HAL open science

Multifractal analysis of some multiple ergodic averages

Ai-Hua Fan, Joerg Schmeling, Meng Wu

▶ To cite this version:

Ai-Hua Fan, Joerg Schmeling, Meng Wu. Multifractal analysis of some multiple ergodic averages. 2012. hal-00763948

HAL Id: hal-00763948

https://hal.science/hal-00763948v1

Preprint submitted on 11 Dec 2012

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

MULTIFRACTAL ANALYSIS OF SOME MULTIPLE ERGODIC AVERAGES

AI-HUA FAN, JÖRG SCHMELING AND MENG WU

Abstract. In this paper we study the multiple ergodic averages

$$\frac{1}{n}\sum_{k=1}^{n}\varphi(x_k, x_{kq}, \cdots, x_{kq^{\ell-1}}), \qquad (x_n)\in\Sigma_m$$

on the symbolic space $\Sigma_m = \{0, 1, \cdots, m-1\}^{\mathbb{N}^*}$ where $m \geq 2, \ell \geq 2, q \geq 2$ are integers. We give a complete solution to the problem of multifractal analysis of the limit of the above multiple ergodic averages. Actually we develop a non-invariant and non-linear version of thermodynamic formalism that is of its own interest. We study a large class of measures (called telescopic measures) and the special case of telescopic measures defined by the fixed points of some non-linear transfer operators plays a crucial role in studying our multiplicatively invariant sets. These measures share many properties with Gibbs measures in the classical thermodynamic formalism. Our work also concerns with variational principle, pressure function and Legendre transform in this new setting.

1. Introduction

Let (X, T) be a topological dynamical system where T is a continuous map on a compact metric space X. Fürstenberg had initiated the study of the *multiple ergodic average*:

$$\frac{1}{n}\sum_{k=1}^{n} f_1(T^k x) f_2(T^{2k} x)\cdots f_s(T^{sk} x) \tag{1}$$

where $f_1, \cdots, f_s$ are s continuous functions on X with $s \geq 2$ when he gave a proof of the existence of arithmetic sequences of arbitrary length amongst sets of integers with positive density ([16]). Later on, the research of such a kind of average has attributed a lot of attentions (see e.g. [4, 6, 1, 17]).

The authors in [10] have recently proposed to analyze such multiple ergodic averages from the point of view of multifractal analysis. They have succeeded in a very special case where (X, T) is the shift dynamics on symbolic space and $f_1, \cdots f_s$ are Rademacher functions on the symbolic space viewed as an additive group. It is a challenge to solve the problem in its generality.

Key words and phrases. Multifractal, multiple ergodic average, Hausdorff dimension.

In the present paper, we shall consider the problem for the shift dynamics and for a class of functions $f_1, \cdots, f_s$. The setting is as follows. Let $S = \{0, \cdots, m-1\}$ be a set of m symbols $(m \geq 2)$. Consider the shift map T on the symbolic space $X = \Sigma_m = S^{\mathbb{N}}$. Fix two integers $q \geq 2$ and $\ell \geq 2$. For any given ℓ continuous functions $g_1, g_2, \cdots, g_\ell$ defined on X, we consider the multiple ergodic average

$$A_n(g_1, g_2 \cdots, g_\ell)(x) = \frac{1}{n} \sum_{k=1}^{n} g_1(T^k x) g_2(T^{kq} x) \cdots g_\ell(T^{kq^{\ell-1}} x).$$

This is a special case of (1) with $s = q^{\ell-1}$, $f_{q^j} = g_{j-1}$ and $f_k = 1$ for other $k \neq q^j$. Furthermore we assume that the functions $f_1, f_2 \cdots, f_\ell$ depend only on the first coordinate of $x = (x_k)_{k \geq 0} \in \Sigma_m$. So, under this assumption of f_j's we have

$$A_n(g_1, g_2 \cdots, g_\ell)(x) = \frac{1}{n} \sum_{k=1}^{n} g_1(x_k) g_2(x_{kq}) \cdots g_\ell(x_{kq^{\ell-1}}). \tag{2}$$

For the time being, there is no idea for the multifractal analysis of (1) in its general form. So we are content with investigating the special case (2). Actually we can do a little more. Given a function $\varphi : S^\ell \to \mathbb{R}$ we shall study

$$A_n \varphi(x) = \frac{1}{n} \sum_{k=1}^{n} \varphi(x_k, x_{qk}, \cdots, x_{q^{\ell-1}}). \tag{3}$$

The average in (2) corresponds to the special case of (3) with $\varphi = g_1 \otimes \cdots \otimes g_\ell$. For $\alpha \in \mathbb{R}$, we define

$$E(\alpha) = \left\{ x \in \Sigma_m : \lim_{n \to \infty} A_n \varphi(x) = \alpha \right\}.$$

Our problem is to determine the Hausdorff dimension of $E(\alpha)$. The problem is classical when $\ell = 1$ and the answers are well known (see e.g. [9, 11, 3, 2]). Let

$$\alpha_{\min} = \min_{a_1, \cdots, a_\ell \in S} \varphi(a_1, \cdots, a_\ell), \quad \alpha_{\max} = \max_{a_1, \cdots, a_\ell \in S} \varphi(a_1, \cdots, a_\ell).$$

We assume that $\alpha_{\min} < \alpha_{\max}$ (otherwise φ is constant and the problem is trivial).

Let $\mathcal{F}(S^{\ell-1}, \mathbb{R}^+)$ be the cone of functions defined on $S^{\ell-1}$ taking non-negative real values. For any $s \in \mathbb{R}$, consider the transfer operator $\mathcal{L}_s$ defined on $\mathcal{F}(S^{\ell-1}, \mathbb{R}^+)$ by

$$\mathcal{L}_s \psi(a) = \sum_{j \in S} e^{s\varphi(a,j)} \psi(Ta, j) \tag{4}$$

where $T : S^{\ell-1} \to S^{\ell-2}$ is defined by $T(a_1, \cdots, a_{\ell-1}) = (a_2, \cdots, a_{\ell-1})$. We also consider the non-linear operator $\mathcal{N}_s$ on $\mathcal{F}(S^{\ell-1}, \mathbb{R}^+)$ defined by

$$\mathcal{N}_s \psi(a) = (\mathcal{L}_s \psi(a))^{1/q}.$$

We shall prove that the equation

$$\mathcal{N}_s\psi_s = \psi_s \tag{5}$$

admits a unique strictly positive solution $\psi_s = \psi_s^{(\ell-1)} : S^{\ell-1} \to \mathbb{R}_+^*$ (see Section 4, Theorem 4.1). The function ψ_s is defined on $S^{\ell-1}$. We extend it on S^k for all $1 \le k \le \ell-2$ by induction:

$$\psi_s^{(k)}(a) = \left(\sum_{j\in S} \psi_s^{(k+1)}(a,j)\right)^{\frac{1}{q}}, \quad (a \in S^k). \tag{6}$$

For simplicity, we will simply write $\psi_s(a) = \psi_s^{(k)}(a)$ for $a \in S^k$ with $1 \le k \le \ell-1$. So, $a \mapsto \psi_s(a)$ is not only defined on $S^{\ell-1}$ but on $\bigcup_{1\le k\le \ell-1} S^k$.

Then we define the pressure function by

$$P_\varphi(s) = (q-1)q^{\ell-2}\log\sum_{j\in S}\psi_s(j). \tag{7}$$

Throughout this paper, log means the natural logarithm.

We will prove that $P_\varphi(s)$ is an analytic convex function of $s \in \mathbb{R}$ and even strictly convex since $\alpha_{\min} < \alpha_{\max}$. The Legendre transform of P_φ is defined as

$$P_\varphi^*(\alpha) = \inf_{s\in\mathbb{R}}(-s\alpha + P_\varphi(s)).$$

We denote by L_φ the set of $\alpha \in \mathbb{R}$ such that $E(\alpha) \neq \emptyset$. One of the main results of the paper is stated as follows.

Theorem 1.1. *We have*

$$L_\varphi = [P'_\varphi(-\infty), P'_\varphi(+\infty)].$$

If $\alpha = P'_\varphi(s_\alpha)$ for some $s_\alpha \in \mathbb{R}\cup\{-\infty,+\infty\}$, then $E(\alpha) \neq \emptyset$ and the Hausdorff dimension of $E(\alpha)$ is equal to

$$\dim_H E(\alpha) = \frac{P_\varphi^*(\alpha)}{q^{\ell-1}\log m}.$$

This result was announced for $\ell = 2$ in [13]. It is obvious that $L_\varphi \subset [\alpha_{\min}, \alpha_{\max}]$. In general, this inclusion is strict. In fact, we have the following criterion for $L_\varphi = [\alpha_{\min}, \alpha_{\max}]$.

Theorem 1.2. *We have the equality*

$$P'_\varphi(-\infty) = \alpha_{\min}$$

if and only if there exist an $x = (x_i)_{i=1}^\infty \in \Sigma_m$ such that

$$\forall k \ge 1,\ \varphi(x_k, x_{k+1}, \cdots, x_{k+\ell-1}) = \alpha_{\min}.$$

We have analogue criterion for $P'_\varphi(+\infty) = \alpha_{\max}$.

Let us look at the definition of

$$A_n\varphi(x) = \frac{1}{n}\sum_{k=1}^{n}\varphi(x_k, x_{kq}, \cdots, x_{kq^{\ell-1}}).$$

One of the key points in our study of the problem is the observation that the coordinates $x_1, \cdots, x_n, \cdots$ of x appearing in the definition of $A_n\varphi(x)$ share the following independence. This observation was first exploited in [10] in order to compute the Box dimension of some subset of $E(\alpha_{\min})$. Consider the following partition of $\mathbb{N}^*$:

$$\mathbb{N}^* = \bigsqcup_{i\geq 1, q\nmid i} \Lambda_i \ \text{ with } \Lambda_i = \{iq^j\}_{j\geq 0}.$$

Observe that if $k = iq^j$ with $q \nmid i$, then $\varphi(x_k, x_{kq}, \cdots, x_{kq^{\ell-1}})$ depends only on $x_{|\Lambda_i}$, the restriction of x on Λ_i. So the summands in the definition of $A_n\varphi(x)$ can be put into different groups, each of which depends on one restriction $x_{|\Lambda_i}$. For this reason, we decompose Σ_m as follows:

$$\Sigma_m = \prod_{i\geq 1, q\nmid i} S^{\Lambda_i}.$$

Let μ be a probability measure on Σ_m. Notice that S^{Λ_i} is nothing but a copy of Σ_m. We consider μ as a measure on S^{Λ_i} for every i with $q \nmid i$. Then we define the infinite product measure $\mathbb{P}_\mu$ on $\prod_{i\geq 1, q\nmid i} S^{\Lambda_i}$ of the copies of μ. More precisely, for any word u of length n we define

$$\mathbb{P}_\mu([u]) = \prod_{i\leq n, q\nmid i} \mu([u_{|\Lambda_i}]),$$

where $[u]$ denotes the cylinder of all sequences starting with u. Then $\mathbb{P}_\mu$ is a probability measure on Σ_m and we call it a *telescopic product measure*. Kenyon, Peres and Solomyak [18, 19] used this kind of measures to compute the Hausdorff dimension of sets like $\{x = (x_n)_{n\geq 1} \in \Sigma_2 : \forall k \geq 1, x_k x_{2k} = 0\}$ which was proposed in [10].

A class of measures $\mathbb{P}_\mu$ will play the same role as Gibbs measures played in the study of simple ergodic averages ($\ell = 1$). Concerning the dimension of $\mathbb{P}_\mu$ (see [8] for the dimension of a measure), we have the following result which is one of the main ingredients of the proof of the main result (Theorem 1.1) and which has its own interest. A measure ν on Σ_m is said to be exact if there exists an $\alpha \in \mathbb{R}$ such that

$$\lim_{n\to\infty} \frac{\log_m \nu([x_{|n}])}{n} = \alpha, \ \nu\text{−a.e.}$$

This value α is the dimension of ν.

Theorem 1.3. *For any given measure μ, the telescopic product measure $\mathbb{P}_\mu$ is exact and its dimension is equal to*

$$\dim_H \mathbb{P}_\mu = \frac{(q-1)^2}{\log m} \sum_{k=1}^{\infty} \frac{H_k(\mu)}{q^{k+1}}$$

where

$$H_k(\mu) = -\sum_{a_1,\cdots,a_k \in S} \mu([a_1 \cdots a_k]) \log \mu([a_1 \cdots a_k]).$$

A similar formula for some special $\mathbb{P}_\mu$ has appeared in [19]. Another ingredient of the proof of Theorem 1.1 is a law of large numbers relative to the probability $\mathbb{P}_\mu$. We consider $(\prod_{i\geq 1, q\nmid i} S^{\Lambda_i}, \mathbb{P}_\mu)$ as a probability space $(\Omega, \mathbb{P}_\mu)$. Let $(F_k)_{k\geq 1}$ be a sequence of functions defined on Σ_m. For each k, there exists a unique integer $i(k)$ such that $k = i(k)q^j$ and $q \nmid i$. Then

$$x \mapsto F_k(x_{|\Lambda_{i(k)}})$$

defines a random variable on Ω. Concerning the sequence of random variables $\left\{F_k(x_{|\Lambda_{i(k)}})\right\}$, we have the following law of large numbers.

Theorem 1.4. *Let $(F_k)_{k\geq 1}$ be a sequence of functions defined on Σ_m. Suppose that there exist $C > 0$ and $0 < \eta < q^{3/2}$ such that for any $i \geq 1$ with $q \nmid i$, any $j_1, j_2 \in \mathbb{N}$, we have*

$$\mathrm{cov}_\mu \left(F_{iq^{j_1}}(x), F_{iq^{j_2}}(x)\right) \leq C\eta^{\frac{j_1+j_2}{2}}. \tag{8}$$

Then for $\mathbb{P}_\mu$−a.e. $x \in \Sigma_m$

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \left(F_k(x_{|\Lambda_{i(k)}}) - \mathbb{E}_\mu F_k(x)\right) = 0.$$

We observe that the set $E(\alpha)$ is not invariant. So it is not a standard set studied from the classical dynamical system point of view. Actually, as we shall see, in general the dimension of the set $E(\alpha)$ can not be described by invariant measures supported on it. This is confirmed by the following result.

Given two real valued functions f_1 and f_2 defined on Σ_m. For $\alpha \in \mathbb{R}$, let $E(\alpha)$ be the set of all points x such that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} f_1(T^k x) f_2(T^{2k} x) = \alpha.$$

We describe the size of the invariant part of $E(\alpha)$ by

$$F_{\mathrm{inv}}(\alpha) = \sup \left\{\dim \mu : \mu \text{ ergodic}, \mu(E(\alpha)) = 1 \right\}.$$

Theorem 1.5. *Let f_1 and f_2 be two Hölder continuous functions on Σ_m. If $E(\alpha)$ supports an ergodic measure, then*

$$F_{\text{inv}}(\alpha) = \sup\left\{\dim\mu : \mu \text{ ergodic}, \int f_1 d\mu \int f_2 d\mu = \alpha\right\}.$$

It is interesting to compare this result with the level sets of V-statistics studied in [14]. We return to the above theorem. A remarkable corollary is that when $f_1 = f_2$, we must have $\alpha \geq 0$ if $E(\alpha)$ supports an ergodic measure, or even an invariant measure (using Jacobs' entropy decomposition). Therefore, it is possible that for some $\alpha < 0$, $E(\alpha)$ has strictly positive Hausdorff dimension but it doesn't carry any invariant measure.

The paper is organized as follows. In Section 2, we first construct a class of measures, called telescopic product measures, part of which will play the same role as Gibbs measures played in the classical theory. This construction is inspired by Kenyon-Peres-Solomyak [18] (also see [19]). Then we establish a law of large numbers relative to such a telescopic product measure. Telescopic product measures constitute a new object of study. In Section 3, we prove that any telescopic product measure is exact and we obtain a formula for its dimension. In Section 4, we study a non–linear transfer operator and we prove the existence and the uniqueness of its positive solution. We also prove the analyticity and the convexity of the solution as a function of its parameter s. Each solution defines a Markov measure associated to which is a telescopic product measure. The last measure plays the role of a Gibbs measure in our study of $E(\alpha)$. Section 5 is devoted to the properties of the pressure function: a Ruelle type formula says that the limit in the law of large numbers is the derivative of the pressure; the pressure function is an analytic and strictly convex function (except the trivial case); the extreme values of the derivative of the pressure are studied. In Section 6, we establish the Gibbs property of the telescopic product measures defined by the solution of the non–linear transfer operator. After all these preparations, many of which have their own interests, we prove the main theorem (Theorem 1.1) in Section 7. In Section 8, we discuss the invariant part of $E(\alpha)$. Some concrete examples are presented in Section 9. In the final section, we make some remarks and present some unsolved problems.

Acknowledgement: The authors would like to thank B. Host for his interests in the work and especially for his remarks, some of which are contained in Section 8.

