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► **To cite this version:**

Pierre-Antoine Guiheneuf. Dynamical properties of spatial discretizations of a generic homeomorphism. 2012. hal-00764571v1

HAL Id: hal-00764571

<https://hal.science/hal-00764571v1>

Preprint submitted on 13 Dec 2012 (v1), last revised 25 Nov 2013 (v4)

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DYNAMICAL PROPERTIES OF SPATIAL DISCRETIZATIONS OF A GENERIC HOMEOMORPHISM

PIERRE-ANTOINE GUIHÉNEUF

ABSTRACT. This paper concerns the link between the dynamical behaviour of a dynamical system and the dynamical behaviour of its numerical simulations. Here, we model numerical truncation as a spatial discretization of the system. Some previous works on well chosen examples (see e.g. [Boy86], [GT83], [GB88], [KMP99] or [Lan98]) show that the dynamical behaviours of dynamical systems and of their discretizations can be quite different. In this paper we are interested in generic homeomorphisms of compact manifolds. So our aim is to tackle the following question: can the dynamical properties of a generic homeomorphism be detected on the spatial discretizations of this homeomorphism? We study both conservative and dissipative homeomorphisms (i.e. with or without the assumption that the homeomorphism preserves a given measure).

We will prove that the dynamics of a single discretization of a generic conservative homeomorphism does not depend on the homeomorphism itself, but rather on the grid used for the discretization. Therefore, dynamical properties of a given generic conservative homeomorphism cannot be detected using a single discretization. Nevertheless, we will also prove that some dynamical features of a generic conservative homeomorphism (such as the set of the periods of all periodic points) can be read on a sequence of finer and finer discretizations. The dissipative case is somehow opposite: we will see that the dynamics of a generic dissipative homeomorphism is well-approximated by the dynamics of any fine enough discretization of this homeomorphism.

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Date: December 13, 2012.

2010 Mathematics Subject Classification. 37M05, 37M25, 37M99, 37B99, 37A05.

Key words and phrases. Conservative generic homeomorphism, dissipative generic homeomorphism, dynamical property, discretization.

1. INTRODUCTION

When one tries to simulate a discrete time dynamical system with a computer, calculations are performed with a finite numerical precision. Numerical errors made at each iteration may add up, so that after a while, the numerically calculated orbit of a point will have nothing in common with the actual one. Nevertheless, a numerically calculated orbit is close to a actual orbit at any time, thus one can hope that the collective behaviour of numerically calculated orbits provides information about the collective behaviour of actual orbits. Interesting properties of dynamical systems are often *asymptotic* (i.e. concern the behaviour of the system in the long term) on the one hand, and *collective* (i.e. concern the collective behaviour of orbits or even the action of the system on open sets and probability measures) on the other. The purpose of our study is to tackle the question: can such properties be detected on numerical simulations?

We will model numerical truncation as a spatial discretization¹. Consider a dynamical system whose phase space is a manifold X and whose evolution law is given by a map f from X into itself. Roughly speaking, a numerical simulation of the system with a precision 10^{-N} replaces the continuous phase space X by a discrete space E_N made of points of X whose coordinates are decimal numbers with at most N decimal places, and replaces the map f by its discretization $f_N : E_N \rightarrow E_N$ that maps $x \in E_N$ to the point (or one of the points) of E_N nearest $f(x)$. We can clarify a bit the question raised in the previous paragraph:

Question. Can the dynamical properties of a homeomorphism f be read on the dynamics of its discretizations $(f_N)_{N \geq 0}$?

Questions about discretizations of dynamical systems have mainly been studied empirically, using numerical simulations of well-chosen examples (see e.g. [Boy86], [GB88], [KMP99] or [Lan98]). In particular, a short article of J.-M. Gambaudo and C. Tresser [GT83] gives the examples of two simple homeomorphisms of $[0, 1]^2$, which both have sinks that attract most of the orbits. However, those sinks are undetectable in practice, simply because the connected components of their basins are much too small. Thus, the actual dynamics of these homeomorphisms does not have much in common with that observed on simulations. From a more theoretical point of view, É. Ghys studied (in [Ghy94]) the dynamical behaviour of discretizations of the Anosov automorphism of the torus \mathbf{T}^2 given by $(x, y) \mapsto (y, x + y)$. É. Ghys notes that the dynamics of these discretizations (he shows that they are permutations with remarkably small orders) has nothing to do with the actual dynamics of the automorphism, which is a paradigm of chaotic dynamics. The very specific arithmetical properties of this example let one hope that such a difference between the actual dynamics of a system and the dynamics of its discretizations might be exceptional. This hope is supported by more recent results of S. Hayashi in [Hay12], roughly speaking, he shows that every C^1 -diffeomorphism of a compact manifold with no positive Lyapunov exponent can be approximated by another diffeomorphism whose sinks are “observable in practise”. Furthermore, T. Miernowski made a fairly comprehensive study of discretizations of circle homeomorphisms [Mie06]. He basically shows that the dynamical properties of a typical (generic or prevalent) circle homeomorphism/diffeomorphism can be read on the dynamics of its discretizations. All these results suggest to tackle the above question not for *arbitrary* homeomorphisms but rather for *typical* homeomorphisms.

In this paper we study dynamical properties of discretizations of generic homeomorphisms in the sense of Baire. We establish properties for both *dissipative*, i.e. arbitrary homeomorphisms of X , and *conservative* homeomorphisms, i.e. homeomorphisms of X that preserve a given good probability measure.

The results we obtained concern generic homeomorphisms of a compact manifold (with boundary) of dimension $n \geq 2$. In this introduction we state the main results for the torus \mathbf{T}^n and for “uniform” discretizations on \mathbf{T}^n . The general framework in which the results are valid is defined in section 2.

¹See page 9 about the accuracy of this model.

Let $n \geq 2$. For all $N \in \mathbf{N}$, let E_N be the finite subset of the torus $\mathbf{T}^n = \mathbf{R}^n/\mathbf{Z}^n$ made of the points whose coordinates are decimal numbers with at most N decimal places:

$$E_N = \left\{ \left(\frac{i_1}{10^N}, \dots, \frac{i_n}{10^N} \right) \in \mathbf{T}^n \mid (i_1, \dots, i_n) \in \mathbf{Z}^n \right\}.$$

Let P_N be a projection of \mathbf{T}^n on E_N minimizing the euclidian distance: for $x \in \mathbf{T}^n$, $P_N(x)$ is one of the points of E_N nearest x . Let $f_N : E_N \rightarrow E_N$ be the discretization of the homeomorphism $f : \mathbf{T}^n \rightarrow \mathbf{T}^n$ according to E_N defined by $f_N = P_N \circ f$.

We will prove many results, concerning various aspects of the dynamics of the discretizations. In the sequel of this introduction, we try to organize those results according to some “lessons”:

1) The dynamics of a single discretization of a generic homeomorphism has in general nothing to do with the dynamics of the initial homeomorphism.

Given a finite set E , the dynamics of any finite map $\sigma : E \rightarrow E$ is quite simple: given $x \in E$, the orbit $(\sigma^k(x))_k$ is preperiodic. Hencefore, the union of periodic orbits of σ , which we will denote by $\Omega(\sigma)$ and called the *maximal invariant set* of σ , is exactly the union of the ω -limits sets of points of E . To study the dynamics of σ one can focus on quantities such as the cardinal of $\Omega(\sigma)$, the stabilization time of σ (i.e. the smallest $t \in \mathbf{N}$ such that $\sigma^t(E) = \Omega(\sigma)$), the number of orbits of $\sigma|_{\Omega(\sigma)}$, their lengths, the period of $\sigma|_{\Omega(\sigma)}$...

Another interesting dynamical quantity for a finite map $\sigma : E \rightarrow E$ is the *recurrence rate*: it is the ratio between the cardinals of $\Omega(\sigma)$ and that of E . The starting point of our article is a question from É. Ghys (see chapter 6 of [Ghy94]): for a generic conservative homeomorphism of the torus, what is the asymptotical behaviour of the sequence of recurrence rates of f_N ? A partial answer to this question was obtained by T. Miernowski in the second chapter of his thesis.

Theorem (Miernowski). *For a generic conservative homeomorphism² f , there are infinitely many integers N such that the discretization f_N is a cyclic permutation.*

To prove this theorem, T. Miernowski combines a genericity argument with a quite classical technique in generic dynamics of homeo(and auto)morphisms: approximation by permutations (see e.g. [KS70], [Alp78], [Hal56]), and more precisely Lax’s theorem (see [Lax71]), which states that any conservative homeomorphism of the torus can be approximated by cyclic permutations of the discretization grids. In fact this proof can be generalized to obtain many results about discretizations. We will establish some variants of Lax theorem; each of them, combined with a genericity argument, leads to a result for discretizations of generic homeomorphisms. For instance:

Theorem. *For a generic conservative homeomorphism f , it exists $C > 0$ such that there are infinitely many integers N such that the cardinal of $\Omega(f_N)$ is smaller than C .*

Note that the combination of these two theorems answer the question of É. Ghys: for a generic homeomorphism f , the sequence of the recurrence rate of f_N accumulates on both 0 and 1, one can even show that it accumulates on the whole segment $[0, 1]$.

Let us move on to the behaviour on the maximal invariant set $\Omega(f_N)$. Another variations of Lax’s theorem lead to a theorem that enlightens the behaviour of the discretizations on their maximal invariant set:

Theorem. *For a generic conservative homeomorphism f , there are infinitely many integers N such that the discretization f_N is a cyclic permutation and for all $M \in \mathbf{N}$, there are infinitely many integers N such that f_N is a permutation of E_N having at least M periodic orbits.*

²I.e. there is a G_δ dense subset of the set of conservative homeomorphisms of the torus on which the conclusion of theorem holds.

To summarize, generically, infinitely many discretizations are cyclic permutations, but also infinitely many discretizations are highly non injective or else permutations with many cycles. In particular, it implies that for all $x \in X$, it exists infinitely many integers N such the orbit of x_N under f_N does not shadow the orbit of x under f : in this sense, generically, the dynamics of discretizations does not reflect that of the homeomorphism.

2) A dynamical property of a generic homeomorphism can not be deduced from the frequency it appears on discretizations either.

The previous theorems express that the dynamics of a single discretization does not reflect the actual dynamics of the homeomorphism. However, one might reasonably expect that the properties of the homeomorphism are transmitted to many discretizations. More precisely, one may hope that given a property (P) about discretizations, if there are many N such that the discretization f_N satisfies (P), then the homeomorphism satisfies a similiary property. It is not so, for instance:

Theorem. *For a generic conservative homeomorphism f , the proportion of integers N between 1 and M such that f_N is a cyclic permutation accumulates on both 0 and 1 when M goes to infinity.*

In fact, for most of the properties considered in the previous paragraph, the frequency they appear on discretizations of orders smaller than M accumulates on both 0 and 1 when M goes to infinity.

3) Some important dynamical features of a generic homeomorphism can be detected by looking at some dynamical features of *all* the discretizations.

We have observed that one can not detect the dynamics of a generic homeomorphism when looking at the dynamics of its discretizations, or even at the frequency some dynamics appears on discretizations. Nevertheless, some dynamical features can be deduced from the analogous dynamical features of *all* the discretizations. For instance, the periods of periodic orbits of a homeomorphism can be read on the periods of periodic orbits of its discretizations:

Proposition. *A generic homeomorphism f has a periodic orbit with period p if and only if there exists infinitely many integers N such that f_N has a periodic orbit with period p .*

We will also prove a theorem in the same vein for invariant measures. It expresses that the set of invariant measures of the homeomorphism can be deduced from the sets of invariant measures of its discretizations. More precisely:

Theorem. *Let \mathcal{M}_N be the set of probability measures on E_N that are invariant under f_N . For a generic conservative homeomorphism f , the upper limit over N (for Hausdorff topology) of the sets \mathcal{M}_N is exactly the set of probability measures that are invariant under f .*

4) Physical measures of a generic homeomorphism can not be detected on discretizations.

Given $x \in \mathbf{T}^n$, the Birkhoff limit of x is defined (when it exists) as the limit of $\frac{1}{m} \sum_{i=0}^{m-1} f_*^i \delta_x$ when m goes to infinity; the *basin* of a measure μ is the set of points x whose Birkhoff limit coincides with μ . Heuristically, the basin of a measure is the set of points that can see the measure. A Borel measure is said *physical* if its basin has positive Lebesgue measure (see e.g. [You02]). The heuristic idea underlying this concept is that physical measures are invariant measure which can be detected “experimentally” (since many initial conditions lead to these measures).

According to this heuristic, one could expect physical measures to be the only invariant measures that can be detected on discretization. This is not the case: for a generic conservative homeomorphism, there exists a unique physical measure, say Lebesgue measure. According to the previous theorem, invariant measures of the discretizations accumulate on all the invariant measures of the homeomorphism and not only on Lebesgue measure.

However, one can still hope to distinguish the physical measure from other invariant measures. For this purpose, we define the canonical physical measure μ_N^f associated to a discretization f_N : it is the limit in the sense of Cesàro of the images of the uniform measure on E_N by the iterates of f_N : if λ_N is the uniform measure on E_N , then

$$\mu_N^f = \lim_{M \rightarrow \infty} \frac{1}{M} \sum_{m=0}^{M-1} (f_N^m)_* \lambda_N.$$

This measure is supported by the maximal invariant set $\Omega(f_N)$; it is uniform on every periodic orbit and the total weight of a periodic orbit is proportional to the size of its basin of attraction. The following theorem expresses that these measures accumulate on the whole set of f -invariant measures: physical measures can not be distinguished from other invariant measures on discretizations, at least for generic homeomorphisms.

Theorem. *For a generic conservative homeomorphism f , the set of accumulations points of the sequence $(\mu_N^f)_{N \in \mathbf{N}}$ is the set of all f -invariant measures.*

5) The dynamics of discretizations of a generic dissipative homeomorphism tends to that of the initial homeomorphism.

We then study properties of discretizations of generic *dissipative* homeomorphisms³. The basic tool is the *shredding lemma* of F. Abdenur and M. Andersson [AA12], which implies that a generic homeomorphism has a “sink dynamics”. This easily transmits to discretizations, for example the basins of attraction of the homeomorphism can be seen on all the fine enough discretizations. Moreover there is *convergence* of the dynamics of discretizations f_N to that of f :

Theorem. *For a generic dissipative homeomorphism f , for all $\varepsilon > 0$ and all $\delta > 0$, there is a full measure dense open subset O of \mathbf{T}^n such that for all $x \in O$, all $\delta > 0$ and all N large enough, the orbit of x_N under f_N δ -shadows the orbit⁴ of x under f .*

Thus, one can detect on discretizations the dynamics of a generic dissipative homeomorphism, which is mainly characterized by position of the sinks and of the corresponding basins of attraction. Note that this behaviour is in the opposite of the conservative case, where the individual behaviour of discretizations does not indicate anything about the actual dynamics of the homeomorphism.

6) In practice, one can not deduce the dynamics of a conservative homeomorphism from its discretizations, and the dynamics of a dissipative homeomorphism can be detected on discretizations only if the basins are large enough.

Finally, we compare our theoretical results with the reality of numerical simulations. Indeed, it is not clear that the behaviour predicted by our results can be observed on computable discretizations of an homeomorphism defined by a simple formula. On the one hand, all our results are valid “for *generic* homeomorphisms”: nothing indicates that these results apply to practical examples of homeomorphisms defined by simple formulas. On the other hand, results such as “there are infinitely many integers N such that the discretization of order N ...” provide no control over the integers N involved, they may be so large that the associated discretizations are not calculable in practice.

For conservative homeomorphisms, our numerical simulations produce mixed results. From a quantitative viewpoint, the behaviour predicted by our theoretical result cannot be observed on our numerical simulations. For example, we can not observe any discretization whose recurrence rate is equal to 1 (i.e. which is a permutation). This is nothing but surprising: the events pointed out by the theorems are a priori very rare. For instance, there is a very little proportion of bijective maps among maps from a given finite set into

³I.e. without assumption of preservation of a given measure.

⁴I.e. for all $k \in \mathbf{N}$, $d(f_N^k(x_N), f^k(x)) < \delta$.

itself. From a more qualitative viewpoint, the behaviour of the simulations is quite in accordance with the predictions of the theoretical results. For example, for a given conservative homeomorphism, the recurrence rate of a discretization does depend a lot on the size of the grid used for the discretization. Similarly, the canonical invariant measure associated with a discretization of a homeomorphism f does depend a lot on the size of the grid used for the discretization.

We also carried out simulations of dissipative homeomorphisms. The results of discretizations of a small perturbation of identity (in C^0 topology) may seem disappointing at first sight: the sinks of the initial homeomorphisms can not be detected, and there is little difference with the conservative case. This behaviour is similar to that highlighted by J.-M. Gambaudo and C. Tresser in [GT83] (see page 1). That is why it seemed to us useful to test an homeomorphism which is C^0 close to the identity, but whose basins are large enough. In this case the simulations point out a behaviour that is very similar to that described by theoretical results, namely that the dynamics converges to the dynamics of the initial homeomorphism. In fact, one have actually observed behaviours as described by theorems only for examples of homeomorphisms with a very few number of sinks.

We carried out many other numerical simulations, they can be found on the web page: <http://www.math.u-psud.fr/~guiheneu/Recherche.html>

In section 2 we will present the framework. Sections 4 to 9 concern conservative homeomorphisms. More precisely, the results concerning the behaviour of a single discretization are set out in section 6, and those concerning the average behaviour of discretizations in section 7. We then come to the results about all the discretizations in section 8 and about physical measures in section 9. The behaviour of discretizations of generic dissipative homeomorphisms is established in section 11. Finally, the results of numerical simulations are presented in part 3.

2. FRAMEWORK

The manifold X and the measure λ . The results stated in the introduction for the torus \mathbf{T}^n and the Lebesgue measure Leb actually extend to any smooth manifold X with dimension $n \geq 2$, compact and possibly with boundary, endowed with a Riemannian metric d . We fix once and for all such a manifold X endowed with the metric d . In the general case, Lebesgue measure on \mathbf{T}^n can be replaced by a *good measure* λ on X :

Definition 1. A Borel probability measure λ on X is called a *good measure*, or an *Oxtoby-Ulam* measure if it is nonatomic, it has total support (it is positive on nonempty open sets) and it is zero on the boundary of X .

We fix once and for all a good measure λ on X .

Notation 2. We denote by $\text{Homeo}(X)$ the set of homeomorphisms of X , endowed by the metric d defined by:

$$d(f, g) = \sup_{x \in X} d(f(x), g(x)).$$

We note $\text{Homeo}(X, \lambda)$ the subset of $\text{Homeo}(X)$ made of the homeomorphisms that preserve the measure λ , endowed with the same metric d . Elements of $\text{Homeo}(X)$ will be called *dissipative* homeomorphisms and which of $\text{Homeo}(X, \lambda)$ *conservative* homeomorphisms.

Generic properties in $\text{Homeo}(X)$ and $\text{Homeo}(X, \lambda)$. The topological spaces $\text{Homeo}(X)$ and $\text{Homeo}(X, \lambda)$ are Baire spaces (see [Gui12]), i.e.: the intersection of every countable collection of dense open sets is dense. We call G_δ a countable intersection of open sets; a property satisfied on at least a G_δ dense set is called *generic*. Note that in a Baire space, generic properties are stable under intersection.

Sometimes we will use the phrase “for a generic homeomorphism $f \in \text{Homeo}(X)$ (resp. $\text{Homeo}(X, \lambda)$), we have the property (P) ”. By that we will mean that “the property (P) is generic in $\text{Homeo}(X)$ (resp. $\text{Homeo}(X, \lambda)$)”, i.e. “it exists G_δ dense subset G of $\text{Homeo}(X)$ (resp. $\text{Homeo}(X, \lambda)$), such that every $f \in G$ satisfy the property (P) ”.

discretization grids, discretizations of a homeomorphism. We now define a more general notion of discretization grid. Some of the assumptions about these grids will be useful later, it will be the subject of the next paragraph.

Definition 3 (discretization grids). A *sequence of discretization grids* on X is a sequence $(E_N)_{N \in \mathbf{N}}$ of discrete subsets of $X \setminus \partial X$, such that the grids are more and more precise: for all $\varepsilon > 0$, it exists $N_0 \in \mathbf{N}$ such that for all $N \geq N_0$, the grid E_N is ε -dense. We denote by q_N the cardinal of E_N .

We fix once and for all a sequence $(E_N)_{N \in \mathbf{N}}$ of discretization grids on X . We can now define discretizations associated to these grids:

Notation 4 (Discretizations). Let P_N be a projection of X on E_N (the projection of $x_0 \in X$ on E_N is some $y_0 \in E_N$ minimizing the distance $d(x_0, y)$ when y runs through E_N). Such a projection is uniquely defined out of the set E'_N made of the points $x \in X$ for which there exists at least two points minimizing the distance between x and E_N . On E'_N the map P_N is chosen arbitrarily (nevertheless mesurably). For $x \in X$ we note x_N the *discretization of order N* of x , defined by $x_N = P_N(x)$. For $f \in \text{Homeo}(X)$ we note $f_N : E_N \rightarrow E_N$ the *discretization of order N* of f , defined by $f_N = P_N \circ f$.

