On the structure of the Galois group of the Abelian closure of a number field

Georges Gras

ON THE STRUCTURE OF THE GALOIS GROUP OF THE ABELIAN CLOSURE OF A NUMBER FIELD

by

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Abstract. — Following a paper by Athanasios Angelakis and Peter Stevenhagen on the determination of imaginary quadratic fields having the same absolute Abelian Galois group A, we study this property for arbitrary number fields. We show that such a property is probably not easily generalizable, apart from imaginary quadratic fields, because of some p-adic obstructions coming from the global units. By restriction to the p-Sylow subgroups of A, we show that the corresponding study is related to a generalization of the classical notion of p-rational fields. However, we obtain some non-trivial information about the structure of the profinite group A, for every number field, by application of results published in our book on class field theory.

Résumé. — A partir d'un article de Athanasios Angelakis et Peter Stevenhagen sur la détermination de corps quadratiques imaginaires ayant le même groupe de Galois Abélien absolu A, nous étudions cette propriété pour les corps de nombres quelconques. Nous montrons qu'une telle propriété n'est probablement pas facilement généralisable, en dehors des corps quadratiques imaginaires, en raison d'obstructions p-adiques provenant des unités globales. En se restreignant aux p-sous-groupes de Sylow de A, nous montrons que l'étude correspondante est liée à une généralisation de la notion classique de corps p-rationnels. Cependant, nous obtenons des informations non triviales sur la structure du groupe profini A, pour tout corps de nombres, par application de résultats publiés dans notre livre sur la théorie du corps de classes.

1. Introduction – Notations

Let K be a number field of signature $(r_1,\, r_2)$ for which $r_1 + 2\,r_2 = [K : \mathbb{Q}]$, and let A_K be the Galois group $\mathrm{Gal}(\overline{K}^{\mathrm{ab}}/K)$ where $\overline{K}^{\mathrm{ab}}$ is the maximal Abelian pro-extension of K. The question that was asked was the following: in what circumstances the groups A_{K_1} and A_{K_2} are isomorphic groups when K_1 and K_2 are two non-conjugate number fields ?

A first paper on this subject was published in [O] by M. Onabe. In [AS], A. Angelakis and P. Stevenhagen show that $A_K \simeq \widehat{\mathbb{Z}}^2 \times \prod\limits_{n\geq 1} \mathbb{Z}/n\,\mathbb{Z}$ for a specific family of imaginary quadratic fields.

In this paper, we prove (under the Leopoldt conjecture) that, for any number field K, the group A_K *contains* a subgroup isomorphic to $\widehat{\mathbb{Z}}^{r_2+1} \times \prod\limits_{n\geq 1} \Big((\mathbb{Z}/2\,\mathbb{Z})^{\delta} \times \mathbb{Z}/\mathbf{w}\,n\,\mathbb{Z} \Big)$, where $\delta = 1$ if $K \cap \mathbb{Q}(\mu_{2^\infty})$ is a non-trivial extension of $\mathbb{Q}$ distinct from $\mathbb{Q}(\mu_4)$, $\delta = 0$ otherwise, and where $\mathbf{w} = \prod\limits_{p} \mathbf{w}_p$ is an integer whose local factors $\mathbf{w}_p$ depend simply on the intersections $K \cap \mathbb{Q}(\mu_{p^\infty})$; then we give a class field theory interpretation of the quotient of A_K by this subgroup, quotient which measures the defect of p-rationality for all p.

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Such isomorphisms are only isomorphisms of Abelian profinite groups for which Galois theory and, a fortiori, arithmetical objects (decomposition and inertia groups) are not effective.

When an isomorphism is canonical (for instance if it is induced by the reciprocity map of class field theory), we shall write $\overset{\text{can}}{\simeq}$ contrary to the non-canonical case denoted $\overset{\text{nc}}{\simeq}$ if necessary.

Let p be a fixed prime number and let

$$H,\ H_p,\ H_{\text{ta}},\ \widetilde{K}_p,$$

be the p-Hilbert class field (in ordinary sense), the maximal p-ramified (i.e., unramified outside p) Abelian pro-p-extension of K (in ordinary sense), the maximal tamely ramified Abelian pro-p-extension of K (in restricted sense), the compositum of the $\mathbb{Z}_p$-extensions of K, respectively. Then let

$$\mathcal{T}_p := \mathrm{Gal}(H_p/\widetilde{K}_p) \ \text{ and } \ \mathcal{C}\ell_p := \mathrm{Gal}(H/K)$$

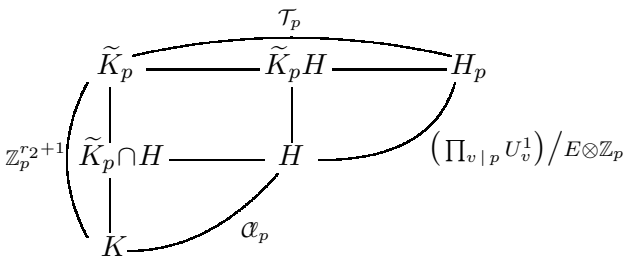
canonically isomorphic to the p-class group of K. The groups $\mathcal{T}_p$ and $\mathcal{C}\ell_p$ are finite groups.

For any finite place v of K, we denote by K_v the corresponding completion [(1)] of K, then by

$$U_v := \{u \in K_v, \,|\, u \,|_v = 1\} \ \text{ and } \ U_v^1 := \{u \in U_v, \,|\, u - 1 \,|_v < 1\},$$

the unit group and principal unit group of K_v, respectively. So, U_v/U_v^1 is isomorphic to the multiplicative group of the residue field F_v of K at v. We shall denote by ℓ the characteristic of F_v; then U_v^1 is a $\mathbb{Z}_\ell$-module. If v is a real infinite place, we put $K_v = \mathbb{R}$, $U_v = \mathbb{R}^\times$, $U_v^1 = \mathbb{R}^{\times +}$, hence $F_v^\times = \{\pm 1\}$, according to [Gr1, I.3.1.2, II.7.1.3].

The structure of $\mathrm{Gal}(H_p/K)$ can be summarized by the following diagram, from [Gr1, III.2.6.1, Fig. 2.2] under the Leopoldt conjecture for p in K:



where E is the group of global units of K and where $E \otimes \mathbb{Z}_p$ is diagonally embedded with the obvious map $i_p := (i_v)_{v\,|\,p}$.

To characterize the notion of p-rationality (see Definition 2.1 and Remarks 2.2), we shall make use of some p-adic logarithms as follows:

(i) We consider the p-adic logarithm $\log_p : K^\times \longrightarrow \prod_{v|p} K_v$ defined by $\log_p = \log \circ\, i_p$ on $K^\times$, where $\log \ : \ \mathbb{C}_p^\times \longrightarrow \mathbb{C}_p$ is the Iwasawa extension of the usual p-adic logarithm.