2. Telescopic product measures and LLN

In this section, we will study telescopic product measures and establish a law of large numbers (LLN). These measures, which take into

account the multiplicative structure of the multiple ergodic averages $A_n\varphi(x)$, will play the same role as Gibbs measures played in the study of simple ergodic averages. In the next section, we will prove that $\mathbb{P}_\mu$ is exact and its dimension is equal to

$$\dim_H \mathbb{P}_\mu = \frac{(q-1)^2}{\log m} \sum_{k=1}^{\infty} \frac{H_k(\mu)}{q^{k+1}}$$

where

$$H_k(\mu) = -\sum_{a_1,\cdots,a_k \in S} \mu([a_1 \cdots a_k]) \log \mu([a_1 \cdots a_k]).$$

We could call H_k the k-th entropy of μ. But we should point out that μ is not assumed to be invariant and that $\mathbb{P}_\mu$ is not invariant either.

2.1. **Telescopic product measures.** Let us recall the definition of the telescopic product measure $\mathbb{P}_\mu$. Consider the following partition of $\mathbb{N}^*$:

$$\mathbb{N}^* = \bigsqcup_{i \geq 1, q \nmid i} \Lambda_i \quad \text{with } \Lambda_i = \{iq^j\}_{j \geq 0}.$$

Then we decompose Σ_m as follows:

$$\Sigma_m = \prod_{i \geq 1, q \nmid i} S^{\Lambda_i}.$$

Let μ be a probability measure on Σ_m. We consider μ as a measure on S^{Λ_i}, which is identified with Σ_m, for every i with $q \nmid i$. Then we define the infinite product measure $\mathbb{P}_\mu$ on $\prod_{i \geq 1, q \nmid i} S^{\Lambda_i}$ of the copies of μ. More precisely, for any word u of length n we define

$$\mathbb{P}_\mu([u]) = \prod_{i \leq n, q \nmid i} \mu([u_{|\Lambda_i}]),$$

where $[u]$ denotes the cylinder of all sequences starting with u.

We consider $(\Sigma_m, \mathbb{P}_\mu)$ as a probability space. Let $X_k(x) = x_k$ be the k-th coordinate projection. For each i with $q \nmid i$, consider the process $Y^{(i)} = (X_k)_{k \in \Lambda_i}$. Then, by the definition of $\mathbb{P}_\mu$, the following fact is obvious.

Lemma 2.1. *The processes $Y^{(i)} = (X_k)_{k \in \Lambda_i}$ for different $i \geq 1$ with $q \nmid i$ are $\mathbb{P}_\mu$-independent and identically distributed with μ as the common probability law.*

As we shall see, the behaviour of $A_n\varphi(x)$ as $n \to \infty$ will be described by measures $\mathbb{P}_\mu$ with particular choices of μ. It is natural that $\mathbb{P}_\mu$ strongly depends on the above partition of $\mathbb{N}^*$. The following is a detail of the partition which will be useful. Fix $n \in \mathbb{N}^*$. Let

$$\Lambda_i(n) = \Lambda_i \cap \{1, \cdots, n\}.$$

We are going to examine the cardinality $\sharp\Lambda_i(n)$, called the length of $\Lambda_i(n)$ and the number $N(n, q, k)$ of $\Lambda_i(n)$'s of a given length k.

Lemma 2.2. *Let $k, n \in \mathbb{N}^*$.*

(1) $\sharp\Lambda_i(n) = k$ *if and only if* $\frac{n}{q^k} < i \leq \frac{n}{q^{k-1}}$. *Consequently we have*

$$\sharp\Lambda_i(n) = \left\lfloor \log_q \frac{n}{i} \right\rfloor.$$

(2) *We have the partition*

$$\{1, \cdots, n\} = \bigsqcup_{k=1}^{\lfloor \log_q n \rfloor} \bigsqcup_{\frac{n}{q^k} < i \leq \frac{n}{q^{k-1}}, q \nmid i} \Lambda_i(n).$$

(3) $N(n, q, k)$ *is the number of i's such that $q \nmid i$ and* $\frac{n}{q^k} < i \leq \frac{n}{q^{k-1}}$. *We have*

$$\left| \frac{N(n, q, k)}{n} - \frac{(q-1)^2}{q^{k+1}} \right| \leq \frac{4}{n}.$$

Proof. (1) It is simply because $\sharp\Lambda_i(n) = k$ means that

$$\Lambda_i(n) = \{i, iq, \cdots, iq^{k-1}\} \text{ with } iq^{k-1} \leq n < iq^k.$$

(2) We have the obvious partition

$$\{1, \cdots, n\} = \bigsqcup_{i \leq n, q \nmid i} \Lambda_i(n).$$

Then we collect $\Lambda_i(n)$ by their lengths. By (1), we have $1 \leq \sharp\Lambda_i(n) \leq \lfloor \log_q n \rfloor$ and

$$\{1, \cdots, n\} = \bigsqcup_{k=1}^{\lfloor \log_q n \rfloor} \bigsqcup_{\substack{i < n, q \nmid i, \\ \sharp\Lambda_i(n) = k}} \Lambda_i(n).$$

(3) By (1), $N(n, q, k)$ is obviously the numbers of i such that $\frac{n}{q^k} < i \leq \frac{n}{q^{k-1}}$ and $q \nmid i$. It is the number of i's such that $\frac{n}{q^k} < i \leq \frac{n}{q^{k-1}}$ minus the i's such that $\frac{n}{q^k} < i \leq \frac{n}{q^{k-1}}$ and $q \mid i$, i.e.

$$N(n, q, k) = \left(\left\lceil \frac{n}{q^{k-1}} \right\rceil - \left\lceil \frac{n}{q^k} \right\rceil \right) - \left(\left\lceil \frac{n}{q^k} \right\rceil - \left\lceil \frac{n}{q^{k+1}} \right\rceil \right).$$

It follows that

$$\left| N(n, q, k) - \left(\frac{n}{q^{k-1}} - \frac{2n}{q^k} + \frac{n}{q^{k+1}} \right) \right| \leq 4.$$

It is the desired estimate for $\frac{1}{q^{k-1}} - \frac{2}{q^k} + \frac{1}{q^{k+1}} = \frac{(q-1)^2}{q^{k+1}}$. □

Now we consider $(\prod_{i \geq 1, q \nmid i} S^{\Lambda_i}, \mathbb{P}_\mu)$ as a probability space $(\Omega, \mathbb{P}_\mu)$. Let $(F_k)_{k \geq 1}$ be a sequence of functions defined on Σ_m. For each k, there exists a unique integer $i(k)$ such that $k = i(k)q^j$ and $q \nmid i$. Then $x \mapsto F_k(x_{|\Lambda_{i(k)}})$ defines a random variable on Ω. Later, we will study the law

of large numbers for the sequence of variables $\{F_k(x_{|\Lambda_{i(k)}})\}_{k\geq 1}$. Notice that if $i(k) \neq i(k')$, then the two variables $F_k(x_{|\Lambda_{i(k)}})$ and $F_{k'}(x_{|\Lambda_{i(k')}})$ are independent. But if $i(k) = i(k')$, they are not independent in general. In order to prove the law of large numbers, we will need the following technical lemma which allows us to compute the expectation of the product of $F_k(x_{|\Lambda_{i(k)}})$'s. The proof of the lemma is based on the independence of $x_{|\Lambda_i}$'s.

Lemma 2.3. *Let $(F_k)_{k\geq 1}$ be a sequence of functions defined on Σ_m. Then for any integer $N \geq 1$, we have*

$$\mathbb{E}_{\mathbb{P}_\mu}\left(\prod_{k=1}^{N} F_k(x_{|\Lambda_{i(k)}})\right) = \prod_{k=1}^{\lfloor \log_q N\rfloor} \prod_{\frac{N}{q^k}<i\leq \frac{N}{q^{k-1}}, q\nmid i} \mathbb{E}_\mu\left(\prod_{h=0}^{k-1} F_{iq^h}(x)\right).$$

In particular, for any function G defined on Σ_m, for any $n \geq 1$,

$$\mathbb{E}_{\mathbb{P}_\mu} G(x_{|\Lambda_{i(n)}}) = \mathbb{E}_\mu G(\cdot).$$

Proof. Let

$$Q_N(x) = \prod_{k=1}^{N} F_k(x_{|\Lambda_{i(k)}}), \quad Q_{N,i}(x) = \prod_{k\in\Lambda_i(N)} F_k(x_{|\Lambda_i}).$$

Since the variables $x_{|\Lambda_i}$ for different $i \geq 1$ with $q \nmid i$ are independent under $\mathbb{P}_\mu$ (by Lemma 2.1), we have

$$\mathbb{E}_{\mathbb{P}_\mu} Q_N = \prod_{i\leq N, q\nmid i} \mathbb{E}_{\mathbb{P}_\mu} Q_{N,i}. \tag{9}$$

Then, by (2) of Lemma 2.2, we can rewrite the right hand side in (9) to get

$$\mathbb{E}_{\mathbb{P}_\mu} Q_N = \prod_{k=1}^{\lfloor \log_q N\rfloor} \prod_{\frac{N}{q^k}<i\leq \frac{N}{q^{k-1}}} \mathbb{E}_{\mathbb{P}_\mu} Q_{N,i}.$$

However, the marginal measures on S^{Λ_i} of $\mathbb{P}_\mu$ is equal to μ and $\Lambda_i(N) = \{i, iq, \cdots, iq^{k-1}\}$ if $\frac{N}{q^k} < i \leq \frac{N}{q^{k-1}}$. So

$$\mathbb{E}_{\mathbb{P}_\mu} Q_{N,i} = \mathbb{E}_\mu\left(\prod_{h=0}^{k-1} F_{iq^h}(x)\right).$$

Now, for any function G defined on Σ_m and any $n \in \mathbb{N}^*$, if we set $F_n = G$ and $F_k = 1$ for $k \neq n$ we have

$$\mathbb{E}_{\mathbb{P}_\mu} G(x_{|\Lambda_{i(n)}}) = \mathbb{E}_\mu G(x).$$

□

2.2. **Law of large numbers.** In order to prove the law of large numbers (LLN), we need the following result.

Recall that the covariance of two bounded functions f, g with respect to μ is defined by

$$\mathrm{cov}_\mu(f,g) = \mathbb{E}_\mu\left[(f-\mathbb{E}_\mu f)(g-\mathbb{E}_\mu g)\right]$$

Proposition 2.4. *Let $(F_k)_{k\geq 1}$ be a sequence of functions defined on Σ_m satisfying*

$$\mathrm{cov}_\mu\left(F_{iq^{j_1}}(x)F_{iq^{j_2}}(x)\right) \leq C\eta^{\frac{j_1+j_2}{2}} \tag{10}$$

for some constants $C>0$ and $0<\eta<q^{3/2}$ and for all $i\geq 1$ with $q\nmid i$ and all $j_1, j_2\in\mathbb{N}$. Let p_0, p_1 and p_2 be three maps from $\mathbb{N}^*$ into $\mathbb{N}^*$ such that

$$\forall n\in\mathbb{N}^*,\quad 1\leq \frac{p_2(n)}{p_1(n)}\leq \alpha;\quad \sum_{n=1}^{\infty}\frac{p_2(n)^{\frac{3}{2}-\epsilon}}{p_0(n)^2}<+\infty. \tag{11}$$

for some $\alpha>1$ and some $0<\epsilon<1/2$ with $q^{3/2-\epsilon}>\eta$. Then for $\mathbb{P}_\mu-$a.e. $x\in\Sigma_m$

$$\lim_{n\to\infty}\frac{1}{p_0(n)}\sum_{k=p_1(n)}^{p_2(n)}\left(F_k(x_{|\Lambda_{i(k)}})-\mathbb{E}_\mu F_k(x)\right)=0.$$

Proof. Without loss of generality, we can assume that $\mathbb{E}_{\mathbb{P}_\mu}F_k(x_{|\Lambda_{i(k)}})=0$ for all $k\in\mathbb{N}^*$. Otherwise, we replace $F_k(x_{|\Lambda_{i(k)}})$ by $F_k(x_{|\Lambda_{i(k)}})-\mathbb{E}_{\mathbb{P}_\mu}F_k(x_{|\Lambda_{i(k)}})$. We denote

$$Z_n=\frac{1}{p_0(n)}\sum_{k=p_1(n)}^{p_2(n)}Y_k\quad\text{with}\quad Y_k=F_k(x_{|\Lambda_{i(k)}}).$$

We have only to show that

$$\sum_{n=1}^{\infty}\mathbb{E}_{\mathbb{P}_\mu}Z_n^2<+\infty.$$

Notice that

$$\mathbb{E}_{\mathbb{P}_\mu}Z_n^2=\frac{1}{p_0^2(n)}\sum_{p_1(n)\leq u,v\leq p_2(n)}\mathbb{E}_{\mathbb{P}_\mu}Y_uY_v.$$

Observe that by Lemma 2.1, $\mathbb{E}_{\mathbb{P}_\mu}Y_uY_v\neq 0$ only if $i(u)=i(v)$, in other words only if u and v are in the same set Λ_i. So

$$\mathbb{E}_{\mathbb{P}_\mu}Z_n^2=\frac{1}{p_0^2(n)}\sum_{\substack{i\geq 1, q\nmid i,\\ \Lambda_i\cap[p_1(n),p_2(n)]\neq\emptyset}}\ \sum_{u,v\in\Lambda_i\cap[p_1(n),p_2(n)]}\mathbb{E}_{\mathbb{P}_\mu}Y_uY_v. \tag{12}$$

However by the hypothesis (10) on the sequence $(F_k)_{k\geq 1}$, for any $u,v\in\Lambda_i\cap[p_1(n),p_2(n)]$ we have

$$\left|\mathbb{E}_{\mathbb{P}_\mu}Y_uY_v\right|=\left|\mathbb{E}_\mu F_u(x)F_v(x)\right|\leq C\eta^{\log_q\frac{p_2(n)}{i}}.$$

Substituting the last estimate into (12), we get

$$\mathbb{E}_{\mathbb{P}_\mu} Z_n^2 \leq \frac{C}{p_0^2(n)} \sum_{\substack{i\geq 1, q\nmid i,\\ \Lambda_i\cap[p_1(n),p_2(n)]\neq\emptyset}} \eta^{\log_q \frac{p_2(n)}{i}} \sharp\left(\Lambda_i \cap [p_1(n), p_2(n)]\right). \tag{13}$$

The cardinality $\sharp\left(\Lambda_i \cap [p_1(n), p_2(n)]\right)$ is estimated as follows:

$$\sharp\left(\Lambda_i \cap [p_1(n), p_2(n)]\right) \leq 1 + \log_q \alpha. \tag{14}$$

In fact, assume that

$$\Lambda_i \cap [p_1(n), p_2(n)] = \{a_1, \cdots, a_k\}$$

with $a_1 < \cdots < a_k$. Then by the definition of Λ_i, we must have $\frac{a_{j+1}}{a_j} \geq q$ for $1 \leq j \leq k-1$ so that

$$\frac{a_k}{a_1} \geq q^{k-1}.$$

On the other hand,

$$\frac{a_k}{a_1} \leq \frac{p_2(n)}{p_1(n)} \leq \alpha.$$

So $q^{k-1} \leq \alpha$, i.e. $k \leq 1 + \log_q \alpha$. Substituting (14) into (13), we get

$$\mathbb{E}_{\mathbb{P}_\mu} Z_n^2 \leq \frac{C(1+\log_q \alpha)^+}{p_0^2(n)} \sum_{\substack{i\geq 1, q\nmid i,\\ \Lambda_i\cap[p_1(n),p_2(n)]\neq\emptyset}} \eta^{\log_q \frac{p_2(n)}{i}}. \tag{15}$$

There are at most $p_2(n) - p_1(n)$ integers i such that $i \geq 1, q \nmid i$ and $\Lambda_i \cap [p_1(n), p_2(n)] \neq \emptyset$. If they are increasingly ordered, then the j-th is bigger than j. We deduce that

$$\sum_{\substack{i\geq 1, q\nmid i,\\ \Lambda_i\cap[p_1(n),p_2(n)]\neq\emptyset}} \eta^{\log_q \frac{p_2(n)}{i}} \leq \sum_{j=1}^{p_2(n)-p_1(n)} \eta^{\log_q \frac{p_2(n)}{j}} \leq \sum_{j=1}^{p_2(n)-p_1(n)} \left(\frac{p_2(n)}{j}\right)^{\frac{3}{2}-\epsilon},$$

where the last inequality is due to the fact that $\log_q \eta < 3/2 - \epsilon$. Since $\epsilon < 1/2$, we have $\sum_{j=1}^{\infty} j^{-(3/2-\epsilon)} < \infty$. Then

$$\mathbb{E}_{\mathbb{P}_\mu} Z_n^2 \leq C \frac{p_2(n)^{3/2-\epsilon}}{p_0(n)^2}.$$

We conclude by the hypothesis which says that the right hand side of the above estimate is the general term of a convergent series. □

The following is the LLN which will be useful for our computation of the dimension of the telescopic product measure $\mathbb{P}_\mu$.

