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NUMBER FIELDS WHOSE ABELIAN CLOSURES HAVE ISOMORPHIC GALOIS GROUPS

by

Georges GRAS

Abstract. — Following a paper by Athanasios Angelakis and Peter Stevenhagen on the determination of imaginary quadratic fields having the same absolute Abelian Galois group (arXiv:1209.6005), we study this property for arbitrary number fields. We show that such a property is probably not easily generalizable to non-imaginary quadratic fields, because of some p -adic obstructions coming from the global units. By restriction to the p -Sylow subgroups of the Galois groups, we show that the corresponding study is related to the classical notion of p -rational fields. All this is an application of results published in our Springer book on class field theory.

Résumé. — A partir d'un article de Athanasios Angelakis et Peter Stevenhagen sur la détermination de corps quadratiques imaginaires ayant le même groupe de Galois Abélien absolu (arXiv:1209.6005), nous étudions cette propriété pour les corps de nombres quelconques. Nous montrons qu'une telle propriété n'est probablement pas facilement généralisable aux corps non quadratiques imaginaires en raison d'obstructions p -adiques provenant des unités globales. En se restreignant aux p -sous-groupes de Sylow des groupes de Galois, nous montrons que l'étude correspondante est liée à la notion classique de corps p -rationnels. Tout ceci est une application de résultats publiés dans notre livre chez Springer sur la théorie du corps de classes.

1. Introduction – Notations

Let K be a number field of degree $n =: r_1 + 2r_2$, and let A_K be the Galois group $\text{Gal}(\overline{K}^{\text{ab}}/K)$ where \overline{K}^{ab} is the maximal Abelian pro-extension of K . The question is the following: in what circumstances the groups A_{K_1} and A_{K_2} are isomorphic groups when K_1 and K_2 are two distinct number fields? Of course such isomorphisms are only isomorphisms of Abelian profinite groups in which the arithmetical invariants (decomposition groups, inertia groups) are not respected. A first work on this subject was published in [O] by Midori Onabe.

Let p be a fixed prime number and let $H, H_p, H_{\text{ta}}, \widetilde{K}_p$, be the p -Hilbert class field (in ordinary sense), the maximal p -ramified (i.e. unramified outside p) Abelian pro- p -extension of K (in ordinary sense), the maximal tamely ramified Abelian pro- p -extension of K (in restricted sense), the compositum of the \mathbb{Z}_p -extensions of K , respectively.

Then let $\mathcal{T}_p := \text{Gal}(H_p/\widetilde{K}_p)$ and $\mathcal{C}_p := \text{Gal}(H/K)$ isomorphic to the p -class group of K . The groups \mathcal{T}_p and \mathcal{C}_p are finite groups.

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For any finite place v of K , we denote by K_v the corresponding completion of K , then by $U_v := \{u \in K_v, |u|_v = 1\}$ and $U_v^1 := \{u \in U_v, |u - 1|_v < 1\}$, the unit group and principal unit group of K_v , respectively.

These definitions can be summarized by the following diagram [Gr1, III.2.6.1, Fig. 2.2]:

$$\begin{array}{ccccc}
 & & \mathcal{T}_p & & \\
 & & \frown & & \\
 \tilde{K}_p & \xrightarrow{\quad} & \tilde{K}_p H & \xrightarrow{\quad} & H_p \\
 \downarrow & & \downarrow & & \nearrow \\
 \tilde{K}_p \cap H & \xrightarrow{\quad} & H & \xrightarrow{\quad} & \bigoplus_{v|p} U_v^1 / E \otimes \mathbb{Z}_p \\
 \downarrow & & \nearrow & & \\
 K & & & & \\
 & & \searrow & & \\
 & & \mathcal{C}_p & &
 \end{array}$$

where E is the group of global units of K and where $E \otimes \mathbb{Z}_p$ is diagonally embedded with obvious maps $i_p = (i_v)_{v|p}$.