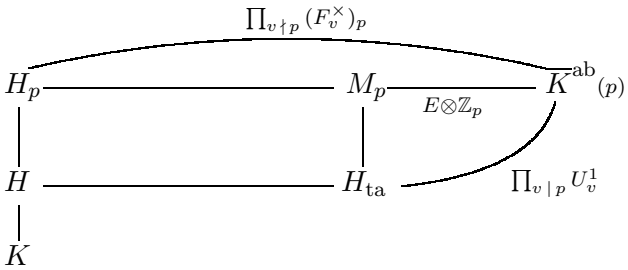
(ii) We then define the quotient $\mathbb{Q}_p$-vector space $\mathcal{L}_p := \Big(\prod_{v|p} K_v\Big)\Big/ \mathbb{Q}_p \log_p(E)$. We have, under the Leopoldt conjecture for p in K, $\dim_{\mathbb{Q}_p}(\mathcal{L}_p) = r_2 + 1$.

(iii) Finally, we denote by Log_p the map, from the group I_p of ideals of K prime to p, to $\mathcal{L}_p$, sending $\mathfrak{a} \in I_p$ to $\mathrm{Log}_p(\mathfrak{a})$ defined as follows. If m is such that $\mathfrak{a}^m = (\alpha)$ with $\alpha \in K^\times$, we set $\mathrm{Log}_p(\mathfrak{a}) := \frac{1}{m}\log_p(\alpha) + \mathbb{Q}_p \log_p(E)$; this does not depend on the choices of m and α.

[(1)] As in [Gr1, I, § 2], we consider that $K_v = i_v(K)\,\mathbb{Q}_\ell \subset \mathbb{C}_\ell$ for a suitable embedding i_v of the number field K in $\mathbb{C}_\ell$, where ℓ is the residue characteristic.

2. Structure of the Galois group of the Abelian closure of a number field

2.1. Class field theory – Fundamental diagram – p-rationality. — Let $\overline{K}^{\rm ab}{}_{(p)}$ be the maximal Abelian pro-p-extension of K. In [Gr1, III.4.4.1], we have given (assuming the Leopoldt conjecture for p in K) the following diagram for the structure of $\mathrm{Gal}(\overline{K}^{\rm ab}{}_{(p)}/K)$, isomorphic to the p-Sylow subgroup of A_K:



where $(F_v^\times)_p$ is the p-Sylow subgroup of the multiplicative group of the residue field F_v of K at v. This also concerns the real infinite places for which $F_v^\times=\{\pm1\}$. In this diagram, M_p is the direct compositum, over H, of H_p and $H_{\rm ta}$.

The diagonal embeddings $i_{\rm ta}:=(i_v)_{v\nmid p}$ and $i_p:=(i_v)_{v\mid p}$ of $E\otimes\mathbb{Z}_p$ in $\prod\limits_{v\nmid p}(F_v^\times)_p$ and $\prod\limits_{v\mid p}U_v^1$, respectively, are injective (under the Leopoldt conjecture for the second one).

Let $\mathcal{U}_p:=\prod\limits_{v\nmid p}(F_v^\times)_p\times\prod\limits_{v\mid p}U_v^1$ be the p-Sylow subgroup of the group of unit idèles $\mathcal{U}:=\prod\limits_v U_v$ of K, and let ρ be the reciprocity map on $\mathcal{U}_p$.

The kernel of ρ is $i(E\otimes\mathbb{Z}_p)$, where $i=(i_{\rm ta},i_p)$; this yields $\mathrm{Gal}(\overline{K}^{\rm ab}{}_{(p)}/H)\simeq\mathcal{U}_p/i(E\otimes\mathbb{Z}_p)$, $\mathrm{Gal}(\overline{K}^{\rm ab}{}_{(p)}/H_p)=\rho\Big(\prod\limits_{v\nmid p}(F_v^\times)_p\times\{1\}\Big)\simeq\prod\limits_{v\nmid p}(F_v^\times)_p$ since $\Big(\prod\limits_{v\nmid p}(F_v^\times)_p\times\{1\}\Big)\cap i\,(E\otimes\mathbb{Z}_p)=1$, and similarly $\mathrm{Gal}(\overline{K}^{\rm ab}{}_{(p)}/H_{\rm ta})=\rho\Big(\{1\}\times\prod\limits_{v\mid p}U_v^1\Big)\simeq\prod\limits_{v\mid p}U_v^1$ (see [Gr1, III.4.4.5.1]).

This will be useful in Section 3.

Definition 2.1. — The number field K is said to be p-rational (see [MN], [GJ], [JN], and [Gr1, IV, § b, 3.4.4]) if it satisfies the Leopoldt conjecture for p and if $\mathcal{T}_p=1$.

Remarks 2.2. — Assuming the Leopoldt conjecture for p in K, we have:

(i) From [Gr1, IV.3.4.5], the p-rationality of K is equivalent to the following three conditions:

- $\prod\limits_{v\mid p}\mu_p(K_v)=i_p(\mu_p(K))$, where $\mu_p(k)$ denotes, for any field k, the group of roots of unity of k of p-power order,
- the p-Hilbert class field H is contained in the compositum $\widetilde{K}_p$ of the $\mathbb{Z}_p$-extensions of K; this is equivalent to $\mathcal{C}\ell_p\overset{\rm can}{\simeq}\mathbb{Z}_p\mathrm{Log}_p(I_p)\Big/\Big(\prod\limits_{v\mid p}\log(U_v^1)+\mathbb{Q}_p\log_p(E)\Big)$, which can be non-trivial,
- $\mathbb{Z}_p\log_p(E)$ is a direct summand in the $\mathbb{Z}_p$-module $\prod\limits_{v\mid p}\log(U_v^1)$, which expresses the minimality of the valuation of the p-adic regulator.

(ii) For a p-rational field K, we have $\mathrm{Gal}(\overline{K}^{\rm ab}{}_{(p)}/K)\overset{\rm nc}{\simeq}\mathbb{Z}_p^{r_2+1}\times\prod\limits_{v\nmid p}(F_v^\times)_p$, with (canonically) $\mathrm{Gal}(\overline{K}^{\rm ab}{}_{(p)}/\widetilde{K}_p)\overset{\rm can}{\simeq}\prod\limits_{v\nmid p}(F_v^\times)_p$.