Let \mathcal{D}_N be the set of homeomorphisms g such that $g(E_N) \cap E'_N = \emptyset$. Subsequently we will restrict the study to the G_δ dense set $\bigcap_{N \in \mathbf{N}} \mathcal{D}_N$, which is the set of homeomorphisms whose N -th discretization is uniquely defined for all $N \in \mathbf{N}$.

If $\sigma : E_N \rightarrow E_N$ and $f \in \text{Homeo}(X)$, we note $d_N(f, \sigma)$ the distance between $f|_{E_N}$ and σ , considered as maps from E_N into X .

Remark 5. One might ask why the points of the discretization grids are supposed to be *inside* X . The reason is simple: a homeomorphism f of X sends ∂X on ∂X . Putting points of some grids on the edge could perturb the dynamics of discretizations f_N . In particular it would introduce at least one orbit⁵ with length smaller than $\text{Card}(E_N \cap \partial X)$.

Probability measures on X . From section 8, we will be interested in *ergodic* properties of discretizations of f . Note \mathcal{P} the set of Borel probability measures on X endowed with the weak-star topology: a sequence $(\nu_m)_{m \in \mathbf{N}}$ of \mathcal{P} tends to $\nu \in \mathcal{P}$ (denoted by $\nu_m \rightarrow \nu$) if for all continuous function $\varphi : X \rightarrow \mathbf{R}$,

$$\lim_{m \rightarrow \infty} \int_X \varphi d\nu_m = \int_X \varphi d\nu.$$

Under these conditions the space \mathcal{P} is metrizable and compact, therefore separable (Prohorov, Banach-Alaoglu-Bourbaki theorem).

To study ergodic properties of homeomorphisms and their discretizations, we define natural invariant probability measures associated with these maps:

Definition 6. For any nonempty open subset U of X , we note λ_U the normalized restriction of λ on U , i.e. $\lambda_U = \frac{1}{\lambda(U)} \lambda|_U$. We also note $\lambda_{N,U}$ the uniform probability measure on $E_N \cap U$ and $\lambda_N = \lambda_{N,X}$. For $x \in X$ we note (when the limit exists) the *Birkhoff limit* of x :

$$\mu_x^f = \lim_{m \rightarrow \infty} \frac{1}{m} \sum_{i=0}^{m-1} f_*^i \delta_x,$$

and similarly for f_N . When it is well defined, we set

$$\mu_U^f = \lim_{m \rightarrow \infty} \frac{1}{m} \sum_{i=0}^{m-1} f_*^i \lambda_U$$

and

$$\mu_{N,U}^f = \lim_{m \rightarrow \infty} \frac{1}{m} \sum_{i=0}^{m-1} (f_N)_*^i \lambda_{N,U}.$$

Finally, we note $\mu_N^f = \mu_{N,X}^f$.

⁵Recall that orbit means forward orbit.

We just define two types of invariant measures: on the one hand from a point x , the other from the uniform measure λ . The link between it is done by the following proposition:

Proposition 7. *When U is an open set whose almost every point admit a Birkhoff limit, the measure μ_U^f is well defined and satisfies, for every continuous map $\varphi : X \rightarrow \mathbf{R}$,*

$$\int_X \varphi d\mu_U^f = \int_U \left(\int_X \varphi d\mu_x^f \right) d\lambda_U.$$

Similarly,

$$\int_X \varphi d\mu_{N,U}^f = \int_U \left(\int_X \varphi d\mu_{N,x}^f \right) d\lambda_{N,U}.$$

Proof of proposition 7. It follows easily from the dominated convergence theorem. \square

Hypothesis on discretization grids. Previously, we have given a very general definition of the concept of sequence of discretization grids. In some cases, we will need additional technical assumptions about these sequences of grids. Of course all of them will be satisfied by the uniform discretization grids on the torus (as defined in the introduction).

The first assumption is useful to prove Lax's theorem (theorem 19), and therefore necessary only in the part concerning conservative homeomorphisms.

Definition 8 (Well distributed and well ordered grids). We say that a sequence of discretization grids $(E_N)_{N \in \mathbf{N}}$ is *well distributed* if one can associate to each $x \in E_N$ a subset $C_{N,x}$ of X , which will be called a *cube of order N* , such that:

- for all N and all $x \in E_N$, $x \in C_{N,x}$,
- for all N , $\{C_{N,x}\}_{x \in E_N}$ is a measurable partition of X : $\bigcup_{x \in E_N} C_{N,x}$ is full measure and for x, y two distinct points of E_N , the intersection $C_{N,x} \cap C_{N,y}$ is null measure,
- the diameter of the cubes of order N tends to 0: $\max_{x \in E_N} \text{diam}(C_{N,x}) \xrightarrow{N \rightarrow +\infty} 0$.

If $(E_N)_{N \in \mathbf{N}}$ is well distributed and if $\{C_{N,x}\}_{N \in \mathbf{N}, x \in E_N}$ is a family of cubes as below, we will say that $(E_N)_{N \in \mathbf{N}}$ is *well ordered* if, for a fixed N , the cubes $\{C_{N,x}\}_{x \in E_N}$ can be indexed by $\mathbf{Z}/q_N\mathbf{Z}$ such that two consecutive cubes (in $\mathbf{Z}/q_N\mathbf{Z}$) are adjacent.

The two following definitions describe assumptions that will be useful especially to obtain properties in average.

Definition 9 (Refining grids). We say that a sequence of discretization grids *refines* if for all $N, N' \in \mathbf{N}$ such that $N \leq N'$, one has $E_N \subset E_{N'}$.

Definition 10 (Self-similar grids). We say that a sequence of discretization grids $(E_N)_{N \in \mathbf{N}}$ is *self similar* if for all $\varepsilon > 0$, there exists $N_0, N_1 \in \mathbf{N}$ such that for all $N \geq N_1$, the set E_N contains disjoint subsets $\tilde{E}_N^1, \dots, \tilde{E}_N^{\alpha_N}$ whose union fills a proportion greater than $1 - \varepsilon$ of E_N , and such that for all j , \tilde{E}_N^j is the image of the grid E_{N_0} by a bijection h_j which is ε -close to identity.

We say that a sequence of discretization grids $(E_N)_{N \in \mathbf{N}}$ is *strongly self similar* if it is self similar and for all $N \geq N_0$, one of the h_j equals to identity.

Remark 11. One easily verify that the hypothesis “being strongly self-similar” implies both “being self-similar” and “refining”.

3. SOME EXAMPLES OF DISCRETIZATION GRIDS

In the previous section we set properties on discretizations — namely being well distributed, being well ordered, refining, being strongly self-similar or self-similar — that will be used subsequently. In this section we give some examples of grids that verify some of these assumptions.

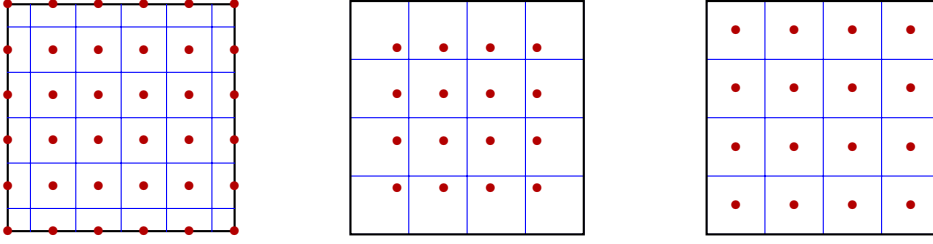


FIGURE 1. Uniform discretization grids of order 5 on the torus \mathbf{T}^2 (E^N left) and on the cube I^2 (E_N^0 middle and E_N^1 right) and their associated cubes

Uniform discretization grids on the torus. The simplest example, which will be used for the simulations, is that of the torus $\mathbf{T}^n = \mathbf{R}^n/\mathbf{Z}^n$ of dimension $n \geq 2$ endowed with discretizations called *uniform* discretizations, defined from the fundamental domain $I^n = [0, 1]^n$ of \mathbf{T}^n : take an increasing sequence of integers $(k_N)_{N \in \mathbf{N}}$ and set

$$E_N = \left\{ \left(\frac{i_1}{k_N}, \dots, \frac{i_n}{k_N} \right) \in I^n \mid \forall j, 0 \leq i_j \leq k_N - 1 \right\},$$

$$C_{N, (i_1/N, \dots, i_n/N)} = \prod_{j=1}^n \left[\frac{i_j}{k_N} - \frac{1}{2k_N}, \frac{i_j}{k_N} + \frac{1}{2k_N} \right].$$

We easily verify that this sequence of grids is well distributed, well ordered and self-similar. If we further assume that for any $N \in \mathbf{N}$, k_N divides k_{N+1} (which is true when $k_N = p^N$ with $p \geq 2$), then the sequence is strongly self-similar (therefore refines). When $k_N = p^N$ with $p = 2$ (resp. $p = 10$) the discretization performs what one can expect from a numerical simulation: doing a binary (resp. decimal) discretization at order N is the same as truncating each binary (resp. decimal) coordinate of the point $x \in I^n$ to the N -th digit, i.e. working with a fixed digital precision⁶.

Uniform discretization grids on the cube. Similarly, one can define uniform discretizations on the cube $I^n = [0, 1]^n$ by setting

$$E_N^0 = \left\{ \left(\frac{i_1}{k_N}, \dots, \frac{i_n}{k_N} \right) \in I^n \mid \forall j, 1 \leq i_j \leq k_N - 1 \right\},$$

still with an increasing sequence $(k_N)_{N \in \mathbf{N}}$ of integers, to which are associated the cubes⁷ (see figure 1)

$$C_{N, (i_1/N, \dots, i_n/N)} = \prod_{j=1}^n \left[\frac{i_j - 1}{k_N - 1}, \frac{i_j}{k_N - 1} \right].$$

As before, one easily verifies that this sequence of grids is well distributed, well ordered and self-similar (see figure 2). If we further assume that for any $N \in \mathbf{N}$, k_N divides k_{N+1} , then the sequence is strongly self-similar (therefore refines).

One can also take discretizations according to the centers of the cubes.

$$E_N^1 = \left\{ \left(\frac{i_1 + 1/2}{k_N}, \dots, \frac{i_n + 1/2}{k_N} \right) \in I^n \mid \forall j, 0 \leq i_j \leq k_N - 1 \right\}.$$

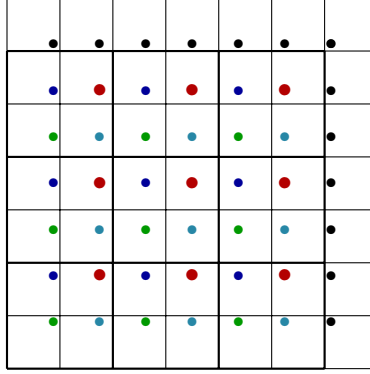
This time the cubes are much more natural (see figure 1):

$$C_{N, (i_1/N, \dots, i_n/N)} = \prod_{j=1}^n \left[\frac{i_j}{k_N}, \frac{i_j + 1}{k_N} \right].$$

Again, we easily verify that this sequence of grids is well distributed, well ordered and self-similar (but does not refine).

⁶In fact, in practice the computer works in floating point format, so that the number of decimal places is not the same when the number is close to 0 or not.

⁷Be careful, these cubes have their vertices on the grid of order $k_N - 1$.

FIGURE 2. Self similarity of grids E_N^0

discretization grids on an arbitrary manifold. In fact, discretizations E_N^0 on I^n can be generalized to an arbitrary manifold X with the Oxtoby-Ulam theorem (see [OU41]):

Theorem 12 (Oxtoby, Ulam). *Under the assumptions that have been made on X and λ , it exists a map $\phi : I^n \rightarrow X$ such that:*

- (1) ϕ is surjective
- (2) $\phi|_{I^n}$ is an homeomorphism on its image
- (3) $\phi(\partial I^n)$ is a closed subset of X with empty interior and disjoint from $\phi(I^n)$
- (4) $\lambda(\phi(\partial I^n)) = 0$
- (5) $\phi^*(\lambda) = \text{Leb}$, where Leb is Lebesgue measure.

This theorem allows us to define discretization grids on X from uniform discretizations on the cube. We obtain the following informal proposition:

Proposition 13. *If there is a sequence $(E_N)_N$ of grids on I^n whose elements are not on the edge of the cube, then its image by ϕ defines a sequence of grids on X which satisfy the same properties as the initial grid on the cube.*

Remark 14. The example also includes the case where $X = I^n$, $\lambda = \text{Leb}$ and where the grids are the images of the grids E_N by a unique homeomorphism of X preserving Lebesgue measure.

Part 1. Discretizations of a generic conservative homeomorphism

Recall that we have fixed once and for all a manifold X , a good measure λ on X and a sequence of discretization grids $(E_N)_{N \in \mathbf{N}}$ (see definition 3). It is further assumed that this sequence of grids is well distributed and well ordered (see definition 8). In this section we will focus on discretizations of elements of $\text{Homeo}(X, \lambda)$, so homeomorphisms will always be supposed conservative.

4. DENSE TYPES OF APPROXIMATIONS

To begin with we define the notion of *type of approximation*.

Definition 15. A *type of approximation* $\mathcal{T} = (\mathcal{T}_N)_{N \in \mathbf{N}}$ is a sequence of subsets of the set $\mathcal{F}(E_N, E_N)$ made of applications from E_N into itself.

Let \mathcal{U} be an open subset of $\text{Homeo}(X, \lambda)$. A type of approximation \mathcal{T} is said to be *dense* in \mathcal{U} if for all $f \in \mathcal{U}$, $\varepsilon > 0$ and all $N_0 \in \mathbf{N}$, there exists $N \geq N_0$ and $\sigma_N \in \mathcal{T}_N$ such that $d_N(f, \sigma_N) < \varepsilon$ (recall that d_N is the distance between $f|_{E_N}$ and σ_N considered as maps from E_N into X).

The goal of this paragraph is to obtain the following result: every dense type of approximation appears on infinitely many discretizations of a generic homeomorphism.

Theorem 16. *Let \mathcal{U} be an open subset⁸ of $\text{Homeo}(X, \lambda)$ and \mathcal{T} be a type of approximation which is dense in \mathcal{U} . Then for a generic homeomorphism $f \in \mathcal{U}$ and for all $N_0 \in \mathbf{N}$, there exists $N \geq N_0$ such that $f_N \in \mathcal{T}_N$.*

When we will try to obtain properties of discretizations of generic homeomorphisms, this theorem will be the second step of the proof, the first being to show that the approximation type we are interested in is dense. In section 6 we will try to establish as many corollaries of this theorem as possible, showing that the approach “prove that such approximation is dense then apply theorem 16” is a general method in the study of properties of discretizations⁹.

To prove of theorem 16 we start from a dense type of approximations — in other words discrete applications — and we want to get properties of homeomorphisms. The tool that allows us to restore a homeomorphism from a finite map $\sigma_N : E_N \rightarrow E_N$, i.e. do the opposite of a discretization, is the finite map extension proposition:

Proposition 17 (Finite map extension). *Let (x_1, x_2, \dots, x_k) and (y_1, y_2, \dots, y_k) be two k -tuples of pairwise distinct points in $X \setminus \partial X$ (i.e. for all $i \neq j$, $x_i \neq x_j$ and $y_i \neq y_j$). Then there exists $f \in \text{Homeo}(X, \lambda)$ such that for all i , $f(x_i) = y_i$. Moreover f can be chosen such that if $\max_i d(x_i, y_i) < \delta$ then $d(f, \text{Id}) < \delta$.*

The idea of the proof is quite simple: it suffice to compose homeomorphisms whose support's size is smaller than ε and which are central symmetries within this support. Then one constructs a sequence $(z_j)_{1 \leq j \leq k}$ such that $z_1 = x_1$, $y_1 = z_k$ and $d(z_j, z_{j+1}) < \varepsilon/10$. Composing $k - 1$ homeomorphisms as above, such that each one sends z_j on z_{j+1} , one obtains a conservative homeomorphism which sends x_0 on y_0 . Implementing these remarks is then essentially technical. A detailed proof can be found in section 2.2 of [Gui12].

With this proposition, we can build homeomorphisms from a dense type of approximation:

Lemma 18. *Let \mathcal{U} be an open subset of $\text{Homeo}(X, \lambda)$ and $\mathcal{T} = (\mathcal{T}_N)_{N \in \mathbf{N}}$ be a type of approximation which is dense in \mathcal{U} . Then for all $N_0 \in \mathbf{N}$, the set of homeomorphisms f such that there exists $N \geq N_0$ such that the discretization f_N belongs to \mathcal{T}_N is dense in \mathcal{U} .*

Proof of lemma 18. Let $f \in \mathcal{U}$, $N_0 \in \mathbf{N}$ and $\varepsilon > 0$. Since \mathcal{T} is a type of approximation which is dense in \mathcal{U} , there exists $N \geq N_0$ and $\sigma_N \in \mathcal{T}_N$ such that $d_N(f, \sigma_N) < \varepsilon$. Let x_1, \dots, x_{q_N} be the elements of E_N . Then for all ℓ , $d(f(x_\ell), \sigma_N(x_\ell)) < \varepsilon$. We modify σ_N into a bijection $\sigma'_N : E_N \rightarrow X$ whose discretization on E_N equals to σ_N : set $\sigma'_N(x_1) = \sigma(x_1)$, σ'_N is defined by induction by choosing $\sigma'_N(x_\ell)$ such that $\sigma'_N(x_\ell)$ is different from $\sigma'_N(x_i)$ for $i \leq \ell$, such that $P_N(\sigma'_N(x_\ell)) = P_N(\sigma_N(x_\ell))$ and that $d(f(x_\ell), \sigma'_N(x_\ell)) < \varepsilon$.

Since σ'_N is a bijection proposition 17 can be applied to $f(x_\ell)$ and $\sigma'_N(x_\ell)$; this gives a measure-preserving homeomorphism φ such that $\varphi(f(x_\ell)) = \sigma'_N(x_\ell)$ for all ℓ and such that $d(\varphi, \text{Id}) < \varepsilon$. Set $f' = \varphi \circ f$, we have $d(f, f') \leq \varepsilon$ and $f'(x_\ell) = \sigma'_N(x_\ell)$ for all ℓ , and therefore $f_N = \sigma_N$. \square

Proof of theorem 16. Let $(x_{N,\ell})_{1 \leq \ell \leq q_N}$ be the elements of E_N and consider the set (where ρ_N is the minimal distance between two distinct points of E_N)

$$\bigcap_{N_0 \in \mathbf{N}} \bigcup_{\substack{N \geq N_0 \\ \sigma_N \in \mathcal{T}_N}} \left\{ f \in \mathcal{U} \mid \forall \ell, d_N(f(x_{N,\ell}), \sigma_N(x_{N,\ell})) < \frac{\rho_N}{2} \right\}.$$

This set is clearly a G_δ set and its density follows directly from lemma 18. Moreover we can easily see that its elements satisfy the conclusions of the theorem. \square

⁸We will often take $\mathcal{U} = \text{Homeo}(X, \lambda)$.

⁹Although some properties can be proved in different ways: for instance corollary 37 can be proved in inserting horseshoes in the dynamics of a given homeomorphism, using the local modification theorem (theorem 30, for a presentation of the technique in another context see section 3.3 of [Gui12]); also a variation of corollary 25 can be shown in perturbing any given homeomorphism such that it has a periodic orbit whose distance to the grid E_N (grids are assumed to refine) is smaller than the modulus of continuity of f , so that the actual periodic orbit and the discrete orbit fit together.

5. LAX'S THEOREM

Now that we have shown theorem 16, we have to obtain dense types of approximation. Again, a theorem will be systematically used: Lax's theorem, and more precisely its improvement stated by S. Alpern [Alp76] which allows us to approach any homeomorphism by a cyclic permutation of the elements of a discretization grid.

Theorem 19 (Lax, Alpern). *Let $f \in \text{Homeo}(X, \lambda)$ and $\varepsilon > 0$. Then there exists $N_0 \in \mathbf{N}$ such that for all $N \geq N_0$, there exists a cyclic permutation σ_N of E_N such that $d(f, \sigma_N) < \varepsilon$.*

As a compact metric set can be seen as a “finite set up to ε ”, Lax's theorem allows us to see an homeomorphism as a “cyclic permutation up to ε ”. In practice this theorem is used to “break” homeomorphisms into small pieces. Genericity of transitivity in $\text{Homeo}(X, \lambda)$ follows easily from Lax's theorem together with finite map extension proposition (again, see [AP00] or part 2.4 of [Gui12]). In our case, applying theorem 16, we deduce that infinitely many discretizations of a generic homeomorphism are cyclic permutations. The purpose of section 6 is to find variations of Lax's theorem (which are at the same time corollaries of it) concerning other properties of discretizations.