We shall make use of some p -adic logarithms as follows:

(i) We consider the p -adic logarithm $\log_p : K^\times \rightarrow \bigoplus_{v|p} K_v$ defined by $\log_p(x) = (\log_v(x))_{v|p}$

where $\log_v(x) := \log(i_v(x))$ for all $v|p$, and where $\log : \mathbb{C}_p^\times \rightarrow \mathbb{C}_p$ is the Iwasawa extension of the usual p -adic logarithm. We have $\log_p = \log \circ i_p$ on K^\times .

(ii) We then define the quotient \mathbb{Q}_p -vector space $\mathcal{L}_p := \left(\bigoplus_{v|p} K_v \right) / \mathbb{Q}_p \log_p(E)$. We have, under the Leopoldt conjecture for p in K , $\dim_{\mathbb{Q}_p}(\mathcal{L}_p) = r_2 + 1$.

(iii) Finally, we denote by Log_p the map, from the group I_p of ideals of K prime to p , to \mathcal{L}_p , sending $\mathfrak{a} \in I_p$ to $\text{Log}_p(\mathfrak{a})$ defined as follows. If m is such that $\mathfrak{a}^m = (\alpha)$ with $\alpha \in K^\times$, we set $\text{Log}_p(\mathfrak{a}) := \frac{1}{m} \log_p(\alpha) \bmod \mathbb{Q}_p \log_p(E)$; this does not depend on the choice of m .

2. Class field theory study of the Abelian closures of number fields

Let $\overline{K}^{\text{ab}}_{(p)}$ be the maximal Abelian pro- p -extension of K . In [Gr1, III.4.4.1], we have given (assuming the Leopoldt conjecture for p) the following fundamental diagram for the structure of $\text{Gal}(\overline{K}^{\text{ab}}_{(p)}/K)$, isomorphic to the p -Sylow subgroup of A_K :

$$\begin{array}{ccccc}
 & & \prod_{v \nmid p} (F_v^\times)_p & & \\
 & & \frown & & \\
 H_p & \xrightarrow{\quad} & M_p & \xrightarrow{\quad} & \overline{K}^{\text{ab}}_{(p)} \\
 \downarrow & & \downarrow & & \nearrow \\
 H & \xrightarrow{\quad} & H_{\text{ta}} & \xrightarrow{\quad} & \bigoplus_{v|p} U_v^1 \\
 \downarrow & & & & \\
 K & & & & \\
 & & \searrow & & \\
 & & \mathcal{E}_p := E \otimes \mathbb{Z}_p & &
 \end{array}$$

where F_v is the residue field of v in K and $(F_v^\times)_p \simeq F_v^\times \otimes \mathbb{Z}_p$ the p -Sylow subgroup of its multiplicative group. This also concerns the real infinite places for which $F_v^\times = \{\pm 1\}$ [Gr1, I.3.1.2]. In this diagram, M_p is the direct compositum over H of H_p and H_{ta} .

For some comments on the diagonal embeddings of $\mathcal{E}_p := E \otimes \mathbb{Z}_p$ in $\prod_{v \nmid p} (F_v^\times)_p$ and in $\bigoplus_{v \mid p} U_v^1$, respectively, see [Gr1, III.4.4.5.1].

Definition 2.1. — The number field K is said to be p -rational (see [MN], [GJ], [JN], and [Gr1, IV, § b, 3.4.4]) if it satisfies the Leopoldt conjecture for p and if $\mathcal{T}_p = 1$.

Remarks 2.2. — (i) By [Gr1, IV.3.4.5], assuming the Leopoldt conjecture for p , the p -rationality of K is equivalent to the following three conditions:

- $\bigoplus_{v \mid p} \mu_p(K_v) = \mu_p(K)$ (diagonal embedding), where $\mu_p(k)$ denotes, for any field k , the group of roots of unity of k of p -power order,
- the p -Hilbert class field H is contained in the compositum \tilde{K}_p of the \mathbb{Z}_p -extensions of K ; this is equivalent to $\mathcal{C}_p \simeq \mathbb{Z}_p \text{Log}_p(I_p) / \left(\bigoplus_{v \mid p} \log(U_v^1) + \mathbb{Q}_p \log_p(E) \right)$, which can be non-trivial,
- $\mathbb{Z}_p \log_p(E)$ is a direct summand in $\bigoplus_{v \mid p} \log(U_v^1)$, which expresses the minimality of the valuation of the p -adic regulator.