Theorem 2.5. *Let $(F_k)_{k\geq 1}$ be a sequence of functions defined on Σ_m. Suppose that there exist $C > 0$ and $0 < \eta < q^{3/2}$ such that for any $i \geq 1$ with $q \nmid i$, any $j_1, j_2 \in \mathbb{N}$,*

$$\mathrm{cov}_\mu\left(F_{iq^{j_1}}(x), F_{iq^{j_2}}(x)\right) \leq C\eta^{\frac{j_1+j_2}{2}}. \tag{16}$$

Then for $\mathbb{P}_\mu-$*a.e.* $x \in \Sigma_m$

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n}\Big(F_k(x_{|\Lambda_{i(k)}}) - \mathbb{E}_\mu F_k(x)\Big) = 0.$$

Proof. Without loss of generality, we can assume that $\mathbb{E}_{\mathbb{P}_\mu} F_k(x_{|\Lambda_{i(k)}}) = 0$ for all $k \in \mathbb{N}^*$. Our aim is to prove $\lim_{n\to\infty} Y_n = 0$ $\mathbb{P}_\mu$-a.e., where

$$Y_n = \frac{1}{n}\sum_{k=1}^{n} X_k \quad \text{with} \quad X_k = F_k(x|_{\Lambda_{i(k)}}).$$

First we claim that it suffices to show

$$\lim_{n\to\infty} Y_{n^2} = 0, \quad \mathbb{P}_\mu - \text{a.e.} \tag{17}$$

In fact, for every $n \in \mathbb{N}$ there exists a unique $k \in \mathbb{N}$ such that $k^2 \leq n < (k+1)^2$. Then we have

$$|Y_n| \leq |Y_{k^2}| + \frac{\big(|X_{k^2+1}| + \cdots + |X_n| + \cdots + |X_{(k+1)^2}|\big)}{k^2}.$$

So, since $Y_{k^2} \to 0$ $\mathbb{P}_\mu$-a.e., we have only to show

$$\lim_{k\to\infty} \frac{\big(|X_{k^2+1}| + \cdots + |X_n| + \cdots + |X_{(k+1)^2}|\big)}{k^2} = 0, \quad \mathbb{P}_\mu - \text{a.e.} \tag{18}$$

Let p_0, p_1 and p_2 be the three maps from $\mathbb{N}^*$ to $\mathbb{N}^*$ defined as follows:

$$p_0(k) = p_1(k) = k^2, \quad p_2(k) = (k+1)^2 \text{ for } k \in \mathbb{N}^*.$$

Then observe that

$$1 \leq \frac{p_2(k)}{p_1(k)} = \frac{(k+1)^2}{k^2} \leq 4 \quad \forall k \in \mathbb{N}^*$$

$$\sum_{k=2}^{\infty} \frac{(p_2(k)^{\frac{3}{2}-\epsilon})}{p_0^2(n)} \leq \sum_{k=2}^{\infty} \frac{((k+1)^2)^{\frac{3}{2}-\epsilon}}{k^4} < +\infty.$$

Thus we have verified that the maps p_0, p_1 and p_2 satisfy the hypothesis of Lemma 2.4. Then (18) is assured by Proposition 2.4.

Now we are going to show

$$\sum_{n=1}^{\infty} \mathbb{E}_{\mathbb{P}_\mu} Y_{n^2}^2 < +\infty, \tag{19}$$

which will imply (17). Notice that

$$\mathbb{E}_{\mathbb{P}_\mu} Y_n^2 = \frac{1}{n^2} \sum_{1\leq u,v\leq n} E_{\mathbb{P}_\mu} X_u X_v.$$

By Lemma 2.1, we have $\mathbb{E}_{\mathbb{P}_\mu} X_u X_v \neq 0$ only if $i(u) = i(v)$. So

$$\mathbb{E}_{\mathbb{P}_\mu} Y_n^2 = \sum_{i\leq n, q\nmid i} \sum_{u,v\in\Lambda_i(n)} \mathbb{E}_{\mathbb{P}_\mu} X_u X_v.$$

By (2) of Lemma 2.2, we can rewrite the above sum as

$$\mathbb{E}_{\mathbb{P}_\mu} Y_n^2 = \sum_{k=1}^{\lfloor \log_q n \rfloor} \sum_{\substack{\frac{n}{q^k} < i \le \frac{n}{q^{k-1}} \\ q \nmid i}} \sum_{u,v \in \Lambda_i(n)} \mathbb{E}_{\mathbb{P}_\mu} X_u X_v. \tag{20}$$

Recall that $\mathbb{E}_{\mathbb{P}_\mu} X_k = \mathbb{E}_\mu F_k$ for all $k \in \mathbb{N}^*$ (Lemma 2.3). For $u, v \in \Lambda_i(n)$, we write $u = iq^{j_1}$ and $v = iq^{j_2}$ with $0 \le j_1, j_2 \le \sharp\Lambda_i(n)$. By the Cauchy-Schwarz inequality and the hypothesis (16), we obtain

$$\left|\mathbb{E}_{\mathbb{P}_\mu} X_u X_v\right| \le \sqrt{\mathbb{E}_\mu F_u^2}\sqrt{\mathbb{E}_\mu F_v^2} \le C\eta^{\sharp\Lambda_i(n)}.$$

This estimate holds for all $u, v \in \Lambda_i(n)$. So

$$\sum_{u,v \in \Lambda_i(n)} \left|\mathbb{E}_{\mathbb{P}_\mu} X_u X_v\right| \le C\left(\sharp\Lambda_i(n)\right)^2 \eta^{\sharp\Lambda_i(n)}.$$

Substituting this estimate into (20) and using (1) of Lemma 2.2, we get

$$\left|\mathbb{E}_{\mathbb{P}_\mu} Y_n^2\right| \le \frac{C}{n^2} \sum_{k=1}^{\lfloor \log_q n \rfloor} \sum_{\substack{\frac{n}{q^k} < i \le \frac{n}{q^{k-1}} \\ q \nmid i}} k^2 \eta^k = \frac{C}{n^2} \sum_{k=1}^{\lfloor \log_q n \rfloor} k^2 \eta^k N(n,q,k),$$

where $N(n,q,k)$ appeared in Lemma 2.2. Then by (3) of Lemma 2.2, the last term is equivalent to

$$\frac{C(q-1)^2}{n} \sum_{k=1}^{\lfloor \log_q n \rfloor} \frac{k^2 \eta^k}{q^{k+1}} = O\left(\frac{1}{n}\left(\frac{\eta}{q}\right)^{\log_q n}\right) = O\left(n^{-1/2-\epsilon}\right)$$

for some $\epsilon > 0$. This implies (19). □

2.3. **A special LLN.** When, in the LLN (Theorem 2.5), the functions $(F_i)_i$ are all the same function F, then we have the following special LLN.

Theorem 2.6. *Let μ be any probability measure μ on Σ_m and let $F \in \mathcal{F}(S^\ell)$. For $\mathbb{P}_\mu$ a.e. $x \in \Sigma_m$ we have*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} F(x_k, \cdots, x_{kq^{\ell-1}}) = (q-1)^2 \sum_{k=1}^{\infty} \frac{1}{q^{k+1}} \sum_{j=0}^{k-1} \mathbb{E}_\mu F(x_j, \cdots, x_{j+\ell-1}).$$

Proof. For any integer k we write $k = i(k)q^j$ with $q \nmid i(k)$. Then we define a function F_k by

$$F_k(x) = F(x_j, \cdots, x_{j+\ell-1}).$$

Therefore we can re-write

$$F(x_k, x_{kq}, \cdots, x_{kq^{\ell-1}}) = F_k(x_{|\Lambda_{i(k)}}).$$

By the law of large numbers, for $\mathbb{P}_{\mu_s}$ a.e. $x \in \Sigma_m$ we have

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} F_k(x_{|_{\Lambda_{i(k)}}}) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \mathbb{E}_\mu F_k(x)$$

if the limit in the right hand side exists. The limit does exists. In fact, by (2) of Lemma 2.2, we have

$$\sum_{k=1}^{n} \mathbb{E}_\mu F_k(x) = \sum_{k=1}^{\lfloor \log_q n \rfloor} \sum_{\substack{\frac{n}{q^k} < i \le \frac{n}{q^{k-1}} \\ q \nmid i}} \sum_{j=0}^{\sharp \Lambda_i(n)-1} \mathbb{E}_\mu F_{iq^j}(x).$$

By the definition of the sequence (F_k), for any $k = iq^j$ with $q \nmid i$ we have

$$\mathbb{E}_\mu F_{iq^j}(x) = \mathbb{E}_\mu F(x_j, \cdots, x_{j+\ell-1}),$$

which is independent of i. Combining the last two equations, we get

$$\sum_{k=1}^{n} \mathbb{E}_\mu F_k(x) = \sum_{k=1}^{\lfloor \log_q n \rfloor} N(n,q,k) \sum_{j=0}^{k-1} \mathbb{E}_\mu F(x_j, \cdots, x_{j+\ell-1}),$$

where $N(n,q,k)$ appeared in Lemma 2.2. Then, by (3) of Lemma 2.2, we get

$$\begin{aligned} \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \mathbb{E}_\mu F_k(x) &= \lim_{n\to\infty} \sum_{k=1}^{\lfloor \log_q n \rfloor} \frac{N(n,q,k)}{n} \sum_{j=0}^{k-1} \mathbb{E}_\mu F(x_j, \cdots, x_{j+\ell-1}) \\ &= (q-1)^2 \sum_{k=1}^{\infty} \frac{1}{q^{k+1}} \sum_{j=0}^{k-1} \mathbb{E}_\mu F(x_j, \cdots, x_{j+\ell-1}). \end{aligned}$$

□

3. Dimensions of telescopic product measures

Let ν be a measure on Σ_m. The lower local dimension of ν at a point $x \in \Sigma_m$ is defined as

$$\underline{D}(\nu, x) := \liminf_{n\to\infty} \frac{-\log_m \nu([x_1^n])}{n}.$$

Similarly, we can define the upper local dimension $\overline{D}(\nu, x)$. If $\underline{D}(\nu, x) = \overline{D}(\nu, x)$, we write $D(\nu, x)$ for the common value and we say that ν admits $D(\nu, x)$ as the exact local dimension at x. See [8] for the dimensions of measures. Recall that the Hausdorff dimension of a Borel measure ν, denoted by $\dim_H \nu$, is the minimal dimension of Borel sets of full measure and is equal to ess $\sup_\nu \underline{D}(\nu, x)$ ([8]).

In this section, as a consequence of the LLN, we will prove that every telescopic product measure $\mathbb{P}_\mu$ admits its exact local dimension for $\mathbb{P}_\mu$-a.e. point in Σ_m, which is a constant.

3.1. **Local dimension of telescopic product measures.** For a measure μ on Σ_m and for $k \geq 1$, we define

$$H_k(\mu) = -\sum_{a_1,\cdots,a_k} \mu([a_1 \cdots a_k]) \log \mu([a_1 \cdots a_k]).$$

We note that for a probability measure μ we have $0 \leq H_k(\mu) \leq k \log m$.

Theorem 3.1. *For $\mathbb{P}_\mu$-a.e. $x \in \Sigma_m$, we have*

$$D(\mathbb{P}_\mu, x) = \frac{(q-1)^2}{\log m} \sum_{k=1}^{\infty} \frac{H_k(\mu)}{q^{k+1}}.$$

Proof. By the definition of $\mathbb{P}_\mu$, we have

$$\log \mathbb{P}_\mu([x_1^n]) = \sum_{i \leq n, q \nmid i} \log \mu([x_1^n|_{\Lambda_i(n)}]) = \sum_{k=1}^{\lfloor \log_q n \rfloor} \sum_{\substack{\frac{n}{q^k} < i \leq \frac{n}{q^{k-1}} \\ q \nmid i}} \log \mu([x_1^n|_{\Lambda_i(n)}]). \tag{21}$$

Recall that $x_1^n|_{\Lambda_i(n)} = x_i x_{iq} x_{iq^2} \cdots x_{iq^{\sharp\Lambda_i - 1}}$. So

$$\mu([x_1^n|_{\Lambda_i(n)}]) = \mu([x_i x_{iq} x_{iq^2} \cdots x_{iq^{\sharp\Lambda_i - 1}}]).$$

Let us write $\mu([x_1^n|_{\Lambda_i(n)}])$ in the following way

$$\mu([x_1^n|_{\Lambda_i(n)}]) = \mu([x_i]) \prod_{j=1}^{\sharp\Lambda_i - 1} \frac{\mu([x_i x_{iq} x_{iq^2} \cdots x_{iq^j}])}{\mu([x_i x_{iq} x_{iq^2} \cdots x_{iq^{j-1}}])}.$$

Now we define a suitable sequence of functions $(F_k)_{k \geq 1}$ on Σ_m in order to express $\mu([x_1^n|_{\Lambda_i(n)}])$. If $k = i$ such that $q \nmid i$, we define

$$F_k(x) = F_i(x) = -\log \mu([x_0]).$$

If $k = iq^j$ with $q \nmid i$ and $j \geq 1$, we define

$$F_k(x) = F_{iq^j}(x) = -\log \frac{\mu([x_0, x_1, \cdots, x_j])}{\mu([x_0, x_1, \cdots, x_{j-1}])}.$$

Then, we have the following relationship between F_k and μ.

$$-\log \mu([x_1^n|_{\Lambda_i}]) = \sum_{k \in \Lambda_i(n)} F_k(x_{|\Lambda_i}).$$

Substituting this expression into (21) we obtain

$$-\log \mathbb{P}_\mu([x_1^n]) = \sum_{k=1}^{n} F_k(x_{|\Lambda_{i(k)}}). \tag{22}$$

Now we check that the sequence $(F_k)_{k\geq 1}$ verifies the hypothesis (16) of the law of large numbers (Theorem 2.5). Notice that for any $x \in \Sigma_m$ and any $j \geq 1$, we have

$$|F_{iq^j}(x)| = \left|\log \frac{\mu([x_0, x_1, \cdots, x_j])}{\mu([x_0, x_1, \cdots, x_{j-1}])}\right| \leq |\log \mu([x_0, x_1, \cdots, x_j])|\,.$$

This is because $\log \frac{x}{y} \leq \log \frac{1}{x}$ when $0 \leq x \leq y \leq 1$. So, for any $i \in \mathbb{N}^*$ with $q \nmid i$ and $j \geq 0$, we have

$$\mathbb{E}_\mu \left(F_{iq^j}(x)\right)^2 \leq \sum_{x_0,\cdots,x_j \in S} \mu([x_0, x_1, \cdots, x_j]) \left(\log \mu([x_0, x_1, \cdots, x_j])\right)^2.$$

Then by Lemma 3.3 stated below, we obtain

$$\mathbb{E}_\mu \left(F_{iq^j}(x)\right)^2 = O(j^2)$$

which implies through Cauchy-Schwarz inequality

$$\mathbb{E}_\mu \left|F_{iq^{j_1}}(x) F_{iq^{j_2}}(x)\right| = O((j_1 + j_2)^2).$$

This quadratic estimate is more than the exponential estimate required by the hypothesis (16). By the law of large numbers, we have

$$D(\mathbb{P}_\mu, x) = \frac{1}{\log m} \lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} F_j = \frac{1}{\log m} \lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} \mathbb{E}_\mu F_j \quad \mathbb{P}_\mu\text{−a.e.} \tag{23}$$

if the limit in the right side hand exists.