We give briefly the beautiful proof of Lax's theorem, which is essentially combinatorial and based on marriage lemma and on a lemma of approximation of permutations by cyclic permutations due to S. Alpern. Readers wishing to find proofs of these lemmas may consult section 2.1 of [Gui12]. This is here that we use the fact that discretizations are well distributed and well ordered.

Lemma 20 (Marriage lemma). *Let E and F be two finite sets and \approx a relation between elements of E and F . Suppose that the number of any subset $E' \subset E$ is smaller than the number of elements in F that are associated with an element of E' , i.e.:*

$$\forall E' \subset E, \quad \text{Card}(E') \leq \text{Card} \{f \in F \mid \exists e \in E' : e \approx f\},$$

then there exists an injection $\Phi : E \rightarrow F$ such that for all $e \in E$, $e \approx \Phi(e)$.

Lemma 21 (Cyclic approximations in \mathfrak{S}_q , [Alp76]). *Let $q \in \mathbf{N}^*$ and $\sigma \in \mathfrak{S}_q$ (\mathfrak{S}_q is seen as the permutation group of $\mathbf{Z}/q\mathbf{Z}$). Then there exists $\tau \in \mathfrak{S}_q$ such that $|\tau(k) - k| \leq 2$ for all k (where $|\cdot|$ is the distance in $\mathbf{Z}/q\mathbf{Z}$) and such that the permutation $\tau\sigma$ is cyclic.*

Proof of theorem 19. Let $f \in \text{Homeo}(X, \lambda)$ and $\varepsilon > 0$. Consider $N_0 \in \mathbf{N}$ such that for all $N \geq N_0$, the diameter of the cubes of order N (given by the hypothesis “being well distributed”) and their images by f is smaller than ε . We define a relation \approx between cubes of order $N - 1$: $C \approx C'$ if and only if $f(C) \cap C' \neq \emptyset$. Since f preserves λ , the image of the union of ℓ cubes intersects at least ℓ cubes, so marriage lemma (lemma 20) applies: there exists an injection Φ_N from the set of cubes of order N into itself (then a bijection) such that for all cube C , $f(C) \cap \Phi_N(C) \neq \emptyset$. Let σ_N be the application that maps the center of any cube C to the center of the cube $\Phi_N(C)$, we obtain:

$$d_N(f, \sigma_N) \leq \sup_C (\text{diam}(C)) + \sup_C (\text{diam}(f(C))) \leq 2\varepsilon.$$

It remains to show that σ_N can be chosen as a cyclic permutation. We number the cubes such that two consecutive cubes are adjacent (this is possible because the grids are well ordered), then we use lemma 21 to obtain a cyclic permutation σ'_N whose distance to σ_N is smaller than $2 \sup(\text{diam}(C))$. Thus we have found $N_0 \in \mathbf{N}$ and for all $N \geq N_0$ a cyclic permutation σ'_N of E_N whose distance to f is smaller than 4ε . \square

6. INDIVIDUAL BEHAVIOUR OF DISCRETIZATIONS

Once we have set theorems 16 and 19, we can establish results concerning the behaviour of discretizations of a generic conservative homeomorphism. Here we study *individual* behaviour of discretizations, i.e. properties about only one order of discretization. As has already been said, applying theorem 16, it suffices to find dense types of approximation to obtain properties about discretizations. In practice, these dense types of approximations are obtained from variations of Lax's theorem (theorem 19).

Recall that the sequence $(E_N)_{n \in \mathbf{N}}$ of discretization grids is well distributed and well ordered (see definition 8), we denote by f_N the discretization of an homeomorphism f and $\Omega(f_N)$ the maximal invariant set of f_N (i.e. the union of periodic orbits of f_N).

We will try to show that for simple dynamical properties (P) about finite maps and for a generic conservative homeomorphism f , infinitely many discretizations f_N verify (P) as well as infinitely many discretizations verify its contrary. For instance, for a generic homeomorphism f , the maximal invariant set $\Omega(f_N)$ is sometimes as large as possible, i.e. $\Omega(f_N) = E_N$ (corollary 22), sometimes very small (corollary 25) and even better sometimes the number of elements of the image of E_N is small (corollary 37). In the same way stabilization time¹⁰ is sometimes zero (corollary 22 for example), sometimes around $\text{Card}(E_N)$ (corollary 27). Finally, concerning the dynamics of $f_{N|\Omega(f_N)}$, sometimes it is a cyclic permutation (corollary 22) or a bicyclic permutation (remark 23, see also corollary 35), sometimes it has many orbits (corollary 39).

Firstly, we deduce directly from Lax's theorem that cyclic permutations of sets E_N form a dense type of approximation in $\text{Homeo}(X, \lambda)$. Combining this with theorem 16, we obtain:

Corollary 22 (Miernowski, [Mie05]). *For a generic homeomorphism $f \in \text{Homeo}(X, \lambda)$, for every $N_0 \in \mathbf{N}$, it exists $N \geq N_0$ such that f_N is a cyclic permutation.*

Remark 23. The same result is obtained for with bicyclic permutations, which are permutations having exactly two orbits whose lengths are relatively prime (see [Gui12]).

The first variation of Lax's theorem concerns the approximation of applications which maximal invariant set is small. It is the contrary to what happens in corollary 22.

Proposition 24 (First variation of Lax's theorem). *Let $f \in \text{Homeo}(X, \lambda)$. Then for all $\varepsilon, \varepsilon' > 0$, it exists $N_0 \in \mathbf{N}$ such that for all $N \geq N_0$, it exists a map $\sigma_N : E_N \rightarrow E_N$ such that $d_N(f, \sigma_N) < \varepsilon$ and*

$$\frac{\text{Card}(\Omega(\sigma_N))}{\text{Card}(E_N)} = \frac{\text{Card}(\Omega(\sigma_N))}{q_N} < \varepsilon',$$

and such that E_N is made of a unique (pre-periodic) orbit of σ_N .

Proof of proposition 24. Let $f \in \text{Homeo}(X, \lambda)$, $\varepsilon > 0$ and x a recurrent point of f . It exists $\tau \in \mathbf{N}^*$ such that $d(x, f^\tau(x)) < \frac{\varepsilon}{8}$; this inequality remains true for fine enough discretizations: it exists $N_1 \in \mathbf{N}$ such that if $N \geq N_1$, then

$$d(x, x_N) < \frac{\varepsilon}{8}, \quad d(f^\tau(x), f^\tau(x_N)) < \frac{\varepsilon}{8} \quad \text{and} \quad \frac{\tau}{q_N} < \varepsilon'.$$

Using the modulus of continuity of f^τ and Lax's theorem (theorem 19), we obtain an integer $N_0 \geq N_1$ such that for all $N \geq N_0$, it exists a cyclic permutation σ_N of E_N such that $d_N(f, \sigma_N) < \frac{\varepsilon}{2}$ and $d_N(f^\tau, \sigma_N^\tau) < \frac{\varepsilon}{8}$. Then

$$\begin{aligned} d(x_N, \sigma_N^\tau(x_N)) &\leq d(x_N, x) + d(x, f^\tau(x)) + d(f^\tau(x), f^\tau(x_N)) \\ &\quad + d(f^\tau(x_N), \sigma_N^\tau(x_N)) \\ &< \frac{\varepsilon}{2}. \end{aligned}$$

We compose σ_N by the (non bijective) application mapping $\sigma_N^{\tau-1}(x_N)$ on x_N and being identity anywhere else (see figure 3), in other words we consider the application

$$\sigma'_N(x) = \begin{cases} x_N & \text{if } x = \sigma_N^{\tau-1}(x_N) \\ \sigma_N(x) & \text{otherwise.} \end{cases}$$

The map σ'_N has a unique injective orbit whose associated periodic orbit $\Omega(\sigma'_N)$ has length τ (it is $(x_N, \sigma_N(x_N), \dots, \sigma_N^{\tau-1}(x_N))$). Since $d(f, \sigma'_N) < \varepsilon$, the map σ'_N verifies the conclusion of the proposition. \square

A direct application of theorem 16 gives:

¹⁰I.e. the smallest integer k such that $f_N^k(E_N) = \Omega(f_N)$.

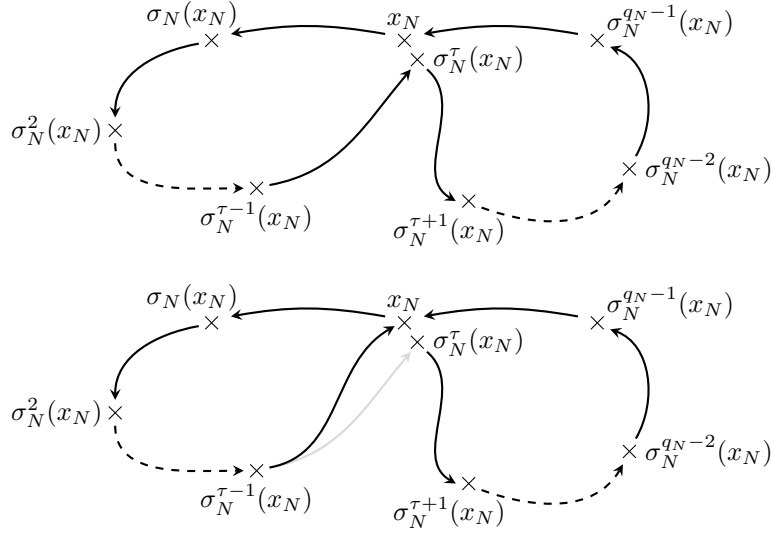


FIGURE 3. Modification of a cyclic permutation

Corollary 25. For a generic homeomorphism $f \in \text{Homeo}(X, \lambda)$,

$$\lim_{N \rightarrow +\infty} \frac{\text{Card}(\Omega(f_N))}{\text{Card}(E_N)} = 0.$$

Specifically for all $\varepsilon > 0$ and all $N_0 \in \mathbf{N}$, it exists $N \geq N_0$ such that $\frac{\text{Card}(\Omega(f_N))}{\text{Card}(E_N)} < \varepsilon$ and such that E_N is made of a unique (pre-periodic) orbit of f_N .

The same kind of idea leads to the following proposition:

Proposition 26. Let $f \in \text{Homeo}(X, \lambda)$ having at least one periodic point of period p . Then for all $\varepsilon > 0$, it exists $N_0 \in \mathbf{N}$ such that for all $N \geq N_0$, it exists an application $\sigma_N : E_N \rightarrow E_N$ with $\text{Card}(\Omega(\sigma_N)) = p$ and $d_N(f, \sigma_N) < \varepsilon$, such that E_N is made of a unique (pre-periodic) orbit of σ_N .

Proof of proposition 26. Simply replace the recurrent point by a periodic point of period p in the proof of proposition 24. \square

We will show that owning a periodic point is a generic property in $\text{Homeo}(X, \lambda)$ (see also [DF00] or part 3.2 of [Gui12]), a proper application of Baire's theorem leads to a refinement of corollary 25:

Corollary 27. Let $\varepsilon > 0$. A generic homeomorphism $f \in \text{Homeo}(X, \lambda)$ has a periodic point whose orbit is ε -dense. Moreover, f_N has a unique periodic orbit for infinitely many integers N , whose period does not depend on $N \in \mathbf{N}$ and equals the smallest period of periodic points of f which are ε -dense; moreover E_N is covered by a single (pre-periodic) orbit of f_N .

To prove corollary 27 we have to introduce the concept of *persistent point*.

Definition 28. Let $f \in \text{Homeo}(X)$. A periodic point x of f with period p is said *persistent* if for all neighbourhood U of x , there exists a neighbourhood \mathcal{V} of f in $\text{Homeo}(X)$ such that every $\tilde{f} \in \mathcal{V}$ has a periodic point $\tilde{x} \in U$ with period p .

Example 29. The endomorphism $h = \text{Diag}(\lambda_1, \dots, \lambda_n)$ of \mathbf{R}^n , with $\prod \lambda_i = 1$ and $\lambda_i \neq 1$ for all i , is measure-preserving and has a persistent fixed point at the origin (see e.g. [KH95], p. 319). Let s be a reflection of \mathbf{R}^n , the application $h \circ f$ is also measure-preserving and has a persistent fixed point at the origin.

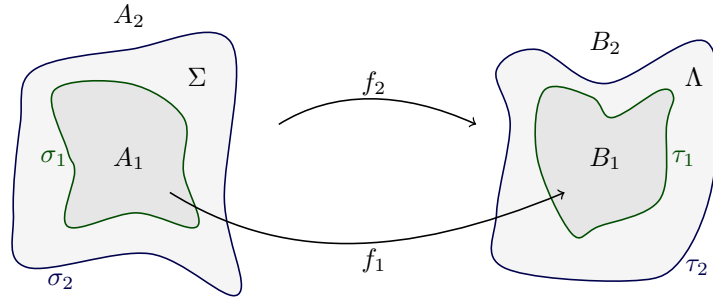


FIGURE 4. Local modification technique

Finally, we use the theorem of local modification of conservative homeomorphisms, which allows us to replace locally a homeomorphism by another. Although it is geometrically “obvious” and it has an elementary proof in dimension two, it is easily deduced from (difficult) *annulus theorem*. For more details one may refer to [DF00] or part 3.1 of [Gui12].

Theorem 30 (Local modification). *Let $\sigma_1, \sigma_2, \tau_1$ and τ_2 be four bicollared embeddings¹¹ of \mathbf{S}^{n-1} in \mathbf{R}^n , such that σ_1 is in the bounded connected component¹² of σ_2 and τ_1 is in the bounded connected component of τ_2 . Let A_1 be the bounded connected component of $\mathbf{R}^n \setminus \sigma_1$ and B_1 the bounded connected component of $\mathbf{R}^n \setminus \tau_1$; Σ be the connected component of $\mathbf{R}^n \setminus (\sigma_1 \cup \sigma_2)$ having $\sigma_1 \cup \sigma_2$ as boundary and Λ the connected component of $\mathbf{R}^n \setminus (\tau_1 \cup \tau_2)$ having $\tau_1 \cup \tau_2$ as boundary; A_2 be the unbounded connected component of $\mathbf{R}^n \setminus \sigma_2$ and B_2 the unbounded connected component of $\mathbf{R}^n \setminus \tau_2$ (see figure 4).*

Suppose that $\text{Leb}(A_1) = \text{Leb}(B_1)$ and $\text{Leb}(\Sigma) = \text{Leb}(\Lambda)$. Let $f_i : A_i \rightarrow B_i$ be two measure-preserving homeomorphisms such that either each one preserves the orientation, or each one reverses it. Then it exists a measure-preserving homeomorphism $f : \mathbf{R}^n \rightarrow \mathbf{R}^n$ whose restriction to A_1 equals f_1 and restriction to A_2 equals f_2 .

Since this theorem is local, it can be applied to an open space O instead of \mathbf{R}^n , or even better, together with Oxtoby-Ulam theorem (theorem 12), to any domain of chart of a manifold X instead of \mathbf{R}^n and measure λ instead of Lebesgue measure

Proof of corollary 27. Let $\varepsilon > 0$ and \mathcal{U}_p the set of homeomorphisms that have at least one periodic persistent point with period p whose orbit is ε -dense (strictly), but no periodic point with period less than p whose orbit is ε -dense (not strictly). The sets \mathcal{U}_p are open and pairwise disjoint. Moreover their union is dense. Indeed, for $f \in \text{Homeo}(X, \lambda)$, if f has at least one periodic point whose orbit is ε -dense, then we choose a periodic point x whose orbit is ε -dense and whose period p is minimal. Then we perturb f such that this point is persistent, avoiding the creation of periodic points of smaller periods: we choose a neighbourhood D of x such that the sets $D, f(D), \dots, f^{p-1}(D)$ are pairwise disjoint and we replace locally f by the map $h \circ f^{-(p-1)}$ (where h is one of the two maps of the example 29, depending of whether $f^{-(p-1)}$ preserves orientation or not) in the neighbourhood of $f^{-1}(x)$, using the theorem of local modification (theorem 30), such that f do not change outside the union of the sets $f^i(D)$. On the contrary, if f does not have any periodic point, we pick a recurrent point and perturb f such that this point is periodic, and then we apply the previous case.

Let $p \in \mathbf{N}^*$. Proposition 26 indicates that the set of maps whose longest periodic orbit has length p is a dense type of approximation in \mathcal{U}_p . So, by theorem 16, it exists open sets $\mathcal{O}_{p,N}$ such that $\bigcap_{N \in \mathbf{N}} \mathcal{O}_{p,N}$ is a dense G_δ of \mathcal{U}_p , made of homeomorphisms of \mathcal{U}_p whose an infinity of discretizations have only one injective orbit whose associated cycle has length p and is ε -dense.

¹¹An embedding i of a manifold M in \mathbf{R}^n is said *bicollared* if it exists an embedding $j : M \times [-1, 1] \rightarrow \mathbf{R}^n$ such that $j_{M \times \{0\}} = i$.

¹²By Jordan-Brouwer theorem the complement of a set which is homeomorphic to \mathbf{S}^{n-1} has exactly two connected components: one bounded and one unbounded.

Set $\mathcal{O}_N = \bigcup_{p \in \mathbf{N}^*} \mathcal{O}_{p,N}$. Since the sets \mathcal{U}_p are pairwise disjoint,

$$\bigcap_{N \in \mathbf{N}} \mathcal{O}_N = \bigcap_{N \in \mathbf{N}} \bigcup_{p \in \mathbf{N}^*} \mathcal{O}_{p,N} = \bigcup_{p \in \mathbf{N}^*} \bigcap_{N \in \mathbf{N}} \mathcal{O}_{p,N}.$$

The right side of the equation forms a dense subset of $\text{Homeo}(X, \lambda)$, so the left side is a dense G_δ of $\text{Homeo}(X, \lambda)$. It is made of homeomorphisms f such that for infinitely many $N \in \mathbf{N}$, the grid E_N is covered by a single forward orbit of f_N whose associated periodic orbit is ε -dense and whose length is the shortest period of periodic points of f whose orbit are ε -dense. \square

Remark 31. The shortest period of periodic points of generic homeomorphisms has not any global upper bound in $\text{Homeo}(X, \lambda)$: for example, for all $p \in \mathbf{N}$, there is a open set of homeomorphisms of the torus without periodic point of period less than p (e.g. the neighbourhood of a nontrivial rotation) and this property remains true for discretizations.

We now want to get a discrete analogue to topological weak mixing (which is generic in $\text{Homeo}(X, \lambda)$, see e.g. [AP00] or part 2.4 of [Gui12]).

Definition 32. A homeomorphism f is said *topologically weakly mixing* if for all nonempty open sets $(U_i)_{i \leq M}$ and $(U'_i)_{i \leq M}$, it exists $m \in \mathbf{N}$ such that $f^m(U_i) \cap U'_i$ is nonempty for all $i \leq M$.