(ii) If K is p -rational, then all its subfields are p -rational.

(iii) For a p -rational field K , we have (non-canonically):

$$\text{Gal}(\overline{K}^{\text{ab}}_{(p)}/K) \simeq \mathbb{Z}_p^{r_2+1} \times \prod_{v \nmid p} (F_v^\times)_p, \text{ with } \text{Gal}(\overline{K}^{\text{ab}}_{(p)}/\tilde{K}_p) \stackrel{\text{can}}{\simeq} \prod_{v \nmid p} (F_v^\times)_p.$$

(iv) Let \tilde{K}_∞ be the compositum of the \tilde{K}_p ; \tilde{K}_∞ is also the maximal $\widehat{\mathbb{Z}}$ -extension of K ; then a necessary and sufficient condition to get $\text{Gal}(\overline{K}^{\text{ab}}/K) \simeq \widehat{\mathbb{Z}}^{r_2+1} \times \prod_v F_v^\times$, with $\text{Gal}(\overline{K}^{\text{ab}}/\tilde{K}_\infty) \stackrel{\text{can}}{\simeq} \prod_v F_v^\times$, is that K be p -rational for all p .

We have, following [AS, Lemma 3.2], the following result valid for any number field K .

Proposition 2.3. — *Let K be a number field. We denote by μ_K the group of roots of unity of K and by \mathfrak{w} its order. Let F_v be the residue field of the place v (finite or real infinite) of K . Then we have the group isomorphism*

$$\prod_v F_v^\times \simeq \prod_{n \geq 1} \mathbb{Z}/\mathfrak{w} n \mathbb{Z}.$$

Proof. — Since $\mathbb{Q}(\mu_K) \subseteq K$, we know that $|F_v^\times| \equiv 0 \pmod{\mathfrak{w}}$ for all $v \nmid \mathfrak{w}$. Then we consider, in the two groups $\prod_{v \nmid \mathfrak{w}} F_v^\times$ and $\prod_{n \geq 1} \mathbb{Z}/\mathfrak{w} n \mathbb{Z}$, the ℓ -Sylow subgroups for fixed prime ℓ . The property giving an isomorphism is that for any given ℓ -power ℓ^k , $k \geq 1$, the two ℓ -Sylow subgroups have 0 or infinitely many cyclic direct components of order ℓ^k . This fact is obvious for $\prod_{n \geq 1} \mathbb{Z}/\mathfrak{w} n \mathbb{Z}$ in which there is no direct component of order ℓ^k , $k \geq 1$, if and only if $\ell^{k+1} \mid \mathfrak{w}$. We have to prove that in $\prod_{v \nmid \mathfrak{w}} F_v^\times$ there exist infinitely many places v of K for which $|F_v^\times|$ is congruent to 0 modulo ℓ^m but not modulo ℓ^{m+1} , where $m \geq 0$ is any integer greater than or equal to the ℓ -valuation λ_ℓ of \mathfrak{w} .

Let ζ_{m+1} be a primitive ℓ^{m+1} th root of unity and $\zeta_m := \zeta_{m+1}^\ell$; then $K(\zeta_{m+1})/K(\zeta_m)$ is a non-trivial extension. There exist infinitely many prime numbers q with a suitable place $v \mid q$ whose inertia group is the cyclic group $\text{Gal}(K(\zeta_{m+1})/K(\zeta_m))$; so we get $F_v = \mathbb{F}_q$ which is such that $|F_v^\times| \equiv 0 \pmod{\ell^m}$ and $|F_v^\times| \not\equiv 0 \pmod{\ell^{m+1}}$.

