + \frac{13\cos\frac{19\varphi}{3}}{76\sqrt{3}} - \frac{1}{4}\sin\frac{13\varphi}{3} - \frac{13}{76}\sin\frac{19\varphi}{3} \\
\phi_{0,8,2} &= -\frac{1}{32}\sqrt{3}\cos\frac{10\varphi}{3} - \frac{1}{19}\sqrt{3}\cos\frac{16\varphi}{3} + \frac{3}{32}\sin\frac{10\varphi}{3} + \frac{3}{19}\sin\frac{16\varphi}{3} \\
\phi_{0,8,3} &= \frac{5\cos\frac{7\varphi}{3}}{128\sqrt{3}} + \frac{83\sqrt{3}\cos\frac{13\varphi}{3}}{2432} + \frac{41\sqrt{3}\cos\frac{19\varphi}{3}}{1672} + \frac{91\cos\frac{25\varphi}{3}}{6688\sqrt{3}} - \frac{5}{128}\sin\frac{7\varphi}{3} \\
&- \frac{249\sin\frac{13\varphi}{3}}{2432} - \frac{123\sin\frac{19\varphi}{3}}{1672} - \frac{91\sin\frac{25\varphi}{3}}{6688} \\
\phi_{0,8,4} &= -\frac{35\cos\frac{4\varphi}{3}}{2048\sqrt{3}} - \frac{295\cos\frac{10\varphi}{3}}{4864\sqrt{3}} - \frac{5371\sqrt{3}\cos\frac{16\varphi}{3}}{214016} - \frac{2309\cos\frac{22\varphi}{3}}{60800\sqrt{3}} + \frac{35\sin\frac{4\varphi}{3}}{2048} \\
&+ \frac{295\sin\frac{10\varphi}{3}}{4864} + \frac{16113\sin\frac{16\varphi}{3}}{214016} + \frac{2309\sin\frac{22\varphi}{3}}{60800} \\
\phi_{0,8,5} &= \frac{21\sqrt{3}\cos\frac{\varphi}{3}}{8192} + \frac{1785\sqrt{3}\cos\frac{7\varphi}{3}}{155648} + \frac{17015\sqrt{3}\cos\frac{13\varphi}{3}}{856064} + \frac{354033\sqrt{3}\cos\frac{19\varphi}{3}}{21401600} \\
&+ \frac{122377\cos\frac{25\varphi}{3}}{6809600\sqrt{3}} + \frac{116909\cos\frac{31\varphi}{3}}{74905600\sqrt{3}} - \frac{63\sin\frac{\varphi}{3}}{8192} - \frac{5355\sin\frac{7\varphi}{3}}{155648} - \frac{51045\sin\frac{13\varphi}{3}}{856064} \\
&- \frac{1062099\sin\frac{19\varphi}{3}}{21401600} - \frac{122377\sin\frac{25\varphi}{3}}{6809600} - \frac{116909\sin\frac{31\varphi}{3}}{74905600} \\
\phi_{0,8,6} &= -\frac{77\sqrt{3}\cos\frac{2\varphi}{3}}{65536} - \frac{987\sqrt{3}\cos\frac{4\varphi}{3}}{155648} - \frac{96145\sqrt{3}\cos\frac{10\varphi}{3}}{6848512} - \frac{4343\sqrt{3}\cos\frac{16\varphi}{3}}{267520} \\
&- \frac{24148337\sqrt{3}\cos\frac{22\varphi}{3}}{2396979200} - \frac{988653\sqrt{3}\cos\frac{28\varphi}{3}}{331724800} - \frac{231\sin\frac{2\varphi}{3}}{65536} + \frac{2961\sin\frac{4\varphi}{3}}{155648} \\
&+ \frac{288435\sin\frac{10\varphi}{3}}{6848512} + \frac{13029\sin\frac{16\varphi}{3}}{267520} + \frac{72445011\sin\frac{22\varphi}{3}}{2396979200} + \frac{2965959\sin\frac{28\varphi}{3}}{331724800}
\end{aligned}
$$