The proof of the genericity of topological weak mixing starts by an approximation of every conservative homeomorphism by another having ε -dense periodic orbits whose lengths are relatively prime. The end of the proof lies primarily in the use of Baire's theorem and Bezout's identity. In the discrete case, the notion of weak mixing is replaced by the following:

Definition 33. Let $\varepsilon > 0$. A finite map σ_N is said *ε -topologically weakly mixing* if for all $M \in \mathbf{N}$ and all balls $(B_i)_{i \leq M}$ and $(B'_i)_{i \leq M}$ with diameter ε , it exists $m \in \mathbf{N}$ such that for all i

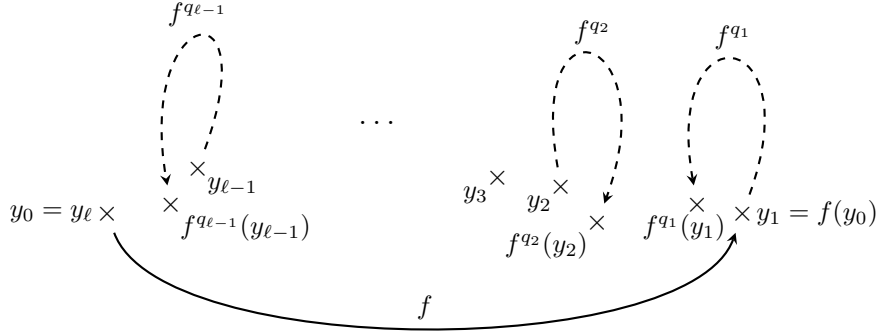
$$\sigma_N^m(B_i \cap E_N) \cap (B'_i \cap E_N) \neq \emptyset.$$

The first step of the proof is replaced by the following variation of Lax's theorem:

Proposition 34 (Second variation of Lax's theorem). *Let $f \in \text{Homeo}(X, \lambda)$ be a homeomorphism whose all iterates are topologically transitive. Then for all $\varepsilon > 0$ and all $M \in \mathbf{N}^*$, there exists $N_0 \in \mathbf{N}$ such that for all $N \geq N_0$, it exists $\sigma_N : E_N \rightarrow E_N$ which has M ε -denses periodic orbits whose lengths are pairwise relatively prime, and such that $d_N(f, \sigma_N) < \varepsilon$.*

Proof of proposition 34. We prove the proposition in the case where $M = 2$, the other cases being easily obtained by an induction. Let $\varepsilon > 0$ and f be an homeomorphism whose all iterates are topologically transitive. Then it exists $x_0 \in X$ and $p \in \mathbf{N}^*$ such that $\{x_0, \dots, f^{p-1}(x_0)\}$ is ε -dense and $d(x_0, f^p(x_0)) < \varepsilon/2$. Since transitive points of f^p form a dense G_δ subset of X , while the orbit of x_0 form a F_σ set with empty interior, the set of points whose orbit under f^p is dense and disjoint from that of x_0 is dense. So we can pick such a transitive point y_0 . Set $y_1 = f(y_0)$. Then there exists a multiple q_1 of p such that the orbit $\{y_1, \dots, f^{q_1-1}(y_1)\}$ is ε -dense and $d(y_1, f^{q_1}(y_1)) < \varepsilon/2$. Again, by density, we can chose a transitive point y_2 whose orbit is disjoint from that of x_0 and y_1 , with $d(y_1, y_2) < \varepsilon/2$ and $d(y_0, y_1) - d(y_0, y_2) > \varepsilon/4$. Then it exists a multiple q_2 of p such that $d(y_2, f^{q_2}(y_2)) < \varepsilon/2$. And so on, we construct a sequence $(y_m)_{1 \leq m \leq \ell}$ such that (see figure 5):

- (i) for all m , it exists $q_m > 0$ such that $p|q_m$ and $d(y_m, f^{q_m}(y_m)) < \varepsilon/2$,
- (ii) the orbits $\{x_0, \dots, f^{p-1}(x_0)\}$ and $\{y_m, \dots, f^{q_m-1}(y_m)\}$ (m going from 0 to $\ell - 1$) are pairwise disjoint,
- (iii) for all m , $d(y_m, y_{m+1}) < \varepsilon/2$ and $d(y_0, y_m) - d(y_0, y_{m+1}) > \varepsilon/4$,
- (iv) $y_\ell = y_0$.


 FIGURE 5. Construction of the sequence $(y_m)_{1 \leq m \leq \ell}$

Let σ_N be a finite map given by Lax's theorem. For all N large enough, σ_N satisfies the same properties (i) to (iii) than f . Changing σ_N at the points $\sigma_N^{q_m-1}((y_m)_N)$ and $\sigma_N^{p-1}((x_0)_N)$, we obtain a finite map σ'_N such that $\sigma'^{q_N}((y_m)_N) = (y_{m+1})_N$ and $\sigma'^p((x_0)_N) = (x_0)_N$. Thus the orbit of $(x_0)_N$ under σ'_N is 2ε -dense and has period p and the orbit of $(y_0)_N$ under σ'_N is 2ε -dense, disjoint from which of $(x_0)_N$ and has period $1 + q_1 + \dots + q_{\ell-1}$ relatively prime to p . \square

Corollary 35. *For a generic homeomorphism $f \in \text{Homeo}(X, \lambda)$, for all $\varepsilon > 0$ and all $N_0 \in \mathbf{N}$, it exists $N \geq N_0$ such that f_N is ε -topologically weakly mixing.*

Proof of corollary 35. Again, we prove the corollary in the case where $M = 2$, other cases being easily obtained by induction. Let $\varepsilon > 0$ and $N_0 \in \mathbf{N}$. All iterates of a generic homeomorphism f are topologically transitive: it is an easy consequence of the genericity of transitivity (see e.g. corollary 22 or theorem 2.11 of [Gui12]); we pick such a homeomorphism. Combining theorem 16 and proposition 34, we obtain $N \geq N_0$ such that f_N has two $\varepsilon/3$ -dense periodic orbits whose lengths p et q are coprime. We now have to prove that f_N is ε -topologically weakly mixing. Let B_1, B_2, B'_1 and B'_2 be balls with diameter ε . Since each one of these orbits is $\varepsilon/3$ -dense, it exists $x_N \in X$ which is in the intersection of the orbit whose length is p and B_1 , and $y_N \in X$ which is in the intersection of the orbit whose length is q and B_2 . Similarly, it exists two integers a and b such that $f_N^a(x_N) \in B'_1$ and $f_N^b(y_N) \in B'_2$.

Recall that we want to find a power of f_N which sends both x_N in B'_1 and y_N in B'_2 . It suffices to pick $m \in \mathbf{N}$ such that $m = a + \alpha p = b + \beta q$. Bezout's identity states that there exists two integers α and β such $\alpha p - \beta q = b - a$. Set $m = a + \alpha p$, adding a multiple of pq if necessary, we can suppose that m is positive. Thus $f_N^m(x_N) \in B'_1$ and $f_N^m(y_N) \in B'_2$. \square

We now establish a new variation of Lax's theorem. It asserts that every homeomorphism can be approximated by a finite map whose image has a small cardinal, unlike what happens in Lax's theorem.

Proposition 36 (Third variation of Lax's theorem). *Let $f \in \text{Homeo}(X, \lambda)$ and $\vartheta : \mathbf{N} \rightarrow \mathbf{R}_+^*$ a map which tends to $+\infty$ in $+\infty$. Then for all $\varepsilon > 0$, it exists $N_0 \in \mathbf{N}$ such that for all $N \geq N_0$, it exists a map $\sigma_N : E_N \rightarrow E_N$ such that $\text{Card}(\sigma_N(E_N)) < \vartheta(N)$ and $d_N(f, \sigma_N) < \varepsilon$.*

Proof of proposition 36. Let $f \in \text{Homeo}(X, \lambda)$, $\vartheta : \mathbf{N} \rightarrow \mathbf{R}_+^*$ a map which tends to $+\infty$ in $+\infty$ and $\varepsilon > 0$. By Lax's theorem (theorem 19) it exists $N_1 \in \mathbf{N}$ such that for all $N \geq N_1$, it exists a cyclic permutation $\sigma_N : E_N \rightarrow E_N$ whose distance to f is smaller than $\varepsilon/2$. For $N \geq N_1$, set $\sigma'_N = P_{N_1} \circ \sigma_N$. Increasing N_1 if necessary we have $d(f, \sigma'_N) < \varepsilon$, regardless of N . Moreover $\text{Card}(\sigma'_N(E_N)) \leq q_{N_1}$, if we choose N_0 large enough such that for all $N \geq N_0$ we have $q_{N_1} < \vartheta(N)$, then $\text{Card}(\sigma'_N(E_N)) \leq \vartheta(N)$. We have shown that the map σ'_N satisfies the conclusions of proposition for all $N \geq N_0$. \square

Corollary 37. *Let $\vartheta : \mathbf{N} \rightarrow \mathbf{R}_+^*$ a map which tends to $+\infty$ in $+\infty$. Then for a generic homeomorphism $f \in \text{Homeo}(X, \lambda)$,*

$$\lim_{N \rightarrow +\infty} \frac{\text{Card}(f_N(E_N))}{\vartheta(N)} = 0.$$

In particular, generically, $\lim_{N \rightarrow +\infty} \frac{\text{Card}(f_N(E_N))}{\text{Card}(E_N)} = 0$.

Proof of corollary 37. Remark that if we replace $\vartheta(N)$ by $\sqrt{\vartheta(N)}$, it suffices to prove that for a generic homeomorphism, $\lim_{N \rightarrow +\infty} \frac{\text{Card}(f_N(E_N))}{\vartheta(N)} \leq 1$. It is easily obtained in combining theorem 16 and proposition 36. \square

So far all variations of Lax's theorem built finite maps with a small number of orbits. With the additional assumption that the sequence of grids is self-similar, we show a last variation of Lax's theorem approaching every homeomorphism with a finite map with a large number of orbits.

Proposition 38 (Fourth variation of Lax's theorem). *Assume that the sequence of grids $(E_N)_{N \in \mathbf{N}}$ is self-similar. Let $f \in \text{Homeo}(X, \lambda)$ and $\vartheta : \mathbf{N} \rightarrow \mathbf{R}$ such that $\vartheta(N) = o(q_N)$. Then for all $\varepsilon > 0$ it exists $N_1 \in \mathbf{N}$ such that for all $N \geq N_1$, it exists a permutation σ_N of E_N such that $d_N(f, \sigma_N) < \varepsilon$ and that the number of cycles of σ_N is greater than $\vartheta(N)$. Moreover these $\vartheta(N)$ cycles of σ_N are conjugated to a cyclic permutation of E_{N_0} by bijections whose distance to identity is smaller than ε .*

Proof of proposition 38. Let $\varepsilon > 0$, for all $N_0 \in \mathbf{N}$ large enough, Lax's theorem gives us a cyclic permutation σ'_{N_0} of E_{N_0} whose distance to f is smaller than ε . Since the grids are self-similar, it exists $N_1 \in \mathbf{N}$ such that for all $N \geq N_1$, the set E_N contains $q_N/q_{N_0} \geq \vartheta(N)$ disjoint subsets \tilde{E}_N^j , each one conjugated to a grid E_{N_0} by a bijection h_j whose distance to identity is smaller than ε . On each \tilde{E}_N^j , we define σ_N as the conjugation of σ'_{N_0} by h_j ; outside these sets we just pick σ_N such that $d_N(f, \sigma_N) < \varepsilon$. Since the distance between h_j and identity is smaller than ε , we have $d_N(f, \sigma_N) < 2\varepsilon$. Moreover, σ_N has at least $\vartheta(N)$ cycles; this completes the proof. \square

The application of theorem 16 gives us:

Corollary 39. *We still assume that the sequence of grids $(E_N)_{N \in \mathbf{N}}$ is self-similar. Let $\vartheta : \mathbf{N} \rightarrow \mathbf{R}$ such that $\vartheta(N) = o(q_N)$. Then for a generic homeomorphism $f \in \text{Homeo}(X, \lambda)$ and for infinitely many integers N , the discretization f_N of f have at least $\vartheta(N)$ cycles which are pairwise conjugated.*

7. AVERAGE BEHAVIOUR OF DISCRETIZATIONS

We now want to study the average behaviour of discretizations of a generic homeomorphism. For example one could imagine that even if for a generic homeomorphism f , the event " f_N is a cyclic permutation" appears for infinitely many orders, it is still quite rare. More precisely, we study the frequency of occurrence of properties related to the discretizations of generic homeomorphisms in the Cesàro sense: given a property (P) concerning discretizations, what is the behaviour of the proportion between 1 and M of discretizations satisfying the property (P) , when M goes to infinity? For this study, we assume that the sequence of discretization grids refines (which is true for example for discretizations upon uniform grids of orders powers of an integer, see section 3). This prevents us from tricky arithmetic problems about overlay of grids.

Definition 40. Let $f \in \text{Homeo}(X, \lambda)$. We say that a property (P) about discretizations is *satisfied in average* if for all $N_0 \in \mathbf{N}$ and all $\varepsilon > 0$, it exists $N \geq N_0$ such that the proportion of integers $M \in \{0, \dots, N\}$ such that f_M satisfies (P) is greater than $1 - \varepsilon$, i.e.

$$\lim_{N \rightarrow +\infty} \frac{1}{N+1} \text{Card} \{M \in \{0, \dots, N\} \mid f_M \text{ satisfies } (P)\} = 1.$$

We will show that most of the dynamical properties studied in the previous section are actually satisfied on average for generic homeomorphisms. To start with we set out a technical lemma:

Lemma 41. *Let \mathcal{T} be a dense type of approximation in $\text{Homeo}(X, \lambda)$. Then for a generic homeomorphism $f \in \text{Homeo}(X, \lambda)$, for all $\varepsilon > 0$ and all $\alpha > 0$, the property (P) : “ E_N contains at least α disjoint subsets which fulfill a proportion greater than $1 - \varepsilon$ of E_N , each one stabilized by f_N and such that the restriction of f_N to each one is conjugated to a map of \mathcal{T} by a bijection whose distance to identity is smaller than ε ” is satisfied in average.*

In practice, this lemma provides many properties satisfied on average, for instance:

- quantitative properties on discretizations, such as owning at least M periodic orbits,
- properties of existence of sub-dynamics on discretizations, such as owning at least one dense periodic orbit.

Proof of lemma 41. Let us consider the set

$$\mathcal{C} = \bigcap_{\substack{\varepsilon > 0 \\ N_0 \in \mathbf{N}}} \bigcup_{N \geq N_0} \left\{ \frac{1}{N+1} \text{Card} \{M \in \{0, \dots, N\} \mid f_M \text{ satisfies (P)}\} > 1 - \varepsilon \right\}.$$

We want to show that \mathcal{C} contains a dense G_δ of $\text{Homeo}(X, \lambda)$. The set \mathcal{C} is a G_δ of the generic set $\bigcap_{N \in \mathbf{N}} \mathcal{D}_N$, it suffices to prove that it is dense in $\bigcap_{N \in \mathbf{N}} \mathcal{D}_N$. Let $f \in \text{Homeo}(X, \lambda)$, $N_0 \in \mathbf{N}$, $\delta > 0$ and $\varepsilon > 0$. To prove it we want to find a homeomorphism g whose distance to f is smaller than δ and an integer $N \geq N_0$ such that

$$\frac{1}{N+1} \text{Card} \{M \in \{0, \dots, N\} \mid g_M \text{ satisfies (P)}\} > 1 - \varepsilon.$$

It is simply obtained in combining the density of the type of approximation \mathcal{T} and the fact that the grids refines. \square

This corollary allows us to obtain some properties about the average behaviour of discretizations. For instance there is an improvement of corollary 27:

Corollary 42. *For a generic homeomorphism $f \in \text{Homeo}(X, \lambda)$, the property “ f_N has a ε -dense periodic orbit and the cardinal of $\Omega(f_N)$ is smaller than $\vartheta(N) = o(q_N)$ ” is satisfied in average.*

Or an improvement of corollary 35:

Corollary 43. *For a generic homeomorphism $f \in \text{Homeo}(X, \lambda)$ and for all $\varepsilon > 0$, the property “ f_N is ε -topologically weakly mixing (see definition 33)” is satisfied in average.*

Or even an improvement of corollary 37:

Corollary 44. *For a generic homeomorphism $f \in \text{Homeo}(X, \lambda)$ and for all $\varepsilon > 0$, the property “ $\frac{\text{Card}(f_N(E_N))}{\text{Card}(E_N)} < \varepsilon$ ” is satisfied in average.*

And an improvement of corollary 39:

Corollary 45. *For a generic homeomorphism $f \in \text{Homeo}(X, \lambda)$ and for all $M \in \mathbf{N}$, property “ f_N has at least M periodic orbits” is satisfied in average.*

However, note that the most simple property about discretizations, i.e. being a cyclic permutation, can not be proved by using lemma 41. To do this, we need a slightly more precise result, that we will not prove here.

Proposition 46. *For a generic homeomorphism $f \in \text{Homeo}(X, \lambda)$, the property “ f_N is a cyclic permutation” is satisfied in average.*

Remark 47. However, the property of approximation by bicyclic permutations in average can not be proven with this technique.

Remark 48. A simple calculation shows that everything that has been done in this section also applies to the behaviour of discretizations in average of Cesàro average, in average of average of Cesàro average etc., i.e. when studying quantities

$$\frac{1}{N_2 + 1} \sum_{N_1=0}^{N_2} \frac{1}{N_1 + 1} \text{Card} \{M \in \{0, \dots, N_1\} \mid f_M \text{ satisfies } (P)\},$$

$$\frac{1}{N_3 + 1} \sum_{N_2=0}^{N_3} \frac{1}{N_2 + 1} \sum_{N_1=0}^{N_2} \frac{1}{N_1 + 1} \text{Card} \{M \in \{0, \dots, N_1\} \mid f_M \text{ satisfies } (P)\} \dots$$

8. BEHAVIOUR OF ALL THE DISCRETIZATIONS

In the previous sections we showed that the dynamical behaviour of discretizations seems very chaotic depending of the order of discretization. In contrast, the dynamics of a generic homeomorphism is well known (see e.g. [Gui12]) and independent from the homeomorphism. One even has a 0-1 law on $\text{Homeo}(X, \lambda)$ (see [GK98] or the last chapter of [Gui12]) which states that either a given ergodic property on conservative homeomorphisms is generic, or its contrary is generic. Thus, the dynamics of a generic homeomorphism and that of its discretizations seem quite of independent. In fact one can deduce some dynamical features of a generic homeomorphism from the corresponding dynamical features of *all* its discretizations. For instance, the following property can be easily deduced from corollary 27.

Proposition 49. *Let f be a generic homeomorphism of $\text{Homeo}(X, \lambda)$ and p be an integer. Then f has a periodic orbit with period p if and only if there exists infinitely many integers N such that f_N has a periodic orbit with period p .*

Proof of proposition 49. An easy variation of corollary 27 shows that if p is a period of a periodic orbit of a generic homeomorphism f , then there exists infinitely many discretizations f_N such that f_N has a periodic orbit with period p . The other implication of the proposition follows easily from a compactness argument. \square

We now try to obtain information about invariant measures of a generic homeomorphism from invariant measures of its discretizations. More precisely, given all the invariant measures of discretizations of a generic homeomorphism, what can be deduced about invariant measures of the initial homeomorphism? A first step in this study was performed by T. Miernowski in 2006 in part 8 of his article [Mie06]:

Proposition 50 (Miernowski). *Let $f : X \rightarrow X$ be a uniquely ergodic homeomorphism and μ^f his unique invariant probability measure. For all $N \in \mathbf{N}$ let $\gamma_N \subset E_N$ be a periodic cycle of f_N and ν_N the uniform probability measure on γ_N . Then the weak convergence $\nu_N \rightharpoonup \mu^f$ occurs independently of the choice of the cycles γ_N .*

The proof of this proposition essentially consists in an appropriate application of Prokhorov's theorem which express the compactness of the set of probability measures on X . We now set a theorem of this kind for *generic* homeomorphisms. Recall that μ_N^f is the limit in the sense of Cesàro of the pushforwards by iterates of f_N of uniform measure on E_N (see definition 6):

$$\mu_N^f = \lim_{m \rightarrow \infty} \frac{1}{m} \sum_{i=0}^{m-1} (f_N)^i_* \lambda_N.$$

The measure μ_N^f is supported by the maximal invariant set of f_N ; it is uniform on every periodic orbit and the total weight of a periodic orbit is proportional to the size of its basin of attraction. The following theorem expresses that we can obtain all the invariant measures of a generic homeomorphism from all the invariant measures of its discretizations:

Theorem 51. *Let $f \in \text{Homeo}(X, \lambda)$ be a generic homeomorphism and suppose that the sequence of grids $(E_N)_{N \in \mathbf{N}}$ is self-similar. Let \mathcal{M}_N be the set of probability measures on E_N that are invariant under f_N . Then the upper limit over N (for Hausdorff metric) of the sets \mathcal{M}_N is exactly the set \mathcal{M} of probability measures that are invariant under f .*

Before giving a detailed proof of theorem 51 let us give its main arguments. To show that the upper limit of sets \mathcal{M}_N is a subset of \mathcal{M} , by a compactness argument, we only have to prove that limit points of a sequence $(\nu_N^f)_{N \in \mathbf{N}}$, such that every measure ν_N^f is invariant under f_N , are invariant under f . This can be easily verified in using uniform convergence of f_N to f and equicontinuity of the measures ν_N^f . Then (again by a compactness argument) we have to show that generically every invariant measure is a limit point of a sequence of discretizations. An ad hoc application of Baire's theorem reduces the proof to which of the following variation of Lax's theorem¹³:

Lemma 52 (Ergodic variation of Lax's theorem). *Suppose that the sequence of grids $(E_N)_{N \in \mathbf{N}}$ is self-similar. For all $f \in \text{Homeo}(X, \lambda)$, for all collection of f -invariant measures ν_1, \dots, ν_ℓ , for all $\varepsilon > 0$ and $k_0, N_0 \in \mathbf{N}$:*

- (1) *It exists $g \in \text{Homeo}(X, \lambda)$ and $N \geq N_0$ such that $d(f, g) < \varepsilon$, $d(\nu_1, \mu_N^g) < \frac{1}{k_0}$ and μ_N^g is g -invariant.*
- (2) *It exists $g \in \text{Homeo}(X, \lambda)$ and $N \geq N_0$ such that $d(f, g) < \varepsilon$, and for all $i \leq \ell$ it exists a measure $\nu_{i,N}^g$ such that $d(\nu_i, \nu_{i,N}^g) < \frac{1}{k_0}$ and that $\nu_{i,N}^g$ is invariant under g and g_N .*

Suppose first that $\ell = 1$ and that ν_1 is ergodic. For this purpose we apply Birkhoff's theorem to f, ν_1, φ and a recurrent point x : for all M large enough,

$$\frac{1}{M} \sum_{k=0}^{M-1} \varphi \circ f^k(x) \simeq \int \varphi d\nu_1.$$

Since x is recurrent we can choose an integer M large enough such that $x \simeq f^M(x)$. First we approximate f by a cyclic permutation σ_N given by Lax's theorem, then we slightly modify σ_N into a map σ'_N by choosing $\sigma'_N(\sigma_N^M(x_N)) = x_N$, as in proposition 24. The measure $\nu_N^{\sigma'_N}$ is the uniform measure on the orbit $x_N, \dots, \sigma_N^{M-1}(x_N)$, so it is close to ν_1 with respect to the test function φ . This property can be obtained for all test functions by a diagonal-like method. The proof of the lemma when ν_1 is not ergodic but only invariant is obtained by approximating the invariant measure by a finite convex combination of ergodic measures; this proves the first point of the lemma. To obtain the latter, it suffices to use the self-similarity hypothesis: on at least ℓ sub-grids we apply the first point to each measure ν_i .

