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# Approximate Lipschitz stability for non-overdetermined inverse scattering at fixed energy 

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Abstract. We prove approximate Lipschitz stability for non-overdetermined inverse scattering at fixed energy with incomplete data in dimension $d \geq 2$. Our estimates are given in uniform norm for coefficient difference and related stability precision efficiently increases with increasing energy and coefficient difference regularity. In addition, our estimates are rather optimal even in the Born approximation.

## 1. Introduction

We consider the Schrödinger equation

$$
\begin{equation*}
-\Delta \psi+v(x) \psi=E \psi, \quad x \in \mathbb{R}^{d}, d \geq 2, E>0 \tag{1.1}
\end{equation*}
$$

where

$$
\begin{equation*}
v \text { is real - valued, } \quad v \in L_{\sigma}^{\infty}\left(\mathbb{R}^{d}\right) \text { for some } \sigma>d, \tag{1.2}
\end{equation*}
$$

where

$$
\begin{align*}
& L_{\sigma}^{\infty}\left(\mathbb{R}^{d}\right)=\left\{u \in L^{\infty}\left(\mathbb{R}^{d}\right):\|u\|_{\sigma}<+\infty\right\} \\
& \|u\|_{\sigma}=\text { ess } \sup _{-\infty}(1+|x|)^{\sigma}|u(x)|, \sigma \geq 0 \tag{1.3}
\end{align*}
$$

For equation (1.1) we consider the scattering amplitude $f$ on $\mathcal{M}_{E}$,

$$
\begin{equation*}
\mathcal{M}_{E}=\left\{k \in \mathbb{R}^{d}, l \in \mathbb{R}^{d}: k^{2}=l^{2}=E\right\}, E>0 \tag{1.4}
\end{equation*}
$$

For definitions of the scattering amplitude, see formula (1.5) below and, for example, reviews given in [F2], [FM]. The scattering amplitude $f$ arises, in particular, as a coefficient with scattered spherical wave $e^{i|k||x|} /|x|^{(d-1) / 2}$ in the asymptotics of the wave solutions $\psi^{+}(x, k)$ describing scattering of incident plan wave $e^{i k x}$ for equation (1.1):

$$
\begin{equation*}
\psi^{+}(x, k)=e^{i k x}+c(d,|k|) \frac{e^{i|k||x|}}{|x|^{(d-1) / 2}} f\left(k,|k| \frac{x}{|x|}\right)+o\left(\frac{1}{|x|^{(d-1) / 2}}\right) \quad \text { as } \quad|x| \rightarrow \infty, \tag{1.5}
\end{equation*}
$$

where $x \in \mathbb{R}^{d}, k \in \mathbb{R}^{d}, k^{2}=E, c(d,|k|)=-\pi i(-2 \pi i)^{(d-1) / 2}|k|^{(d-3) / 2}$.
Given $v$, to determine $f$ one can use, in particular, the Lippmann-Schwinger integral equation

$$
\begin{align*}
\psi^{+}(x, k) & =e^{i k x}+\int_{\mathbb{R}^{d}} G^{+}(x-y, k) v(y) \psi^{+}(y, k) d y \\
G^{+}(x, k) & =-\left(\frac{1}{2 \pi}\right)^{d} \int_{\mathbb{R}^{d}} \frac{e^{i \xi x} d \xi}{\xi^{2}-k^{2}-i 0} \tag{1.6}
\end{align*}
$$

and the formula

$$
\begin{equation*}
f(k, l)=\left(\frac{1}{2 \pi}\right)^{d} \int_{\mathbb{R}^{d}} e^{-i l y} v(y) \psi^{+}(y, k) d y \tag{1.7}
\end{equation*}
$$

where $x, k, l \in \mathbb{R}^{d}, k^{2}=l^{2}=E$; see, for example, $[\mathrm{BS}]$, $[\mathrm{F} 2]$.
In the present work, in addition to $f$ on $\mathcal{M}_{E}$, we consider $\left.f\right|_{\Gamma_{E}}$ and $\left.f\right|_{\Gamma_{E}^{\tau}}$, where