(iii) Let $\widetilde{\mathcal{K}}_\infty$ be the compositum of the $\widetilde{K}_p$. By assumption, it is the maximal $\widehat{\mathbb{Z}}$-extension of K for which $\mathrm{Gal}(\widetilde{\mathcal{K}}_\infty/K) \overset{\mathrm{nc}}{\simeq} \widehat{\mathbb{Z}}^{r_2+1}$. A sufficient condition to get $\mathrm{Gal}(\overline{K}^{\mathrm{ab}}/K) \overset{\mathrm{nc}}{\simeq} \widehat{\mathbb{Z}}^{r_2+1} \times \prod_v F_v^\times$ is that K be p-rational for all p. The stronger condition $H_p = \widetilde{K}_p$ for all p (i.e., p-rationality of K for all p) is equivalent to the class field theory isomorphism $\mathrm{Gal}(\overline{K}^{\mathrm{ab}}/\widetilde{\mathcal{K}}_\infty) \overset{\mathrm{can}}{\simeq} \prod_v F_v^\times$.

2.2. Structure of $\prod_v F_v^\times$. — Let $(F_v)_v$ be the family of residue fields of K for its finite or real infinite places v. We intend to give, for all prime numbers p, the structure of the p-Sylow subgroup of $\prod_v F_v^\times$. If $v \mid p$, then $(F_v^\times)_p = 1$; so we can restrict ourselves to $\prod_{v \nmid p} (F_v^\times)_p$.
We shall prove that there exist integers $\delta \in \{0,1\}$ and $\mathbf{w} \geq 1$ such that

$$\prod_{v \nmid p} (F_v^\times)_p \overset{\mathrm{nc}}{\simeq} \Big(\prod_{n \geq 1} \Big((\mathbb{Z}/2\,\mathbb{Z})^\delta \times \mathbb{Z}/\mathbf{w}\,n\,\mathbb{Z} \Big) \Big)_p .$$

The property giving such an isomorphism is that for any given p-power p^k, $k \geq 1$, the two p-Sylow subgroups have 0 or infinitely many (countable) cyclic direct components of order p^k.
It is obvious that for any p, $\Big(\prod_{n \geq 1} \Big((\mathbb{Z}/2\,\mathbb{Z})^\delta \times \mathbb{Z}/\mathbf{w}\,n\,\mathbb{Z} \Big) \Big)_p$ has 0 or infinitely many cyclic direct components of order p^k for any $k \geq 1$; moreover, in $\Big(\prod_{n \geq 1} \big(\mathbb{Z}/\mathbf{w}\,n\,\mathbb{Z} \big) \Big)_p$ there is no direct component of order p^k, $k \geq 1$, if and only if $p^{k+1} \mid \mathbf{w}$.

Remark 2.3. — Write $\mathbf{w} = \prod_{q \text{ prime}} q^{\lambda_q}$ and put $\mathbf{w}^1 := \prod_{\lambda_q \geq 2} q^{\lambda_q}$ so that $\mathbf{w}^0 := \mathbf{w}/\mathbf{w}^1$ and $\mathbf{w}^1$ are coprime integers. Then in the above expressions we can replace $\mathbf{w}$ by $\mathbf{w}^1$. Indeed, in the two writings $\prod_{n \geq 1} \mathbb{Z}/\mathbf{w}^0.\mathbf{w}^1 n\,\mathbb{Z}$ and $\prod_{n \geq 1} \mathbb{Z}/\mathbf{w}^1 n\,\mathbb{Z}$, for all $q \mid \mathbf{w}^0$ the direct summands of order q are infinite in number. Then we can ensure that $\mathbf{w}$ will be defined in such a way that $\mathbf{w}^0 = 1$.

Definitions 2.4. — (i) We denote by $\mu(K)$ (resp. $\mu_p(K)$) the group of roots of unity of K (resp. of p-power order) and for any $e \geq 1$ we denote by μ_e the group (of order e) of eth roots of unity in a field of characteristic 0 or ℓ prime to e.
(ii) Let Q_ν, $\nu \geq 1$, be for any p the unique subfield, of degree p^ν over $\mathbb{Q}$, of the cyclotomic $\mathbb{Z}_p$-extension of $\mathbb{Q}$. Then let Q'_ν, $\nu \geq 1$, be for $p = 2$ the unique non-real subfield of $\mathbb{Q}(\mu_{2^\infty})$ of degree 2^ν over $\mathbb{Q}$. We put $Q'_0 = Q_0 = \mathbb{Q}$ in any case.

2.2.1. Analysis of the case $p \neq 2$. — In the study of $\prod_{v \nmid p} (F_v^\times)_p$, we can assume that $|F_v^\times| \equiv 0 \pmod p$, which is equivalent to the splitting of v in $K(\mu_p)/K$ (this includes the case where K contains μ_p).
a) If K contains μ_p, then $\mu_p(K) = \mu_{p^{\nu+1}}$ where $\nu \geq 0$ is the maximal integer such that $Q_\nu \subseteq K$, and we get necessarily $|F_v^\times| \equiv 0 \pmod{p^{\nu+1}}$ for all these places. We obtain the following tower of extensions (where $\subset$ means a stric inclusion)

$$K = K(\mu_{p^{\nu+1}}) \subset K(\mu_{p^{\nu+2}}) \subset \cdots$$

From Chebotarev's theorem, for any $m \geq \nu + 1$ there exist infinitely many places v of K, whose inertia group in $K(\mu_{p^{m+1}})/K$ is $\mathrm{Gal}(K(\mu_{p^{m+1}})/K(\mu_{p^m}))$, cyclic of order p; so we get $|F_v^\times| \equiv 0 \pmod{p^m}$ and $|F_v^\times| \not\equiv 0 \pmod{p^{m+1}}$ for these places.
b) If K does not contain μ_p and if $K \cap \mathbb{Q}(\mu_{p^\infty}) = Q_\nu$, $\nu \geq 0$, we have the tower of extensions

$$K \subset K(\mu_p) = \cdots = K(\mu_{p^{\nu+1}}) \subset K(\mu_{p^{\nu+2}}) \subset \cdots$$

Since by assumption the places v considered are split in $K(\mu_p)/K$, we still have $|F_v^\times| \equiv 0 \pmod{p^{\nu+1}}$. From Chebotarev's theorem, for any $m \geq \nu+1$ there exist infinitely many places v, whose inertia group in $K(\mu_{p^{m+1}})/K$ is $\mathrm{Gal}(K(\mu_{p^{m+1}})/K(\mu_{p^m}))$, cyclic of order p; thus, $|F_v^\times| \equiv 0 \pmod{p^m}$ and $|F_v^\times| \not\equiv 0 \pmod{p^{m+1}}$ for these places.

In conclusion, the case $p \neq 2$ leads to identical results from the knowledge of the integer ν.