Proof of theorem 51. For all $\delta > 0$, for all N large enough, $\mathcal{M}_N \subset B(\mathcal{M}, \delta)$. For a set A and $\delta > 0$, we note $B(A, \delta)$ the set of elements whose distance to A is smaller than δ . Let $f \in \text{Homeo}(X, \lambda)$ and ν^f a limit point of a sequence $(\nu_N^f)_{N \in \mathbf{N}}$. By compactness of \mathcal{P} , it suffices to show that ν^f is invariant under f , i.e. that for all continuous map φ ,

$$\int \varphi \circ f d\nu^f = \int \varphi d\nu^f.$$

By the triangle inequality we have:

$$\begin{aligned} \left| \int \varphi \circ f d\nu^f - \int \varphi d\nu^f \right| &\leq \left| \int \varphi \circ f d\nu^f - \int \varphi \circ f d\nu_N^f \right| + \left| \int \varphi \circ f d\nu_N^f - \int \varphi \circ f_N d\nu_N^f \right| \\ &\quad + \left| \int \varphi \circ f_N d\nu_N^f - \int \varphi d\nu_N^f \right| + \left| \int \varphi d\nu_N^f - \int \varphi d\nu^f \right|. \end{aligned}$$

Since $\nu_N^f \rightharpoonup \nu^f$ (for a subsequence) the first and the last terms tend to 0 when N goes to infinity (for a subsequence). The third term is zero because ν_N^f is invariant under f_N , and the second one tends to 0 because $\varphi \circ f_N$ tends to $\varphi \circ f$ uniformly and $(\nu_N^f)_N$ is equicontinuous. Then

$$\left| \int \varphi \circ f d\nu^f - \int \varphi d\nu^f \right| = 0;$$

¹³The first point of this lemma is somehow a weak version of the latter. It will be useful in the next section.

it shows that the measure ν^f is invariant under f .

Generically, for all $\delta > 0$ and all $N_0 \in \mathbf{N}$, it exists $N \geq N_0$ such that $\mathcal{M} \subset B(\mathcal{M}_N, \delta)$. Recall that we denote by \mathcal{P} the set of all Borel probability measures on X . Set d a distance on \mathcal{P} defining the weak-* topology. Thereafter homeomorphisms will be taken in the generic set

$$\bigcap_{N \in \mathbf{N}} \mathcal{D}_N,$$

made of homeomorphisms whose N th discretization is uniquely defined for all N . Consider

$$\mathcal{A} = \bigcap_{(N_0, k_0) \in \mathbf{N}^2} \mathcal{O}_{N_0, k_0},$$

where

$$\mathcal{O}_{N_0, k_0} = \left\{ f \in \bigcap_{N \in \mathbf{N}} \mathcal{D}_N \mid \exists N \geq N_0 \text{ s.t. } \mathcal{M} \subset B(\mathcal{M}_N, \frac{3}{k_0}) \right\}.$$

Trivially, if $f \in \mathcal{A}$, then the upper limit of the sets \mathcal{M}_N contains \mathcal{M} .

We now show that the sets \mathcal{O}_{N_0, k_0} are open, which is equivalent to show that their complements are closed. Set $N_0, k_0 \in \mathbf{N}$. Let $(f^m)_{m \in \mathbf{N}}$ be a sequence of homeomorphisms which tend to a homeomorphism f , such that for all $m \geq 0$, $f^m \notin \mathcal{O}_{N_0, k_0}$. We want to show that $f \notin \mathcal{O}_{N_0, k_0}$. Let $N \geq N_0$, for all m , $\mathcal{M}^m \not\subset B(\mathcal{M}_N^m, \frac{3}{k_0})$. So it exists ν^m which is f^m invariant such that $\nu^m \notin B(\mathcal{M}_N^m, \frac{3}{k_0})$. Taking a subsequence if necessary, the sequence $(\nu^m)_m$ tends to a measure ν which is invariant under f . Since $f \in \mathcal{D}_N$, $(f^m)_N = f_N$ for all m great enough, so $\mathcal{M}_N^m = \mathcal{M}_N$ for all m great enough. Therefore, $\nu \notin B(\mathcal{M}_N, \frac{3}{k_0})$ and thus $f \notin \mathcal{O}_{N_0, k_0}$.

It remains to show that the sets \mathcal{O}_{N_0, k_0} are dense; it follows from the second point of the variation of Lax's theorem (lemma 52): by semi-continuity of $g \mapsto \mathcal{M}^g$, one has $\mathcal{M}^g \subset B(\mathcal{M}^f, \frac{1}{k_0})$ for all g close enough to f . Moreover, by compactness, it exists f -invariant measures ν_1, \dots, ν_ℓ such that $\mathcal{M}^f \subset \bigcup_i B(\nu_i, \frac{1}{k_0})$. So it suffices to find g close to f such that for a $N \geq N_0$, it exists g_N -invariant measures $\nu_{1,N}^g, \dots, \nu_{\ell,N}^g$ such that $d(\nu_i, \nu_{i,N}^g) < \frac{1}{k_0}$ for all i , but this is exactly the conclusion of the lemma. \square

Proof of lemma 52. To begin with we prove the first point of the lemma. Suppose first that the measure $\nu = \nu_1$ is ergodic. We want to show that there exists a homeomorphism g whose distance to f is smaller than ε , and an integer $N \geq N_0$ such that $d(\nu, \mu_N^g) < \frac{2}{k_0}$. Let $(\varphi_j)_{j \in \mathbf{N}}$ be a sequence which is dense in the set of continuous maps from X into \mathbf{R} . By Prokhorov's theorem, there exists $i \in \mathbf{N}$ such that if we have

$$\left| \int \varphi_j d\nu - \int \varphi_j d\mu \right| \leq \frac{1}{i}$$

for all $j \leq i$, then $d(\nu, \mu) < \frac{1}{k_0}$.

Since ν is ergodic, for all continuous map φ , by Birkhoff's theorem,

$$\frac{1}{M} \sum_{m=0}^{M-1} \varphi \circ f^m(x) \xrightarrow{M \rightarrow +\infty} \int \varphi d\nu \quad (1)$$

for ν -a.e. x . Let $x \in X$ be a recurrent point for f satisfying equation (1) for all φ_j (such points form ν -full measure set). For all j there exists $M_j \in \mathbf{N}$ such that for all $M \geq M_j$ one has

$$\left| \frac{1}{M} \sum_{m=0}^{M-1} \varphi_j \circ f^m(x) - \int \varphi_j d\nu \right| \leq \frac{1}{2i}. \quad (2)$$

Let $M'_i = \max_{j \leq i} M_j$. Since x is recurrent, there exists $\tau \geq M'_i$ such that $d(x, f^\tau(x)) \leq \varepsilon/4$. Let σ_N be a map from E_N into itself given by Lax's theorem: it is a cyclic permutation and its distance to f is smaller than $\varepsilon/2$. For all N large enough the orbit $(\sigma_N^m(x_N))_{0 \leq m \leq \tau}$

shadows the orbit $(f^m(x))_{0 \leq m \leq \tau}$, thus $d(x_N, \sigma_N^\tau(x_N)) < \varepsilon/2$. Then we “close” the orbit of x_N between the points x_N and $\sigma_N^\tau(x_N)$, i.e. we set (as in proposition 24, see also figure 3)

$$\sigma'_N(y_N) = \begin{cases} x_N & \text{if } y_N = \sigma_N^{\tau-1}(x_N) \\ \sigma_N(y_N) & \text{otherwise.} \end{cases}$$

Then $d_N(\sigma'_N, f) < \varepsilon$ and $(x_N, \dots, \sigma_N^{\tau-1}(x_N))$ is a periodic orbit for σ'_N whose basin of attraction is the whole set E_N .

Since the periodic orbit $(\sigma_N^m(x))_{0 \leq m \leq \tau}$ attracts E_N , for all M' large enough and $y_N \in E_N$ we have

$$\frac{1}{M'} \sum_{m=0}^{M'-1} \varphi_j \circ \sigma_N^m(y_N) \xrightarrow{M' \rightarrow \infty} \frac{1}{\tau} \sum_{m=0}^{\tau-1} \varphi_j \circ \sigma_N^m(x_N). \quad (3)$$

With proposition 17, the same way as in lemma 18, we construct a homeomorphism g^N from the map σ'_N such that the discretization $(g^N)_N$ and σ'_N fit together, and such that g and the restriction of $(g^N)_N$ to the orbit of x_N fit together. Since $d_N(f, \sigma'_N) < \varepsilon$, we can furthermore assume that $d(f, g^N) < \varepsilon$. Thus, equation (3) implies that

$$\mu_N^{g^N} = \frac{1}{\tau} \sum_{m=0}^{\tau-1} \delta_{\sigma_N^m(x_N)}. \quad (4)$$

Since φ_j is continuous, increasing N (if necessary) such that the orbit $(\sigma_N^m(x_N))_{0 \leq m \leq \tau}$ shadows $(f^m(x))_{0 \leq m \leq \tau}$ better, we have for all $j \leq i$:

$$\left| \frac{1}{\tau} \sum_{m=0}^{\tau-1} \varphi_j \circ f^m(x) - \frac{1}{\tau} \sum_{m=0}^{\tau-1} \varphi_j \circ \sigma_N^m(x_N) \right| \leq \frac{1}{2i} \quad (5)$$

We now have all the necessary estimations to compute the distance between $\mu_N^{g^N}$ and ν : by equation (4) and triangle inequality,

$$\begin{aligned} \left| \int \varphi_j d\mu_N^{g^N} - \int \varphi_j d\nu \right| &= \left| \frac{1}{\tau} \sum_{m=0}^{\tau-1} \varphi_j \circ \sigma_N^m(x_N) - \int \varphi_j d\nu \right| \\ &\leq \left| \frac{1}{\tau} \sum_{m=0}^{\tau-1} \varphi_j \circ \sigma_N^m(x_N) - \frac{1}{\tau} \sum_{m=0}^{\tau-1} \varphi_j \circ f^m(x_N) \right| \\ &\quad + \left| \frac{1}{\tau} \sum_{m=0}^{\tau-1} \varphi_j \circ f^m(x_N) - \int \varphi_j d\nu \right| \end{aligned}$$

Hence, with equation (2) applied to $M = \tau$ and equation (5),

$$\left| \int \varphi_j d\mu_N^{g^N} - \int \varphi_j d\nu \right| \leq \frac{1}{i}.$$

We obtain $d(\mu_N^{g^N}, \nu) < \frac{1}{k_0}$. Since $\mu_N^{g^N}$ is invariant under g^N (g^N is a cyclic permutation of the orbit of x_N), g^N is the desired homeomorphism.

In the general case the measure ν is only invariant (and not ergodic). It reduces to the ergodic case by the fact that the set of invariant measures is a compact convex set whose extremal points are exactly the ergodic measures: by Krein-Milman theorem, for all $M \geq 1$ there exists an f -invariant measure ν' which is a finite convex combination of ergodic measures:

$$\nu' = \sum_{j=0}^r \lambda_j \nu_j^e,$$

and whose distance to ν is smaller than $\frac{1}{k_0}$.

Then we use the hypothesis of self-similarity of the grids: for all $\varepsilon > 0$ and all $P \in \mathbf{N}$ there exists a map σ_N whose distance to f is smaller than ε and whose at least $p \geq P$ cycles are conjugated to a permutation of a coarser subdivision, via the composition of a fixed homothety by a translation. Moreover the union on these p cycles fulfills the set E_N

with a proportion of at least $1 - \frac{1}{2k_0}$. Take $P \in \mathbb{N}$ such that $\frac{2r}{P} \leq \frac{1}{2k_0}$. For all j between 1 and r we set

$$\lambda'_j = \frac{\lfloor \lambda_j p \rfloor}{p}$$

and

$$\lambda'_0 = 1 - \sum_{j=1}^r \lambda'_j.$$

Then for all $j > 0$ we have $|\lambda_j - \lambda'_j| \leq \frac{1}{P}$ and $|\lambda_0 - \lambda'_0| \leq \frac{r}{P}$. Since $\sum \lambda'_j = 1$ we can associate bijectively a set Λ_j made of $p\lambda'_j$ cycles to each integer j ; the union of the sets Λ_j fills the set E_N with a proportion of at least $1 - \frac{1}{2k_0}$. We do the previous construction concerning ergodic measures for each j : increasing N if necessary, the cycles of each Λ_j can be changed such that each of them carries a (unique) invariant probability measure $\nu_j^{e'}$ satisfying $d(\nu_j^e, \nu_j^{e'}) \leq \frac{1}{2k_0}$. These modifications transform the map σ_N into a map σ'_N satisfying $d(\sigma'_N, \sigma_N) < \varepsilon$. Then the measure $\mu_N^{\sigma'_N}$ associated with the map σ'_N satisfies:

$$d\left(\mu_N^{\sigma'_N}, \sum_{j=0}^r \lambda'_j \nu_j^{e'}\right) \leq \frac{1}{2k_0}$$

and we have:

$$\begin{aligned} d(\nu, \mu_N^{\sigma'_N}) &\leq \sum_{j=0}^r d(\lambda_j \nu_j^e, \lambda'_j \nu_j^{e'}) + \frac{1}{2k_0} \\ &\leq \sum_{j=0}^r d(\lambda_j \nu_j^e, \lambda'_j \nu_j^e) + \sum_{j=0}^r d(\lambda'_j \nu_j^e, \lambda'_j \nu_j^{e'}) + \frac{1}{2k_0} \\ &\leq \sum_{j=0}^r |\lambda_j - \lambda'_j| + \sum_{j=0}^r \lambda'_j d(\nu_j^e, \nu_j^{e'}) + \frac{1}{2k_0} \\ &\leq \frac{2r}{P} + \frac{1}{2k_0} + \frac{1}{2k_0} \leq \frac{3}{2k_0}. \end{aligned}$$

Thus $d(\nu, \mu_N^{\sigma'_N}) \leq \frac{2}{k_0}$. As above the desired homeomorphism is obtained by extending σ'_N to a homeomorphism of X . We have proven the first point of the lemma.

Finally we handle the second point of the lemma. To do that, it suffices to apply the first point on ℓ different sub-grids of a given grid E_{N_0} . On the i -th sub-grid, define the discretization g_N such that it has a single invariant measure which is close to ν_i . This proves the lemma. \square

There is a kind of duality between invariant measures and invariant compact sets, thus the previous theorem is also true for invariant compact sets:

Proposition 53. *Let $f \in \text{Homeo}(X, \lambda)$ be a generic homeomorphism and suppose that the sequence of grids $(E_N)_{N \in \mathbb{N}}$ is self-similar. Set \mathcal{I}_N the set of subsets of E_N that are invariant under f_N . Then the upper limit over N (for Hausdorff metric) of the sets \mathcal{I}_N is exactly the set of compacts that are invariant under f .*

Sketch of proof of proposition 53. To show that all limit point K of a sequence (K_N) of compact invariant sets is a compact invariant set, it suffices to calculate $d_H(K, f(K))$. For the other inclusion, as in the case of invariant measures, it follows from an approximation lemma whose proof is the same as for invariant measures: \square

Lemma 54 (Compact variation of Lax's theorem). *Suppose that the sequence of grids $(E_N)_{N \in \mathbb{N}}$ is self-similar. For all $f \in \text{Homeo}(X, \lambda)$, for all collection of f -invariant compacts K_1, \dots, K_ℓ , for all $\varepsilon > 0$ and $k_0, N_0 \in \mathbb{N}$:*

- (1) *It exists $g \in \text{Homeo}(X, \lambda)$ and $N \geq N_0$ such that $d(f, g) < \varepsilon$, $d_H(K_1, \Omega(f_N)) < \frac{1}{k_0}$ and $\Omega(f_N)$ is g -invariant.*

- (2) It exists $g \in \text{Homeo}(X, \lambda)$ and $N \geq N_0$ such that $d(f, g) < \varepsilon$, and for all $i \leq \ell$, it exists a compact $K_{i,N}^g$ such that $d_H(K_i, K_{i,N}^g) < \frac{1}{k_0}$ and that $K_{i,N}^g$ is invariant under g and g_N .

9. PHYSICAL MEASURES

In the previous part we proved that generically, the upper limit of the sets of invariant measures of discretizations is the set of invariant measures of the initial homeomorphism. It expresses that the sets of invariant measures of discretizations behave “as chaotically as possible”. However, one might expect that physical measures (Borel measures μ such that $\mu = \mu_x^f$ is verified on a set of points x with λ positive measure, see e.g. [You02]) play a specific part: their definition expresses that they are the measures that can be observed in practice, because they governs the ergodic behaviour of λ a.e. point. So one can hope that the natural invariant measures μ_N^f of f_N , which can be seen as the physical measures of f_N , converge to the physical measures of f . This expectation is supported by the following variation of proposition 50, obtained by replacing ν_N by μ_N^f : *if f is uniquely ergodic, then the measures μ_N^f converge weakly to the only measure μ^f that is invariant under f .*

In this section we show that this is not at all the case: the sequence of measures (μ_N^f) accumulates of the whole set of f -invariant measures¹⁴. More precisely one has the following theorem:

Theorem 55. *If the sequence of grids $(E_N)_{N \in \mathbb{N}}$ is self-similar, for a generic homeomorphism $f \in \text{Homeo}(X, \lambda)$, the set of limit points of the sequence $(\mu_N^f)_{N \in \mathbb{N}}$ is exactly the set of f -invariant measures.*

This theorem can be seen as a discretized version of the following conjecture:

Conjecture 56 (F. Abdenur, M. Andersson, [AA12]). *A homeomorphism f which is generic in the set of homeomorphisms of X (without measure preserving hypothesis) that do not have any open trapping set is wicked, i.e. it is not uniquely ergodic and the measures*

$$\frac{1}{m} \sum_{k=0}^{m-1} f_*^k(\text{Leb})$$

accumulate on the whole set of invariant measures under f .

The behaviour described in this conjecture is the opposite of that consisting of possessing a physical measure.

Sketch of proof of theorem 55. The proof is similar to which of theorem 51: the set \mathcal{A} is replaced by

$$\mathcal{A}' = \bigcap_{(\ell, N_0, k_0) \in \mathbb{N}^3} \mathcal{O}_{\ell, N_0, k_0},$$

where

$$\mathcal{O}_{\ell, N_0, k_0} = \left\{ f \in \bigcap_{N \in \mathbb{N}} \mathcal{D}_N \left| \begin{array}{l} \left(\exists \nu f\text{-inv.} : d(\nu, \tilde{\nu}_\ell) \leq \frac{1}{k_0} \right) \implies \\ \left(\exists N \geq N_0 : d(\tilde{\nu}_\ell, \mu_N^f) < \frac{2}{k_0} \right) \end{array} \right. \right\},$$

and the second point of lemma 52 is replaced by the first. \square

Remark 57. Taking over the proof of the theorem, if we set $x \notin \bigcap_N E'_N$, then generically the measures $\mu_{N,x}^f$ accumulate on the whole set of f -invariant measures. This seems to contradict the empirical observations made by A. Boyarsky in 1986 (see [Boy86] or [GB88]): when a homeomorphism f has a unique ergodic measure μ^f which is absolutely continuous with respect to Lebesgue measure “most of” the measures $\mu_{N,x}^f$ tend to measure μ^f . However, the author does not specify in what sense he means “most of the points”, or if his remark is based on a tacit assumption of regularity for f .

¹⁴It is easy to see that if $(\nu_N)_{N \in \mathbb{N}}$ is a sequence of measures, with ν_N invariant under f_N for all N , then its limit set is included in the set of f -invariant measures. Again, this reflects the fact that the behaviour of discretizations of a generic homeomorphism is “as chaotic as possible”.