$$
\begin{gather*}
\Gamma_{E}=\left\{k=\frac{p}{2}+\eta_{E}(p), l=-\frac{p}{2}+\eta_{E}(p): p \in \mathcal{B}_{2 \sqrt{E}}\right\} \\
\Gamma_{E}^{\tau}=\left\{k=\frac{p}{2}+\eta_{E}(p), l=-\frac{p}{2}+\eta_{E}(p): p \in \mathcal{B}_{2 \tau \sqrt{E}}\right\}  \tag{1.8}\\
\mathcal{B}_{r}=\left\{p \in \mathbb{R}^{d}:|p| \leq r\right\}, \quad r>0 \tag{1.9}
\end{gather*}
$$

where $E>0, \tau \in] 0,1], \eta_{E}$ is a piecewise continuous vector-function of $p \in \mathcal{B}_{2 \sqrt{E}}$ such that

$$
\begin{equation*}
\eta_{E}(p) p=0, \quad \frac{p^{2}}{4}+\left(\eta_{E}(p)\right)^{2}=E, \quad p \in \mathcal{B}_{2 \sqrt{E}} \tag{1.10}
\end{equation*}
$$

Note that

$$
\begin{align*}
& \Gamma_{E}^{\tau} \subseteq \Gamma_{E} \subset \mathcal{M}_{E}  \tag{1.11}\\
& \operatorname{dim} \mathcal{M}_{E}=2 d-2, \quad \operatorname{dim} \Gamma_{E}^{\tau}=\operatorname{dim} \Gamma_{E}=d,
\end{align*}
$$

where $E>0, \tau \in] 0,1], d \geq 2$.
We consider the following inverse scattering problems for equation (1.1) under assumptions (1.2):

Problem 1.1. Given $f$ on $\mathcal{M}_{E}$ at fixed $E>0$, find $v$ on $\mathbb{R}^{d}$ (at least approximately).
Problem 1.2. Given $f$ on $\Gamma_{E}^{\tau}$ at fixed $\left.\left.E>0, \tau \in\right] 0,1\right]$, find $v$ on $\mathbb{R}^{d}$ (at least approximately).

Using (1.11) one can see that Problem 1.1 is overdetermined for $d \geq 3$, whereas Problem 1.2 is non-overdetermined.

There are many important results on Problem 1.1, see $[A B R],[B],[B A R],[E],[E R 2]$, [F1], [G], [HH], [HN], [I], [IN2], [N1]-[N5], [S1], [VW], [W], [WY] and references therein. On the other hand, to our knowledge, Problem 1.2 was not yet considered explicitly in the literature. Concerning known results for some other non-overdetermined multi-dimensional coefficient inverse problems, see [BK], [ER1], [HN], [K], [N6], [S2] and references therein.

Problems 1.1, 1.2 can be also considered as examples of ill-posed problems; see [BK], [LRS] for an introduction to this theory.

In the present work we obtain approximate Lipschitz stability estimates for Problem 1.2 (with $\tau=\tau(E)=\varepsilon E^{(1-d) /(2 d)}$ for $E \geq 1$ ) in dimension $d \geq 2$, see Theorem 2.1 of Section 2. Our estimates are given in uniform norm for coefficient difference and related stability precision efficiently increases with increasing energy and coefficient difference regularity. In addition, at the end of Section 2, we show that our estimates of Theorem 2.1 are rather optimal even for the case of the Born approximation (that is in the linear

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approximation near zero potential). Our new estimates are much different but coherent with respect to results of [N4], [N5] for Problem 1.1.

## 2. Stability estimates

Let

$$
\begin{align*}
& W^{n, 1}\left(\mathbb{R}^{d}\right)=\left\{u: \partial^{J} u \in L^{1}\left(\mathbb{R}^{d}\right),|J| \leq n\right\} \\
& \|u\|_{n, 1}=\max _{|J| \leq n}\left\|\partial^{J} u\right\|_{L^{1}\left(\mathbb{R}^{d}\right)} \tag{2.1}
\end{align*}
$$

where

$$
J \in(\mathbb{N} \cup 0)^{d},|J|=\sum_{i=1}^{d} J_{i}, \partial^{J} u(x)=\frac{\partial^{|J|} u(x)}{\partial x_{1}^{J_{1}} \ldots \partial x_{d}^{J_{d}}}, n \in(\mathbb{N} \cup 0) .
$$