This limit does exist. We are going to compute it. By (2) of Lemma 2.2, we have

$$\sum_{k=1}^{n} \mathbb{E}_\mu F_k = \sum_{k=1}^{\lfloor \log_q n \rfloor} \sum_{\substack{\frac{n}{q^k} < i \leq \frac{n}{q^{k-1}} \\ q \nmid i}} \sum_{j=0}^{k-1} \mathbb{E}_\mu F_{iq^j}. \tag{24}$$

By the definition of the sequence $(F_k)_{k\geq 1}$, we have

$$\sum_{j=0}^{k-1} F_{iq^j}(x) = -\log \mu([x_0, \cdots, x_{k-1}])$$

which implies immediately

$$\sum_{j=0}^{k-1} \mathbb{E}_\mu F_{iq^j} = -\mathbb{E}_\mu \log \mu([x_0, \cdots, x_{k-1}]) = H_k(\mu).$$

Then substituting this into (24) we get

$$\sum_{k=1}^{n} \mathbb{E}_\mu F_k = \sum_{k=1}^{\lfloor \log_q n \rfloor} \sum_{\substack{\frac{n}{q^k} < i \leq \frac{n}{q^{k-1}} \\ q \nmid i}} H_k(\mu) = \sum_{k=1}^{\lfloor \log_q n \rfloor} N(n, q, k) H_k(\mu)$$

where $N(n,q,k)$ is the number of i's such that $\frac{n}{q^k} < i \leq \frac{n}{q^{k-1}}$ and $q \nmid i$. So, by (3) of Lemma 2.2, we obtain

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \mathbb{E}_\mu F_k = \lim_{n\to\infty} \sum_{k=1}^{\lfloor \log_q n \rfloor} \frac{N(n,q,k)}{n} H_k(\mu) = (q-1)^2 \sum_{k=1}^{\infty} \frac{H_k(\mu)}{q^{k+1}} < \infty.$$

□

Remark 3.2. *Even if the measure μ itself is not exact dimensional the telescopic measure $\mathbb{P}_\mu$ is. This is because the $\mathbb{P}_\mu$-measure of a cylinder of length N is governed by the measure μ on short pieces $\Lambda_i(N)$ while the non-exactness of μ can be seen only on long cylinders. These short pieces are independent.*

3.2. **An elementary inequality.** In the last proof we have used the following elementary estimation. For $n \geq 1$, let

$$\mathcal{P}_n := \left\{ p = (p_1, \cdots, p_n) \in \mathbb{R}_+^n, \sum_{i=1}^{n} p_i = 1 \right\}$$

be the set of probability vectors. We define $L_n : \mathcal{P}_n \longrightarrow \mathbb{R}^+$ by

$$L_n(p) = \sum_{i=1}^{n} p_i (\log p_i)^2.$$

Lemma 3.3. *There exists a constant $D > 0$ such that*

$$\max_{p\in\mathcal{P}_n} L_n(p) \leq (\log n)^2 + D \log n.$$

Proof. The function $x \mapsto x(\log x)^2$ is bounded on $[0,1]$ and attains its maximal values $4e^{-2}$ at $x = e^{-2}$. Hence the inequality holds for $n = 2$ with $D = 8e^{-2}$. Now we prove the inequality by induction on n. Suppose that the inequality holds for $n \leq N$. Let $p \in \mathcal{P}_{N+1}$ be a maximal point of L_{N+1}. If p is on the boundary of $\mathcal{P}_{N+1}$, then there exists at least one component p_{i_0} of p such that $p_{i_0} = 0$. So

$$L_{N+1}(p) = \sum_{1\leq i\leq N+1, i\neq i_0} p_i (\log p_i)^2 = L_N(p')$$

where $p' = (p_1, \cdots, p_{i_0-1}, p_{i_0+1}, \cdots, p_{N+1})$ is in $\mathcal{P}_N$. In this case, we can conclude by the hypothesis of induction. Now we suppose that p is not on the boundary of $\mathcal{P}_{N+1}$. We use the method of Lagrange multiplier. Differentiating $L_{N+1}(p)$ yields

$$\frac{\partial L_{N+1}}{\partial p_i}(p) = (\log p_i)^2 + 2\log p_i, \quad (1 \leq i \leq N+1).$$

So we have

$$(\log p_i)^2 + 2\log p_i = \lambda, \quad (1 \leq i \leq N+1) \tag{25}$$

for some real number λ. Let a, b be the two solutions of the equation

$$(\log x)^2 + 2\log x = \lambda.$$

The components of the maximal point $p = (p_1, \cdots, p_{N+1})$ have two choices: a or b. So

$$L_{N+1}(p) = ka(\log a)^2 + (N+1-k)b(\log b)^2, \tag{26}$$

where k $(0 \le k \le N+1)$ is the number of a's taken by the components of p. Recall that $ka + (N+1-k)b = 1$. Notice that

$$ka(\log a)^2 = ka(\log ka - \log k)^2 = ka(\log ka)^2 + ka(\log k)^2 - 2ka(\log ka)\log k. \tag{27}$$

Since $\max_{x\in[0,1]} -x\log x = \frac{1}{e}$ and $\max_{x\in[0,1]} x(\log x)^2 = \frac{4}{e^2}$, we get

$$ka(\log a)^2 \le \frac{4}{e^2} + ka(\log k)^2 + \frac{2}{e}\log k \le \frac{4}{e^2} + ka(\log(N+1))^2 + \frac{2}{e}\log(N+1).$$

A similar estimate holds for $(N+1-k)b(\log b)^2$. Put these two estimates into (26), we get

$$P_{N+1}(p) \le \frac{8}{e^2} + (\log(N+1))^2 + \frac{4}{e}\log(N+1).$$

We conclude that the inequality holds with $D = \frac{8}{e^2} + \frac{4}{e}$. □

4. Non-linear transfer equation

Our study of $A_n\varphi(x)$ will depend upon a class of special telescopic product measures $\mathbb{P}_\mu$ where μ is a $(\ell-1)$-Markov measure. Our $(\ell-1)$-Markov measures are nothing but Markov measures with S^ℓ as state space. The transition probability of such a $(\ell-1)$-Markov measure will be determined by the solution of a non-linear transfer equation. In this section, we will study this non-linear transfer equation, find its positive solution and construct the $(\ell-1)$-Markov measure and the corresponding telescopic product measure.

4.1. **Non-linear transfer equation.** Let $\mathcal{F}(S^{\ell-1}, \mathbb{R}^+)$ denote the cone of functions defined on $S^{\ell-1}$ taking non-negative real values. It is identified with a subset in the Euclidean space $\mathbb{R}^{m^{\ell-1}}$. Let $A : S^\ell \to \mathbb{R}^+$ be a given function. We define a non-linear operator $\mathcal{N} : \mathcal{F}(S^{\ell-1}, \mathbb{R}^+) \to \mathcal{F}(S^{\ell-1}, \mathbb{R}^+)$ by

$$\mathcal{N}y(a_1, a_2, \cdots, a_{\ell-1}) = \left(\sum_{j\in S} A(a_1, a_2, \cdots, a_{\ell-1}, j)y(a_2, \cdots, a_{\ell-1}, j)\right)^{\frac{1}{q}}. \tag{28}$$

We are interested in positive fixed points of the operator $\mathcal{N}$. That means we are interested in $y \in \mathcal{F}(S^{\ell-1}, \mathbb{R}^+)$ such that $\mathcal{N}y = y$ and $y(a) > 0$ for all $a \in S^{\ell-1}$. In general, such fixed points of $\mathcal{N}$ may not exist. If $\mathcal{N}$ admits a positive fixed point, then for each $(a_1, \cdots, a_{\ell-1}) \in$

$S^{\ell-1}$, there exists at least one $j \in S$ such that $A(a_1, \cdots, a_{\ell-1}, j)$ is strictly positive. In fact, this is also a sufficient condition.

Theorem 4.1. *Suppose that A is non-negative and that for every $(a_1, \cdots, a_{\ell-1}) \in S^{\ell-1}$ there exists at least one $j \in S$ such that $A(a_1, \cdots, a_{\ell-1}, j) > 0$. Then $\mathcal{N}$ has a unique positive fixed point.*

Proof. We define a partial order on $\mathcal{F}(S^{\ell-1}, \mathbb{R}^+)$, denoted by $\leq$, as follows:

$$y_1 \leq y_2 \;\Leftrightarrow\; y_1(a) \leq y_2(a),\ \forall a \in S^{\ell-1}.$$

It is obvious that $\mathcal{N}$ is increasing with respect to this partial order, i.e.,

$$y_1 \leq y_2 \Rightarrow \mathcal{N}(y_1) \leq \mathcal{N}(y_2).$$

Uniqueness. We first prove the uniqueness of the positive fixed point by contradiction. Suppose that there are two distinct positive fixed points y_1 and y_2 for $\mathcal{N}$. Without loss of generality we can suppose that $y_1 \not\leq y_2$. Let

$$\xi = \inf\{\gamma > 1,\ y_1 \leq \gamma y_2\}.$$

It is clear that ξ is a well defined real number and $y_1 \leq \xi y_2$. Since $y_1 \not\leq y_2$, we must have $\xi > 1$. On the other hand, by the definition of $\mathcal{N}$, the operator $\mathcal{N}$ is homogeneous in the sense that

$$\mathcal{N}(cy) = c^{\frac{1}{q}}\mathcal{N}(y),\ \forall y \in \mathcal{F}(S^{\ell-1}, \mathbb{R}^+),\ \forall c \in \mathbb{R}^+.$$

It follows that

$$y_1 = \mathcal{N}(y_1) \leq \mathcal{N}(\xi y_2) = \xi^{\frac{1}{q}}\mathcal{N}(y_2) = \xi^{\frac{1}{q}} y_2.$$

This is a contradiction to the minimality of ξ for $\xi^{\frac{1}{q}} < \xi$.

Existence. Now we prove the existence. Let

$$\theta_1 = \left(\min_{a \in S^\ell} A(a)\right)^{\frac{1}{q-1}}, \quad \theta_2 = \left(m \max_{a \in S^\ell} A(a)\right)^{\frac{1}{q-1}}.$$

Consider the restriction of $\mathcal{N}$ on the compact set $\mathcal{F}(S^{\ell-1}, [\theta_1, \theta_2])$ consisting of functions on $S^{\ell-1}$ taking values in $[\theta_1, \theta_2]$. By the definitions of θ_1 and θ_2, the compact set $\mathcal{F}(S^{\ell-1}, [\theta_1, \theta_2])$ is $\mathcal{N}$-invariant, i.e.,

$$\mathcal{N}\left(\mathcal{F}(S^{\ell-1}, [\theta_1, \theta_2])\right) \subset \mathcal{F}(S^{\ell-1}, [\theta_1, \theta_2]).$$

In fact, let $y \in \mathcal{F}(S^{\ell-1}, [\theta_1, \theta_2])$ and let $y_{j_0} = \min_j y_j$. Then $y_{j_0} \geq \theta_1$ and $A(a, j_0) \geq \theta_1^{q-1}$ for all $a \in S^{\ell-1}$, so that

$$\mathcal{N}y(a) \geq (A(a, j_0) y_{j_0})^{1/q} \geq \theta_1.$$

The verification of $\mathcal{N}y(a) \leq \theta_2$ is even easier.

Now take any function y_0 from the compact set $\mathcal{F}(S^{\ell-1}, [\theta_1, \theta_2])$. By the monotonicity of $\mathcal{N}$, we get an increasing sequence

$$y_0 \leq \mathcal{N}(y_0) \leq \mathcal{N}^2(y_0) \leq \cdots.$$

Since $\mathcal{F}(S^{\ell-1},[\theta_1,\theta_2])$ is compact, the limit $g=\lim_{n\to\infty}\mathcal{N}^n(y_0)$ exists. It is a fixed point of $\mathcal{N}$. $\square$

From now on, we concentrate on the following special case:

$$A(a)=e^{s\varphi(a)},\quad (a\in S^\ell)$$

where $s\in\mathbb{R}$ is a parameter. The corresponding operator will be denoted by $\mathcal{N}_s$. By Theorem 4.1, there exists a unique positive fixed point for $\mathcal{N}_s$. We denote this fixed point by ψ_s. In the following, we are going to study the analyticity and the convexity of the functions $s\mapsto\psi_s(a)$.

4.2. **Analyticity of $s\mapsto\psi_s(a)$.**

Proposition 4.2. *For every $a\in S^{\ell-1}$, the function $s\to\psi_s(a)$ is analytic on $\mathbb{R}$.*

Proof. We consider the map $G:\mathbb{R}\times\mathbb{R}_+^{*m^{\ell-1}}\to\mathbb{R}^{m^{\ell-1}}$ defined by

$$G(s,(z_a)_{a\in S^{\ell-1}})=(G_b(s,(z_a)_{a\in S^{\ell-1}}))_{b\in S^{\ell-1}},$$

where

$$G_{(b_1,\cdots,b_{\ell-1})}(s,(z_a)_{a\in S^{\ell-1}})=z^q_{(b_1,\cdots,b_{\ell-1})}-\sum_{j\in S}e^{s\varphi(b_1,\cdots,b_{\ell-1},j)}z_{(b_2,\cdots,b_{\ell-1},j)}.$$

It is clear that G is analytic. By Theorem 4.1, we have

$$G(s,(\psi_s(a))_{a\in S^{\ell-1}})=0.$$

Moreover the uniqueness in Theorem 4.1 implies that for any fixed $s\in\mathbb{R}$, $(\psi_s(a))_{a\in S^{\ell-1}}$ is the unique positive vector satisfying the above equation. For practice, in the following we will write $\underline{\psi_s}=(\psi_s(a))_{a\in S^{\ell-1}}$ and $\underline{z}=(z_a)_{a\in S^{\ell-1}}$.

By the implicit function theorem, if the Jacobian matrix

$$D(s)=\left(\frac{\partial G_a}{\partial z_b}(s,\underline{\psi_s})\right)_{(a,b)\in S^{\ell-1}\times S^{\ell-1}}$$

is invertible on a point $s_0\in\mathbb{R}$, then there exist a neighbourhood (s_0-r_0,s_0+r_0) of s_0, a neighbourhood V of $\underline{\psi_s}$ in $\mathbb{R}^{m^{\ell-1}}$ and a analytic function f on (s_0-r_0,s_0+r_0) taking values in V such that for any $(t,\underline{z})\subset(s_0-r_0,s_0+r_0)\times V$, we have

$$G(t,\underline{z})=0\ \Leftrightarrow\ f(t)=\underline{z}.$$

Then by the uniqueness of ψ_s for fixed s, we have $\underline{\psi_t}=f(t)$. So the functions $s\to\psi_s(a)$ $(a\in S^{\ell-1})$, which are coordinate functions of f, are analytic in (s_0-r_0,s_0+r_0).

We now prove that the matrix $D(s)$ is invertible for any $s\in\mathbb{R}$. To this end, we consider the following matrix

$$\widetilde{D}(s)=\left(\psi_s(b)\frac{\partial G_a}{\partial z_b}(s,\underline{\psi_s})\right)_{(a,b)\in S^{\ell-1}\times S^{\ell-1}},$$

which is the one obtained by multiplying the b-th column of $D(s)$ by $\psi_s(b)$ for each $b \in S^{\ell-1}$. Then we have the following relation between the determinants of $D(s)$ and $\widetilde{D}(s)$:

$$\det(\widetilde{D}(s)) = \left(\prod_{a \in S^{\ell-1}} \psi_s(a) \right) \det(D(s)).$$

So we only need to prove that $\widetilde{D}(s)$ is invertible. We will prove this by showing that $\widetilde{D}(s)$ is strictly diagonal dominating and by applying the Gershgorin circle theorem (also called Levy-Desplanques Theorem) (see e.g. [24]). Recall that a matrix is said to be strictly diagonal dominating if for every row of the matrix, the modulus of the diagonal entry in the row is strictly larger than the sum of the modulus of all the other (non-diagonal) entries in that row.

Let $a = (a_1, \cdots, a_{\ell-1})$ be fixed. The function $G_a(s, \cdot)$ depends only on z_a and z_b's with $b = (a_2, \cdots, a_{\ell-1}, j)$. So

$$\frac{\partial G_a}{\partial z_b}(s, \underline{\psi_s}) \neq 0$$

only if $b = a$ or $b = (a_2, \cdots, a_{\ell-1}, j)$ for some $j \in S$. It is possible that $a = (a_2, \cdots, a_{\ell-1}, j)$ for some $j \in S$ and it is actually the case if and only if $a = (j, j, \cdots, j)$. To effectively apply the implicit function theorem, we only need to show that for any $a = (a_1, \cdots, a_{\ell-1})$, we have

$$\left| \psi_s(a) \frac{\partial G_a}{\partial z_a}(s, \underline{\psi_s}) \right| - \sum_{\substack{j \in S, \\ b=(a_2, \cdots, a_{\ell-1}, j) \neq a}} \left| \psi_s(b) \frac{\partial G_a}{\partial z_b}(s, \underline{\psi_s}) \right| > 0. \tag{29}$$

In fact, we have

$$\frac{\partial G_a}{\partial z_a}(s, \underline{\psi_s}) = \begin{cases} q\psi_s^{q-1}(a) - e^{s\varphi(a,j)} & \text{if } a = (j, \cdots, j) \text{ for some } j \in S, \\ q\psi_s^{q-1}(a) & \text{otherwise.} \end{cases}$$

and for $b = (a_2, \cdots, a_{\ell-1}, j) \neq a$, we have

$$\frac{\partial G_a}{\partial z_b}(s, \underline{\psi_s}) = e^{s\varphi(a,j)}.$$

Then, substituting the last two expressions into (29), we obtain that the member at the left hand side of (29) is equal to

$$q\psi_s^q(a) - \sum_{j \in S} e^{s\varphi(a,j)} \psi_s(a_2, \cdots, a_{\ell-1}, j) = (q-1)\psi_s^q(a) > 0.$$

For the last equality we have used the fact that ψ_s is the solution of $\mathcal{N}_s \psi_s = \phi_s$. □

Our function ψ_s is defined on $S^{\ell-1}$. We extend it on S^k for all $1 \le k \le \ell-2$ by induction on k as follows

$$\psi_s(a) = \left(\sum_{j\in S} \psi_s(a,j)\right)^{\frac{1}{q}}, \quad (\forall a \in S^k).$$

It is clear that all these functions ψ_s are strictly positive for all $s \in \mathbb{R}$.

Corollary 4.3. *For any $a \in \bigcup_{1\le k\le \ell-1} S^k$, the function $s \to \psi_s(a)$ is analytic on $\mathbb{R}$.*

4.3. **Convexity of $s \mapsto \psi_s(a)$.** In this subsection, we prove that the functions $s \to \psi_s(a)$ for $a \in \bigcup_{1\le k\le \ell-1} S^k$ and the pressure function $P_\varphi(s)$ are convex functions on $\mathbb{R}$.

The following lemma is nothing but the Cauchy-Schwarz inequality. We will use it in this form several times in the proof of the convexity.