2.2.2. Analysis of the case $p = 2$. — In that case, we always have $|F_v^\times| \equiv 0 \pmod 2$ in the study of $\prod_{v\nmid 2}(F_v^\times)_2$ (v finite or real infinite). So we consider $K(\mu_4)/K$.

a) If K contains μ_4 and if $K \cap \mathbb{Q}(\mu_{2^\infty}) =: \mathbb{Q}(\mu_{4.2^\nu})$, $\nu \geq 0$, we have $|F_v^\times| \equiv 0 \pmod{4.2^\nu}$ for all places, and the tower of extensions

$$K = K(\mu_{4.2^\nu}) \subset K(\mu_{4.2^{\nu+1}}) \subset \cdots$$

From Chebotarev's theorem, for any $m \geq \nu$ there exist infinitely many places v whose inertia group, in $K(\mu_{4.2^{m+1}})/K$, is $\mathrm{Gal}(K(\mu_{4.2^{m+1}})/K(\mu_{4.2^m}))$, cyclic of order 2; so $|F_v^\times| \equiv 0 \pmod{4.2^m}$ and $|F_v^\times| \not\equiv 0 \pmod{4.2^{m+1}}$ for these places.

b) If K does not contain μ_4, we have two possible towers depending on $K \cap \mathbb{Q}(\mu_{2^\infty})$:

- $K \cap \mathbb{Q}(\mu_{2^\infty}) = \mathbb{Q}$: $\quad K \subset K(\mu_4) \subset K(\mu_8) \subset \cdots$
- $K \cap \mathbb{Q}(\mu_{2^\infty}) \in \{Q_\nu, Q'_\nu\},\ \nu \geq 1$: $\quad K \subset K(\mu_4) = \cdots = K(\mu_{4.2^\nu}) \subset K(\mu_{4.2^{\nu+1}}) \subset \cdots$

(i) In the first case ($\nu = 0$), for any $m \geq 1$ Chebotarev's theorem gives infinitely many places v whose inertia group in $K(\mu_{2^{m+1}})/K$ is $\mathrm{Gal}(K(\mu_{2^{m+1}})/K(\mu_{2^m}))$, cyclic of order 2; so we get $|F_v^\times| \equiv 0 \pmod{2^m}$ and $|F_v^\times| \not\equiv 0 \pmod{2^{m+1}}$ for these places (the real infinite places are solution, taking $m = 1$).

(ii) In the second case ($\nu \geq 1$), we will have two disjoint sets of places of K for the structure of the product $\prod_{v\nmid 2}(F_v^\times)_2$:

– There exist infinitely many places v inert in $K(\mu_4)/K$. Then $|F_v^\times| \equiv 0 \pmod 2$ and $|F_v^\times| \not\equiv 0 \pmod 4$ for these places (this includes the real infinite places, if any).

– For any $m \geq \nu$ ($\nu \geq 1$), Chebotarev's theorem gives infinitely many places v whose inertia group in $K(\mu_{4.2^{m+1}})/K$ is $\mathrm{Gal}(K(\mu_{4.2^{m+1}})/K(\mu_{4.2^m}))$, cyclic of order 2; a fortiori, these places are split in $K(\mu_4)/K$. So we get $|F_v^\times| \equiv 0 \pmod{4.2^m}$ and $|F_v^\times| \not\equiv 0 \pmod{4.2^{m+1}}$.

We see that in the exceptional case $K \cap \mathbb{Q}(\mu_{2^\infty}) \in \{Q_\nu, Q'_\nu\}$ with $\nu \geq 1$, we have a group isomorphism of the form $\prod\limits_{v\nmid 2}(F_v^\times)_2 \simeq \big(\mathbb{Z}/2\,\mathbb{Z}\big)^{\mathbb{N}} \times \prod\limits_{m\geq\nu} \mathbb{Z}/4.2^m\,\mathbb{Z}$.

Definition 2.5. — From the above discussion about the number field K, we define the integers $\delta \in \{0,1\}$ and $\mathbf{w} := \prod\limits_p \mathbf{w}_p$, where $\mathbf{w}_p$, depending on $K \cap \mathbb{Q}(\mu_{p^\infty})$, is given as follows:

(i) Case $p \neq 2$. Let $\nu \geq 0$ be the maximal integer such that $Q_\nu \subseteq K$ (thus $\mu_{p^{\nu+1}}$ is the maximal group of roots of unity of p-power order contained in $K(\mu_p)$ whether K contains μ_p or not); we put $\mathbf{w}_p = p^{\nu+1}$ if $\nu \geq 1$ and $\mathbf{w}_p = 1$ otherwise (from the use of Remark 2.3).

(ii) If, in the case $p = 2$, K contains μ_4, we put $\mathbf{w}_2 = 4.2^\nu$, where $\nu \geq 0$ is the maximal integer such that $Q_\nu \subseteq K$ (in this case, the reasonning with Q'_ν gives the same integer ν).

(iii) If, in the case $p = 2$, K does not contain μ_4, let $\nu \geq 0$ be the maximal integer such that $Q_\nu \subseteq K$ or $Q'_\nu \subseteq K$ (thus $\mu_{4.2^\nu}$ is the maximal group of roots of unity of 2-power order of $K(\mu_4)$); we put $\mathbf{w}_2 = 4.2^\nu$ if $\nu \geq 1$ and $\mathbf{w}_2 = 1$ otherwise (from the use of Remark 2.3).

(iv) We put $\delta = 1$ in the case (iii) when $\nu \geq 1$, and $\delta = 0$ otherwise.

We can state the following result correcting an error discovered by Peter Stevenhagen in the first draft of [AS, Lemma 3.2] as well as in [O] and in the previous versions of our paper reproducing this Lemma; this will also be corrected in the final paper [AS] in the proceedings volume of ANTS-X, San Diego 2012. We refer to Definitions 2.4 and 2.5 giving δ and $\mathbf{w}$.

Proposition 2.6. — *Let K be a number field. We have a group isomorphism of the form*

$$\prod_v F_v^\times \overset{\text{nc}}{\simeq} \prod_{n\geq 1} \Big((\mathbb{Z}/2\,\mathbb{Z})^\delta \times \mathbb{Z}/\mathbf{w}\,n\,\mathbb{Z} \Big).$$

We have $\delta = 1$ if and only if K does not contain μ_4 and $8 \,|\, \mathbf{w}$.
When $\mathbf{w} = 1$ (the most usual case), then $\delta = 0$ and $\prod_v F_v^\times \overset{\text{nc}}{\simeq} \prod_{n\geq 1} \mathbb{Z}/n\,\mathbb{Z}$.

2.2.3. Examples. — (i) Example with $p = 3$. Let K be the maximal real subfield of $\mathbb{Q}(\mu_9)$; we have $\mathbf{w} = 9$. The prime $\ell = 5$ is totally inert in $\mathbb{Q}(\mu_9)$; then for $v \,|\, \ell$, F_v does not contain μ_3 since $\ell^3 = 125 \not\equiv 1 \pmod 3$. But for $\ell = 7$, inert in K and split in $\mathbb{Q}(\mu_3)$, we get $F_v^\times = \mathbb{F}_{343}^\times$ which contains μ_9 as expected.