$$
\begin{aligned}
\phi_{0,9,0} &= \cos 6\varphi \\
\phi_{0,9,1} &= -\frac{1}{4}\cos 5\varphi - \frac{5}{28}\cos 7\varphi \\
\phi_{0,9,2} &= \frac{3}{32}\cos 4\varphi + \frac{9}{56}\cos 6\varphi \\
\phi_{0,9,3} &= -\frac{5}{128}\cos 3\varphi - \frac{93}{896}\cos 5\varphi - \frac{135\cos 7\varphi}{1792} - \frac{25\cos 9\varphi}{1792} \\
\phi_{0,9,4} &= \frac{35\cos 2\varphi}{2048} + \frac{55}{896}\cos 4\varphi + \frac{315\cos 6\varphi}{4096} + \frac{5}{128}\cos 8\varphi \\
\phi_{0,9,5} &= -\frac{63\cos\varphi}{8192} - \frac{285\cos 3\varphi}{8192} - \frac{6975\cos 5\varphi}{114688} - \frac{835\cos 7\varphi}{16384} - \frac{19\cos 9\varphi}{1024} - \frac{23\cos 11\varphi}{14336}
\end{aligned}
$$

$$
\begin{aligned}
\phi_{0,10,0} &= \sin\frac{20\varphi}{3} + \frac{1}{\sqrt{3}}\cos\frac{20\varphi}{3} \\
\phi_{0,10,1} &= -\frac{\cos\frac{17\varphi}{3}}{4\sqrt{3}} - \frac{17\cos\frac{23\varphi}{3}}{92\sqrt{3}} - \frac{1}{4}\sin\frac{17\varphi}{3} - \frac{17}{92}\sin\frac{23\varphi}{3} \\
\phi_{0,10,2} &= \frac{1}{32}\sqrt{3}\cos\frac{14\varphi}{3} + \frac{5}{92}\sqrt{3}\cos\frac{20\varphi}{3} + \frac{3}{32}\sin\frac{14\varphi}{3} + \frac{15}{92}\sin\frac{20\varphi}{3} \\
\phi_{0,10,3} &= -\frac{5\cos\frac{11\varphi}{3}}{128\sqrt{3}} - \frac{103\sqrt{3}\cos\frac{17\varphi}{3}}{2944} - \frac{245\sqrt{3}\cos\frac{23\varphi}{3}}{9568} - \frac{17\cos\frac{29\varphi}{3}}{1196\sqrt{3}} - \frac{5}{128}\sin\frac{11\varphi}{3} \\
&- \frac{309\sin\frac{17\varphi}{3}}{2944} - \frac{735\sin\frac{23\varphi}{3}}{9568} - \frac{17\sin\frac{29\varphi}{3}}{1196} \\
\phi_{0,10,4} &= \frac{35\cos\frac{8\varphi}{3}}{2048\sqrt{3}} + \frac{365\cos\frac{14\varphi}{3}}{5888\sqrt{3}} + \frac{7987\sqrt{3}\cos\frac{20\varphi}{3}}{306176} + \frac{3413\cos\frac{26\varphi}{3}}{85376\sqrt{3}} + \frac{35\sin\frac{8\varphi}{3}}{2048} \\
&+ \frac{365\sin\frac{14\varphi}{3}}{5888} + \frac{23961\sin\frac{20\varphi}{3}}{306176} + \frac{3413\sin\frac{26\varphi}{3}}{85376}
\end{aligned}
$$

$$
\begin{aligned}
\phi_{0,11,0} &= \sin\frac{22\varphi}{3} - \frac{1}{\sqrt{3}}\cos\frac{22\varphi}{3} \\
\phi_{0,11,1} &= \frac{\cos\frac{19\varphi}{3}}{4\sqrt{3}} + \frac{19\cos\frac{25\varphi}{3}}{100\sqrt{3}} - \frac{1}{4}\sin\frac{19\varphi}{3} - \frac{19}{100}\sin\frac{25\varphi}{3} \\
\phi_{0,11,2} &= -\frac{1}{32}\sqrt{3}\cos\frac{16\varphi}{3} - \frac{11}{200}\sqrt{3}\cos\frac{22\varphi}{3} + \frac{3}{32}\sin\frac{16\varphi}{3} + \frac{33}{200}\sin\frac{22\varphi}{3} \\
\phi_{0,11,3} &= \frac{5\cos\frac{13\varphi}{3}}{128\sqrt{3}} + \frac{113\sqrt{3}\cos\frac{19\varphi}{3}}{3200} + \frac{583\sqrt{3}\cos\frac{25\varphi}{3}}{22400} + \frac{323\cos\frac{31\varphi}{3}}{22400\sqrt{3}} - \frac{5}{128}\sin\frac{13\varphi}{3} \\
&- \frac{339\sin\frac{19\varphi}{3}}{3200} - \frac{1749\sin\frac{25\varphi}{3}}{22400} - \frac{323\sin\frac{31\varphi}{3}}{22400} \\
\phi_{0,11,4} &= -\frac{35\cos\frac{10\varphi}{3}}{2048\sqrt{3}} - \frac{\cos\frac{16\varphi}{3}}{16\sqrt{3}} - \frac{9487\sqrt{3}\cos\frac{22\varphi}{3}}{358400} - \frac{4043\cos\frac{28\varphi}{3}}{99200\sqrt{3}} + \frac{35\sin\frac{10\varphi}{3}}{2048} \\
&+ \frac{1}{16}\sin\frac{16\varphi}{3} + \frac{28461\sin\frac{22\varphi}{3}}{358400} + \frac{4043\sin\frac{28\varphi}{3}}{99200}
\end{aligned}
$$