Note that as in the previous section, we have a compact counterpart of theorem 55:

Proposition 58. *If the sequence of grids $(E_N)_{N \in \mathbf{N}}$ is self-similar, for a generic homeomorphism $f \in \text{Homeo}(X, \lambda)$, the set of limit points of the sequence $(\Omega(f_N))_{N \in \mathbf{N}}$ is exactly the set of f -invariant compact sets.*

Part 2. Discretizations of a generic dissipative homeomorphism

Throughout this section, we fix a manifold X and a good measure λ as defined in section 2. Now homeomorphisms are no longer supposed conservative. More formally, we recall that we note $\text{Homeo}(X)$ the set of all *dissipative* homeomorphisms of X (without assumption of conservation of a given measure), we are interested in properties of discretizations of generic elements of $\text{Homeo}(X)$. The manifold X is equipped with a sequence $(E_N)_{N \in \mathbf{N}}$ of discretization grids.

Generic topological properties of dissipative homeomorphisms have been discussed in a survey of E. Akin, M. Hurley and J. Kennedy [AHK03]. However, ergodic properties of generic homeomorphisms had not been studied until the work of F. Abdenur and M. Andersson in [AA12], in which the authors are interested in the typical behaviour of Birkhoff averages. For this purpose they establish a technical lemma they call *shredding lemma* which allows them to show that a generic homeomorphism is *weird*.

Definition 59. A homeomorphism f is said *weird* if almost every point $x \in X$ (for λ) has a Birkhoff's limit μ_x^f , and if f is *totally singular* (i.e. there exists a Borel set with total measure whose image by f is null measure) and does not admit any physical measure.

In opposition to the conservative case, the lack of conservation of a given measure allows us to perturb any diffeomorphism to create attractors whose basins of attraction cover almost the entire space X . This is an open property, so such basins of attraction can be seen on discretizations.

With what has just been said, we can already note that properties of discretizations of generic dissipative and conservative homeomorphisms are quite different: if before (almost) all possible dynamical properties are observed on the discretizations of a homeomorphism, the dissipative case reveals a much more regular behaviour: “the dynamics of discretizations of a homeomorphism f converges to the dynamics of f ”. For example, the measures μ_N^f accumulate on a single measure, namely μ_X^f , and not on all the invariant measures under f .

10. THE SHREDDING LEMMA AND HIS DISCRETE COUNTERPART

In their article [AA12], F. Abdenur and M. Andersson try to identify the generic ergodic properties of continuous maps and homeomorphisms of compact manifolds. More precisely, they study the behaviour of Birkhoff limits μ_x^f for a generic homeomorphism $f \in \text{Homeo}(X)$ and almost every point x for Lebesgue measure. To do this they define some interesting behaviours of homeomorphisms related to Birkhoff limits, including one they call *weird* (see def 59). This definition is supported by their proof, based on the shredding lemma, that a generic homeomorphism is weird. We outline an improvement of this lemma, whose main consequence is that a generic homeomorphism has many open attractive sets, all of small measure and decomposable into a small number of small diameter open sets:

Lemma 60 (Shredding lemma, F. Abdenur, M. Andersson, [AA12]). *For all homeomorphism $f \in \text{Homeo}(X)$, for all $\delta > 0$, there exists a family of regular pairwise disjoint open sets¹⁵ U_1, \dots, U_ℓ such that for all $\varepsilon' > 0$, it exists $g \in \text{Homeo}(X)$ such that $d(f, g) < \delta$ and:*

- (i) $g(\overline{U_j}) \subset U_j$,
- (ii) $\lambda(U_j) < \varepsilon$,
- (iii) $\lambda\left(\bigcup_{j=1}^{\ell} U_j\right) > 1 - \varepsilon$,
- (iv) $\lambda(g(U_j)) < \varepsilon \lambda(U_j)$,

¹⁵An open set is said *regular* if it is equal to the interior of its closure.

(v) there exists open sets $W_{j,1}, \dots, W_{j,\ell_j}$ such that:

- (a) $\text{diam}(W_{j,i}) < \varepsilon'$ for all $i \in \{1, \dots, \ell_j\}$,
- (b) $g(\overline{W_{j,i}}) \subset W_{j,i+1}$, for every $i \in \{1, \dots, \ell_j - 1\}$ and $g(\overline{W_{j,\ell_j}}) \subset W_{j,1}$,
- (c)

$$\overline{U_j} \subset \bigcup_{m \geq 0} g^{-m} \left(\bigcup_{i=1}^{\ell_j} W_{j,i} \right).$$

The lemma as stated by F. Abdenur and M. Andersson in their paper can be obtained making $\varepsilon = \varepsilon'$. To get this improvement, it suffices to consider the sets $W_{j,i}$ given by the weak version of the lemma and to crush them with a contraction mapping, as the authors do in the first part of the proof of the lemma.

Remark 61. It can be easily seen that the lemma remains true on a neighbourhood of the homeomorphism g .

Remark 62. We can further assume that:

- (1) The open sets $W_{j,i}$ have disjoint attractive sets, *i.e.* for all $j \in \mathbf{N}$ and all $i \neq i'$, we have¹⁶

$$\left(\bigcap_{m \geq 0} \bigcup_{m' \geq m} f^{m'}(W_{j,i}) \right) \cap \left(\bigcap_{m \geq 0} \bigcup_{m' \geq m} f^{m'}(W_{j,i'}) \right) = \emptyset.$$

For this it suffices to replace $W_{j,i}$ by $W'_{j,i}$ defined by induction by:

$$\begin{aligned} W'_{j,1} &= W_{j,1} \\ W'_{j,i+1} &= W_{j,i+1} \setminus \left(\bigcup_{i' < i} \bigcap_{m \geq 0} \bigcup_{m' \geq m} f^{m'}(W_{j,i'}) \right). \end{aligned}$$

- (2) Each set $W_{j,i}$ contains a periodic attractive point. Indeed, for all $\varepsilon > 0$, it exists $x \in W_{j,i}$ and $m > 0$ such that $d(x, f^m(x)) < \varepsilon$. Then we perturb f in order to make x m -périodic and attractive, and this remains true in a neighborhood of f .
- (3) The set $g(\overline{U_j})$ is “independent” from the choice of ε' : it exists compact sets V_j such that $g(\overline{U_j}) \subset V_j$ and $V_j \subset U_j$, $\lambda(V_j) < \varepsilon \lambda(U_j)$.

This lemma tells us a lot about the dynamics of a generic homeomorphism, which becomes quite clear: there are many sinks whose basins of attraction are small and attract almost all the manifold X . Moreover there is convergence of the attractive sets of the shredding lemma to the closure of the sinks of f :

Notation 63. For an homeomorphism $f \in \text{Homeo}(X)$, we note A_0 the set of *sinks* of f , *i.e.* the sets of periodic points whose orbits attract nonempty open sets.

Corollary 64. Let $f \in \text{Homeo}(X)$ verifying the conclusions of the shredding lemma for all $\varepsilon = \varepsilon' > 0$ and $W_{j,i,\varepsilon}$ be the corresponding open sets. Such homeomorphisms form a G_δ dense subset of $\text{Homeo}(X)$. Then the sets

$$A_\varepsilon = \overline{\bigcup_{j,i} W_{j,i,\varepsilon}}$$

converge for Hausdorff distance when ε tends to 0 to a closed set. This is a null set¹⁷ which coincides generically with the closure of the set A_0 .

Proof of corollary 64. Let f verifying the hypothesis of the corollary. We want to show that the sets A_ε tend to A_0 when ε goes to 0. This is equivalent to show that for all $\delta > 0$, it exists $\varepsilon_0 > 0$ such that for all $\varepsilon < \varepsilon_0$, $A_0 \subset B(A_\varepsilon, \delta)$ and $A_\varepsilon \subset B(A_0, \delta)$ (where $B(A, \delta)$ denotes the set of points of X whose distance to A is smaller than δ). Subsequently we will note $U_{j,\varepsilon}$ and $W_{i,j,\varepsilon}$ the open sets given by the shredding lemma for the parameters $\varepsilon = \varepsilon'$.

¹⁶These two sets are decreasing intersections of compact sets, so they are compact and nonempty.

¹⁷And better, if we are given a countable family $(\lambda_m)_{m \in \mathbf{N}}$ of good measures, generically $\lambda_m(A_0) = 0$.

We start by taking $x \in X$ whose orbit is periodic and attracts a nonempty open set, say O . Then there exists $\varepsilon_0 > 0$ such that for all $\varepsilon \in]0, \varepsilon_0[$, there exists $j \in \mathbf{N}$ such that the intersection between O and $U_{j,\varepsilon}$ is nonempty. Let y be an element of this intersection. Then there exists a subsequence of $(f^m(y))_{m \in \mathbf{N}}$ which tends to x ; moreover $f^m(y) \in \bigcup_i W_{j,i}$ eventually. We deduce that $x \in \overline{\bigcup_{j,i} W_{j,i,\varepsilon_0}}$ for all ε_0 small enough. Now, let $\delta > 0$. Since $\overline{A_0}$ is compact, it is covered by a finite number of balls of radius δ centered in some points x_i whose orbits attract nonempty open sets. Taking ε'_0 the minimum of all the ε_0 associated to the x_i , the inclusion $A_0 \subset B(A_\varepsilon, \delta)$ takes place for all $\varepsilon < \varepsilon'_0$.

In the other way, let $\delta > 0$, $\varepsilon < \delta$ and focus on the set $W_{j,i,\varepsilon}$. By point 2. of the remark 62, we can suppose that there exists $x \in W_{j,i,\varepsilon}$ whose orbit is periodic and attracts a nonempty open set. Thus $x \in A_0$ and since the diameter of $W_{j,i,\varepsilon}$ is smaller than δ , $W_{j,i,\varepsilon} \subset B(A_0, \delta)$. The corollary is proved. \square

Now we establish a discrete counterpart of the shredding lemma. This result shows that the general dynamics of a generic dissipative homeomorphism is quite simple; thus the dynamics of the discretizations these homeomorphisms is also quite simple. More precisely, since having basins of attraction is stable by perturbation, we have a similar statement for discretizations of a generic homeomorphism. The following develops these arguments.

To each point $x_N \in E_N$, we associate a closed set $\overline{P_N^{-1}(\{x_N\})}$, made of the points in X whose one of the projections on E_N is x_N . The closed sets $\overline{P_N^{-1}(\{x_N\})}$ form a basis of the topology of X when N runs through \mathbf{N} and x_N runs through E_N . Let $f \in \text{Homeo}(X)$, $N \in \mathbf{N}$ and $\vartheta : \mathbf{N} \rightarrow \mathbf{R}_+^*$ be a function that tends to $+\infty$ in $+\infty$. Let $\delta, \varepsilon > 0$ and U_j be the sets obtained by the shredding lemma for f , δ and ε .

For all $j \leq \ell$ we note \tilde{U}_j^N the union over $x_N \in E_N$ of the closed sets $\overline{P_N^{-1}(\{x_N\})}$ whose intersection with U_j are nontrivial. Then \tilde{U}_j^N tends to U_j for the metric $d(A, B) = \lambda(A\Delta B)$, and for the Hausdorff metric. By point (3) of remark 62, these convergences are independent from the choice of ε' . Thus, for all k big enough, properties (i) to (iii) of the shredding lemma remain true for the discretizations g_N (for arbitrary g satisfying the properties of the lemma). Taking ε' small enough and modifying a little g if necessary, it exists sets $W_{j,i}$ and $g \in \text{Homeo}(X)$ such that $d(f, g) < \delta$, $W_{j,i} \subset \overline{P_N^{-1}(\{x_N\})}$ and $\text{Card}(\bigcup_{j,i} W_{j,i} \cap E_N) \leq \vartheta(N)$. The others estimations over the sizes of the sets involved in the lemma are obtained similarly. Finally we have:

Lemma 65 (Discrete shredding lemma). *For all $f \in \text{Homeo}(X)$, for all $\varepsilon, \delta > 0$ and all function $\vartheta : \mathbf{N} \rightarrow \mathbf{R}_+^*$ that tends to $+\infty$ in $+\infty$, it exists $N_0 \in \mathbf{N}$ such that for all $N \geq N_0$, it exists a family of subsets U_1^N, \dots, U_ℓ^N of E_N and $g \in \text{Homeo}(X)$ such that $d(f, g) < \delta$ and:*

- (i) $g_N(U_j^N) \subset U_j^N$,
- (ii) $\text{Card}(U_j^N) < \varepsilon q_N$,
- (iii) $\text{Card}\left(\bigcup_{j=1}^\ell U_j^N\right) > (1 - \varepsilon)q_N$,
- (iv) $\text{Card}(g_N(U_j^N)) < \varepsilon \text{Card}(U_j^N)$,
- (v) for all j , it exists $w_{j,1}^N, \dots, w_{j,\ell_j}^N \in E_N$ such that
 - (a) $\text{Card}(\bigcup_{j,i} \{w_{j,i}\}) \leq \vartheta(N)$,
 - (b) $g_N(w_{j,i}^N) = w_{j,i+1}^N$, for every $i \in \{1, \dots, \ell_j - 1\}$ and $g_N(w_{j,\ell_j}^N) = w_{j,1}^N$,
 - (c)

$$U_j^N \subset \bigcup_{m \geq 0} g_N^{-m} \left(\bigcup_{i=1}^{\ell_j} w_{j,i}^N \right),$$

- (vi) for all j and all i , it exists sets U_j and $W_{j,i}$ satisfying properties (i) to (v) of the shredding lemma such that $U_j \subset P_N^{-1}(U_j^N)$ and $W_{j,i} \subset P_N^{-1}(w_{j,i}^N)$.
- (vii) if $N \geq N_0$, then for all j and all i , we have

$$\sum_j d_H(\overline{U_j}, \overline{P_N^{-1}(U_j^N)}) < \varepsilon \quad \text{and} \quad \sum_{i,j} d_H(\overline{W_{j,i}}, \overline{P_N^{-1}(w_{j,i}^N)}) < \varepsilon$$

(where d_H is the Hausdorff metric), and

$$\sum_j \lambda(U_j \Delta P_N^{-1}(U_j^N)) < \varepsilon \quad \text{and} \quad \sum_{i,j} \lambda(W_{j,i} \Delta P_N^{-1}(w_{j,i}^N)) < \varepsilon.$$

Remark 66. Properties (i) to (v) are discrete counterparts of properties (i) to (v) of the continuous shredding lemma, but the two last ones reflect the convergence of the dynamics of discretizations to which of the original homeomorphism.

This lemma implies that we can theoretically deduce the behaviour of a generic homeomorphism from the dynamics of its discretizations. The next section details this remark.

11. DYNAMICS OF DISCRETIZATIONS OF A GENERIC HOMEOMORPHISM

To begin with, we deduce from the shredding lemma that the dynamics of discretizations f_N tends to that of the homeomorphism f . More precisely almost all orbits of the homeomorphism are δ -shadowed by the orbits of the corresponding discretizations.

Definition 67. Let f and g be two maps from a metric space X into itself, $x, y \in X$ and $\delta > 0$. We say that the orbit of x by f δ -shadows the orbit of y by g if for all $m \in \mathbf{N}$, $d(f^m(x), g^m(y)) < \delta$.

Corollary 68. For a generic homeomorphism $f \in \text{Homeo}(X)$, for all $\varepsilon > 0$ and all $\delta > 0$, it exists an open set A such that $\lambda(A) > 1 - \varepsilon$ and $N_0 \in \mathbf{N}$ such that for all $N \geq N_0$ and all $x \in A$, the orbit of x_N by f_N δ -shadows that of x by f .

Therefore, for a generic homeomorphism f , it exists a full measure dense open set O such that for all $x \in O$, all $\delta > 0$ and all N large enough, the orbit of x_N by f_N δ -shadows that of x by f .

Proof of corollary 68. This easily follows from the discrete shredding lemma, and especially from the fact that the points $w_{j,i}^N$ tend to the sets $W_{j,i}$ for Hausdorff metric. \square

Therefore, we can “see” the dynamics of f on which of f_N , with arbitrarily high precision provided that N is large enough. Among other things, this allows us to observe the basins of attraction of sinks of f on discretizations. Better yet, to each family of sinks $(W_{j,i})_i$ of the basin U_j of the homeomorphism correspond a unique family of points $(w_{j,i}^N)_i$ that are permuted cyclically by f_N and which attracts a neighbourhood of U_j . Thus sinks are shadows by cyclic orbits of f_N and we can detect the “period” of the sink (i.e. the integer k_j) on discretizations. This behaviour is the opposite of what happens in the conservative case, where discretized orbits and true orbits are very different for most points.

Again, in order to show that the dynamics of discretizations converge to that of the initial homeomorphism, we establish the convergence of attractive sets of f_N to that of f . Recall that A_0 is the closure of the sinks of f (see notation 63).

Proposition 69. For a generic homeomorphism $f \in \text{Homeo}(X)$, the sets $\Omega(f_N)$ tend weakly to A_0 in the following sense: for all $\varepsilon > 0$, it exists $N_0 \in \mathbf{N}$ such that for all $N \geq N_0$, it exists a subset \tilde{E}_N of E_N , stabilized by f_N such that, noting $\tilde{\Omega}(f_N)$ the corresponding maximal invariant set, we have $\frac{\text{Card}(\tilde{E}_N)}{\text{Card}(E_N)} > 1 - \varepsilon$ and $d_H(A_0, \tilde{\Omega}(f_N)) < \varepsilon$.

Proof of proposition 69. Let $\varepsilon > 0$. For all $N \in \mathbf{N}$, set \tilde{E}_N the union of the sets U_j^N of lemma 65 for the parameter ε . This lemma ensures that \tilde{E}_N is stable by f_N and fills a proportion greater than $1 - \varepsilon$ of E_N . We also note $\tilde{\Omega}(f_N)$ the associated maximal invariant set:

$$\tilde{\Omega}(f_N) = \bigcup_{x \in \bigcup_{j=1}^k \tilde{U}_j^N} \omega_{f_N}(x).$$

Property (viii) of lemma 65 ensures that

$$\overline{\lim}_{N \rightarrow +\infty} d_H(A_\varepsilon, \tilde{\Omega}(f_N)) < \varepsilon.$$

To conclude, it suffices to observe that $A_\varepsilon \rightarrow A_0$ for Hausdorff distance. \square

Finally, we conclude this section with a last consequence of lemma 65 which reflects that the ratio between the cardinal of the image of discretizations and which of the grid is smaller and smaller:

Corollary 70. *For a generic homeomorphism $f \in \text{Homeo}(X)$,*

$$\lim_{N \rightarrow +\infty} \frac{\text{Card}(f_N(E_N))}{\text{Card}(E_N)} = 0;$$

particularly for all $\delta > 0$, it exists $N_0 \in \mathbf{N}$ such that if $N \geq N_0$, then the cardinal of each orbit of f_N is smaller than $\delta \text{Card}(E_N)$.

This corollary can be seen as a discrete analogue of the fact that a generic homeomorphism is totally singular, *i.e.* that there exists a Borel set of full measure whose image under f is zero measure. Again, it reflects the regularity of the behaviour of the discretizations of a dissipative homeomorphism: generically, we can describe the behaviour of *all* (fine enough) discretizations. This is very different from the conservative case, where sometimes $f_N(E_N) = E_N$ and sometimes $\text{Card}(f_N(E_N)) \leq \vartheta(N)$ where $\vartheta : \mathbf{R}_+ \rightarrow \mathbf{R}_+^*$ is a given map that tends to $+\infty$ in $+\infty$. Moreover, we have the following estimation on the maximal invariant set (easily deduced from the discrete shredding lemma):

Corollary 71. *Let $\vartheta : \mathbf{R}_+ \rightarrow \mathbf{R}_+^*$ be a function that tends to $+\infty$ in $+\infty$. Then for a generic homeomorphism $f \in \text{Homeo}(X)$,*

$$\lim_{N \rightarrow +\infty} \text{Card}(\Omega(f_N)) \leq \vartheta(N).$$

It remains to study the behaviour of measures $\mu^{f_N, U}$ (see definition 6). Again, the results are very different from the conservative case: for any open set U , the measures $\mu_{N, U}^f$ tend to a single measure, say μ_U^f .

Theorem 72. *For a generic homeomorphism $f \in \text{Homeo}(X)$ and an open subset U of X , the measure μ_U^f is well defined¹⁸ and is supported by the closure of the set of sinks A_0 . Moreover the measures $\mu_{N, U}^f$ tend weakly to μ_U^f .*

Sketch of proof of theorem 72. The proof of this theorem is based on the shredding lemma: the set of homeomorphisms which satisfy the conclusions of the lemma is a G_δ dense, so it suffices to prove that such homeomorphisms f satisfy the conclusion of the proposition. Let U be an open subset of X and $\varphi : X \rightarrow \mathbf{R}$ be a continuous function.