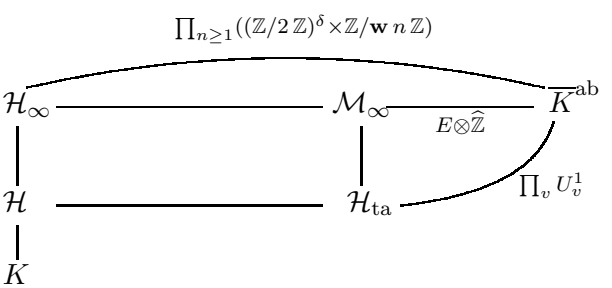
(ii) Examples with $p = 2$. For $K = \mathbb{Q}(\sqrt{2}\,)$, we have $\delta = 1$ and $\mathbf{w} = 8$. The prime $\ell = 7$ splits in K and is inert in $\mathbb{Q}(\mu_4)$; so for $v \,|\, \ell$, $F_v = \mathbb{F}_7$ does not contain μ_4. But for the prime $\ell = 5 \equiv 1 \pmod 4$, inert in K and split in $\mathbb{Q}(\mu_4)$, we get $F_v = \mathbb{F}_{25}$ which contains μ_8. For $K = \mathbb{Q}(\sqrt{2}\,)$, we get the extra factor $(\mathbb{Z}/2\,\mathbb{Z})^{\mathbb{N}}$ and there do not exist any cyclic direct component of order 4.

For $K = \mathbb{Q}(\mu_4)$, we have $\mathbf{w} = 4$ and $F_v = \mathbb{F}_\ell$ ($\ell \equiv 1 \pmod 4$) or $F_v = \mathbb{F}_{\ell^2}$ ($\ell \equiv -1 \pmod 4$); so the 2-Sylow subgroup of $F_v^\times$ is at least of order 4.

2.3. Consequences for the structure of $\mathrm{Gal}(\overline{K}^{\mathrm{ab}}/K)$. — From Proposition 2.6 and the fundamental diagram (Subsection 2.1), we can state:

Proposition 2.7. — *Let $\mathcal{H}_\infty$ be the compositum of the fields H_p (maximal p-ramified Abelian pro-p-extensions of K) for all prime numbers p. Then, under the Leopoldt conjecture in K for all p, we have a group isomorphism of the form $\mathrm{Gal}(\overline{K}^{\mathrm{ab}}/\mathcal{H}_\infty) \overset{\text{nc}}{\simeq} \prod_{n\geq 1} \Big((\mathbb{Z}/2\,\mathbb{Z})^\delta \times \mathbb{Z}/\mathbf{w}\,n\,\mathbb{Z} \Big)$.*
If $\mathbf{w} = 1$, then $\mathrm{Gal}(\overline{K}^{\mathrm{ab}}/\mathcal{H}_\infty) \overset{\text{nc}}{\simeq} \prod_{n\geq 1} \mathbb{Z}/n\,\mathbb{Z}$.

We have obtained the following globalized diagram (under the Leopoldt conjecture for all p), where $\mathcal{H}_{\mathrm{ta}}$ (compositum of the H_{ta}) is the maximal Abelian tamely ramified extension of K and $\mathcal{M}_\infty = \mathcal{H}_\infty \mathcal{H}_{\mathrm{ta}}$ (direct compositum over the Hilbert class field $\mathcal{H}$):



Let $\mathcal{F}_\infty$ be the compositum of some finite extensions F_p of K such that $H_p = \widetilde{K}_p\, F_p$ (direct compositum over K). When they are non-trivial, the extensions F_p/K are non-unique p-ramified extensions. We then have $\mathrm{Gal}(F_p/K) \simeq \mathcal{T}_p$ and $\mathrm{Gal}(\mathcal{F}_\infty/K) \simeq \prod_p \mathcal{T}_p$.

The extension $\mathcal{F}_\infty/K$ is in general non-canonical and conjecturally infinite; its Galois group measures a mysterious degree of complexity of $\mathrm{Gal}(\overline{K}^{\mathrm{ab}}/K)$; it is trivial if and only if K is p-rational for all p (Remark 2.2 (ii)). But we have $\mathrm{Gal}(\mathcal{H}_\infty/\widetilde{\mathcal{K}}_\infty) \overset{\mathrm{can}}{\simeq} \prod_p \mathcal{T}_p$ and $\mathrm{Gal}(\mathcal{H}_\infty/K) \overset{\mathrm{nc}}{\simeq} \widehat{\mathbb{Z}}^{r_2+1} \times \prod_p \mathcal{T}_p$.

Theorem 2.8. — *Let K be a number field and let $\overline{K}^{\mathrm{ab}}$ be the maximal Abelian pro-extension of K. We assume that the p-adic Leopoldt conjecture is verified in K for all prime number p. Then there exists an Abelian extension $\mathcal{F}_\infty$ of K, with $\mathrm{Gal}(\mathcal{F}_\infty/K) \overset{\mathrm{can}}{\simeq} \prod_p \mathcal{T}_p$, such that $\mathcal{H}_\infty$ is the direct compositum of $\mathcal{F}_\infty$ and the maximal $\widehat{\mathbb{Z}}$-extension $\widetilde{\mathcal{K}}_\infty$ of K, and such that*

$$\mathrm{Gal}(\overline{K}^{\mathrm{ab}}/\mathcal{F}_\infty) \overset{\mathrm{nc}}{\simeq} \widehat{\mathbb{Z}}^{r_2+1} \times \prod_{n\geq 1} \Big((\mathbb{Z}/2\,\mathbb{Z})^\delta \times \mathbb{Z}/\mathbf{w}\,n\,\mathbb{Z} \Big),$$

with $\mathrm{Gal}(\overline{K}^{\mathrm{ab}}/\mathcal{H}_\infty) \overset{\mathrm{nc}}{\simeq} \prod_{n\geq 1} \Big((\mathbb{Z}/2\,\mathbb{Z})^\delta \times \mathbb{Z}/\mathbf{w}\,n\,\mathbb{Z} \Big)$, where δ, $\mathbf{w}$ are defined in Definition 2.5.

If $\mathbf{w}=1$, we have a group isomorphism of the form $\mathrm{Gal}(\overline{K}^{\mathrm{ab}}/\mathcal{F}_\infty) \overset{\mathrm{nc}}{\simeq} \widehat{\mathbb{Z}}^{r_2+1} \times \prod_{n\geq 1} \mathbb{Z}/n\,\mathbb{Z}$, with $\mathrm{Gal}(\overline{K}^{\mathrm{ab}}/\mathcal{H}_\infty) \overset{\mathrm{nc}}{\simeq} \prod_{n\geq 1} \mathbb{Z}/n\,\mathbb{Z}$.