Let $C\left(\mathcal{M}_{E}\right)$ denote continuous functions on $\mathcal{M}_{E}$ and $C\left(\Gamma_{E}\right), C\left(\Gamma_{E}^{\tau}\right)$ denote the restrictions of $C\left(\mathcal{M}_{E}\right)$ on $\Gamma_{E}$ and $\Gamma_{E}^{\tau}$. Let

$$
\begin{align*}
& \|f\|_{C\left(\Gamma_{E}^{\tau}\right)}=\|f\|_{C\left(\Gamma_{E}^{\tau}\right), 0} \\
& \|f\|_{C\left(\Gamma_{E}^{\tau}\right), \sigma}=\sup _{(k, l) \in \Gamma_{E}^{\tau}}(1+|k-l|)^{\sigma}|f(k, l)|, \tag{2.2}
\end{align*}
$$

where $E>0,0<\tau \leq 1, \sigma \geq 0$.
Let

$$
\begin{equation*}
s_{0}=\frac{n-d}{n}, s_{1}=\frac{n-d}{d}, s_{2}=n-d . \tag{2.3}
\end{equation*}
$$

Theorem 2.1. Let $v_{1}, v_{2} \in L_{\sigma}^{\infty}\left(\mathbb{R}^{d}\right)$ for some $\sigma>d$, $v_{1}-v_{2} \in W^{n, 1}\left(\mathbb{R}^{d}\right)$ for some $n>d$, $\operatorname{supp}\left(v_{1}-v_{2}\right) \subset D$, where $D$ is an open bounded domain in $\mathbb{R}^{d}, d \geq 2$. Let $\left\|v_{j}\right\|_{\sigma} \leq N_{1},\left\|v_{1}-v_{2}\right\|_{n, 1} \leq N_{2}$, where $\|\cdot\|_{\sigma},\|\cdot\|_{n, 1}$ are defined in (1.3), (2.1). Let $f_{1}, f_{2}$ denote the scattering amplitudes for $v_{1}, v_{2}$, respectively. Then:

$$
\begin{align*}
& \left\|v_{1}-v_{2}\right\|_{L^{\infty}(D)} \leq C_{1} \sqrt{E}\left\|f_{1}-f_{2}\right\|_{C\left(\Gamma_{E}^{\tau(E)}\right)}+C_{2}(\sqrt{E})^{-s_{1}}  \tag{2.4}\\
& \left\|v_{1}-v_{2}\right\|_{L^{\infty}(D)} \leq \tilde{C}_{1}\left\|f_{1}-f_{2}\right\|_{C\left(\Gamma_{E}^{\tau(E)}\right), n_{0}}+C_{2}(\sqrt{E})^{-s_{1}}, d<n_{0} \leq n \tag{2.5}
\end{align*}
$$

where $\|\cdot\|_{C\left(\Gamma_{E}^{\tau}\right)},\|\cdot\|_{C\left(\Gamma_{E}^{\tau}\right), n_{0}}$ are defined by (2.2), $\tau(E)=\varepsilon(\sqrt{E})^{(1-d) / d}, \varepsilon=\varepsilon\left(N_{1}, D, \sigma\right)$, $C_{1}=C_{1}\left(N_{1}, D, \sigma\right), C_{2}=C_{2}\left(N_{1}, N_{2}, D, \sigma, n\right), \tilde{C}_{1}=\tilde{C}_{1}\left(N_{1}, D, \sigma, n_{0}\right), s_{1}$ is defined in (2.3), $E \geq 1$.

In Theorem 2.1, $\varepsilon, C_{1}, C_{2}, \tilde{C}_{1}$ denote appropriate positive constants (independent of $E)$. In addition, in particular, $0<\varepsilon \leq 1$.

Theorem 2.1 is proved in Section 4. There is a considerable similarity between this proof and the proof of recent stability estimates of [IN1].

Note that the old approach to inverse scattering at high energies based on formula (3.3) of Section 3 yields estimates like (2.4), (2.5) with $s_{0}$ only instead of $s_{1}$ in the error term. In addition, due to (2.3), we have that

$$
s_{0} \leq 1 \text { even for } n \rightarrow+\infty \text {, whereas } s_{1} \rightarrow+\infty \text { for } n \rightarrow+\infty
$$

In Theorem 2.1, we have that $\tau(E) \rightarrow 0$ as $E \rightarrow+\infty$. Therefore, $\Gamma_{E}^{\tau(E)}$ is a very small part of $\Gamma_{E}^{\tau_{1}}$ for any fixed $\left.\left.\tau_{1} \in\right] 0,1\right]$ for sufficiently high energy $E$. Therefore, estimates (2.4), (2.5) of Theorem 2.1 can be considered as a stability result for Problem 1.2 with incomplete data.