Consider the term $\prod_{v \mid \mathfrak{w}} F_v^\times$; if $v \mid q \mid \mathfrak{w}$ (q prime), then $|F_v^\times| \equiv 0 \pmod{\frac{\mathfrak{w}}{q^{\lambda_q}}}$, where λ_q is the q -valuation of \mathfrak{w} ; for all $\ell \neq q$, the ℓ -Sylow subgroup of F_v^\times is of order ℓ^m , $m \geq \lambda_\ell$ (the ℓ -valuation of \mathfrak{w}), so we do not modify the structure of the product $\prod_{n \geq 1} \mathbb{Z}/\mathfrak{w} n \mathbb{Z}$. \square

Corollary 2.4. — *Let \mathcal{H}_∞ be the compositum of the fields H_p (maximal p -ramified Abelian pro- p -extensions of K) for all prime numbers p . Then, under the Leopoldt conjecture for all p , we have the group isomorphism*

$$\text{Gal}(\overline{K}^{\text{ab}}/\mathcal{H}_\infty) \simeq \prod_{n \geq 1} \mathbb{Z}/\mathfrak{w} n \mathbb{Z}.$$

So we have obtained the following diagram (under the Leopoldt conjecture for all p):

$$\begin{array}{ccccc}
 & & \prod_{n \geq 1} \mathbb{Z}/\mathfrak{w} n \mathbb{Z} & & \\
 & \text{---} & \text{---} & \text{---} & \\
 \mathcal{H}_\infty & \text{---} & \mathcal{M}_\infty & \text{---} & \overline{K}^{\text{ab}} \\
 | & & | & \text{---} & \mathcal{E} := E \otimes \widehat{\mathbb{Z}} \\
 \mathcal{H} & \text{---} & \mathcal{H}_{\text{ta}} & \text{---} & \prod_v U_v^1 \\
 | & & & & \\
 K & & & &
 \end{array}$$

where \mathcal{H}_{ta} is the compositum of the H_{ta} and $\mathcal{M}_\infty = \mathcal{H}_\infty \mathcal{H}_{\text{ta}}$ (direct compositum over the Hilbert class field \mathcal{H}).

Let \mathcal{F}_∞ be the compositum of some finite extensions F_p of K such that $H_p = \widetilde{K}_p F_p$; we then have $\text{Gal}(F_p/K) \simeq \mathcal{T}_p$ and $\text{Gal}(\mathcal{F}_\infty/K) \simeq \prod_p \mathcal{T}_p$. The extension \mathcal{F}_∞/K is in general non-canonical and conjecturally infinite; it measures a mysterious degree of complexity of $\text{Gal}(\overline{K}^{\text{ab}}/K)$. But we have

$$\text{Gal}(\mathcal{H}_\infty/\widetilde{K}_\infty) \stackrel{\text{can}}{\simeq} \prod_p \mathcal{T}_p \quad \text{and} \quad \text{Gal}(\mathcal{H}_\infty/K) \simeq \widehat{\mathbb{Z}}^{r_2+1} \times \prod_p \mathcal{T}_p.$$

Theorem 2.5. — *Let K be a number field and \overline{K}^{ab} the maximal Abelian pro-extension of K . We suppose that the p -adic Leopoldt conjecture is verified in K for all prime number p .*

Then there exists an Abelian extension \mathcal{F}_∞ of K such that \mathcal{H}_∞ (the compositum of the H_p) is the direct compositum of \mathcal{F}_∞ and the maximal $\widehat{\mathbb{Z}}$ -extension \widetilde{K}_∞ of K , and such that

$$\text{Gal}(\overline{K}^{\text{ab}}/\mathcal{F}_\infty) \simeq \widehat{\mathbb{Z}}^{r_2+1} \times \prod_{n \geq 1} \mathbb{Z}/\mathfrak{w} n \mathbb{Z}, \quad \text{with} \quad \text{Gal}(\overline{K}^{\text{ab}}/\mathcal{H}_\infty) \simeq \prod_{n \geq 1} \mathbb{Z}/\mathfrak{w} n \mathbb{Z},$$

where \mathfrak{w} is the number of roots of unity of K .

Remark 2.6. — Write $\mathfrak{w} = \prod_{q \text{ prime}} q^{\lambda_q}$ and put $\mathfrak{w}^1 := \prod_{\lambda_q \geq 2} q^{\lambda_q}$, so that $\mathfrak{w}^0 := \mathfrak{w}/\mathfrak{w}^1$ and \mathfrak{w}^1 are coprime integers. Then we can replace \mathfrak{w} by \mathfrak{w}^1 in the above formulas.