We want to show that on one hand the integral $\int_X \varphi d\mu_U^f$ is well defined, *i.e.* that the Birkhoff limits for the function φ

$$\lim_{m \rightarrow +\infty} \frac{1}{m} \sum_{i=0}^{m-1} \varphi(f^i(x))$$

are well defined for a.e. $x \in U$; and on the other hand we have the convergence

$$\int_X \varphi d\mu_{N, U}^f \xrightarrow{N \rightarrow +\infty} \int_X \varphi d\mu_U^f,$$

For the first step, the idea of the proof is that most of the points (for λ) go in a $W_{j,i}$. Since the iterates of the sets $W_{j,i}$ have small diameter, by uniform continuity, the function φ is almost constant on the sets $f^m(W_{j,i})$. Thus the measure μ_x^f is well defined and almost constant on the set of points which goes in $W_{j,i}$. And by the same construction, since the dynamics of f_N converge to that of f , and in particular that the sets U_j^N and $\{w_{j,i}^N\}$ converge to the sets U_j and $W_{j,i}$, the measures $\mu_{N, U}^f$ tend to the measures μ_U^f . \square

¹⁸In other words a generic homeomorphism is weird, see also [AA12].

Part 3. Numerical simulations

12. CONSERVATIVE HOMEOMORPHISMS

Now we present the results of numerical simulations of conservative homeomorphisms we conducted. We sought whether it is possible to observe behaviours such as those obtained in section 6 or that described by theorem 55 on actual simulations: it is not clear that the orders of discretization described by these results can be reached in practice, or if simple examples of homeomorphisms behave the same way as generic homeomorphisms.

Recall that we simulate homeomorphisms $f(x, y) = P \circ Q \circ P(x, y)$, where P and Q are both homeomorphisms of the torus that modify only one coordinate:

$$P(x, y) = (x, y + p(x, y)) \quad \text{and} \quad Q(x, y) = (x + q(x, y), y),$$

and that we discretize according to the uniform grids on the torus:

$$E_N = \left\{ \left(\frac{i_1}{N}, \dots, \frac{i_n}{N} \right) \in \mathbf{T}^n \mid \forall j, 0 \leq i_j \leq N - 1 \right\}.$$

To obtain conservative homeomorphisms, we choose $p(x, y) = p(x)$ and $q(x, y) = q(y)$. We perform simulations of two conservative homeomorphisms which are small perturbations of identity or of the standard Anosov automorphism $A : (x, y) \mapsto (x + y, x + 2y)$:

- To begin with we study $f_1 = P \circ Q \circ P$, with

$$p(x) = \frac{1}{259} \cos(2\pi \times 227x) + \frac{1}{271} \sin(2\pi \times 253x),$$

$$q(y) = \frac{1}{287} \cos(2\pi \times 241y) + \frac{1}{263} \sin(2\pi \times 217y).$$

This conservative homeomorphism is a small C^0 perturbation of identity. Experience shows that even dynamical systems with fairly simple definitions behave chaotically (see e.g. [GT83]). We can expect that a homeomorphism such as f_1 has a complex dynamical behaviour and even behaves essentially as a generic homeomorphism, at least for small orders of discretization. Note that we choose coefficients that have (virtually) no common divisors to avoid arithmetical phenomena such as periodicity or resonance.

- We also simulate f_2 the composition of f_1 with the standard Anosov automorphism A , say $f_2 = P \circ Q \circ P \circ A$. This makes it a small C^0 -perturbation of A . Thus f_2 is semi-conjugated to A but not conjugated: to each periodic orbit of A corresponds many periodic orbits of f_2 . As for f_1 , we define f_2 in the hope that the behaviour of its discretizations is fairly close to that of discretizations of a generic homeomorphism.

Note that the homeomorphisms f_1 and f_2 have at least one fixed point (for f_1 , simply make simultaneously $p(x)$ and $q(y)$ equal to 0; for f_2 , note that 0 is a persistent fixed point of A). Thus, the theoretical results indicate that for a generic homeomorphism f which has a fixed point, infinitely many discretizations has a unique fixed point; moreover a subsequence of $(\mu_N^f)_N$ tends to an invariant measure under f supported by a fixed point. We can test if this is true on simulations.

From a practical point of view, we restricted ourselves to grids of sizes smaller than $2^{15} \times 2^{15}$: the initial data become quickly very large, and the algorithm creates temporary variables that are of size of the order of five times the size of the initial data. For example, for a grid $2^{15} \times 2^{15}$, the algorithm takes between 25 and 30 Go of RAM on the machine.

12.1. Combinatorial behaviour. As a first step, we are interested in some quantities related to the combinatorial behaviour of discretizations of homeomorphisms. These quantities are:

- the cardinal of the maximal invariant set $\Omega(f_N)$,
- the number of periodic orbits of the map f_N ,
- the maximal size of a periodic orbit of f_N .

Recall that according to corollaries 22, 27 and 39, for a generic homeomorphism, the ratio $\frac{\text{Card}(\Omega(f_N))}{q_N}$ must fluctuate between 0 and 1 depending on N , the number of orbits must fluctuate between 1 and (e.g.) $\sqrt{q_N}$ and the maximal size of a periodic orbit must fluctuate between 1 and q_N .

We calculated these quantities for discretizations of orders $128k$, for k from 1 to 100 and have represented them graphically. For information, if $N = 128 \times 100$, then $q_N \simeq 1.6 \cdot 10^8$.

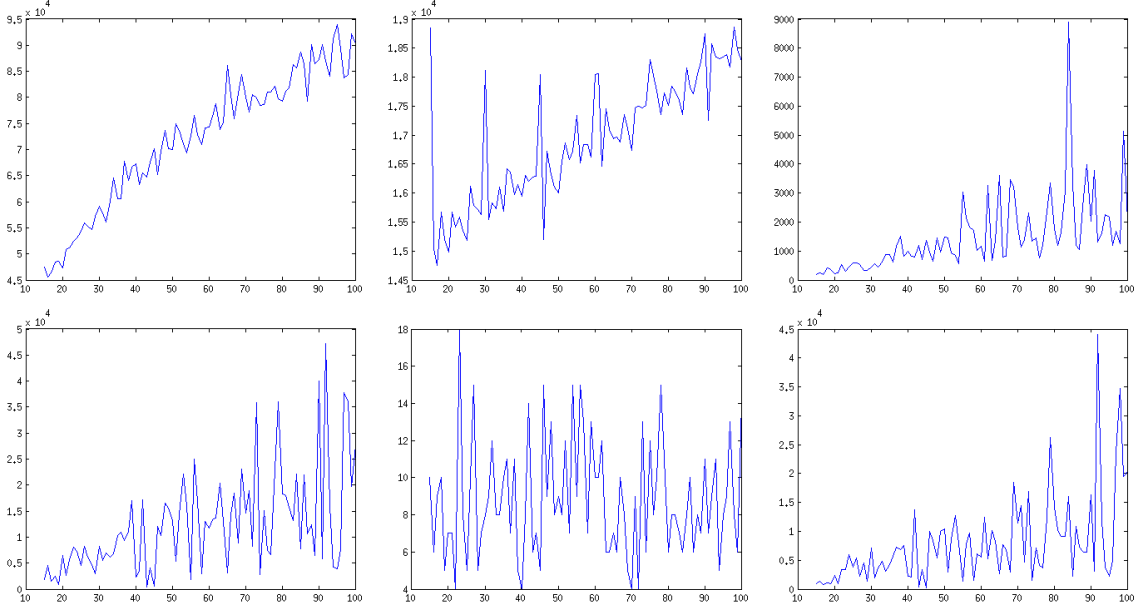


FIGURE 6. Size of the maximal invariant set $\Omega((f_i)_N)$ (left), number of periodic orbits of $(f_i)_N$ (middle) and length of the largest periodic orbit of $(f_i)_N$ (right) depending on N , for f_1 (top) and f_2 (bottom), on the grids E_N with $N = 128k$, $k = 15, \dots, 100$

We begin with the cardinal of the maximal invariant set $\Omega(f_N)$ (figure 6). Contrary to what the theoretical results provide for a generic homeomorphism, for all simulations the ratio $\frac{\text{Card}(\Omega(f_N))}{q_N}$ tends to 0 when N increases. More specifically, the cardinal of $\Omega(f_N)$ evolves much more regularly for f_1 than for f_2 : for f_1 the value of this cardinal seems to be the sum of a smooth increasing function and a random noise, but for f_2 this value seems to be the product of a smooth increasing function with a random noise. We have no explanation to the parabolic shape of the curve for f_1 : it reflects the fact that the cardinal of $\Omega((f_1)_N)$ evolves in the same way as $\sqrt[4]{q_N}$ (whereas for a random map of a finite set with q elements into itself it evolves the same way as \sqrt{q}). Finally, it is interesting to note that for f_2 , the size of the maximal invariant set is distributed more or less around the size of the maximal invariant set of a random map of a set with q_N elements into itself, which depends (asymptotically) linearly of N and is worth about 16,000 for $N = 128 \times 100$ (see [Bol01]).

According to the results of the previous sections, for a generic homeomorphism f , the number of periodic orbits of f_N must fluctuate between 1 and (e.g.) $\sqrt{q_N}$. It is clearly not the case for the simulations. In addition, its behaviour is not the same for f_1 and for f_2 (figure 6): for f_1 the number of orbits reaches quickly a value around $2 \cdot 10^4$ to increase slightly thereafter, while for f_2 it oscillates between 1 and 18, regardless the order of discretization. These graphics can be compared with those representing the size of the maximal invariant set $\Omega((f_i)_N)$: the number of periodic orbits and the size of the maximal invariant set are of the same order of magnitude for f_1 (up to a factor 5), which means that the average period of a periodic orbit is small (which is not surprising, since f_1 is a small perturbation of identity). In contrast, they differ by a factor roughly equal to 10^3 for f_2 , which means that this time the average period of a periodic orbit is very large. This can be

explained partly by the fact that the standard Anosov automorphism tends to mix what happens in the neighborhood of identity. Fact remains that these simulations (such as the size of the maximal invariant set $\Omega((f_i)_N)$) suggest that the behaviour in the neighborhood of identity and of the standard Anosov automorphism are quite different, at least for small orders of discretization.

Regarding the maximum size of a periodic orbit of f_N (figure 6), again its behaviour does not correspond to that of a generic homeomorphism: it should oscillate between 1 and q_N , which is not the case. However, it varies widely depending on N , especially when N is large. The qualitative behaviours are similar for both simulations, but there are some quantitative differences: the maximum of the maximal length of a periodic orbit is greater for f_2 than for f_1 .

12.2. Behaviour of invariant measures. We also simulated the measure $\mu_N^{f_i}$ for the two examples of conservative homeomorphisms f_1 and f_2 as defined in page 31. Our purpose is to test whether phenomena as described by theorem 55 can be observed in practice or not. We obviously can not expect to see the sequence $(\mu_N^{f_i})_{N \in \mathbb{N}}$ accumulating on *all* the invariant probability measures of f , since these measures generally form an infinite-dimensional convex set, but we can still test if it seems to converge or not.

We present images of sizes 128×128 pixels representing in logarithmic scale the density of the measures $\mu_N^{f_i}$: each pixel is coloured according to the measure carried by the set of points of E_N it covers. The blue corresponds to a pixel with very small measure and the red to a pixel with very high measure. Scales on the right of each image corresponds to the measure of one pixel in logarithmic scale to base 10: if the green codes -3 , then a green pixel will have measure 10^{-3} for $\mu_N^{f_i}$. For information, when Lebesgue measure is represented, all the pixels have a value about -4.2 .

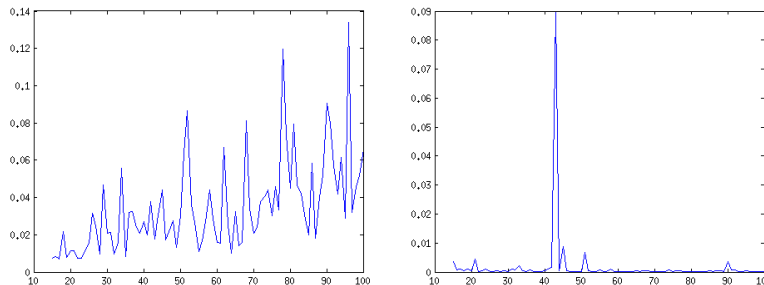


FIGURE 7. Maximal value of the measure μ_N on E_N depending on N for f_1 (left) and f_2 (right), on the grids E_N with $N = 128k$, $k = 15, \dots, 100$

Theoretically, the maximal value of the measure μ_N (figure 7) should oscillate between $1/q_N$ and 1, but this is not the case for these examples. Again the behaviour is quite different for f_1 and for f_2 : for f_1 it seems that the values of μ_N are well distributed between null function and a linear function of N while for f_2 we observe peaks: there are a few values of N for which μ_N is much higher than elsewhere. For f_1 the maximal value of μ_N seems globally grow with N , but with a quite irregular behaviour, apparently confirming theorem 55. Finally, we note that the maximum value of μ_N is much lower for f_2 than for f_1 .

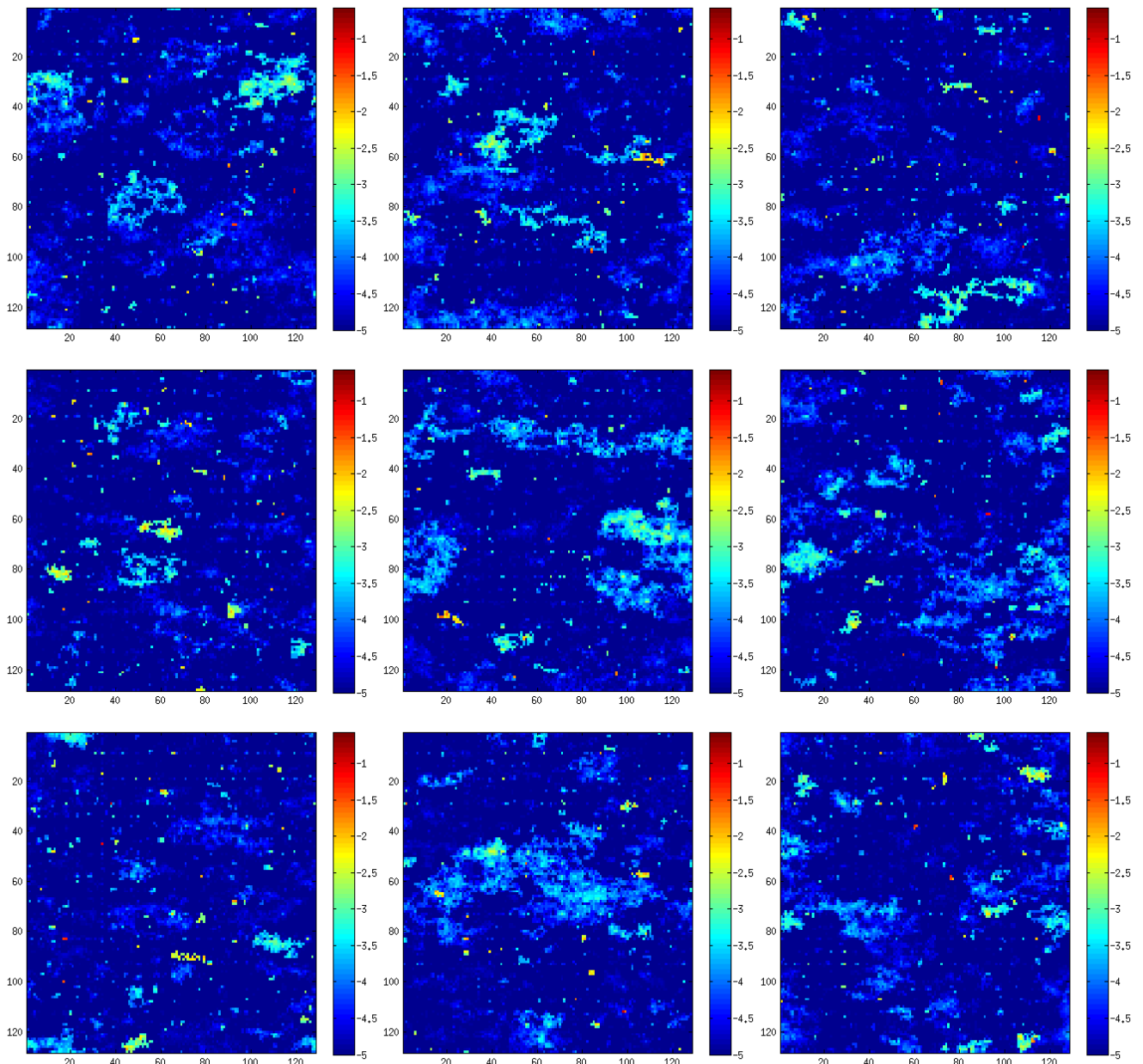


FIGURE 8. Simulations of invariant measures $\mu_N^{f_1}$ on grids of size $N \times N$, with $N = 20000, \dots, 20008$ (from left to right and top to bottom)

The results of simulations of invariant measures of discretizations of f_1 (which is a C^0 conservative perturbation of identity) are quite positive: they agree with theoretical results about discretizations of generic conservative homeomorphisms, in particular with theorem 55. When f_1 is discretized, we observe that the measure is first fairly well distributed. When the step of the discretization improves we can observe places where the measure accumulates; moreover these places changes a lot when the orders of discretizations varies (see figure 8). This fairly agrees with what happens in the C^0 generic case, where the measure μ_N^f depends very much on the order of discretization rather than on the homeomorphism itself. There is also an other phenomenon: when the size of the grid is large enough (around $10^{12} \times 10^{12}$), it appears areas uniformly charged by the measure μ_N^f ; their sizes seems to be inversely proportional to the common mass of the pixels of the area.

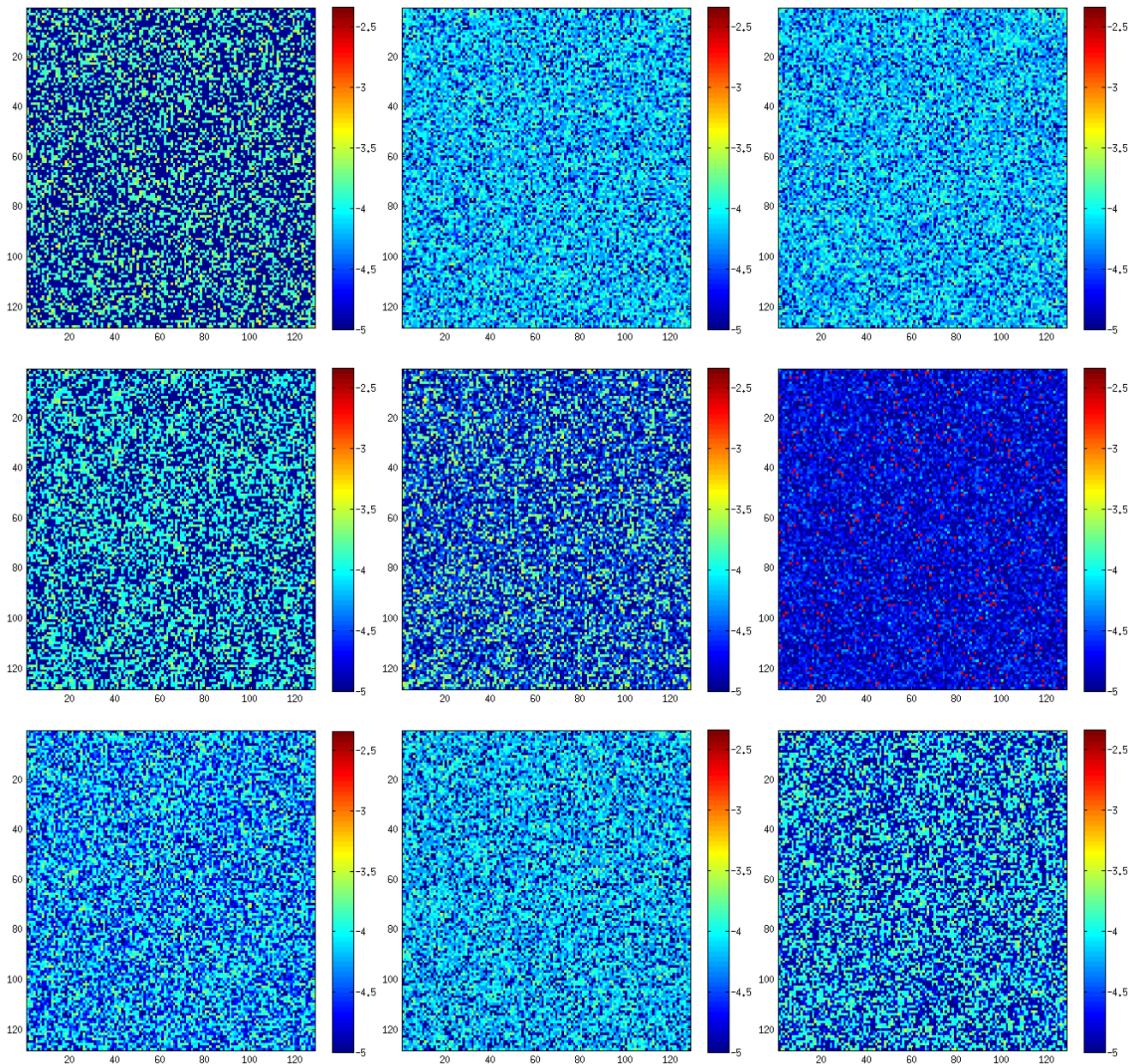


FIGURE 9. Simulations of invariant measures $\mu_N^{f_2}$ on grids of size $N \times N$, with $N = 20005, \dots, 20013$ (from left to right and top to bottom)

For the discretizations of f_2 , which is a C^0 conservative perturbation of the standard Anosov automorphism, the simulations on grids of size $2^k \times 2^k$ might suggest that the measures $\mu_N^{f_2}$ tend to Lebesgue measure. In fact, making a large number of simulations, we realize that there is also strong variations of the behaviour of measures (figure 9): the measure is often well distributed in the torus and sometimes quite singular with respect to Lebesgue measure (as in figure 7). For example when we discretize on the grid of size 20010×20010 (middle right of figure 9), one observe an orbit with length 369 which mass 84% of the total measure. In fact the behaviour of discretizations looks the same that in the neighbourhood of identity, modulo the fact that the standard Anosov automorphism tends to spread the attractive periodic orbits of discretizations on the entire torus: for many values of N composing by A spreads the behaviour of the measure $\mu_N^{f_2}$, but sometimes (in fact seldom) a fixed point of $(f_1)_N$ which attracts a large part of E_N is located around one of the few periodic points of small period for A . This then creates a periodic orbit for $(f_2)_N$ with a big measure for $\mu_N^{f_2}$.