Let

$$
\begin{equation*}
\left.\left.\mathcal{M}_{E}^{\tau}=\left\{(k, l) \in \mathcal{M}_{E}: k-l \in \mathcal{B}_{2 \tau \sqrt{E}}\right\}, E>0, \tau \in\right] 0,1\right] . \tag{2.6}
\end{equation*}
$$

Let

$$
\begin{align*}
& \|f\|_{C\left(\mathcal{M}_{E}^{\tau}\right)}=\|f\|_{C\left(\mathcal{M}_{E}^{\tau}\right), 0} \\
& \|f\|_{C\left(\mathcal{M}_{E}^{\tau}\right), \sigma}=\sup _{(k, l) \in \mathcal{M}_{E}^{\tau}}(1+|k-l|)^{\sigma}|f(k, l)| \tag{2.7}
\end{align*}
$$

where $E>0,0<\tau \leq 1, \sigma \geq 0$.
To our knowledge, estimates (2.4), (2.5) are completely new even with $\left\|f_{1}-f_{2}\right\|_{C\left(\mathcal{M}_{E}^{\tau(E)}\right)},\left\|f_{1}-f_{2}\right\|_{C\left(\mathcal{M}_{E}^{\tau(E)}\right), n_{0}}$ in place of $\left\|f_{1}-f_{2}\right\|_{C\left(\Gamma_{E}^{\tau(E)}\right)},\left\|f_{1}-f_{2}\right\|_{C\left(\Gamma_{E}^{\tau(E)}\right), n_{0}}$ (respectively).

On the other hand, for the case of Problem 1.1 with complete data, estimates (2.4), (2.5) with $\left\|f_{1}-f_{2}\right\|_{C\left(\mathcal{M}_{E}^{1}\right)},\left\|f_{1}-f_{2}\right\|_{C\left(\mathcal{M}_{E}^{1}\right), n_{0}}$ in place of $\left\|f_{1}-f_{2}\right\|_{C\left(\Gamma_{E}^{\tau(E)}\right)},\left\|f_{1}-f_{2}\right\|_{C\left(\Gamma_{E}^{\tau(E)}\right), n_{0}}$ (respectively) are less precise than related results of [N4], [N5] with the error term estimated as $O\left(E^{-s_{2} / 2}\right), E \rightarrow+\infty$, where $s_{2}$ is defined in (2.3).

In addition, for Problem 1.2 with the scattering amplitude $f$ given on $\Gamma_{E}^{\tau(E)}$ only, estimates (2.4), (2.5) are rather optimal even for the case of the Born approximation (that is in the linear approximation near zero potential). We recall that, in the Born approximation,

$$
\begin{equation*}
f(k, l) \approx \hat{v}(k-l), \quad(k, l) \in \mathcal{M}_{E} \tag{2.8}
\end{equation*}
$$

where

$$
\begin{equation*}
\hat{v}(p)=\left(\frac{1}{2 \pi}\right)^{d} \int_{\mathbb{R}^{d}} e^{i p x} v(x) d x, \quad p \in \mathbb{R}^{d} \tag{2.9}
\end{equation*}
$$

Let

$$
\begin{align*}
& \|\hat{v}\|_{C\left(\mathcal{B}_{r}\right)}=\|\hat{v}\|_{C\left(\mathcal{B}_{r}\right), 0} \\
& \|\hat{v}\|_{C\left(\mathcal{B}_{r}\right), \sigma}=\sup _{p \in \mathcal{B}_{r}}(1+|p|)^{\sigma}|\hat{v}(p)| r>0, \sigma \geq 0 . \tag{2.10}
\end{align*}
$$

Born approximation analogs of (2.4), (2.5) can be written as

$$
\begin{align*}
& \left\|v_{1}-v_{2}\right\|_{L^{\infty}(D)} \leq c_{1}(d) \varepsilon^{d} \sqrt{E}\left\|\hat{v}_{1}-\hat{v}_{2}\right\|_{C\left(\mathcal{B}_{2 \varepsilon(\sqrt{E})^{1 / d}}\right)}+  \tag{2.11}\\
& c_{2}(d, n) N_{2} \varepsilon^{-(n-d)}(\sqrt{E})^{-s_{1}}, \\
& \left\|v_{1}-v_{2}\right\|_{L^{\infty}(D)} \leq \tilde{c}_{1}\left(d, n_{0}\right)\left\|\hat{v}_{1}-\hat{v}_{2}\right\|_{C\left(\mathcal{B}_{\left.2 \varepsilon(\sqrt{E})^{1 / d}\right), n_{0}}+\right.}+  \tag{2.12}\\
& c_{2}(d, n) N_{2} \varepsilon^{-(n-d)}(\sqrt{E})^{-s_{1}},
\end{align*}
$$

where $s_{1}, n, n_{0}, d, N_{2}$ are the same that in (2.3)-(2.5), $0<\varepsilon<1, E \geq 1$.