If $\mathbf{w}^1 = 1$ (\mathbf{w} square-free), we have the isomorphism

$$\mathrm{Gal}(\overline{K}^{\mathrm{ab}}/\mathcal{F}_\infty) \simeq \widehat{\mathbb{Z}}^{r_2+1} \times \prod_{n \geq 1} \mathbb{Z}/n\mathbb{Z}.$$

Corollary 2.7. — *The Galois groups $\mathrm{Gal}(\overline{K}^{\mathrm{ab}}/\mathcal{F}_\infty)$ (up to non-canonical isomorphisms) are independent of the number fields K as soon as these fields satisfy the Leopoldt conjecture for all p , have the same number of complex places and the same \mathbf{w}^1 .*

Thus, for all totally real number fields K (satisfying the Leopoldt conjecture for all p) we have

$$\mathrm{Gal}(\overline{K}^{\mathrm{ab}}/\mathcal{F}_\infty) \simeq \widehat{\mathbb{Z}} \times \prod_{n \geq 1} \mathbb{Z}/n\mathbb{Z}.$$

We have $\mathcal{F}_\infty = K$ if and only if K is p -rational for all p .

From Remarks 2.2 (i), we see that the first two conditions of p -rationality involve a finite number of primes p , but that the third condition is the most ugly.⁽¹⁾ So, we are mainly concerned with the imaginary quadratic fields studied in [AS] for which the third condition is empty; the first and second ones can be verified (for all p) probably for infinitely many imaginary quadratic fields as suggested in [AS, Conjecture 7.1].

As we shall see now, we can strengthen a few the previous results about the first condition involved in the definition of p -rationality, which can be removed for any number field.

This concerns the globalization of the finite groups $\bigoplus_{v|p} \mu_p(K_v)/\mu_p(K)$. This group measures the gap between the regular and Hilbert kernels in $K_2(K)$ [Gr1, II.7.6.1].

Consider the subgroup

$$T_p^1 = \bigoplus_{v|p} \mu_v^1 \subset \bigoplus_{v|p} U_v^1 \simeq \mathrm{Gal}(\overline{K}^{\mathrm{ab}}(p)/H_{\mathrm{ta}}),$$

where we have put $\mu(K_v) \simeq F_v^\times \bigoplus \mu_v^1$ [Gr1, II.7.1.3]; we also have $\mu_v^1 = \mu_p(K_v)$ for all $v|p$, but with this notation the globalisation is more obvious. The places (finite in number) such that $\mu_v^1 \neq 1$ are called irregular places.

Let $\Gamma_p := \prod_{v \nmid p} (F_v^\times)_p \times T_p^1 \simeq \prod_v \mu(K_v)$; we have $\Gamma_p \simeq (\prod_{n \geq 1} \mathbb{Z}/\mathbf{w}^1 n \mathbb{Z})_p$ since, in T_p^1 , each μ_v^1 is cyclic of order multiple of $|\mu_p(K)|$, giving for Γ_p the same group as for $\prod_{v \nmid p} (F_v^\times)_p$.

Let H_p^1 be the subfield of H_p fixed by $\rho(\Gamma_p)$, where ρ is the reciprocity map on the idèle group Γ_p . Then from the local-global characterization of the Leopoldt conjecture at p (see [Gr1, III.3.6.6] or [Ja, § 2.3]), we get $\rho(\Gamma_p) \simeq \Gamma_p/\mathcal{E}_p \cap \Gamma_p = \Gamma_p/\mu_p(K)$.

Taking, as in [AS, Corollary 3.3, Lemma 3.4], $v_0 \nmid \mathbf{w}$ such that the residue image of $\mu_p(K)$ is a direct summand in $(F_{v_0}^\times)_p$, we still get $\mathrm{Gal}(\overline{K}^{\mathrm{ab}}(p)/H_p^1) \simeq \Gamma_p/\mu_p(K) \simeq (\prod_{n \geq 1} \mathbb{Z}/\mathbf{w}^1 n \mathbb{Z})_p$.

We note that $\mathrm{Gal}(H_p/H_p^1) \simeq \bigoplus_{v|p} \mu_v^1/\mu_K$.