$$
\begin{aligned}
\phi_{0,12,0} &= \cos 8\varphi \\
\phi_{0,12,1} &= -\frac{1}{4}\cos 7\varphi - \frac{7}{36}\cos 9\varphi \\
\phi_{0,12,2} &= \frac{3}{32}\cos 6\varphi + \frac{1}{6}\cos 8\varphi \\
\phi_{0,12,3} &= -\frac{5}{128}\cos 5\varphi - \frac{41}{384}\cos 7\varphi - \frac{19}{240}\cos 9\varphi - \frac{7}{480}\cos 11\varphi
\end{aligned}
$$

$$
\begin{aligned}
\phi_{0,13,0} &= \sin\frac{26\varphi}{3} + \frac{1}{\sqrt{3}}\cos\frac{26\varphi}{3} \\
\phi_{0,13,1} &= -\frac{\cos\frac{23\varphi}{3}}{4\sqrt{3}} - \frac{23\cos\frac{29\varphi}{3}}{116\sqrt{3}} - \frac{1}{4}\sin\frac{23\varphi}{3} - \frac{23}{116}\sin\frac{29\varphi}{3} \\
\phi_{0,13,2} &= \frac{1}{32}\sqrt{3}\cos\frac{20\varphi}{3} + \frac{13}{232}\sqrt{3}\cos\frac{26\varphi}{3} + \frac{3}{32}\sin\frac{20\varphi}{3} + \frac{39}{232}\sin\frac{26\varphi}{3}
\end{aligned}
$$

$$
\begin{aligned}
\phi_{0,14,0} &= \sin\frac{28\varphi}{3} - \frac{1}{\sqrt{3}}\cos\frac{28\varphi}{3} \\
\phi_{0,14,1} &= \frac{\cos\frac{25\varphi}{3}}{4\sqrt{3}} + \frac{25\cos\frac{31\varphi}{3}}{124\sqrt{3}} - \frac{1}{4}\sin\frac{25\varphi}{3} - \frac{25}{124}\sin\frac{31\varphi}{3} \\
\phi_{0,14,2} &= -\frac{1}{32}\sqrt{3}\cos\frac{22\varphi}{3} - \frac{7}{124}\sqrt{3}\cos\frac{28\varphi}{3} + \frac{3}{32}\sin\frac{22\varphi}{3} + \frac{21}{124}\sin\frac{28\varphi}{3}
\end{aligned}
$$

$$
\begin{aligned}
\phi_{0,15,0} &= \cos 10\varphi \\
\phi_{0,15,1} &= -\frac{1}{4}\cos 9\varphi - \frac{9}{44}\cos 11\varphi
\end{aligned}
$$

$$
\phi_{0,16,0} = \sin\frac{32\varphi}{3} + \frac{1}{\sqrt{3}}\cos\frac{32\varphi}{3}
$$

$$
\phi_{0,17,0} = \sin\frac{34\varphi}{3} - \frac{1}{\sqrt{3}}\cos\frac{34\varphi}{3}
$$