13. DISSIPATIVE HOMEOMORPHISMS

Let us finish by presenting the results of some numerical simulations of dissipative homeomorphisms. Again, our aim is to compare the theoretical results with the reality of numerical simulations: for simple homeomorphisms and reasonable orders of discretization does one have convergence of the dynamics of discretizations to that of the homeomorphism, as suggested by the above theorems?

Recall that we simulate homeomorphisms such that

$$f(x, y) = Q \circ P(x, y) \quad \text{or} \quad f(x, y) = P \circ Q \circ P(x, y),$$

where P and Q are two homeomorphisms of the torus that modify only one coordinate:

$$P(x, y) = (x, y + p(x, y)) \quad \text{and} \quad Q(x, y) = (x + q(x, y), y).$$

Unlike the conservative case, p and q depends on both x and y . Again, we tested two homeomorphisms:

- To begin with we studied $f_3 = Q \circ P$, with

$$p(x, y) = \frac{1}{259} \cos(2\pi \times 227y) + \frac{1}{271} \sin(2\pi \times 233x),$$

$$q(x, y) = \frac{1}{287} \cos(2\pi \times 241y) + \frac{1}{263} \sin(2\pi \times 217y) + \frac{1}{263} \cos(2\pi \times 271x)$$

This dissipative homeomorphism is a small C^0 perturbation of identity, whose derivative has many oscillations whose amplitudes are close to 1. That creates many fixed points which are sinks, sources or saddles.

- It has also seemed useful to us to simulate a homeomorphism close to the identity in C^0 topology, but with a small number of sinks. Indeed, as explained heuristically by J.-M. Gambaudo and C. Tresser in [GT83], a homeomorphism such that f_3 can have a large number of sinks whose basins of attraction are small. It turns out that the dissipative behaviour of f_3 can not be detected in orders discretization that can be achieved in practice. We therefore defined another homeomorphism close to the identity in C^0 topology, but with much less sinks, say $f_4 = P \circ Q \circ P$, with

$$p(x, y) = \frac{1}{259} \operatorname{th}(50 \cos(2\pi \times y)) + \frac{1}{271} \operatorname{th}(50 \cos(2\pi \times 5x)),$$

$$q(x, y) = \frac{1}{287} \operatorname{th}(50 \cos(2\pi \times y)) + \frac{1}{263} \operatorname{th}(50 \cos(2\pi \times 7y)) + \frac{1}{263} \operatorname{th}(50 \cos(2\pi \times 3x)).$$

13.1. Combinatorial behaviour. We simulated some quantities related to the combinatorial behaviour of discretizations of homeomorphisms, namely:

- the cardinal of the maximal invariant set $\Omega(f_N)$,
- the number of periodic orbits of f_N ,
- the maximal size of a periodic orbit of f_N .

We calculated these quantities for discretizations of orders $128k$ for k from 1 to 100 and represented it graphically. For information, if $N = 128 \times 100$, then $q_N \simeq 1.6 \cdot 10^8$.

Theoretically, the ratio between the cardinal of $\Omega(f_N)$ and q_N must tend to 0, this is what can be seen in simulations. This is not really surprising: we already observed that for discretizations of conservative homeomorphisms. In this context it is interesting to compare the behaviour of $\Omega(f_N)$ in the conservative and the dissipative case. The result is a bit disappointing: for f_3 , the graphics in the case of conservative and dissipative homeomorphisms are as alike as two peas in a pod, while in theory they should be very different. Still, for f_4 , there are very few sinks, and around each one there are just a few attractive points of the discretization (each sink is “sharp”) so that the cardinal of $\Omega((f_4)_N)$ is more or less constant. For this case, the behaviour of discretizations seems clearly typical of the dissipative case.

The behaviour of the number of periodic orbits of f_N is not provided by the theoretical study. However, we can hope that it reflects the fact that the dynamics converges to that of the initial homeomorphism: among others, we can test if it is of the same order as

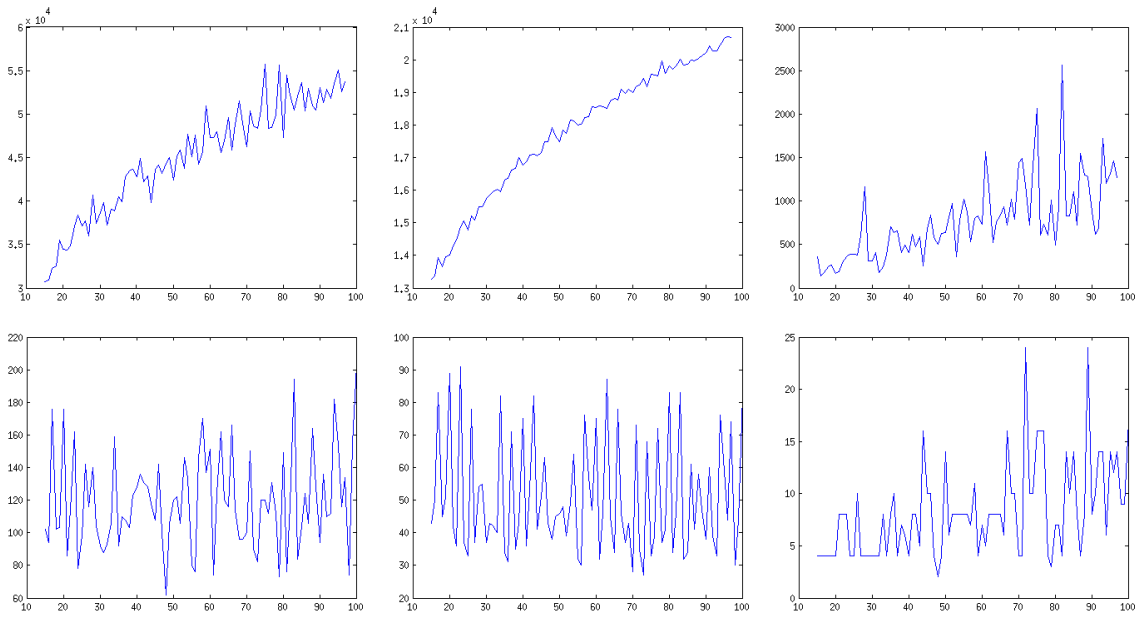


FIGURE 10. Size of the maximal invariant set $\Omega((f_i)_N)$ (left), number of periodic orbits of $(f_i)_N$ (middle) and length of the largest periodic orbit of $(f_i)_N$ (right) depending on N , for f_3 (top) and f_4 (bottom), on the grids E_N with $N = 128k$, $k = 15, \dots, 100$

the number of sinks of the homeomorphism. In practice, its behaviour is similar to that of the cardinal of the maximal invariant set $\Omega(f_N)$: its evolution is very regular for f_3 but oscillates (from $N = 15$) between 30 and 90 for f_4 . As in the conservative case, the comparison with figure 10 indicates that the average length of a periodic cycle is about 3 for f_3 and f_4 .

Since the dynamics of discretizations is assumed to converge to that of the initial homeomorphism, one could expect that the length of the longest periodic orbit of discretizations $(f_i)_N$ is almost always a multiple of that of an attractive periodic orbit of f_i . The graphic of this length for f_3 looks like the conservative case, so one can say that the dissipative behaviour of this homeomorphism is not detected for reasonable orders of discretizations. Despite this, the values of the lengths of the longest periodic orbit are much smaller (up to a factor 10) than in the conservative case. It is not obvious if it is more a dissipative effect than a coincidence. For F_4 , the length of the longest orbit is smaller than 25, which is normal because we are supposed to detect the sinks of the homeomorphism. The changes of the length of the longest orbit are probably due to errors of discretization if the neighborhood of the sinks of f_4 . Furthermore, we note that some values of the size of the longest orbit are more achieved: for example for 30 different integers N the longest periodic orbit has length 4, for 19 different integers N this length is 8 and for 10 different integers N it is 10. This is certainly due to the shadowing property of periodic orbits of f_4 by those of the discretizations.

13.2. Behaviour of invariant measures. As in the conservative case we calculated the invariant measures $\mu_N^{f_i}$ of dissipative homeomorphisms f_3 and f_4 as defined on page 36. Our aim is to test whether theorem 72 apply in practice or if there are technical constraints such that this behaviour can not be observed on these examples. For a presentation of the representations of the measures, see page 33.

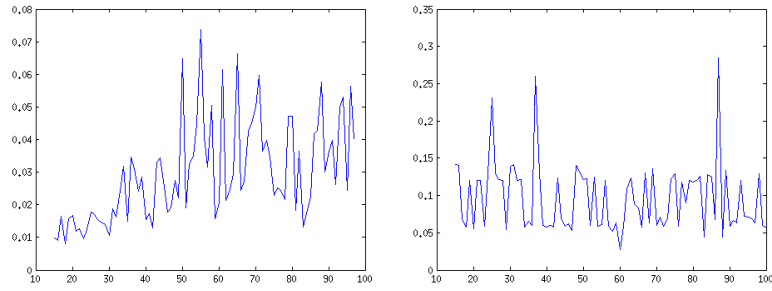


FIGURE 11. Maximal value of μ_N on E_N depending on N for f_3 (left) and f_4 (right), on the grids E_N with $N = 128k$, $k = 15, \dots, 100$

There is no clear theoretical results concerning the behaviour of the maximal value of the measure μ_N^f (figure 11), but we can at least observe that a convergence of this amount would be a good indication that the considered homeomorphism is dissipative. As in the conservative case (in the neighbourhood of identity), the behaviour of $\mu_N^{f_3}$ seems very chaotic. While $\mu_N^{f_3}$ is always smaller than 0.07, the maximum value of $\mu_N^{f_4}$ is very high and fluctuates between 0.02 and 0.3, with many values around 0.06 and 0.12 and some values around 0.25. The obvious arithmetic relationships between these values can be explained by the fact that some sinks of discretizations merge for some values of N . Thus, there are orbits which attract many points; it seems to reflect the typical behaviour of generic homeomorphisms.

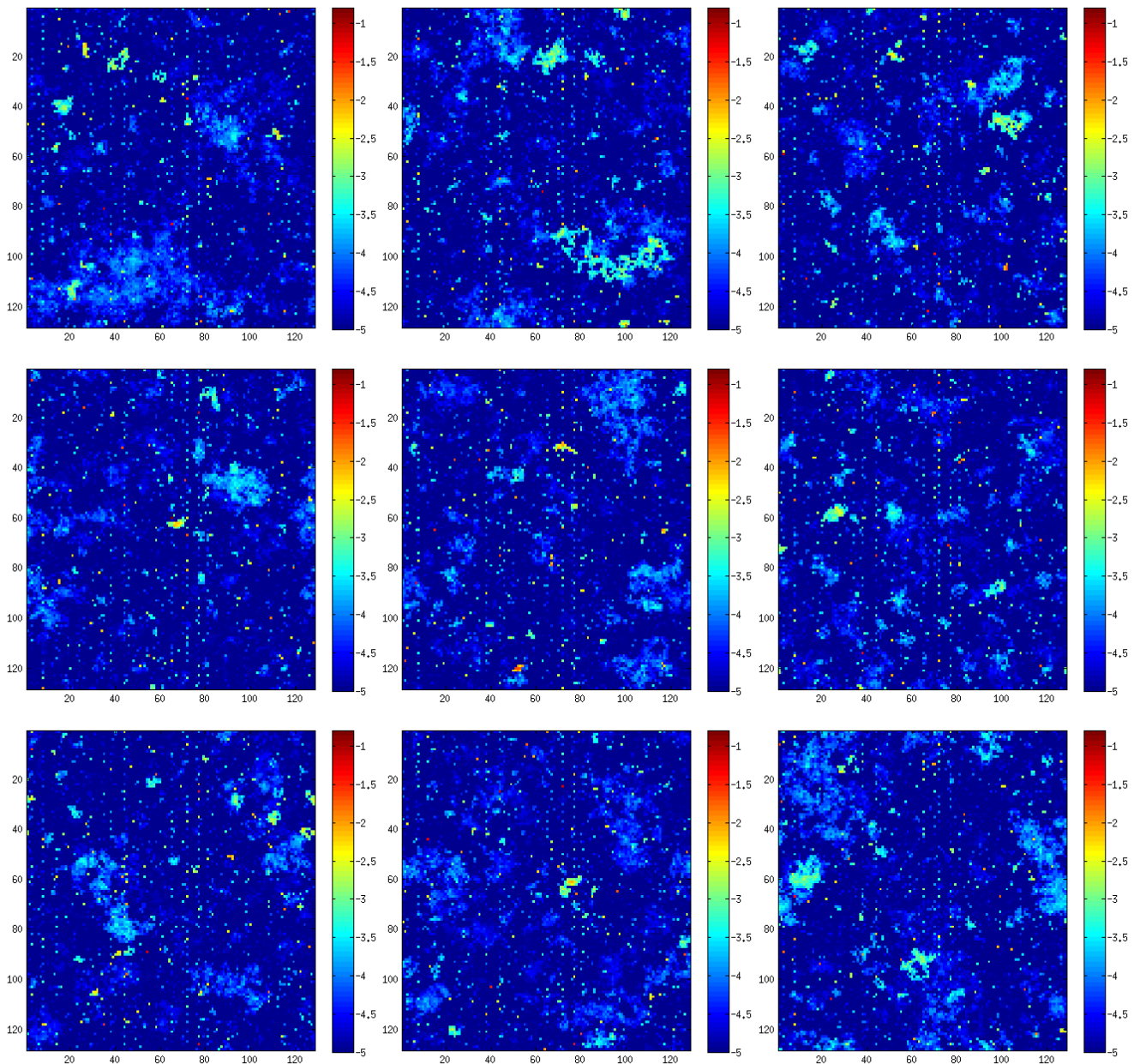


FIGURE 12. Simulations of $\mu_N^{f_3}$ on grids E_N , with $N = 20000, \dots, 20008$ (from left to right and top to bottom)

The behaviour of invariant measures for f_3 , which is a small C^0 dissipative perturbation of identity, is relatively similar to that of invariant measures for f_1 i.e. the corresponding conservative case: when the order discretization is large enough, there is a strong variation of the measure $\mu_N^{f_3}$. Moreover this measure has a significant absolutely continuous component with respect to Lebesgue measure. Nevertheless, there are differences with the conservative case: the maximum value of $\mu_N^{f_3}$ is very large, much more than for f_1 , and there are orbits which attract many points. On the other hand, one can observe the following phenomenon (we hope that it is anecdotic): these orbits are located on vertical stripes. This is probably related to specific arithmetic phenomena due to the specific form of the homeomorphism f_3 .

It is normal not to be able to observe all the sinks of f_3 on these simulations: as noted by J.-M. Gambaudo and C. Tresser in [GT83], the size of these sinks can be very small compared to the numbers involved in the definition of f_3 . So even in orders discretization such as 2^{15} , the dissipative nature of the homeomorphism can not be detected on discretizations. For this reason, it looks as if the discretizations of f_3 are very similar to those of f_1 .

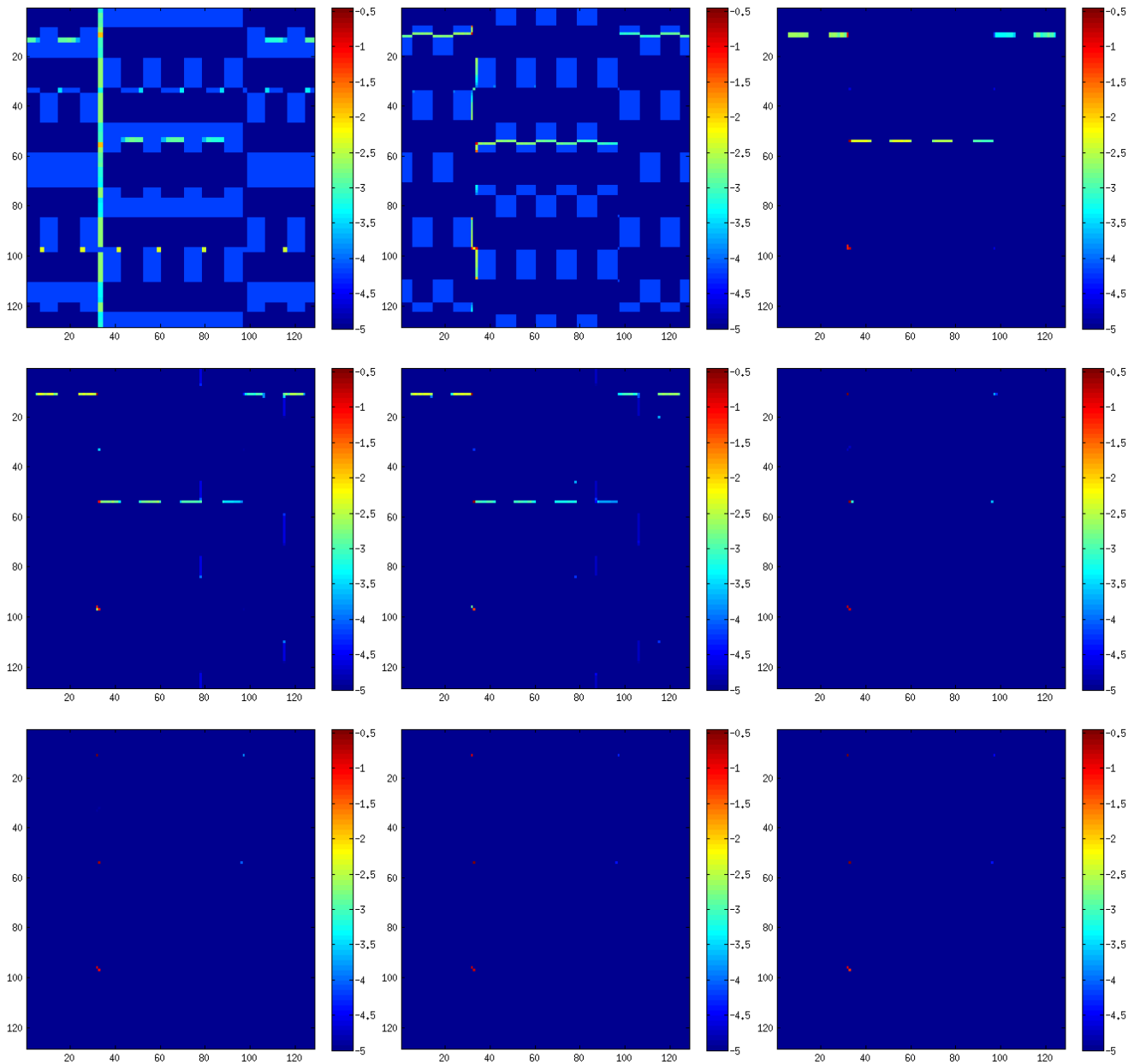


FIGURE 13. Simulations of $\mu_N^{f_4}$ on the grids E_N , with $N = 2^k$ and $k = 6, \dots, 14$ (from left to right and top to bottom)

Recall that what happens for f_3 is pretty close to what happens for f_1 . For their part, simulations of invariant measures for f_4 on grids of size $2^k \times 2^k$ highlight that we expect from a generic dissipative homeomorphism: the measures $\mu_N^{f_4}$ tend quickly to a single measure (that is also observed on a series of simulations), which is carried by the sinks of f_4 . The fact that f_4 has few sinks allows the discretizations of reasonable orders (typically 2^{11}) to find the actual sinks of the initial homeomorphism, contrary to what we had observed for f_3 .

Acknowledgments. I really would like to thank François Béguin for the great attention he paid to this work.

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