Of course, the globalization gives $\mathrm{Gal}(\overline{K}^{\mathrm{ab}}/\mathcal{H}_\infty^1) \simeq \prod_{n \geq 1} \mathbb{Z}/\mathbf{w}^1 n \mathbb{Z}$, where \mathcal{H}_∞^1 is the compositum of the H_p^1 (we have $\widetilde{\mathcal{K}}_\infty \subseteq \mathcal{H}_\infty^1 \subseteq \mathcal{H}_\infty$).

⁽¹⁾For instance, for $K = \mathbb{Q}(\sqrt{2})$, the third condition is not satisfied for $p = 13, 31, 1546463, \dots$ and perhaps for infinitely many primes p depending on Fermat quotients of the fundamental unit [Gr1, III.4.14]. Note that from [Gr2, III] or [Gr1, IV.3.5.1], for $p = 2$, the 2-rational Abelian 2-extensions of \mathbb{Q} are the subfields of the fields $\mathbb{Q}(\mu_{2^\infty})\mathbb{Q}(\sqrt{-\ell})$, $-\ell \equiv 5 \pmod{8}$, or of the fields $\mathbb{Q}(\mu_{2^\infty})\mathbb{Q}\left(\sqrt{\sqrt{\ell} \frac{a-\sqrt{\ell}}{2}}\right)$, $\ell = a^2 + 4b^2 \equiv 5 \pmod{8}$.

We have an analog of Corollary 2.7 with smaller extensions \mathcal{F}_∞ , but the problem for the non-imaginary quadratic fields is unchanged since in the global exact sequence

$$1 \rightarrow \prod_v F_v^\times \simeq \prod_{n \geq 1} \mathbb{Z}/\mathfrak{w}^1 n \mathbb{Z} \longrightarrow \text{Gal}(\overline{K}^{\text{ab}}/\tilde{\mathcal{K}}_\infty) \longrightarrow \prod_p \mathcal{T}_p \rightarrow 1,$$

where $\tilde{\mathcal{K}}_\infty$ is the maximal $\widehat{\mathbb{Z}}$ -extension of K , we do not know if the structure of $\text{Gal}(\overline{K}^{\text{ab}}/\tilde{\mathcal{K}}_\infty)$ can be the same for various number fields K because of the unknown group $\prod_p \mathcal{T}_p$ (which non-trivially depends on the p -adic properties of the classes and units of K), and the nature of the corresponding group extension. We have

$$\text{Gal}(\overline{K}^{\text{ab}}/K) \simeq \widehat{\mathbb{Z}}^{r_2+1} \times \prod_{n \geq 1} \mathbb{Z}/\mathfrak{w}^1 n \mathbb{Z}$$

as soon as the second and third condition of p -rationality (Remarks 2.2 (i)) are satisfied for all p , which defines a weaker version of p -rationality which may have some interest.

For imaginary quadratic fields $K \neq \mathbb{Q}(\sqrt{-1})$, we find again that $\text{Gal}(\overline{K}^{\text{ab}}/K) \simeq \widehat{\mathbb{Z}}^2 \times \prod_{n \geq 1} \mathbb{Z}/n \mathbb{Z}$ as soon as, for all p dividing the class number, the p -Hilbert class field is contained in the compositum of the \mathbb{Z}_p -extensions of K , which is equivalent to $\mathcal{C}_p \simeq \mathbb{Z}_p \log_p(I_p) \Big/ \bigoplus_{v|p} \log_p(U_v^1)$.⁽²⁾

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⁽²⁾From [Gr1, III.2.6.6] or [Gr3, Theorem 2.3], for an imaginary quadratic field K , the 2-Hilbert class field is contained in the compositum of its \mathbb{Z}_2 -extensions if and only if K is one of the following fields: $\mathbb{Q}(\sqrt{-1})$, $\mathbb{Q}(\sqrt{-2})$, $\mathbb{Q}(\sqrt{-\ell})$ (ℓ prime $\equiv 3, 5, 7 \pmod{8}$), $\mathbb{Q}(\sqrt{-2\ell})$ (ℓ prime $\equiv 3, 5 \pmod{8}$), $\mathbb{Q}(\sqrt{-\ell q})$ (ℓ, q primes, $\ell \equiv -q \equiv 3 \pmod{8}$).