Lemma 4.4. *Let $(a_j)_{j=0}^{m-1}$ and $(b_j)_{j=0}^{m-1}$ be two sequences of non-negative real numbers. Then*

$$\left(\sum_{j=0}^{m-1} a_j b_j\right)^2 \le \left(\sum_{j=0}^{m-1} a_j b_j^2\right)\left(\sum_{j=0}^{m-1} a_j\right).$$

Proof. We write $a_j b_j = \sqrt{a_j} b_j \cdot \sqrt{a_j}$ and then use the Cauchy-Schwarz inequality. □

Let $\theta_1^s = \left(\min_{a\in S^\ell} e^{s\varphi(a)}\right)^{\frac{1}{q-1}}$. In the proof of Theorem 4.1, we have shown that

$$\psi_s = \lim_{n\to\infty} \mathcal{N}_s^n(\underline{\theta_1^s}),$$

where $\underline{\theta_1^s}$ the function on $S^{\ell-1}$ which is constantly equal to θ_1^s. By the definition of $\mathcal{N}_s$, it is obvious that

$$\mathcal{N}_s^n(\underline{\theta_1^s}) = (\theta_1^s)^{\frac{1}{q^n}} \mathcal{N}^n(\underline{1}),$$

where $\underline{1}$ is the function constantly equal to 1. However, for any $s \in \mathbb{R}$, we have $\lim_{n\to\infty} (\theta_1^s)^{\frac{1}{q^n}} = 1$, so that

$$\psi_s = \lim_{n\to\infty} \mathcal{N}_s^n(\underline{1}).$$

The above convergence is actually uniform for s in any compact set of $\mathbb{R}$. Let

$$\psi_{s,n} = \mathcal{N}_s^n(\underline{1}).$$

In order to prove convexity of the functions

$$s \mapsto \psi_s(a), \quad \log \sum_{j\in S} \psi_s(b,j), \quad (a \in S^{\ell-1}, b \in S^{\ell-2})$$

we have only to show those of

$$s \mapsto \psi_{s,n}(a), \quad \log \sum_{j\in S} \psi_{s,n}(b,j).$$

Actually we will make a proof by induction on n.

Recall that a function H of class C^2 is convex if $H'' \geq 0$. A function H of class C^2 is log-convex if $\log H$ is convex or equivalently $H''H \geq (H')^2$.

First we have the following initiation of the induction.

Lemma 4.5. *For any $a \in S^{\ell-1}$, the function $s \mapsto \mathcal{L}_s\underline{1}(a)$ is log-convex.*

Proof. The log-convexity of $s \mapsto \mathcal{L}_s\underline{1}(a)$ is equivalent to

$$(\mathcal{L}_s\underline{1}(a))^2 \leq (\mathcal{L}_s\underline{1}(a))''(\mathcal{L}_s\underline{1}(a)).$$

Recall the definition of $\mathcal{L}_s\underline{1}(a)$:

$$\mathcal{L}_s\underline{1}(a) = \sum_{j\in S} e^{s\varphi(Ta,j)}.$$

Notice that

$$\left(e^{s\varphi(a,b)}\right)' = e^{s\varphi(a,b)}\varphi(a,b), \quad \left(e^{s\varphi(a,b)}\right)'' = e^{s\varphi(a,b)}\varphi^2(a,b).$$

Then log-convexity of $s \mapsto \mathcal{L}_s\underline{1}(a)$ is equivalent to

$$\left(\sum_{j\in S} e^{s\varphi(Ta,j)}\varphi(Ta,j)\right)^2 \leq \left(\sum_{j\in S} e^{s\varphi(Ta,j)}\varphi(Ta,j)^2\right)\left(\sum_{j\in S} e^{s\varphi(Ta,j)}\right).$$

This is nothing but the Cauchy-Schwarz inequality (see Lemma 4.4). □

The induction will be based on the following recursive relation

$$\psi_{s,n+1}(a) = \mathcal{N}_s\psi_{s,n}(a), \quad \text{equivalently} \quad (\psi_{s,n+1}(a))^q = \mathcal{L}_s\psi_{s,n}(a).$$

We are going to show that if $s \mapsto \mathcal{L}_s\psi_{s,n}(a)$ is log-convex, then so is $s \mapsto \mathcal{L}_s\psi_{s,n+1}(a)$ and even $s \mapsto \mathcal{N}_s\psi_{s,n}(a) = \psi_{s,n+1}(a)$ is convex and

$$\sum_{j\in S} \psi_{s,n+1}(b,j)$$

is log-convex.

Lemma 4.6. *Let $(u_s)_{s\in\mathbb{R}}$ be a family of functions in $\mathcal{F}(S^{\ell-1})$. We suppose that for $a \in S^{\ell-1}$, $s \mapsto u_s(a)$ is twice differentiable with respect to $s \in \mathbb{R}$. Let*

$$v_s(a) = \mathcal{N}_s u_s(a).$$

Suppose that for any $a \in S^{\ell-1}$, $s \mapsto \mathcal{L}_s u_s(a)$ is log-convex. Then

(1) *For all $a \in S^{\ell-1}$, $s \mapsto v_s(a)$ is convex.*
(2) *For all $b \in S^{\ell-2}$, $s \mapsto \sum_{j\in S} v_s(b,j)$ is log-convex.*
(3) *For all $a \in S^{\ell-1}$, $s \mapsto \mathcal{L}_s v_s(a)$ is log-convex.*

Proof. By the hypothesis, for each $a \in S^{\ell-1}$, the function $s \mapsto \mathcal{L}_s u_s(a)$ is log-convex. That is to say, if we let $H_s(a) = \mathcal{L}_s u_s(a)$, we have

$$H_s''(a)H_s(a) \geq (H_s'(a))^2, \tag{30}$$

where, as well as in the following, $'$ and $''$ will refer to the derivatives with respect to s.

(1) Since $v_s(a) = (H_s(a))^{1/q}$, we have

$$(v_s(a))' = \frac{1}{q}(H_s(a))^{\frac{1}{q}-1}H_s'(a).$$

In other words,

$$(v_s(a))' = v_s(a)R_s(a) \tag{31}$$

with

$$R_s(a) = \frac{1}{q}\frac{H_s'(a)}{H_s(a)}.$$

Furthermore we have

$$\begin{aligned}(v_s(a))'' &= \frac{1}{q}\left(\frac{1}{q}-1\right)(H_s(a))^{\frac{1}{q}-2}[H_s'(a)]^2 + \frac{1}{q}(H_s(a))^{\frac{1}{q}-1}H_s''(a)\\ &= \frac{1}{q^2}(H_s(a))^{\frac{1}{q}-2}[H_s'(a)]^2 + \frac{1}{q}(H_s(a))^{\frac{1}{q}-2}[H_s(a)H_s''(a) - (H_s'(a))^2].\end{aligned}$$

By the hypothesis (30), $(v_s(a))'' \geq 0$. Thus we have proved (1). The last equality implies

$$(v_s(a))'' \geq \frac{1}{q^2}(H_s(a))^{\frac{1}{q}-2}[H_s'(a)]^2.$$

In other words,

$$(v_s(a))'' \geq v_s(a)[R_s(a)]^2. \tag{32}$$

The relations (31) and (32) will be useful later.

(2) By (32), we have

$$\left(\sum_{j\in S}(v_s(b,j))''\right)\left(\sum_{j\in S}v_s(b,j)\right) \geq \left(\sum_{j\in S}v_s(b,j)R_s(b,j)^2\right)\left(\sum_{j\in S}v_s(b,j)\right).$$

Then, by the Cauchy-Schwarz inequality in the form of Lemma 4.4, we have

$$\begin{aligned}\left(\sum_{j\in S}(v_s(b,j))''\right)\left(\sum_{j\in S}v_s(b,j)\right) &\geq \left(\sum_{j\in S}v_s(b,j)R_s(b,j)\right)^2\\ &= \left(\sum_{j\in S}(v_s(b,j))'\right)^2\end{aligned}$$

where the last equality is due to (31). Thus we have proved (2).

(3) Recall that

$$\mathcal{L}v_s(a) = \sum_{j\in S} e^{s\varphi(Ta,j)}v_s(Ta,j).$$

Notice that

$$\frac{d}{ds}e^{s\varphi(a,b)}v_s(Ta,j)$$
$$= e^{s\varphi(a,j)}\left[\varphi(a,j)v_s(Ta,b) + (v_s(Ta,j))'\right],$$
$$\frac{d^2}{ds^2}e^{s\varphi(a,b)}v_s(Ta,j)$$
$$= e^{s\varphi(a,j)}\left[\varphi^2(a,j)v_s(Ta,j) + 2\varphi(a,j)(v_s(Ta,j))' + (v_s(Ta,j))''\right].$$

By using (31), we can write

$$\frac{d}{ds}e^{s\varphi(a,b)}v_s(Ta,j) = e^{s\varphi(a,j)}v_s(Ta,j)\left[\varphi(a,j) + R_s(Ta,j)\right].$$

By using (31) and (32), we get

$$\varphi^2(a,j)v_s(Ta,j)+2\varphi(a,j)(v_s(Ta,j))'+(v_s(Ta,j))'' \geq \left[\varphi(a,j) + R_s(Ta,j)\right]^2,$$

so that

$$\frac{d^2}{ds^2}e^{s\varphi(a,b)}v_s(Ta,j) \geq e^{s\varphi(a,j)}v_s(Ta,j)\left[\varphi(a,j) + R_s(Ta,j)\right]^2.$$

There

$$(\mathcal{L}_s v_s(a))''\mathcal{L}_s v_s(a) \geq \left(\sum_{j\in S} C_s(Ta,j)D_s(a,j)^2\right)\left(\sum_{j\in S} C_s(a,j)\right).$$

where

$$C_s(a,j) = e^{s\varphi(a,j)}v_s(Ta,j), \qquad D_s(a,j) = \varphi(a,j) + R_s(Ta,j).$$

Then, by the Cauchy inequality (see Lemma 4.4), we finally get

$$(\mathcal{L}_s v_s(a))''\mathcal{L}_s v_s(a) \geq \left(\sum_{j\in S} C_s(a,j)D_s(a,j)\right)^2 = [(\mathcal{L}_s v_s(a))']^2.$$

That is the log-convexity of $s \mapsto \mathcal{L}_s v_s(a)$. □

Theorem 4.7. *For any $a \in \bigcup_{1\leq j\leq \ell} S^{\ell-j}$, the function $s \mapsto \psi_s(a)$ is convex. The pressure function $P_\varphi(s)$ is also convex.*

Proof. We prove convexity of $s \mapsto \psi_s(a)$ for $a \in S^{\ell-1}$ by showing those of $s \mapsto \psi_{s,n}(a)$ by induction on n. The induction is based on Lemma 4.5 and Lemma 4.6 (only the points (1) and (3) are used).

Now we prove convexity of $s \mapsto \psi_s(a)$ for $a \in S^{\ell-k}$ $(2 \leq k \leq \ell)$ by induction on k and by using what we have just proved above (as the initiation of induction). We can do that because of the following recursive relation: for $a \in S^{\ell-k}$ $(2 \leq k \leq \ell)$, we have

$$\psi_s(a)^q = \sum_{j\in S} \psi_s(a,j).$$

The right hand side is the operator $\mathcal{L}_s$ defined by the φ which is identically zero. So the log-convexity of $\psi_s(a,j)$ implies that of $\psi_s(a)$ just as the log-convexity of $\psi_{s,n}$ implies that of $\psi_{s,n+1}(a)$.

Recall that the pressure function is proportional to

$$s \mapsto \log \psi_s(\emptyset) = \log \sum_{j\in S} \psi_s(j).$$

The convexity of the pressure is just the log-convexity of $\sum_{j\in S}\psi_s(j)$, which is implied by Lemma 4.6 (3) and the log-convexity of $\psi_s(j)$. □

4.4. **Construction of the measures μ_s and $\mathbb{P}_{\mu_s}$.** Below we construct a class of $(\ell-1)$-Markov measure μ_s whose transition probability and initial law are determined by the fixed point ψ_s of the operator $\mathcal{N}_s$. The corresponding telescopic product measure $\mathbb{P}_{\mu_s}$ will play the same role as Gibbs measure played in the study of simple ergodic averages.

Fix $s \in \mathbb{R}$. Let ψ_s be the function mentioned above. Recall that ψ_s was first defined on $S^{\ell-1}$ as follows

$$(\psi_s(a))^q = \sum_{b\in S} e^{s\varphi(a,b)}\psi_s(Ta,b), \quad (a \in S^{\ell-1}).$$

Then it was extended on S^k by induction on $1 \le k \le \ell-2$ as follows

$$\psi_s(a) = \left(\sum_{b\in S}\psi_s(a,b)\right)^{\frac{1}{q}}, \quad (a \in S^k).$$

These functions defined on words of length varying from 1 to $\ell-1$ allow us to define a $(\ell-1)$-step Markov measure on Σ_m, which will be denoted by μ_s, with the initial law

$$\pi_s([a_1,\cdots,a_{\ell-1}]) = \prod_{j=1}^{\ell-1}\frac{\psi_s(a_1,\cdots,a_j)}{\psi_s^q(a_1,\cdots,a_{j-1})} \tag{33}$$

and the transition probability

$$Q_s\left([a_1,\cdots,a_{\ell-1}],[a_2,\cdots,a_\ell]\right) = e^{s\varphi(a_1,\cdots,a_\ell)}\frac{\psi_s(a_2,\cdots,a_\ell)}{\psi_s^q(a_1,\cdots,a_{\ell-1})}. \tag{34}$$

Here we have identified Σ_m with $(S^{\ell-1})^{\mathbb{N}}$. Actually, π_s is a probability vector because

$$\sum_{a_j\in S}\frac{\psi_s(a_1,\cdots,a_j)}{\psi_s^q(a_1,\cdots,a_{j-1})} = 1$$

and Q is a transition probability because $\mathcal{N}_s\psi_s = \psi_s$.

As usual, $\mathbb{P}_{\mu_s}$ will denote the telescopic product measure associated to μ_s. See §2.1 for its definition and its general properties.

5. Properties of the pressure function

We have seen in the previous section that the pressure function is real analytic and convex on $\mathbb{R}$. In this section we continue to discuss some of its further properties. These properties mainly concern its strict convexity when $\alpha_{\min} < \alpha_{\max}$ and a Ruelle type formula relating

the expected limit of the multiple ergodic average with respect to the measure $\mathbb{P}_{\mu_s}$ and the derivative of P_φ.

5.1. **Ruelle type formula.** We state here the following identity which can be regarded as an analogue of Ruelle's derivative formula concerning the classical Gibbs measure and pressure function, its proof will be given in Section 7.4 (Proposition 7.8).

Theorem 5.1. *We have*

$$(q-1)^2 \sum_{k=1}^{\infty} \frac{1}{q^{k+1}} \sum_{j=0}^{k-1} \mathbb{E}_{\mu_s} \varphi(x_j, \cdots, x_{j+\ell-1}) = P'_\varphi(s).$$

As an applications of Theorem 5.1, we give the following formula concerning the value $P'_\varphi(0)$.

Proposition 5.2.

$$P'_\varphi(0) = \frac{\sum_{a\in S^\ell} \varphi(a)}{m^\ell}.$$

Proof. By Theorem 5.1, we have

$$P'_\varphi(0) = (q-1)^2 \sum_{k=1}^{\infty} \frac{1}{q^{k+1}} \sum_{j=0}^{k-1} \mathbb{E}_{\mu_0} \varphi(x_j, \cdots, x_{j+\ell-1}). \tag{35}$$

First of all, we need to determine μ_0. It is straightforward to verify that the constant function $\psi_0 \equiv m^{\frac{1}{q-1}}$ is a solution of the following equations when $s = 0$.

$$(\psi_s(a))^q = \sum_{b\in S} e^{s\varphi(a,b)} \psi_s(Ta, b), \quad (a \in S^{\ell-1}).$$

Actually, the function ψ_0 is the only positive solution by uniqueness of the positive solution (Theorem 4.1). The measure μ_0 defined by this solution as in (33) and (34) is the Lebesgue measure. So, for any $j \geq 0$ we have

$$\begin{aligned}
\mathbb{E}_{\mu_0} \varphi(x_j, \cdots, x_{j+\ell-1}) &= \sum_{x_0,\cdots,x_{j+\ell-1}} \mu_0([x_0^{j+\ell-1}]) \varphi(x_j, \cdots, x_{j+\ell-1}) \\
&= \sum_{x_0,\cdots,x_{j+\ell-1}} m^{-(j+\ell)} \varphi(x_j, \cdots, x_{j+\ell-1}) \\
&= \sum_{x_0,\cdots,x_{\ell-1}} m^{-\ell} \varphi(x_0, \cdots, x_{\ell-1}) \\
&= \frac{\sum_{a\in S^\ell} \varphi(a)}{m^\ell}.
\end{aligned}$$

Now we get the desired result by substituting the above expression in (35) and by an elementary calculation. □

5.2. **Translation via linearity.**

Theorem 5.3. *For any $\beta \in \mathbb{R}$, we have*

$$P_\varphi(s) - \beta s = P_{\varphi-\beta}(s),$$

where $P_{\varphi-\beta}(s)$ is the pressure function associated to the potential $\varphi-\beta$.