4. Dual functions and dual shadows for 90° V-notch $-\pi \le \varphi \le \pi/2$

4.1. First singular exponent ($j = 1$).

$$
\begin{aligned}
\psi_{0,1,0} &= \sin\frac{2\varphi}{3} + \frac{1}{\sqrt{3}}\cos\frac{2\varphi}{3}\\
\psi_{0,1,1} &= -\frac{1}{4\sqrt{3}}\cos\frac{5\varphi}{3} - \frac{1}{4}\sin\frac{5\varphi}{3} - \frac{5}{4}\sin\frac{\varphi}{3} + \frac{5}{4\sqrt{3}}\cos\frac{\varphi}{3}\\
\psi_{0,1,2} &= -\frac{\sqrt{3}}{8}\cos\frac{2\varphi}{3} - \frac{3}{8}\sin\frac{2\varphi}{3} + \frac{\sqrt{3}}{32}\cos\frac{8\varphi}{3} + \frac{3}{32}\sin\frac{8\varphi}{3}\\
\psi_{0,1,3} &= \frac{5\sqrt{3}}{128}\cos\frac{\varphi}{3} - \frac{15}{128}\sin\frac{\varphi}{3} + \frac{7\sqrt{3}}{128}\cos\frac{5\varphi}{3} + \frac{21}{128}\sin\frac{5\varphi}{3} - \frac{5}{128\sqrt{3}}\cos\frac{11\varphi}{3} - \frac{5}{128}\sin\frac{11\varphi}{3}\\
&+ \frac{25}{128\sqrt{3}}\cos\frac{7\varphi}{3} - \frac{25}{128}\sin\frac{7\varphi}{3}\\
\psi_{0,1,4} &= -\frac{65\sqrt{3}}{2048}\cos\frac{2\varphi}{3} - \frac{195}{2048}\sin\frac{2\varphi}{3} - \frac{85}{896\sqrt{3}}\cos\frac{4\varphi}{3} + \frac{85}{896}\sin\frac{4\varphi}{3} - \frac{5}{64\sqrt{3}}\cos\frac{8\varphi}{3} - \frac{5}{64}\sin\frac{8\varphi}{3} + \frac{35}{2048\sqrt{3}}\cos\frac{14\varphi}{3}\\
&+ \frac{35}{2048}\sin\frac{14\varphi}{3}
\end{aligned}
$$

$$
\begin{aligned}
\psi_{2,1,0} &= -\frac{1}{4}\sqrt{3}\cos\frac{2\varphi}{3} - \frac{3}{4}\sin\frac{2\varphi}{3}\\
\psi_{2,1,1} &= \frac{1}{32}\sqrt{3}\cos\frac{\varphi}{3} + \frac{3}{16}\sqrt{3}\cos\frac{5\varphi}{3} + \frac{31}{224}\sqrt{3}\cos\frac{7\varphi}{3} - \frac{3}{32}\sin\frac{\varphi}{3} + \frac{9}{16}\sin\frac{5\varphi}{3} - \frac{93}{224}\sin\frac{7\varphi}{3}\\
\psi_{2,1,2} &= -\frac{15}{256}\sqrt{3}\cos\frac{2\varphi}{3} - \frac{15}{448}\sqrt{3}\cos\frac{4\varphi}{3} - \frac{15}{128}\sqrt{3}\cos\frac{8\varphi}{3} - \frac{45}{256}\sin\frac{2\varphi}{3} + \frac{45}{448}\sin\frac{4\varphi}{3} - \frac{45}{128}\sin\frac{8\varphi}{3}\\
\psi_{2,1,3} &= \frac{3}{112}\sqrt{3}\cos\frac{\varphi}{3} + \frac{15}{256}\sqrt{3}\cos\frac{5\varphi}{3} + \frac{167\sqrt{3}\cos\frac{7\varphi}{3}}{35840} + \frac{35}{512}\sqrt{3}\cos\frac{11\varphi}{3} + \frac{16659\sqrt{3}\cos\frac{13\varphi}{3}}{465920} - \frac{9}{112}\sin\frac{\varphi}{3} + \frac{45}{256}\sin\frac{5\varphi}{3}\\
&- \frac{501\sin\frac{7\varphi}{3}}{35840} + \frac{105}{512}\sin\frac{11\varphi}{3} - \frac{49977\sin\frac{13\varphi}{3}}{465920}\\
\psi_{2,1,4} &= -\frac{241\sqrt{3}\cos\frac{2\varphi}{3}}{8192} - \frac{1857\sqrt{3}\cos\frac{4\varphi}{3}}{286720} - \frac{385\sqrt{3}\cos\frac{8\varphi}{3}}{8192} - \frac{341\sqrt{3}\cos\frac{10\varphi}{3}}{46592} - \frac{315\sqrt{3}\cos\frac{14\varphi}{3}}{8192} - \frac{723\sin\frac{2\varphi}{3}}{8192} + \frac{5571\sin\frac{4\varphi}{3}}{286720}\\
&- \frac{1155\sin\frac{8\varphi}{3}}{8192} + \frac{1023\sin\frac{10\varphi}{3}}{46592} - \frac{945\sin\frac{14\varphi}{3}}{8192}
\end{aligned}
$$