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## 3. Some results of direct scattering

We recall that, under assumptions (1.2), the Lippmann-Schwinger integral equation (1.6) is uniquely solvable for $\psi^{+}(\cdot, k) \in L^{\infty}\left(\mathbb{R}^{d}\right)$ for fixed $k \in \mathbb{R}^{d} \backslash\{0\}$; see [BS], [F2] and references therein.

We recall that the following estimate holds:

$$
\begin{align*}
& \left\|\Lambda^{-s} G^{+}(k) \Lambda^{-s}\right\|_{L^{2}\left(\mathbb{R}^{d}\right) \rightarrow L^{2}\left(\mathbb{R}^{d}\right)}=O\left(|k|^{-1}\right),  \tag{3.1}\\
& \text { as }|k| \rightarrow \infty, k \in \mathbb{R}^{d}, \text { for } s>1 / 2,
\end{align*}
$$

where $G^{+}(k)$ denotes the integral operator with the Schwartz kernel $G^{+}(x-y, k)$ of (1.6), $\Lambda$ denotes the multiplication operator by the function $\left(1+|x|^{2}\right)^{1 / 2}$; see [E], [J] and references therein.

As a corollary of (1.6), (3.1), under assumptions (1.2), we have that

$$
\begin{equation*}
\left\|\Lambda^{-\sigma / 2} \psi^{+}(\cdot, k)-\Lambda^{-\sigma / 2} \psi_{0}^{+}(\cdot, k)\right\|_{L^{2}\left(\mathbb{R}^{d}\right)} \leq a_{1}(d, \sigma)\|v\|_{\sigma}|k|^{-1} \tag{3.2}
\end{equation*}
$$

for $|k| \geq \rho_{1}(d, \sigma)\|v\|_{\sigma}, k \in \mathbb{R}^{d}$, where $\psi_{0}^{+}(x, k)=e^{i k x}$.
As a corollary of (1.7), (3.2), under assumptions (1.2), we have that

$$
\begin{equation*}
|f(k, l)-\hat{v}(k-l)| \leq a_{2}(d, \sigma)\left(\|v\|_{\sigma}\right)^{2}|k|^{-1} \tag{3.3}
\end{equation*}
$$

for $k, l \in \mathbb{R}^{d},|k|=|l| \geq \rho_{1}(d, \sigma)\|v\|_{\sigma}$, where $\hat{v}$ is defined by (2.9).
We recall also that, under assumptions (1.2) for $v=v_{j}, j=1,2$, the following formula holds:

$$
\begin{align*}
& f_{2}(k, l)-f_{1}(k, l)= \\
& \left(\frac{1}{2 \pi}\right)^{d} \int_{\mathbb{R}^{d}} \psi_{1}^{+}(x,-l)\left(v_{2}(x)-v_{1}(x)\right) \psi_{2}^{+}(x, k) d x, k, l \in \mathbb{R}^{d}, k^{2}=l^{2}>0 \tag{3.4}
\end{align*}
$$

where $f_{j}, \psi_{j}^{+}$denote $f$ and $\psi^{+}$for $v=v_{j}, j=1,2$; see $[\mathrm{S} 2]$.
In addition, in the proof of Theorem 2.1 we use, in particular, the following lemma:
Lemma 3.1. Let $v=v_{j}$ satisfy (1.2), $\left\|v_{j}\right\|_{\sigma} \leq N$, where $j=1$, 2. Let $\operatorname{supp}\left(v_{1}-v_{2}\right) \subset D$, where $D$ is an open bounded domain in $\mathbb{R}^{d}$. Then the following estimate holds:

$$
\begin{align*}
& \left|\left(f_{2}(k, l)-f_{1}(k, l)\right)-\left(\hat{v}_{2}(k-l)-\hat{v}_{1}(k-l)\right)\right| \leq \\
& a_{3}(D, \sigma) N\left\|v_{2}-v_{1}\right\|_{L^{\infty}(D)}|k|^{-1} \tag{3.5}
\end{align*}
$$

for $k, l \in \mathbb{R}^{d},|k|=|l| \geq \rho_{1}(d, \sigma) N$.
Lemma 3.1 follows from formula (3.4), estimate (3.2) and the property that $\inf _{x \subset D}\left(1+|x|^{2}\right)^{-\sigma / 4}>0$.