Proof. Let $\mathcal{N}_{\varphi-\beta,s}$ be the operator as defined in (28) with

$$A(a) = e^{s(\varphi(a)-\beta)}, \quad (a \in S^\ell).$$

By Theorem 4.1, the operator $\mathcal{N}_{\varphi-\beta,s}$ admits a unique positive fixed function $g_s \in \mathcal{F}(S^{\ell-1})$. We have seen that g_s is given by

$$g_s = \lim_{n\to\infty} \mathcal{N}^n_{\varphi-\beta,s}(\underline{1}).$$

By the definitions of $\mathcal{N}_s$ and $\mathcal{N}_{\varphi-\beta,s}$, it is obvious that

$$\mathcal{N}_{\varphi-\beta,s} = e^{-\frac{s\beta}{q}}\mathcal{N}_s.$$

By induction we get that

$$\mathcal{N}^n_{\varphi-\beta,s} = e^{-s\beta(\frac{1}{q}+\cdots+\frac{1}{q^n})}\mathcal{N}^n_s.$$

Thus

$$g_s = \lim_{n\to\infty} \mathcal{N}^n_{\varphi-\beta,s}(\underline{1}) = e^{-s\beta(\sum_{n=1}^\infty \frac{1}{q^n})}\psi_s = e^{-\frac{s\beta}{q-1}}\psi_s.$$

Since for $u \in \bigcup_{1\le k\le \ell-2} S^k$, $g_s(u)$ is defined by

$$g_s(u) = \left(\sum_{j=0}^{m-1} g_s(u,j)\right)^{\frac{1}{q}},$$

we deduce that for $u \in S^k$ with $1 \le k \le \ell-2$ we have

$$g_s(u) = e^{-\frac{s\beta}{(q-1)q^{\ell-1-k}}}\psi_s(u).$$

Thus

$$P_{\varphi-\beta}(s) = (q-1)q^{\ell-2}\log\sum_{j=0}^{m-1} g_s(j) = -s\beta + P_\varphi(s).$$

□

Remark 5.4. *Note that when $\beta = \alpha_{min}$ (resp. $\beta = \alpha_{max}$), the function*

$$s \longmapsto \mathcal{N}_{\varphi-\beta,s}$$

is increasing (resp. decreasing). Then in this case, the function $s \mapsto g_s$ is also increasing (resp. decreasing) and so is the pressure function $s \mapsto P_{\varphi-\beta}(s)$.

As an application of Theorem 5.3 and Remark 5.4 we have the following consequence.

Proposition 5.5. *If $s \mapsto P'_\varphi(s)$ is constant on $\mathbb{R}$, then φ is constant on S^ℓ.*

Proof. Suppose that P'_φ is constant on $\mathbb{R}$. Then

$$P'_\varphi(s) \equiv P'_\varphi(0) = \frac{\sum_{a\in S^\ell} \varphi(a)}{m^\ell} := \overline{\varphi}.$$

By Theorem 5.3, we have

$$P_\varphi(s) = \overline{\varphi}s + P_{\varphi-\overline{\varphi}}(s).$$

The last two equations imply that

$$P'_{\varphi-\overline{\varphi}}(s) \equiv 0.$$

This is equivalent to that

$$\sum_{j=0}^{m-1} g'_s(j) \equiv 0, \tag{36}$$

where g_s is the positive fixed point of $\mathcal{N}_{\varphi-\overline{\varphi},s}$. By Theorem 4.7, the function $s \mapsto g_s$ is convex, so $g'_s(j)$ is increasing for all $j \in S$. This, with (36) imply that $g'_s(j)$ is constant for all $j \in S$. So for every j the function $g_s(j)$ is affine. But these functions are strictly positive on $\mathbb{R}$, they are therefore necessarily constant on $\mathbb{R}$. So

$$g'_s(j) \equiv 0, \quad \forall j \in S.$$

For $u \in \bigcup_{1\le k\le \ell-2}$, $g_s(u)$ is defined by the following inductive relation.

$$g_s(u)^q = \sum_{j=0}^{m-1} g_s(uj), \quad u \in \bigcup_{1\le k\le \ell-2} S^k.$$

Differentiating these equations, we get

$$qg_s^{q-1}(u)g'_s(u) = \sum_{j=0}^{m-1} g'_s(uj), \quad u \in \bigcup_{1\le k\le \ell-2} S^k.$$

For any $i \in S$, since $g'_s(i) \equiv 0$, we get

$$\sum_{j=0}^{m-1} g'_s(ij) \equiv 0.$$

With the same argument used for proving that $g_s(j)$ is constant for all $j \in S$, we can also prove that $g_s(ij)$ is constant for all $(i,j) \in S^2$. By induction, we can show that $g_s(u)$ are constant for all $u \in \bigcup_{1\le k\le \ell-1} S^k$. By the definition of g_s, for $u \in S^{\ell-1}$, we have

$$g_s^q(u) = \sum_{j=0}^{m-1} e^{s(\varphi(uj)-\overline{\varphi})} g_s(Tu, j). \tag{37}$$

We now suppose that φ is not constant on S^ℓ, i.e., $\alpha_{\min} < \alpha_{\max}$. Then there exists $a \in S^\ell$ such that

$$\varphi(a) > \overline{\varphi}.$$

Let us write $a = (u, j)$ with $u \in S^{\ell-1}$ and $j \in S$. By (37), we have

$$g_s^q(u) > e^{s(\varphi(u,j)-\overline{\varphi})} g_s(Tu, j), \quad \forall s \in \mathbb{R}.$$

As $g_s(u)$ and $g_s(Tu, j)$ are strictly positive constants, this is impossible when s tend to $+\infty$. Then we conclude that φ is constant on S^ℓ. □

5.3. **Strict convexity of the pressure function.**

Theorem 5.6. *Suppose that $\alpha_{\min} < \alpha_{\max}$. Then*
(i) *$P'_\varphi(s)$ is strictly increasing on $\mathbb{R}$.*
(ii) *$\alpha_{\min} \le P'_\varphi(-\infty) < P'_\varphi(+\infty) \le \alpha^f_{\max}$.*

Proof. (i) *$P'_\varphi(s)$ is strictly increasing on $\mathbb{R}$.* We know that P'_φ is increasing on $\mathbb{R}$ as P_φ is convex on $\mathbb{R}$. Suppose that P'_φ is not strictly increasing on $\mathbb{R}$. Then there exists an interval $[a, b]$ with $a < b$ such that P'_φ is constant on $[a, b]$. On the other hand, we know that P_φ is analytic and so is P'_φ. Therefore P'_φ must be constant on the whole line $\mathbb{R}$. It is impossible by Proposition 5.5 as φ is supposed to be no constant on S^ℓ.

(ii) $\alpha_{\min} \le P'_\varphi(-\infty) < P'_\varphi(+\infty) \le \alpha_{\max}$. The strict inequality $P'_\varphi(-\infty) < P'_\varphi(+\infty)$ is implied by (i). Let us prove the first inequality. The third inequality can be similarly proved. By Theorem 5.3, we have

$$P_\varphi(s) = \alpha_{\min} s + P_{\varphi-\alpha_{\min}}(s).$$

By Remark 5.4, the function $s \mapsto P_{\varphi-\alpha_{\min}}(s)$ is increasing. Thus we have

$$P'_\varphi(s) = \alpha_{\min} + P'_{\varphi-\alpha_{\min}}(s) \ge \alpha_{\min}$$

which holds for all $s \in \mathbb{R}$. Letting $s \to -\infty$, we get

$$\alpha_{\min} \le P'_\varphi(-\infty).$$

□

To finish this section, we announce the following results concerning the extremal values of P'_φ at infinite. Its proof will be given in Section 7.5.

Theorem 5.7. *We have the equality*

$$P'_\varphi(-\infty) = \alpha_{\min}$$

if and only if there exists an $x = (x_i)_{i=1}^\infty \in \Sigma_m$ such that

$$\varphi(x_k, x_{k+1}, \cdots, x_{k+\ell-1}) = \alpha_{\min}, \ \forall k \ge 1.$$

We have an analogue criterion for $P'_\varphi(+\infty) = \alpha_{\max}$.

Remark 5.8. *We have a proof of three pages by combinatorially analyzing P_φ. But we would like to give another proof in Section 7.5 (see Proposition 7.9), which is shorter, more intuitive and easier to understand.*

6. Gibbs property of $\mathbb{P}_{\mu_s}$

In the following we are going to establish a relation between the mass $\mathbb{P}_{\mu_s}([x_1^n])$ and the multiple ergodic sum $\sum_{j=1}^{J}\varphi(x_j\cdots x_{jq^{\ell-1}})$. This can be regarded as the Gibbs property of the measure $\mathbb{P}_{\mu_s}$.

6.1. Dependence of the Local behavior of $\mathbb{P}_{\mu_s}$ on $\varphi(x_j\cdots x_{jq^{\ell-1}})$.

There is an explicit relation between the mass $\mathbb{P}_{\mu_s}([x_1^n])$ and the multiple ergodic sum $\sum_{j=1}^{\lfloor\frac{n}{q^{\ell-1}}\rfloor}\varphi(x_j\cdots x_{jq^{\ell-1}})$. Before stating this relation, we introduce some notation.

Recall that for any integer $k\in\mathbb{N}^*$ we denote by $i(k)$ the unique integer such that

$$k=i(k)q^j,\quad q\nmid i(k).$$

We associate to k a finite set of integers λ_k defined by

$$\lambda_k\ :=\begin{cases}\{i(k),i(k)q,\cdots,i(k)q^j\} & \text{if}\quad j<\ell-1\\ \{i(k)q^{j-(\ell-1)},\cdots,i(k)q^j\} & \text{if}\quad j\geq\ell-1.\end{cases}$$

We define λ_α to be the empty set if α is not an integer. For any sequence $x=(x_i)_{i=1}^\infty\in\Sigma_m$, we denote by $x_{|\lambda_k}$ the restriction of x on λ_k.

For $x\in\Sigma_m$, we define

$$B_n(x)=\sum_{j=1}^{n}\psi_s(x_{|\lambda_j}).$$

The following basic formula is a consequence of the definitions of μ_s and $\mathbb{P}_{\mu_s}$.

Proposition 6.1. *We have*

$$\log\mathbb{P}_{\mu_s}([x_1^n])=s\sum_{j=1}^{\lfloor\frac{n}{q^{\ell-1}}\rfloor}\varphi(x_j\cdots x_{jq^{\ell-1}})-(n-\lfloor n/q\rfloor)q\log\psi_s(\emptyset)-qB_{\frac{n}{q}}(x)+B_n(x).$$

Proof. By the definition of $\mathbb{P}_{\mu_s}$, we have

$$\log\mathbb{P}_{\mu_s}([x_1^n])=\sum_{q\nmid i,i\leq n}\log\mu_s([x_1^n|_{\Lambda_i(n)}]).\tag{38}$$

However, by the definition of μ_s, if $\sharp\Lambda_i(n)\leq\ell-1$, we have

$$\log\mu_s([x_1^n|_{\Lambda_i(n)}])=\sum_{j=0}^{\sharp\Lambda_i(n)-1}\log\frac{\psi_s(x_i,\cdots,x_{iq^j})}{\psi_s^q(x_i,\cdots,x_{iq^{j-1}})}=\sum_{k\in\Lambda_i(n)}\log\frac{\psi_s(x_{|\lambda_k})}{\psi_s^q(x_{|\lambda_{k/q}})}.\tag{39}$$

If $\sharp\Lambda_i(n)\geq\ell$, $\log\mu_s([x_1^n|_{\Lambda_i(n)}])$ is equal to

$$\sum_{j=0}^{\ell-2}\log\frac{\psi_s(x_i,\cdots,x_{iq^j})}{\psi_s^q(x_i,\cdots,x_{iq^{j-1}})}+\sum_{j=\ell-1}^{\sharp\Lambda_i(n)-1}\log\frac{\psi_s(x_{iq^{j-\ell+2}},\cdots,x_{iq^j})e^{s\varphi(x_{iq^{j-\ell+1}},\cdots,x_{iq^j})}}{\psi_s^q(x_{iq^{j-\ell+1}},\cdots,x_{iq^{j-1}})}$$

$$= \sum_{j=0}^{\sharp\Lambda_i(n)-1} \log \frac{\psi_s(x_i, \cdots, x_{iq^j})}{\psi_s^q(x_i, \cdots, x_{iq^{j-1}})} + s \sum_{j=\ell-1}^{\sharp\Lambda_i(n)-1} \varphi(x_{iq^{j-\ell+1}}, \cdots, x_{iq^j}),$$

in other words,

$$\mu_s([x_1^n|_{\Lambda_i(n)}]) = \sum_{k\in\Lambda_i(n)} \log \frac{\psi_s(x_{|\lambda_k})}{\psi_s^q(x_{|\lambda_{\frac{k}{q}}})} + s \sum_{k\in\Lambda_i(n), k\le n} \varphi(x_{|\lambda_k}). \tag{40}$$

Substituting (39) and (40) into (38), we get

$$\log \mathbb{P}_{\mu_s}([x_1^n]) = S_n' + sS_n'' \tag{41}$$

where

$$S_n' = \sum_{q\nmid i, i\le n} \sum_{k\in\Lambda_i(n)} \log \frac{\psi_s(x_{|\lambda_k})}{\psi_s^q(x_{|\lambda_{\frac{k}{q}}})}$$

$$S_n'' = \sum_{q\nmid i, i\le n} \sum_{k\in\Lambda_i(n), k\le n} \varphi(x_{|\lambda_k}).$$

For any fixed i with $q \nmid i$, we write

$$\sum_{k\in\Lambda_i(n)} \log \frac{\psi_s(x_{|\lambda_k})}{\psi_s^q(x_{|\lambda_{\frac{k}{q}}})} = \sum_{k\in\Lambda_i(n)} \log \psi_s(x_{|\lambda_k}) - q \sum_{k\in\Lambda_i(n)} \log \psi_s(x_{|\lambda_{\frac{k}{q}}}).$$

Recall that if we denote $j_0 = \lfloor \log_q \frac{n}{i} \rfloor$ the largest integer such that $iq^{j_0} \le n$, then

$$\Lambda_i(n) = \{i, iq, iq^2, \cdots, iq^{j_0}\}.$$

If $k = i$, we have $x_{k/q} = \emptyset$. If $k = iq^j$ with $1 \le j \le j_0$, we have $k/q = iq^{j-1}$ which belongs to $\Lambda_i(n)$. In the following we formally write

$$\Lambda_i(n/q) = \{i, iq, iq^2, \cdots, iq^{j_0-1}\}.$$

Then we can write

$$\sum_{k\in\Lambda_i(n)} \log \frac{\psi_s(x_{|\lambda_k})}{\psi_s^q(x_{|\lambda_{\frac{k}{q}}})} = (1-q) \sum_{k\in\Lambda_i(\frac{n}{q})} \psi_s(x_{|\lambda_k}) - q \log \psi_s(\emptyset) + \sum_{k\in\Lambda_i(n), kq>n} \psi_s(x_{|\lambda_k}).$$

Notice that there is only one term in the last sum, which corresponds to $k = iq^{j_0}$. Now we take sum over i to get

$$S_n' = (1-q) \sum_{k\le\frac{n}{q}} \psi_s(x_{|\lambda_k}) - q(n - \lfloor n/q \rfloor) \log \psi_s(\emptyset) + \sum_{k>\frac{n}{q}} \psi_s(x_{|\lambda_k}),$$

because $\sharp\{i \le n, q \nmid i\} = n - \lfloor n/q \rfloor$ and

$$\sum_{i\le n, q\nmid i} \sum_{k\in\Lambda_i(\frac{n}{q})} \psi_s(x_{|\lambda_k}) = \sum_{k\le\frac{n}{q}} \psi_s(x_{|\lambda_k}),$$

$$\sum_{i\le n, q\nmid i} \sum_{k\in\Lambda_i(n), kq>n} \psi_s(x_{|\lambda_k}) = \sum_{k>\frac{n}{q}} \psi_s(x_{|\lambda_k}).$$

Recall that $B_n(x) = \sum_{j=1}^n \psi_s(x_{|\lambda_j})$. We can rewrite

$$\begin{aligned}(1-q)\sum_{k\le \frac{n}{q}} \psi_s(x_{|\lambda_k}) + \sum_{k>\frac{n}{q}} \psi_s(x_{|\lambda_k}) &= -q\sum_{k\le \frac{n}{q}} \psi_s(x_{|\lambda_k}) + \sum_{k\le n} \psi_s(x_{|\lambda_k}) \\ &= -qB_{\frac{n}{q}}(x) + B_n(x).\end{aligned}$$

Thus

$$S_n' = -q(n - \lfloor n/q \rfloor)\log \psi_s(\emptyset) - qB_{\frac{n}{q}}(x) + B_n(x).$$

On the other hand, we have

$$S_n'' = \sum_{q\nmid i, i\le n} \sum_{k\in\Lambda_i(n), k\le n} \varphi(x_{|\lambda_k}) = \sum_{k\le n} \varphi(x_{|\lambda_k}) = \sum_{j=1}^{\left\lfloor \frac{n}{q^{\ell-1}} \right\rfloor} \varphi(x_j \cdots x_{jq^{\ell-1}}).$$

Substituting these expressions of S_n' and S_n'' into (41), we get the desired result. □

7. Proof of theorem 1.1: computation of $\dim_H E(\alpha)$

We will use the measure $\mathbb{P}_{\mu_s}$ to estimate the dimensions of levels sets $E(\alpha)$. Actually, for a given α, there is some s such that $\mathbb{P}_{\mu_s}$ is a nice Frostman type measure sitting on $E(\alpha)$. First of all, let us calculate the local dimensions of $\mathbb{P}_{\mu_s}$.