$$
\begin{aligned}
\psi_{4,1,0} &= \frac{3}{128}\sqrt{3}\cos\frac{2\varphi}{3} + \frac{9}{128}\sin\frac{2\varphi}{3}\\
\psi_{4,1,1} &= -\frac{9}{512}\sqrt{3}\cos\frac{\varphi}{3} - \frac{15}{512}\sqrt{3}\cos\frac{5\varphi}{3} - \frac{93\sqrt{3}\cos\frac{7\varphi}{3}}{8960} + \frac{9\sqrt{3}\cos\frac{13\varphi}{3}}{1280} + \frac{27}{512}\sin\frac{\varphi}{3} - \frac{45}{512}\sin\frac{5\varphi}{3} + \frac{279\sin\frac{7\varphi}{3}}{8960}\\
&- \frac{27\sin\frac{13\varphi}{3}}{1280}\\
\psi_{4,1,2} &= \frac{15}{512}\sqrt{3}\cos\frac{2\varphi}{3} + \frac{2601\sqrt{3}\cos\frac{4\varphi}{3}}{143360} + \frac{105\sqrt{3}\cos\frac{8\varphi}{3}}{4096} + \frac{159\sqrt{3}\cos\frac{10\varphi}{3}}{23296} + \frac{45}{512}\sin\frac{2\varphi}{3} - \frac{7803\sin\frac{4\varphi}{3}}{143360} + \frac{315\sin\frac{8\varphi}{3}}{4096}\\
&- \frac{477\sin\frac{10\varphi}{3}}{23296}\\
\psi_{4,1,3} &= -\frac{3051\sqrt{3}\cos\frac{\varphi}{3}}{114688} - \frac{525\sqrt{3}\cos\frac{5\varphi}{3}}{16384} - \frac{3051\sqrt{3}\cos\frac{7\varphi}{3}}{212992} - \frac{315\sqrt{3}\cos\frac{11\varphi}{3}}{16384} - \frac{190557\sqrt{3}\cos\frac{13\varphi}{3}}{29818880} + \frac{1248279\sqrt{3}\cos\frac{19\varphi}{3}}{566558720}\\
&+ \frac{9153\sin\frac{\varphi}{3}}{114688} - \frac{1575\sin\frac{5\varphi}{3}}{16384} + \frac{9153\sin\frac{7\varphi}{3}}{212992} - \frac{945\sin\frac{11\varphi}{3}}{16384} + \frac{571671\sin\frac{13\varphi}{3}}{29818880} - \frac{3744837\sin\frac{19\varphi}{3}}{566558720}\\
\psi_{4,1,4} &= \frac{10317\sqrt{3}\cos\frac{2\varphi}{3}}{327680} + \frac{9219\sqrt{3}\cos\frac{4\varphi}{3}}{425984} + \frac{945\sqrt{3}\cos\frac{8\varphi}{3}}{32768} + \frac{418479\sqrt{3}\cos\frac{10\varphi}{3}}{36700160} + \frac{3465\sqrt{3}\cos\frac{14\varphi}{3}}{262144} + \frac{3825\sqrt{3}\cos\frac{16\varphi}{3}}{1011712}\\
&+ \frac{30951\sin\frac{2\varphi}{3}}{327680} - \frac{27657\sin\frac{4\varphi}{3}}{425984} + \frac{2835\sin\frac{8\varphi}{3}}{32768} - \frac{1255437\sin\frac{10\varphi}{3}}{36700160} + \frac{10395\sin\frac{14\varphi}{3}}{262144} - \frac{11475\sin\frac{16\varphi}{3}}{1011712}
\end{aligned}
$$