Corollary 2.9. — *The Galois groups $\mathrm{Gal}(\overline{K}^{\mathrm{ab}}/\mathcal{F}_\infty)$ (up to non-canonical isomorphisms) are independent of the number fields K as soon as these fields satisfy the Leopoldt conjecture for all p, have the same number r_2 of complex places and the same parameters $\delta, \mathbf{w}$.*
Thus, for all totally real number fields K (satisfying the Leopoldt conjecture for all p) which do not contain $\sqrt{2}$, we have $\mathrm{Gal}(\overline{K}^{\mathrm{ab}}/\mathcal{F}_\infty) \overset{\mathrm{nc}}{\simeq} \widehat{\mathbb{Z}} \times \prod_{n\geq 1} \mathbb{Z}/n\,\mathbb{Z}$.

Of course, the groups $\mathrm{Gal}(\mathcal{F}_\infty/K)$ strongly depend on K, even if the parameters $r_2, \delta, \mathbf{w}$ are constant. From Remark 2.2 (i), we see that the first two conditions of p-rationality involve a finite number of primes p, but that the third condition is the most ugly. [(2)]

So, we are mainly concerned with the imaginary quadratic fields, studied in [AS], for which the third condition is empty; the first and second ones can be verified (for all p) probably for infinitely many imaginary quadratic fields as suggested in [AS, Conjecture 7.1].

3. A generalization of p-rationality

As we shall see now, we can strengthen a few the previous results about the first condition involved in the definition of p-rationality, condition which can be removed for all number fields. This concerns the finite p-groups $\big(\prod_{v\,|\,p} \mu_p(K_v)\big)/i_p(\mu_p(K))$ whose globalization measures the gap between the regular and Hilbert kernels in $\mathrm{K}_2(K)$ (see [Gr1, II.7.6.1]).

For all finite place v of K we have $\mu(K_v) \simeq F_v^\times \times \mu_v^1$, where μ_v^1 is the torsion subgroup of U_v^1 (it is a finite ℓ-group where ℓ is the residue characteristic); if v is real infinite, we have $F_v^\times = \{\pm 1\}$ and $\mu_v^1 = 1$.

The places (finite in number) such that $\mu_v^1 \neq 1$ are called the irregular places of K.

[(2)] For instance, for $K=\mathbb{Q}(\sqrt{2}\,)$, the third condition is not satisfied for $p = 13, 31, 1546463, \ldots$ and perhaps for infinitely many primes p depending on Fermat quotients of the fundamental unit [Gr1, III.4.14]. Note that from [Gr2, III] or [Gr1, IV.3.5.1], for $p=2$, the 2-rational Abelian 2-extensions of $\mathbb{Q}$ are the subfields of the fields $\mathbb{Q}(\mu_{2^\infty})\mathbb{Q}(\sqrt{-\ell}\,)$, $-\ell \equiv 5 \pmod 8$, or of the fields $\mathbb{Q}(\mu_{2^\infty})\mathbb{Q}\Big(\sqrt{\sqrt{\ell}\,\frac{a-\sqrt{\ell}}{2}}\Big)$, $\ell = a^2+4b^2 \equiv 5 \pmod 8$.

We have $\mu_p(K_v) = \mu_v^1$ if and only if $v \,|\, p$ and $\mu_p(K_v) \simeq (F_v^\times)_p$ if and only if $v \nmid p$. Let

$$\Gamma_p := \prod_{v \nmid p} (F_v^\times)_p \times \prod_{v \,|\, p} \mu_v^1 \simeq \prod_v \mu_p(K_v).$$

To study the influence of the cyclic factors $\mu_v^1 = \mu_p(K_v)$ for $v \,|\, p$, on $\prod_{v \nmid p} (F_v^\times)_p$, we refer to Definition 2.5 for the definitions of ν, δ, $\mathbf{w}_p$, and to Proposition 2.6.

(i) Case $p \neq 2$. If K contains μ_p, then $\mathbf{w}_p = p^{\nu+1} = |\mu_p(K)|$ divides $|\mu_p(K_v)|$; so the cyclic factor $\mu_p(K_v)$ does not modify the structure.

If K does not contain μ_p, we have only to look at the case $\nu \geq 1$ for which $\mathbf{w}_p = p^{\nu+1}$. If $\mu_p(K_v)$ is non-trivial ($v \,|\, p$ is split in $K(\mu_p)$), $|\mu_p(K_v)|$ is a multiple of $p^{\nu+1}$, giving the result.

(ii) Case $p = 2$. If K contains μ_4, $\mathbf{w}_2 = 4.2^\nu = |\mu_2(K)|$ divides $|\mu_2(K_v)|$, hence the result.

If K does not contain μ_4, we have only to consider the case $K \cap \mathbb{Q}(\mu_{2^\infty}) \in \{Q_\nu, Q'_\nu\}$, $\nu \geq 1$. Then $\delta = 1$ and $\mathbf{w}_2 = 4.2^\nu$; so $\mu_2(K_v) = \mu_2$ (if $v \,|\, 2$ is not split in $K(\mu_4)$) or $\mu_{4.2^m}$, $m \geq \nu$ (if v is split in $K(\mu_4)$), hence the result.

We then have

$$\Gamma_p \overset{\text{nc}}{\simeq} \Big(\prod_{n \geq 1} \Big((\mathbb{Z}/2\,\mathbb{Z})^\delta \times \mathbb{Z}/\mathbf{w}\,n\,\mathbb{Z} \Big) \Big)_p.$$

Let H_p^1 be the subfield of H_p fixed by $\rho(\Gamma_p)$, where ρ is the reciprocity map on the p-Sylow subgroup $\mathcal{U}_p := \prod_{v \nmid p} (F_v^\times)_p \times \prod_{v \,|\, p} U_v^1 \supset \Gamma_p$ of the group of unit idèles of K. The kernel of ρ is $i(E \otimes \mathbb{Z}_p)$ (see Subsection 2.1).