## 4. Proof of Theorem 2.1

We have that

$$
\begin{align*}
& \left\|v_{2}-v_{1}\right\|_{L^{\infty}(D)} \leq \sup _{x \in D}\left|\int_{\mathbb{R}^{d}} e^{-i p x}\left(\hat{v}_{2}(p)-\hat{v}_{1}(p)\right) d p\right| \leq  \tag{4.1}\\
& I_{1}(\kappa)+I_{2}(\kappa) \text { for any } \kappa>0,
\end{align*}
$$

where

$$
\begin{align*}
& I_{1}(\kappa)=\int_{|p| \leq \kappa}\left|\hat{v}_{2}(p)-\hat{v}_{1}(p)\right| d p \\
& I_{2}(\kappa)=\int_{|p| \geq \kappa}\left|\hat{v}_{2}(p)-\hat{v}_{1}(p)\right| d p \tag{4.2}
\end{align*}
$$

Due to Lemma 3.1, we have that

$$
\begin{align*}
& \left|\hat{v}_{2}(p)-\hat{v}_{1}(p)\right| \leq\left|f_{2}\left(k_{E}(p), l_{E}(p)\right)-f_{1}\left(k_{E}(p), l_{E}(p)\right)\right|+ \\
& a_{3}(D, \sigma) N_{1}\left\|v_{2}-v_{1}\right\|_{L^{\infty}(D)}(\sqrt{E})^{-1} \tag{4.3}
\end{align*}
$$

for $p \in \mathcal{B}_{2 \sqrt{E}}, \sqrt{E} \geq \rho_{1}(d, \sigma) N_{1}$, where

$$
\begin{equation*}
k_{E}(p)=\frac{p}{2}+\eta_{E}(p), \quad l_{E}(p)=-\frac{p}{2}+\eta_{E}(p) \tag{4.4}
\end{equation*}
$$

where $\eta_{E}$ is the function of (1.8), (1.10).
Using (4.2), (4.3), (2.2), we obtain that

$$
\begin{gather*}
I_{1}(2 \tau \sqrt{E}) \leq\left|\mathcal{B}_{1}\right|(2 \tau \sqrt{E})^{d}\left(\left\|f_{2}-f_{1}\right\|_{C\left(\Gamma_{E}^{\tau}\right)}+\frac{a_{3}(D, \sigma) N_{1}\left\|v_{2}-v_{1}\right\|_{L^{\infty}(D)}}{\sqrt{E}}\right)  \tag{4.5}\\
I_{1}(2 \tau \sqrt{E}) \leq \frac{\left|\mathbb{S}^{d-1}\right|}{n_{0}-d}\left\|f_{2}-f_{1}\right\|_{C\left(\Gamma_{E}^{\tau}\right), n_{0}}+\left|\mathcal{B}_{1}\right| a_{3}(D, \sigma) N_{1} \frac{(2 \tau \sqrt{E})^{d}}{\sqrt{E}}\left\|v_{2}-v_{1}\right\|_{L^{\infty}(D)} \tag{4.6}
\end{gather*}
$$

for $\left.\left.\sqrt{E} \geq \rho_{1}(d, \sigma) N_{1}, \tau \in\right] 0,1\right]$, where $\left|\mathcal{B}_{1}\right|$ and $\left|\mathbb{S}^{d-1}\right|$ denote standard Euclidean volumes of $\mathcal{B}_{1}$ and $\mathbb{S}^{d-1}$ (respectively), $n_{0}$ is the number of (2.5).

The assumptions that $v_{1}-v_{2} \in W^{n, 1}\left(\mathbb{R}^{d}\right),\left\|v_{1}-v_{2}\right\|_{n, 1} \leq N_{2}$ for some $n>d$, imply that

$$
\begin{equation*}
\left|\hat{v}_{2}(p)-\hat{v}_{1}(p)\right| \leq a_{4}(n, d) N_{2}(1+|p|)^{-n}, \quad p \in \mathbb{R}^{d} . \tag{4.7}
\end{equation*}
$$

Using (4.2), (4.7) we obtain that

$$
\begin{equation*}
I_{2}(2 \tau \sqrt{E}) \leq \frac{\left|\mathbb{S}^{d-1}\right| a_{4}(n, d) N_{2}}{n-d} \frac{1}{(2 \tau \sqrt{E})^{n-d}} \tag{4.8}
\end{equation*}
$$

$\sqrt{E}>0, \tau \in] 0,1]$.