7.1. Upper bounds of local dimensions of $\mathbb{P}_{\mu_s}$ on level sets.

We define

$$E^+(\alpha) := \left\{ x \in \Sigma_m : \limsup_{n\to\infty} \frac{1}{n}\sum_{k=1}^n \varphi(x_k, x_{kq}, \cdots, x_{kq^{\ell-1}}) \le \alpha \right\},$$

and

$$E^-(\alpha) := \left\{ x \in \Sigma_m : \liminf_{n\to\infty} \frac{1}{n}\sum_{k=1}^n \varphi(x_k, x_{kq}, \cdots, x_{kq^{\ell-1}}) \ge \alpha \right\}.$$

It is clear that

$$E(\alpha) = E^+(\alpha) \cap E^-(\alpha).$$

In this subsection we will obtain upper bounds for local dimensions of $\mathbb{P}_{\mu_s}$ on the sets $E^+(\alpha)$ and $E^-(\alpha)$. The following elementary result will be useful for the estimation of local dimensions of $\mathbb{P}_{\mu_s}$.

Lemma 7.1. *Let $(a_n)_{n\ge1}$ be a bounded sequence of non-negative real numbers. Then*

$$\liminf_{n\to\infty} \left(a_{\lfloor n/q \rfloor} - a_n\right) \le 0.$$

Proof. Let $b_l = a_{q^{l-1}} - a_{q^l} = a_{\frac{q^l}{q}} - a_{q^l}$ for $l \in \mathbb{N}^*$. Then the boundedness implies

$$\lim_{l\to\infty} \frac{b_1 + \cdots + b_l}{l} = \lim_{l\to\infty} \frac{a_1 - a_{q^l}}{l} = 0.$$

This in turn implies $\liminf_{l\to\infty} b_l \leq 0$ so that

$$\liminf_{l\to\infty} \left(a_{\lfloor n/q \rfloor} - a_n\right) \leq \liminf_{l\to\infty} b_l \leq 0.$$

□

Proposition 7.2. *For every $x \in E^+(\alpha)$, we have*

$$\forall s \leq 0, \quad \underline{D}(\mathbb{P}_{\mu_s}, x) \leq \frac{P(s) - \alpha s}{q^{\ell-1} \log m}.$$

For every $x \in E^-(\alpha)$, we have

$$\forall s \geq 0, \quad \underline{D}(\mathbb{P}_{\mu_s}, x) \leq \frac{P(s) - \alpha s}{q^{\ell-1} \log m}.$$

Consequently, for every $x \in E(\alpha)$, we have

$$\forall s \in \mathbb{R}, \quad \underline{D}(\mathbb{P}_{\mu_s}, x) \leq \frac{P(s) - \alpha s}{q^{\ell-1} \log m}.$$

Proof. The proof is based on Proposition 6.1, which implies that for any $x \in \Sigma_m$ and any $n \geq 1$ we have

$$-\frac{\log \mathbb{P}_{\mu_s}([x_1^n])}{n} = -\frac{s}{n} \sum_{j=1}^{\lfloor \frac{n}{q^{\ell-1}} \rfloor} \varphi(x_j \cdots x_{jq^{\ell-1}}) + q\frac{n - \lfloor n/q \rfloor}{n} \log \psi_s(\emptyset)$$
$$+ \frac{B_{\frac{n}{q}}(x)}{\frac{n}{q}} - \frac{B_n(x)}{n}.$$

Since the function ψ_s is bounded, so is the sequence $(B_n(x)/n)_n$. Then, by Lemma 7.1, we have

$$\liminf_{n\to\infty} \frac{B_{\frac{n}{q}}(x)}{\frac{n}{q}} - \frac{B_n(x)}{n} \leq 0.$$

Therefore

$$\underline{D}(\mathbb{P}_{\mu_s}, x) \leq \liminf_{n\to\infty} -\frac{s}{n \log m} \sum_{j=1}^{\lfloor \frac{n}{q^{\ell-1}} \rfloor} \varphi(x_j \cdots x_{jq^{\ell-1}}) + (q-1) \log_m \psi_s(\emptyset).$$

Now suppose that $x \in E^+(\alpha)$ and $s \leq 0$. Since

$$\liminf_{n\to\infty} \frac{1}{n} \sum_{j=1}^{\lfloor \frac{n}{q^{\ell-1}} \rfloor} \varphi(x_j \cdots x_{jq^{\ell-1}}) \leq \limsup_{n\to\infty} \frac{1}{n} \sum_{j=1}^{\lfloor \frac{n}{q^{\ell-1}} \rfloor} \varphi(x_j \cdots x_{jq^{\ell-1}}) \leq \frac{\alpha}{q^{\ell-1}},$$

we have

$$\begin{aligned}\liminf_{n\to\infty} -\frac{s}{n}\sum_{j=1}^{\lfloor\frac{n}{q^{\ell-1}}\rfloor}\varphi(x_j\cdots x_{jq^{\ell-1}}) &= -s\liminf_{n\to\infty}\frac{1}{n}\sum_{j=1}^{\lfloor\frac{n}{q^{\ell-1}}\rfloor}\varphi(x_j\cdots x_{jq^{\ell-1}})\\ &\le \frac{-s\alpha}{q^{\ell-1}},\end{aligned}$$

so that

$$\underline{D}(\mathbb{P}_{\mu_s},x)\le -\frac{\alpha s}{q^{\ell-1}\log m}+(q-1)\log_m\psi_s(\emptyset)=\frac{P(s)-\alpha s}{q^{\ell-1}\log m},$$

where the last equation is due to

$$P(s)=(q-1)q^{\ell-2}\log\sum_{j\in S^\ell}\psi_s(j)=(q-1)q^{\ell-2}q\log\psi_s(\emptyset).$$

By an analogue argument, we can prove the same result for $x\in E^-(\alpha)$ and $s\ge 0$. □

7.2. **Range of L_φ.** Recall that L_φ is the set of α such that $E(\alpha)\neq\emptyset$.

Proposition 7.3. *We have $L_\varphi\subset[P'_\varphi(-\infty),P'_\varphi(+\infty)]$.*

Proof. We prove it by contradiction. Suppose that $E(\alpha)\neq\emptyset$ for some $\alpha<P'_\varphi(-\infty)$. Let $x=(x_i)_{i=1}^\infty\in E(\alpha)$. Then by Proposition 7.2, we have

$$\liminf_{n\to\infty}\frac{-\log_m\mathbb{P}_{\mu_s}([x_1^n])}{n}\le\frac{P_\varphi(s)-\alpha s}{q^{\ell-1}\log m},\ \forall s\in\mathbb{R}. \tag{42}$$

On the other hand, by the mean value theorem, we have

$$P_\varphi(s)-\alpha s=P_\varphi(s)-P_\varphi(0)-\alpha s+P_\varphi(0)=P'_\varphi(\eta_s)s-\alpha s+P_\varphi(0) \tag{43}$$

for some real number η_s between 0 and s. As P_φ is convex, P'_φ is increasing on $\mathbb{R}$. If we assume $s<0$, then we have

$$P'_\varphi(\eta_s)s-\alpha s+P_\varphi(0)\le P'_\varphi(-\infty)s-\alpha s+P_\varphi(0)=\left(P'_\varphi(-\infty)-\alpha\right)s+P_\varphi(0).$$

As $P'_\varphi(-\infty)-\alpha>0$, we deduce from (43) that for s small enough (close to $-\infty$), we have $P_\varphi(s)-\alpha s<0$. Then by (42), for s small enough we obtain

$$\liminf_{n\to\infty}\frac{-\log_m\mathbb{P}_{\mu_s}([x_1^n])}{n}<0$$

which implies $\mathbb{P}_{\mu_s}([x_1^n])>1$ for an infinite number of n's. This is a contradiction to the fact that $\mathbb{P}_\mu$ is a probability measure on Σ_m. Thus we have prove that for α such that $E(\alpha)\neq\emptyset$, we have $\alpha\ge P'(-\infty)$. Similarly we can also prove $\alpha\le P'(+\infty)$. □

As we shall show, we will have the equality $L_\varphi=[P'_\varphi(-\infty),P'_\varphi(+\infty)]$.

7.3. **Upper bounds of Hausdorff dimensions of level sets.** A upper bound of the Hausdorff dimensions of levels set is a direct consequence of the Billingsley lemma and of Proposition 7.2. The Billingsley lemma is stated as follows.

Lemma 7.4 (see Prop.4.9 in [7])**.** *Let E be a Borel set in Σ_m and let ν be a finite Borel measure on Σ_m.*

(i) *We have $\dim_H(E) \geq d$ if $\nu(E) > 0$ and $\underline{D}(\nu, x) \geq d$ for ν-a.e x.*
(ii) *We have $\dim_H(E) \leq d$ if $\underline{D}(\nu, x) \leq d$ for all $x \in E$.*

Recall that

$$P_\varphi^*(\alpha) = \inf_{s\in\mathbb{R}}(P_\varphi(s) - \alpha s).$$

Proposition 7.5. *For any $\alpha \in (P'_\varphi(-\infty), P'_\varphi(0))$, we have*

$$\dim_H E^+(\alpha) \leq \inf_{s\leq 0} \frac{1}{q^{\ell-1}\log m}[-\alpha s + P_\varphi(s)]$$

For any $\alpha \in (P'_\varphi(0), P'_\varphi(+\infty))$, we have

$$\dim_H E^-(\alpha) \leq \inf_{s\geq 0} \frac{1}{q^{\ell-1}\log m}[-\alpha s + P_\varphi(s)]$$

In particular, we have

$$\dim_H E(\alpha) \leq \frac{P_\varphi^*(\alpha)}{q^{\ell-1}\log m}.$$

7.4. **Ruelle type formula.** This subsection is mainly devoted to proving the following identity which was announced in Theorem 5.1.

$$(q-1)^2 \sum_{k=1}^{\infty} \frac{1}{q^{k+1}} \sum_{j=0}^{k-1} \mathbb{E}_{\mu_s}\varphi(x_j, \cdots, x_{j+\ell-1}) = P'_\varphi(s).$$

This formula will be useful for estimating the lower bounds of $\dim_H E(\alpha)$.

We need to do some preparations for proving this result. First of all, we deduce some identities concerning the functions ψ_s.

Recall that $\psi_s(a)$ are defined for $a \in \bigcup_{1\leq k\leq \ell-1} S^k$. They verify the following equations. For $a \in S^{\ell-1}$, we have

$$\psi_s^q(a) = \sum_{b\in S} e^{s\varphi(a,b)}\psi_s(Ta, b)$$

and for $a \in S^k$ $(1 \leq k \leq \ell-2)$ we have

$$\psi_s^q(a) = \sum_{b\in S} \psi_s(a, b).$$

Differentiating the two sides of each of the above two equations with respect to s, we get for all $a \in S^{\ell-1}$

$$q\psi_s^{q-1}(a)\psi'_s(a) = \sum_{b\in S} e^{s\varphi(a,b)}\varphi(a, b)\psi_s(Ta, b) + \sum_{b\in S} e^{s\varphi(a,b)}\psi'_s(Ta, b)$$

and for all $a \in \bigcup_{1\le k\le \ell-2} S^k$

$$q\psi_s^{q-1}(a)\psi_s'(a) = \sum_{b\in S} \psi_s'(a,b).$$

Dividing these equations by $\psi_s^q(a)$ (for different a respectively), we get

Lemma 7.6. *For any $a \in S^{\ell-1}$, we have*

$$q\frac{\psi_s'(a)}{\psi_s(a)} = \sum_{b\in S} \frac{e^{s\varphi(a,b)}\varphi(a,b)\psi_s(Ta,b)}{\psi_s^q(a)} + \sum_{b\in S} \frac{e^{s\varphi(a,b)}\psi_s'(Ta,b)}{\psi_s^q(a)}, \tag{44}$$

and for any $a \in \bigcup_{1\le k\le \ell-2} S^k$

$$q\frac{\psi_s'(a)}{\psi_s(a)} = \sum_{b\in S} \frac{\psi_s'(a,b)}{\psi_s(a,b)}. \tag{45}$$

We denote

$$w(a) = \frac{\psi_s'(a)}{\psi_s(a)}, \qquad v(a) = \sum_{b\in S} \frac{e^{s\varphi(a,b)}\psi_s'(Ta,b)}{\psi_s^q(a)}, (\forall a \in S^{\ell-1}).$$

Then we have the following identities.

Lemma 7.7. *For any $n \in \mathbb{N}$, we have*

$$\mathbb{E}_{\mu_s}\varphi(x_n^{n+\ell-1}) = q\mathbb{E}_{\mu_s} w(x_n^{n+\ell-2}) - \mathbb{E}_{\mu_s} v(x_n^{n+\ell-2}), \quad (\forall n \ge 0). \tag{46}$$

$$\mathbb{E}_{\mu_s} w(x_n^{n+\ell-2}) = \mathbb{E}_{\mu_s} v(x_{n-1}^{n+\ell-3}), \quad (\forall n \ge 1). \tag{47}$$

$$\mathbb{E}_{\mu_s} w(x_0^{\ell-2}) = \frac{1}{q(q-1)} P_\varphi'(s). \tag{48}$$

Proof. The Markov property of μ_s can be stated as follows (see(34))

$$\mu_s([x_0^{n+\ell-1}]) = \mu_s([x_0^{n+\ell-2}])Q_s(x_n^{n+\ell-1})$$

where

$$Q_s(x_n^{n+\ell-1}) = \frac{e^{s\varphi(x_n^{n+\ell-1})}\psi_s(x_{n+1}^{n+\ell-1})}{\psi_s^q(x_n^{n+\ell-2})}.$$

By the Markov property, we have

$$\begin{aligned}\mathbb{E}_{\mu_s}\varphi(x_n^{n+\ell-1}) &= \sum_{x_0,\cdots,x_{n+\ell-1}} \mu_s([x_0^{n+\ell-1}])\varphi(x_n^{n+\ell-1}) \\ &= \sum_{x_0,\cdots,x_{n+\ell-2}} \mu_s([x_0^{n+\ell-2}]) \sum_{x_{n+\ell-1}} Q_s(x_n^{n+\ell-1})\varphi(x_n^{n+\ell-1}).\end{aligned}$$

However, by the definition of Q_s and using (44), it is straightforward to check that

$$\sum_{x_{n+\ell-1}} Q_s(x_n^{n+\ell-1})\varphi(x_n^{n+\ell-1}) = qw(x_n^{n+\ell-2}) - v(x_n^{n+\ell-2}).$$

So (46) is a combination of the above two equations.

To obtain (47), we still use the Markov property of μ_s, to get

$$\mathbb{E}_{\mu_s} w(x_n^{n+\ell-2}) = \sum_{x_0,\cdots,x_{n+\ell-2}} \mu_s([x_0^{n+\ell-2}]) w(x_n^{n+\ell-2})$$

$$= \sum_{x_0,\cdots,x_{n+\ell-3}} \mu_s([x_0^{n+\ell-3}]) \sum_{x_{n+\ell-2}} \frac{e^{s\varphi(x_{n-1}^{n+\ell-2})}\psi_s(x_n^{n+\ell-2})}{\psi_s^q(x_{n-1}^{n+\ell-3})} \frac{\psi_s'(x_n^{n+\ell-2})}{\psi_s(x_n^{n+\ell-2})}$$

$$= \sum_{x_0,\cdots,x_{n+\ell-3}} \mu_s([x_0^{n+\ell-3}]) v(x_{n-1}^{n+\ell-3}) = \mathbb{E}_{\mu_s} v(x_{n-1}^{n+\ell-3}).$$

Now let us treat (48). First of all, by the definition of w and μ_s we get

$$\psi_s'(x_0^{\ell-3}) = \sum_{x_{\ell-2}} \psi_s'(x_0^{\ell-2}),$$

hence

$$\begin{aligned} \mathbb{E}_{\mu_s} w(x_0^{\ell-2}) &= \sum_{x_0,\cdots,x_{\ell-2}} \mu_s([x_0^{\ell-2}]) w(x_0^{\ell-2}) \\ &= \sum_{x_0,\cdots,x_{\ell-3}} \mu_s([x_0^{\ell-3}]) \sum_{x_{\ell-2}} \frac{\psi_s'(x_0^{\ell-2})}{\psi_s(x_0^{\ell-3})}. \end{aligned}$$

By (45), the last sum is equal to $q\frac{\psi_s'(x_0^{\ell-3})}{\psi_s(x_0^{\ell-3})}$. So

$$\mathbb{E}_{\mu_s} w(x_0^{\ell-2}) = q \sum_{x_0,\cdots,x_{\ell-3}} \mu_s([x_0^{\ell-3}]) \frac{\psi_s'(x_0^{\ell-3})}{\psi_s(x_0^{\ell-3})}.$$

Repeating the same argument, we obtain by induction on j that

$$\mathbb{E}_{\mu_s} w(x_0^{\ell-2}) = q^{\ell-2-j} \sum_{x_0,\cdots,x_j} \mu_s([x_0^j]) \frac{\psi_s'(x_0^j)}{\psi_s(x_0^j)}.$$

So finally when $j = 0$ we get

$$\mathbb{E}_{\mu_s} w(x_0^{\ell-2}) = q^{\ell-2} \sum_{b\in S} \mu_s([b]) \frac{\psi_s'(b)}{\psi_s(b)} = q^{\ell-2} \frac{\sum_{b\in S} \psi_s'(b)}{\sum_{b\in S} \psi_s(b)} = \frac{1}{q(q-1)} P_\varphi'(s)$$

where we used the fact that

$$\mu_s([b]) = \frac{\psi_s(b)}{\sum_{b\in S} \psi_s(b)}.$$

□

Now, we can prove the Ruelle type formula which was announced in Theorem 5.1. We restate it as the following proposition.