4.2. **Higher order singular exponent** ($j = 2, 4$)**.**

$$
\begin{aligned}
\psi_{0,2,0} &= \sin\frac{4\varphi}{3} - \frac{1}{\sqrt{3}}\cos\frac{4\varphi}{3}\\
\psi_{0,2,1} &= \frac{1}{4\sqrt{3}}\cos\frac{7\varphi}{3} - \frac{1}{4}\sin\frac{7\varphi}{3} - \frac{7}{4}\sin\frac{\varphi}{3} + \frac{7}{4\sqrt{3}}\cos\frac{\varphi}{3}\\
\psi_{0,2,2} &= -\frac{\sqrt{3}}{4}\cos\frac{4\varphi}{3} + \frac{3}{4}\sin\frac{4\varphi}{3} - \frac{\sqrt{3}}{32}\cos\frac{10\varphi}{3} + \frac{3}{32}\sin\frac{10\varphi}{3}\\
\psi_{0,2,3} &= \frac{\sqrt{3}}{32}\cos\frac{\varphi}{3} - \frac{3}{32}\sin\frac{\varphi}{3} + \frac{17\sqrt{3}}{128}\cos\frac{7\varphi}{3} - \frac{51}{128}\sin\frac{7\varphi}{3} + \frac{5}{128\sqrt{3}}\cos\frac{13\varphi}{3} - \frac{5}{128}\sin\frac{13\varphi}{3} + \frac{7}{16}\sin\frac{5\varphi}{3} + \frac{7}{16\sqrt{3}}\cos\frac{5\varphi}{3}\\
\psi_{0,2,4} &= -\frac{91}{640\sqrt{3}}\cos\frac{2\varphi}{3} - \frac{29\sqrt{3}}{1024}\cos\frac{4\varphi}{3} - \frac{55}{256\sqrt{3}}\cos\frac{10\varphi}{3} - \frac{35}{2048\sqrt{3}}\cos\frac{16\varphi}{3} - \frac{91}{640}\sin\frac{2\varphi}{3} + \frac{87}{1024}\sin\frac{4\varphi}{3} + \frac{55}{256}\sin\frac{10\varphi}{3}\\
&+ \frac{35}{2048}\sin\frac{16\varphi}{3}
\end{aligned}
$$

$$
\begin{aligned}
\psi_{0,4,0} &= \sin\frac{8}{3}\varphi + \frac{1}{\sqrt{3}}\cos\frac{8}{3}\varphi\\
\psi_{0,4,1} &= -\frac{1}{4\sqrt{3}}\cos\frac{11\varphi}{3} - \frac{1}{4}\sin\frac{11\varphi}{3} - \frac{11}{20}\sin\frac{5\varphi}{3} - \frac{11}{20\sqrt{3}}\cos\frac{5\varphi}{3}\\
\psi_{0,4,2} &= \frac{1}{10}\sqrt{3}\cos\frac{8\varphi}{3} + \frac{1}{32}\sqrt{3}\cos\frac{14\varphi}{3} + \frac{3}{10}\sin\frac{8\varphi}{3} + \frac{3}{32}\sin\frac{14\varphi}{3}\\
\psi_{0,4,3} &= \frac{11\cos\frac{\varphi}{3}}{160\sqrt{3}} - \frac{7}{80}\sqrt{3}\cos\frac{5\varphi}{3} - \frac{37}{640}\sqrt{3}\cos\frac{11\varphi}{3} - \frac{5\cos\frac{17\varphi}{3}}{128\sqrt{3}} - \frac{11}{160}\sin\frac{\varphi}{3} - \frac{21}{80}\sin\frac{5\varphi}{3} - \frac{111}{640}\sin\frac{11\varphi}{3} - \frac{5}{128}\sin\frac{17\varphi}{3}\\
\psi_{0,4,4} &= -\frac{53\cos\frac{2\varphi}{3}}{640\sqrt{3}} + \frac{427\sqrt{3}\cos\frac{8\varphi}{3}}{5120} + \frac{25\cos\frac{14\varphi}{3}}{256\sqrt{3}} + \frac{35\cos\frac{20\varphi}{3}}{2048\sqrt{3}} - \frac{53}{640}\sin\frac{2\varphi}{3} + \frac{1281\sin\frac{8\varphi}{3}}{5120} + \frac{25}{256}\sin\frac{14\varphi}{3} + \frac{35\sin\frac{20\varphi}{3}}{2048}
\end{aligned}
$$

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