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Let

$$
\begin{align*}
& \tau(E)=\varepsilon(\sqrt{E})^{(1-d) / d} \\
& \varepsilon=\min \left(\frac{1}{2}\left(\frac{1}{2\left|\mathcal{B}_{1}\right| a_{3}(D, \sigma) N_{1}}\right)^{1 / d}, 1\right) . \tag{4.9}
\end{align*}
$$

Due to (4.9), we have, in particular, that

$$
\begin{align*}
& \left|\mathcal{B}_{1}\right| a_{3}(D, \sigma) N_{1}(2 \tau(E) \sqrt{E})^{d}(\sqrt{E})^{-1} \leq \frac{1}{2},  \tag{4.10}\\
& \tau(E) \leq 1, \quad E \geq 1
\end{align*}
$$

Using (4.1), (4.5), (4.6), (4.8), (4.10), we obtain that

$$
\begin{align*}
& \left\|v_{2}-v_{1}\right\|_{L^{\infty}(D)} \leq \frac{\sqrt{E}}{a_{3}(D, \sigma) N_{1}}\left\|f_{2}-f_{1}\right\|_{C\left(\Gamma_{E}^{\tau(E)}\right)}+  \tag{4.11}\\
& \frac{1}{2}\left\|v_{2}-v_{1}\right\|_{L^{\infty}(D)}+\frac{\left|\mathbb{S}^{d-1}\right| a_{4}(n, d) N_{2}}{(n-d)(2 \varepsilon)^{n-d}} \frac{1}{(\sqrt{E})^{(n-d) / d}}, \\
& \left\|v_{2}-v_{1}\right\|_{L^{\infty}(D)} \leq \frac{\left|\mathbb{S}^{d-1}\right|}{n_{0}-d}\left\|f_{2}-f_{1}\right\|_{C\left(\Gamma_{E}^{\tau(E)}\right), n_{0}}+ \\
& \frac{1}{2}\left\|v_{2}-v_{1}\right\|_{L^{\infty}(D)}+\frac{\left|\mathbb{S}^{d-1}\right| a_{4}(n, d) N_{2}}{(n-d)(2 \varepsilon)^{n-d}} \frac{1}{(\sqrt{E})^{(n-d) / d}} \tag{4.12}
\end{align*}
$$

for $\sqrt{E} \geq \max \left(\rho_{1}(d, \sigma) N_{1}, 1\right)$.
Estimates (2.4), (2.5) with

$$
\begin{equation*}
C_{1}=\frac{2}{a_{3}(D, \sigma) N_{1}}, \quad C_{2}=\frac{2\left|\mathbb{S}^{d-1}\right| a_{4}(n, d) N_{2}}{(n-d)(2 \varepsilon)^{n-d}}, \quad \tilde{C}_{1}=\frac{2\left|\mathbb{S}^{d-1}\right|}{n_{0}-d} \tag{4.13}
\end{equation*}
$$

for $\sqrt{E} \geq \max \left(\rho_{1}(d, \sigma) N_{1}, 1\right)$, follow from (4.11), (4.12).
Using also that $\left\|v_{2}-v_{1}\right\|_{L^{\infty}(D)} \leq 2 N_{1}$ we obtain estimates (2.4), (2.5) with

$$
\begin{align*}
C_{1} & =\frac{2}{a_{3}(D, \sigma) N_{1}}, \quad \tilde{C}_{1}=\frac{2\left|\mathbb{S}^{d-1}\right|}{n_{0}-d}  \tag{4.14}\\
C_{2} & =\max \left(\frac{2\left|\mathbb{S}^{d-1}\right| a_{4}(n, d) N_{2}}{(n-d)(2 \varepsilon)^{n-d}}, 2 N_{1}\left(\max \left(\rho_{1}(d, \sigma) N_{1}, 1\right)\right)^{(n-d) / d}\right)
\end{align*}
$$

for $E \geq 1$.
This completes the proof of Theorem 2.1.

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