We consider $\rho(\Gamma_p) = \mathrm{Gal}(\overline{K}^{\mathrm{ab}(p)}/H_p^1)$. Then from the local-global characterization of the Leopoldt conjecture at p (see [Ja, § 2.3] or [Gr1, III.3.6.6]), we get (omitting the embedding i)

$$\rho(\Gamma_p) \overset{\text{can}}{\simeq} \Gamma_p/(E \otimes \mathbb{Z}_p) \cap \Gamma_p = \Gamma_p/\mu_p(K).$$

Taking, as in [AS, Lemmas 3.3, 3.4], v_0 such that the residue image of $\mu_p(K)$ is equal to $(F_{v_0}^\times)_p$, we still get $\mathrm{Gal}(\overline{K}^{\mathrm{ab}(p)}/H_p^1) \overset{\text{can}}{\simeq} \Gamma_p/\mu_p(K) \overset{\text{nc}}{\simeq} \Big(\prod_{n \geq 1} \Big((\mathbb{Z}/2\,\mathbb{Z})^\delta \times \mathbb{Z}/\mathbf{w}\,n\,\mathbb{Z} \Big) \Big)_p$.

We note that $\mathrm{Gal}(H_p/H_p^1) \overset{\text{can}}{\simeq} \big(\prod_{v \,|\, p} \mu_v^1 \big)/\mu_p(K)$ (see also [Gr1, III.4.15.3]).

Of course, if $\mathcal{H}_\infty^1 \subseteq \mathcal{H}_\infty$ is the compositum of the H_p^1, the globalization gives

$$\mathrm{Gal}(\overline{K}^{\mathrm{ab}}/\mathcal{H}_\infty^1) \overset{\text{nc}}{\simeq} \prod_{n \geq 1} \Big((\mathbb{Z}/2\,\mathbb{Z})^\delta \times \mathbb{Z}/\mathbf{w}\,n\,\mathbb{Z} \Big), \text{ with } \mathrm{Gal}(\mathcal{H}_\infty/\mathcal{H}_\infty^1) \overset{\text{can}}{\simeq} \Big(\prod_v \mu_v^1 \Big) \Big/ \mu(K).$$

In other words we have obtained (to be compared with Theorem 2.8 using the extension $\mathcal{F}_\infty$):

Theorem 3.1. — *Let K be a number field and let $\overline{K}^{\mathrm{ab}}$ be the maximal Abelian pro-extension of K. We assume that the Leopoldt conjecture is verified in K for all prime numbers.*

Then there exists an Abelian extension $\mathcal{F}_\infty^1 \subseteq \mathcal{F}_\infty$ of K such that $\mathcal{H}_\infty^1$ is the direct compositum over K of $\mathcal{F}_\infty^1$ and the maximal $\widehat{\mathbb{Z}}$-extension $\widetilde{\mathcal{K}}_\infty$ of K, and such that

$$\mathrm{Gal}(\overline{K}^{\mathrm{ab}}/\mathcal{F}_\infty^1) \overset{\text{nc}}{\simeq} \widehat{\mathbb{Z}}^{r_2+1} \times \prod_{n \geq 1} \Big((\mathbb{Z}/2\,\mathbb{Z})^\delta \times \mathbb{Z}/\mathbf{w}\,n\,\mathbb{Z} \Big),$$

with $\mathrm{Gal}(\overline{K}^{\mathrm{ab}}/\mathcal{H}_\infty^1) \overset{\text{nc}}{\simeq} \prod_{n \geq 1} \Big((\mathbb{Z}/2\,\mathbb{Z})^\delta \times \mathbb{Z}/\mathbf{w}\,n\,\mathbb{Z} \Big)$.

If $\mathbf{w} = 1$, *then* $\mathrm{Gal}(\overline{K}^{\mathrm{ab}}/\mathcal{F}_\infty^1) \overset{\text{nc}}{\simeq} \widehat{\mathbb{Z}}^{r_2+1} \times \prod_{n \geq 1} \mathbb{Z}/n\,\mathbb{Z}$, *with* $\mathrm{Gal}(\overline{K}^{\mathrm{ab}}/\mathcal{H}_\infty^1) \overset{\text{nc}}{\simeq} \prod_{n \geq 1} \mathbb{Z}/n\,\mathbb{Z}$.

The problem for the non-imaginary quadratic fields is unchanged since in the following global exact sequence, where $\mathcal{T}_p^1 := \mathrm{Gal}(H_p^1/\widetilde{K}_p)$ for all p,

$$0 \to \prod_{n\geq 1} \Big((\mathbb{Z}/2\,\mathbb{Z})^\delta \times \mathbb{Z}/\mathbf{w}\,n\,\mathbb{Z} \Big) \longrightarrow \mathrm{Gal}(\overline{K}^{\mathrm{ab}}/\widetilde{\mathcal{K}}_\infty) \longrightarrow \prod_p \mathcal{T}_p^1 \to 1,$$

we do not know if the structure of $\mathrm{Gal}(\overline{K}^{\mathrm{ab}}/\widetilde{\mathcal{K}}_\infty)$ can be the same for various number fields K because of the unknown groups $\prod\limits_p \mathcal{T}_p^1$ (which non-trivially depend on the p-adic properties of the classes and units of the fields K) and the nature of the corresponding group extension. We have

$$\mathrm{Gal}(\overline{K}^{\mathrm{ab}}/K) \overset{\mathrm{nc}}{\simeq} \widehat{\mathbb{Z}}^{r_2+1} \times \prod_{n\geq 1} \Big((\mathbb{Z}/2\,\mathbb{Z})^\delta \times \mathbb{Z}/\mathbf{w}\,n\,\mathbb{Z} \Big)$$

as soon as the second and third condition of p-rationality (Remark 2.2 (i)) are satisfied for all p, which defines a weaker version of p-rationality which may have some interest.

For imaginary quadratic fields $K \neq \mathbb{Q}(\sqrt{-1}\,), \mathbb{Q}(\sqrt{-2}\,)$, we find again (since $\delta = 0$ and $\mathbf{w} = 1$) that $\mathrm{Gal}(\overline{K}^{\mathrm{ab}}/K) \overset{\mathrm{nc}}{\simeq} \widehat{\mathbb{Z}}^2 \times \prod\limits_{n\geq 1} \mathbb{Z}/n\,\mathbb{Z}$, as soon as, for all p dividing the class number, the p-Hilbert class field is contained in the compositum of the $\mathbb{Z}_p$-extensions [3] of K, which is equivalent to $\mathcal{C}\ell_p \overset{\mathrm{can}}{\simeq} \mathbb{Z}_p \log_p(I_p) \Big/ \prod\limits_{v\,|\,p} \log(U_v^1)$.

Note that the arithmetical invariant $\prod\limits_p \mathcal{T}_p$ (or $\prod\limits_p \mathcal{T}_p^1$) is one of the deepest invariant of class field theory over K; the duality properties of each component $\mathcal{T}_p$ are related to p-class groups, p-regular kernels, ... via reflection theorems and Galois cohomology; in the totally real case, $\mathcal{T}_p$ is connected with the p-adic ζ-function of K (see [Se] and [Gr1, III.2.6.5]).

Remark 3.2. — Let F_p^1 be any extension of K such that H_p^1 is the direct compositum over K of $\widetilde{K}_p$ and F_p^1. From [Gr1, III.4.15.8], we know that when $F_p^1 \neq K$, all non-trivial cyclic extensions $F_{p,i}^1 \subseteq F_p^1$ of K can be embedded in a cyclic p-extension of arbitrarily large degree (except perhaps in the special case $p = 2$, $K \cap \mathbb{Q}(\mu_{2^\infty}) = Q_\nu$, $\nu \geq 2$).

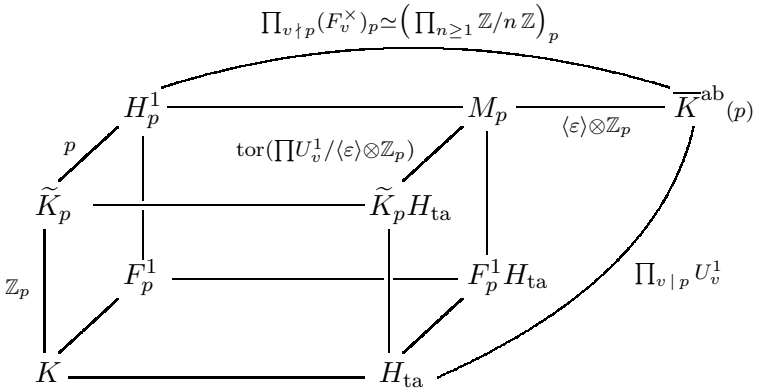
Recall that the subgroup corresponding to the compositum of the p-cyclically embeddable fields (compositum which of course contains $\widetilde{K}_p$) is equal to the group $\prod_{v\,|\,p} \mu_v^1/\mu_p(K)$, except perhaps in the special case where $\prod_{v\,|\,p} \mu_v^1/\mu_p(K)$ may be of index 2 in this group. The quotient of $\mathcal{T}_p$ by this group is called the Bertrandias–Payan module.

So this property shows, when $F_p^1 \neq K$, that $\mathrm{Gal}(\overline{K}^{\mathrm{ab}}{}^{(p)}/H_p^1)$ cannot be a direct summand in $\mathrm{Gal}(\overline{K}^{\mathrm{ab}}{}^{(p)}/\widetilde{K}_p)$, since $\mathcal{T}_p^1$ is finite. In other words, for any power p^k, taking a suitable set of cyclic extensions $F_{p,i}^1 \subseteq F_p^1$, there exists a field $L_k \subset \overline{K}^{\mathrm{ab}}{}^{(p)}$, such that $\widetilde{K}_p \subseteq H_p^1 \subseteq L_k$, with $\mathrm{Gal}(L_k/\widetilde{K}_p)$ of exponent p^k. We can even assume that $\mathrm{Gal}(L_k/\widetilde{K}_p) \simeq (\mathbb{Z}/p^k\mathbb{Z})^r$, where r is the p-rank of $\mathcal{T}_p^1$. However, for distinct values of k, the fields L_k may not follow any specific rule. So it is possible that only numerical computations may help to precise the structure of $\mathrm{Gal}(\overline{K}^{\mathrm{ab}}{}^{(p)}/\widetilde{K}_p)$.

[3] From [Gr1, III.2.6.6] or [Gr3, Theorem 2.3], for an imaginary quadratic field K, the 2-Hilbert class field is contained in the compositum of its $\mathbb{Z}_2$-extensions if and only if K is one of the following fields: $\mathbb{Q}(\sqrt{-1}\,)$, $\mathbb{Q}(\sqrt{-2}\,)$, $\mathbb{Q}(\sqrt{-\ell}\,)$ (ℓ prime $\equiv 3, 5, 7 \pmod 8$), $\mathbb{Q}(\sqrt{-2\ell}\,)$ (ℓ prime $\equiv 3, 5 \pmod 8$), $\mathbb{Q}(\sqrt{-\ell q}\,)$ (ℓ, q primes, $\ell \equiv -q \equiv 3 \pmod 8$). For numerical studies on the groups $\mathcal{T}_p$, see [Cha] and [AS, § 7].

An interesting case is that of $K = \mathbb{Q}(\sqrt{2}\,)$ for $p = 13$; in this case, $H_{13} = H_{13}^1$ is cyclic of degree 13 over $\widetilde{K}_{13}$ since $\varepsilon = 1 + \sqrt{2}$ is such that $-\varepsilon^{14}$ is, modulo 13^3, of the form $1 + 13^2\, a\, \sqrt{2}$ with $a \not\equiv 0 \pmod{13}$, which gives $\mathcal{T}_{13} \simeq \mathbb{Z}/13\,\mathbb{Z}$; indeed, use the reasonning of [Gr1, III.4.14] for real quadratic fields, or the formula given in [Gr1, III.2.6.1 (ii_2)] with $\mathcal{C}\ell_{13} = 1$, $\prod\limits_{v\,|\,13} U_v^1 = U_{13}^1 = 1 + 13\,(\mathbb{Z}_{13} \oplus \mathbb{Z}_{13}\,\sqrt{2}\,)$.

With such similar numerical data for a real quadratic field $\mathbb{Q}(\sqrt{d}\,)$ ($p \neq 2$, class number prime to p, $H_p^1 = H_p$ of degree p over $\widetilde{K}_p$, $\pm\varepsilon^{p+1}$ (p inert) or $\pm\varepsilon^{p-1}$ (p split) is, modulo p^3, of the form $1 + p^2\, a\, \sqrt{d}$ with a rational $a \not\equiv 0 \pmod{p}$), we get the following diagram:



For $K = \mathbb{Q}(\sqrt{2}\,)$, $p = 13$, we have no more information likely to give a result on the structure of the profinite group $\mathrm{Gal}(\overline{K}^{\mathrm{ab}(p)}/\widetilde{K}_p)$ containing a subgroup, of index p, isomorphic (non-canonically) to $\left(\prod\limits_{n\geq 1}\mathbb{Z}/n\,\mathbb{Z}\right)_p$.

Despite the previous class field theory study, it remains possible that $\mathrm{Gal}(\overline{K}^{\mathrm{ab}}/K)$ be always non-canonically isomorphic to $\widehat{\mathbb{Z}}^{r_2+1} \times \prod\limits_{n\geq 1}\left((\mathbb{Z}/2\,\mathbb{Z})^{\delta} \times \mathbb{Z}/\mathbf{w}\,n\,\mathbb{Z}\right)$, independently of additional arithmetic considerations about the group $\prod_p \mathcal{T}_p^1$. If not (more probable), a description of the profinite group $\mathrm{Gal}(\overline{K}^{\mathrm{ab}}/K)$ may be very tricky. Any information will be welcome.

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