Proposition 7.8. *For any $s \in \mathbb{R}$, we have*

$$(q-1)^2 \sum_{k=1}^{\infty} \frac{1}{q^{k+1}} \sum_{j=0}^{k-1} \mathbb{E}_{\mu_s} \varphi(x_j, \cdots, x_{j+\ell-1}) = P'_\varphi(s).$$

Proof. By (46) in Lemma 7.7, for any $k \in \mathbb{N}^*$, we have

$$\begin{aligned}\sum_{j=0}^{k-1} \mathbb{E}_{\mu_s} \varphi(x_j, \cdots, x_{j+\ell-1}) &= \sum_{j=0}^{k-1} \Big(q\mathbb{E}_{\mu_s} w(x_j^{j+\ell-2}) - \mathbb{E}_{\mu_s} v(x_j^{j+\ell-2}) \Big) \\ &= q\mathbb{E}_{\mu_s} w(x_0^{\ell-2}) + q \sum_{j=1}^{k-1} \mathbb{E}_{\mu_s} w(x_j^{j+\ell-2}) \\ &\quad - \sum_{j=0}^{k-1} \mathbb{E}_{\mu_s} v(x_j^{j+\ell-2}).\end{aligned}$$

Let

$$S_k = \sum_{j=0}^{k-1} \mathbb{E}_{\mu_s} v(x_j^{j+\ell-2}).$$

Then by (47) in Lemma 7.7, we have

$$\sum_{j=1}^{k-1} \mathbb{E}_{\mu_s} w(x_j^{j+\ell-2}) = S_{k-1}.$$

Using the above equality and (48) in Lemma 7.7, we can write

$$\sum_{j=0}^{k-1} \mathbb{E}_{\mu_s} \varphi(x_j, \cdots, x_{j+\ell-1}) = \frac{P'_\varphi(s)}{q-1} + qS_{k-1} - S_k.$$

The facts $S_0 = 0$ and $S_k = o(k)$ imply

$$\sum_{k=1}^{\infty} \frac{1}{q^{k+1}} (qS_{k-1} - S_k) = 0.$$

Then

$$(q-1)^2 \sum_{k=1}^{\infty} \frac{1}{q^{k+1}} \sum_{j=0}^{k-1} \mathbb{E}_{\mu_s} \varphi(x_j, \cdots, x_{j+\ell-1}) = (q-1)^2 \sum_{k=1}^{\infty} \frac{1}{q^{k+1}} \frac{P'_\varphi(s)}{q-1}.$$

which is equal to $P'_\varphi(s)$, because $\sum_{k=1}^{\infty} 1/q^{k+1} = 1/(q-1)$. □

7.5. **When $P'_\varphi(-\infty) = \alpha_{\min}$ and when $P'_\varphi(+\infty) = \alpha_{\max}$.** We now give the proof of the statement announced in Theorem 5.7 concerning the extremal values of P'_φ at infinity.

Theorem 7.9. *We have the equality*

$$P'_\varphi(-\infty) = \alpha_{\min}$$

if and only if there exist an $x = (x_i)_{i=0}^\infty \in \Sigma_m$ such that

$$\varphi(x_k, x_{k+1}, \cdots, x_{k+\ell-1}) = \alpha_{\min}, \ \forall k \geq 0.$$

We have an analogue criterion for $P'_\varphi(+\infty) = \alpha_{\max}$.

Proof. We give the proof of the criterion for $P'_\varphi(-\infty) = \alpha_{\min}$, the one for $P'_\varphi(+\infty) = \alpha_{\max}$ is similar.

(1). *Sufficient condition.* Suppose that there exists a $(z_j)_{j=0}^\infty \in \Sigma_m$ such that

$$\varphi(z_j, \cdots, z_{j+\ell-1}) = \alpha_{\min}, \quad \forall j \geq 0.$$

We are going to prove that $P'_\varphi(-\infty) = \alpha_{\min}$. By Theorem 5.6 (ii), we have $P'_\varphi(-\infty) \geq \alpha_{\min}$, thus we only need to show that $P'_\varphi(-\infty) \leq \alpha_{\min}$. Actually we only need to find a $(x_j)_{j=1}^\infty \in \Sigma_m$ such that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} \varphi(x_j, \cdots, x_{jq^{\ell-1}}) = \alpha_{\min},$$

then by Proposition 7.3, $\alpha_{\min} \in [P'_\varphi(-\infty), P'_\varphi(+\infty)]$, so $P'_\varphi(-\infty) \leq \alpha_{\min}$. We can do this by choosing the sequence $(x_j)_{j=1}^\infty = \prod_{i\geq 1,\ q\nmid i} (x_{iq^j})_{j=0}^\infty$ with

$$(x_{iq^j})_{j=0}^\infty \ = \ (z_j)_{j=0}^\infty.$$

(2). *Necessary condition.* Suppose that there is no $(x_j)_{j=0}^\infty \in \Sigma_m$ such that

$$\varphi(x_j, \cdots, x_{j+\ell-1}) = \alpha_{\min}, \quad \forall j \geq 0.$$

We are going to show that there exists an $\epsilon > 0$ such that

$$P'_\varphi(s) \geq \alpha_{\min} + \epsilon, \quad \forall s \in \mathbb{R}.$$

And this will imply that $P'_\varphi(-\infty) \geq \alpha_{\min} + \epsilon$.

From the hypothesis, we deduce that there exist no words $x_0^{n+\ell-1}$ with $n \geq m^\ell$ such that

$$\varphi(x_j, \cdots, x_{j+\ell-1}) = \alpha_{\min}, \quad \forall 0 \leq j \leq n. \tag{49}$$

Indeed, as $x_j^{j+\ell-1} \in S^\ell$ for all $0 \leq j \leq n$ there are at most m^ℓ choices for $x_j^{j+\ell-1}$. So for any word $x_0^{n+\ell-1}$ with $n \geq m^\ell$, there exist at least two $j_1, j_2 \in \{0, \cdots, n\}$ such that

$$x_{j_1}^{j_1+\ell-1} = x_{j_2}^{j_2+\ell-1}.$$

Then if the word $x_0^{n+\ell-1}$ satisfies (49), the infinite sequence

$$(y_j)_{j=0}^\infty \ = \ (x_{j_1}, \cdots, x_{j_2-1})^\infty$$

would verify that

$$\varphi(y_j, \cdots, y_{j+\ell-1}) = \alpha_{\min}, \quad \forall j \geq 0.$$

This is a contradicts the hypothesis. We conclude then that for any word $x_0^{m^\ell+\ell-1} \in S^{m^\ell+\ell-1}$ there exists at lest one $0 \leq j \leq m^\ell$ such that

$$\varphi(x_j, \cdots, x_{j+\ell-1}) \geq \alpha'_{\min} > \alpha_{\min}$$

where $\alpha'_{\min}$ is the second smallest value of φ over S^ℓ, i.e., $\alpha'_{\min} = \min_{a\in S^\ell}\{\varphi(a) \ : \ \varphi(a) > \alpha_{\min}\}$.

We deduce from the above discussions that for any $(x_j)_{j=0}^\infty \in \Sigma_m$ and any $k \geq 0$ we have

$$\sum_{j=k}^{k+m^\ell} \varphi(x_j, \cdots, x_{j+\ell-1}) \geq m^\ell \alpha_{\min} + \alpha'_{\min} = (m^\ell + 1)\alpha_{\min} + \delta,$$

where we denote $\delta = \alpha'_{\min} - \alpha_{\min}$. This implies that for any $(x_j)_{j=0}^\infty \in \Sigma_m$ and any $n \geq 1$, we have

$$\sum_{j=0}^{n-1} \varphi(x_j, \cdots, x_{j+\ell-1}) \geq n\alpha_{\min} + \left\lfloor \frac{n}{m^\ell + 1} \right\rfloor \delta. \tag{50}$$

Now, we will use the above inequality and Proposition 7.8 to show the existence of an $\epsilon > 0$ such that

$$P'_\varphi(s) \geq \alpha_{\min} + \epsilon, \quad \forall s \in \mathbb{R}.$$

By Proposition 7.8, we have

$$P'_\varphi(s) = (q-1)^2 \sum_{k=1}^{\infty} \frac{1}{q^{k+1}} \sum_{j=0}^{k-1} \mathbb{E}_{\mu_s} \varphi(x_j, \cdots, x_{j+\ell-1}). \tag{51}$$

We can rewrite the term $\sum_{j=0}^{k-1} \mathbb{E}_{\mu_s} \varphi(x_j, \cdots, x_{j+\ell-1})$ as

$$\mathbb{E}_{\mu_s} \sum_{j=0}^{k-1} \varphi(x_j, \cdots, x_{j+\ell-1}).$$

By (50), we have for any $(x_j)_{j=0}^\infty \in \Sigma_m$

$$\sum_{j=0}^{k-1} \varphi(x_j, \cdots, x_{j+\ell-1}) \geq k\alpha_{\min} + \left\lfloor \frac{k}{m^\ell + 1} \right\rfloor \delta.$$

As μ_s is a probability measure, we have

$$\mathbb{E}_{\mu_s} \sum_{j=0}^{k-1} \varphi(x_j, \cdots, x_{j+\ell-1}) \geq k\alpha_{\min} + \left\lfloor \frac{k}{m^\ell + 1} \right\rfloor \delta.$$

Substituting this in (51), we get

$$\begin{aligned} P'_\varphi(s) &= (q-1)^2 \sum_{k=1}^{\infty} \frac{1}{q^{k+1}} \left(k\alpha_{\min} + \left\lfloor \frac{k}{m^\ell+1} \right\rfloor \delta \right) \\ &= \alpha_{\min} + \delta(q-1)^2 \sum_{k=1}^{\infty} \frac{1}{q^{k+1}} \left(\left\lfloor \frac{k}{m^\ell+1} \right\rfloor \right). \end{aligned}$$

As

$$\delta(q-1)^2 \sum_{k=1}^{\infty} \frac{1}{q^{k+1}} \left(\left\lfloor \frac{k}{m^\ell+1} \right\rfloor \right) = \delta(q-1)^2 \sum_{k \geq m^\ell+1} \frac{1}{q^{k+1}} \left(\left\lfloor \frac{k}{m^\ell+1} \right\rfloor \right) > 0$$

we have proved the existence of an $\epsilon > 0$ such that

$$P'_\varphi(s) \geq \alpha_{\min} + \epsilon, \quad \forall s \in \mathbb{R}.$$

□

7.6. **Lower bounds of** $\dim_H E(\alpha)$**.** First, as an easy application of Proposition 7.8, we get the following formula for $\dim_H \mathbb{P}_{\mu_s}$.

Proposition 7.10. *For any $s \in \mathbb{R}$, we have*

$$\dim_H \mathbb{P}_{\mu_s} = \frac{1}{q^{\ell-1}}[-sP'_\varphi(s) + P_\varphi(s)].$$

Proof. By Proposition 6.1, we have

$$\begin{aligned} -\frac{\log \mathbb{P}_{\mu_s}([x_1^n])}{n} = -\frac{s}{n} \sum_{j=1}^{\lfloor \frac{n}{q^{\ell-1}} \rfloor} \varphi(x_j \cdots x_{jq^{\ell-1}}) + \frac{n - \lfloor n/q \rfloor}{n} \log \psi_s^q(\emptyset) \\ + \frac{B_{\frac{n}{q}}(x)}{\frac{n}{q}} - \frac{B_n(x)}{n} \end{aligned} \tag{52}$$

Applying the law of large numbers to the function ψ_s, we get the $\mathbb{P}_{\mu_s}$-a.e. existence of the following limit $\lim_{n\to\infty} \frac{B_n(x)}{n}$. So

$$\lim_{n\to\infty} \frac{B_{\frac{n}{q}}(x)}{\frac{n}{q}} - \frac{B_n(x)}{n} = 0, \quad \mathbb{P}_{\mu_s} - \text{a.e.}$$

On the other hand, by Proposition 7.8 and Theorem 2.6, we have

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{\lfloor \frac{n}{q^{\ell-1}} \rfloor} \varphi(x_j \cdots x_{jq^{\ell-1}}) = \frac{1}{q^{\ell-1}} P'_\varphi(s).$$

So we obtain that for $\mathbb{P}_{\mu_s}$-a.e. $x \in \Sigma_m$

$$\lim_{n\to\infty} -\frac{\log \mathbb{P}_{\mu_s}([x_1^n])}{n} = \frac{1}{q^{\ell-1}}[-sP'_\varphi(s) + P_\varphi(s)],$$

where we have used the fact that

$$P(s) = (q-1)q^{\ell-2} \log \sum_{j \in S^\ell} \psi_s(j) = (q-1)q^{\ell-2} q \log \psi_s(\emptyset).$$

□

By Proposition 7.8, Proposition 7.10 and Billingsley's lemma (Lemma 7.4) we get the following lower bound for $\dim_H E(P'_\varphi(s))$.

Proposition 7.11. *For any $s \in \mathbb{R}$, we have*

$$\dim_H E(P'_\varphi(s)) \geq \frac{1}{q^{\ell-1} \log m}[-\alpha P'_\varphi(s) + P_\varphi(s)].$$

By the above proposition and Proposition 7.5 we obtain the following theorem about the exact Hausdorff dimension of $\dim_H(\alpha)$ for $\alpha \in (P'_\varphi(-\infty), P'_\varphi(+\infty))$.

Theorem 7.12. (i) *If $\alpha = P'_\varphi(s_\alpha)$ for some $s_\alpha \in \mathbb{R}$, then*

$$\dim_H E(\alpha) = \frac{1}{q^{\ell-1} \log m}[-P'_\varphi(s_\alpha)s_\alpha + P_\varphi(s_\alpha)] = \frac{P^*_\varphi(\alpha)}{q^{\ell-1} \log m}.$$

(ii) *For $\alpha \in (P'_\varphi(-\infty), P'_\varphi(0)]$, we have*

$$\dim_H E^+(\alpha) = \dim_H E(\alpha).$$

For $\alpha \in [P'_\varphi(0), P'_\varphi(+\infty))$, we have

$$\dim_H E^-(\alpha) = \dim_H E(\alpha).$$

7.7. **Dimension of level sets corresponding to the extreme points in L_φ.** So far, we have calculated $\dim_H E(\alpha)$ for α in $(P'_\varphi(-\infty), P'_\varphi(+\infty))$. Now we turn to the case when $\alpha = P'_\varphi(-\infty)$ or $P'_\varphi(+\infty)$. The aim of this subsection is to prove the following result.

Theorem 7.13. *If $\alpha = P'_\varphi(-\infty)$ or $P'_\varphi(+\infty)$, then $E(\alpha) \neq \emptyset$ and*

$$\dim_H E(\alpha) = \frac{P^*_\varphi(\alpha)}{q^{\ell-1} \log m}.$$

We will give the proof of Theorem 7.13 for $\alpha = P'_\varphi(-\infty)$. The proof for $\alpha = P'_\varphi(-\infty)$ is similar.

7.7.1. *Accumulation points of μ_s when s tends to $-\infty$.* We view the vector π_s defined by (33) and the matrix Q_s defined by (34) as functions of s taking values in finite dimensional Euclidean spaces. As all components of π_s and Q_s are non-negative and bounded by 1, the set $\{(\pi_s, Q_s), s \in \mathbb{R}\}$ is pre-compact in a Euclidean space. So there exists a sequence $(s_n)_{n\in\mathbb{N}}$ of real numbers with $\lim_{n\to\infty} s_n = -\infty$ such that the limits

$$\lim_{n\to\infty} \pi_{s_n}, \quad \lim_{n\to\infty} Q_{s_n}$$

exist. Using these limits as initial law and transition probability, we construct a $(\ell-1)$-step Markov measure which we denote by $\mu_{-\infty}$. It is clear that the Markov measure μ_{s_n} corresponding to π_{s_n} and Q_{s_n} converges to $\mu_{-\infty}$ with respect to the weak-star topology.

Proposition 7.14. *We have*

$$\mathbb{P}_{\mu_{-\infty}}(E(P'_\varphi(-\infty))) = 1.$$

In particular, $E(P'_\varphi(-\infty)) \neq \emptyset$.

Proof. First, we introduce a functional on the space of probability measures which is defined by

$$M(\nu) = (q-1)^2 \sum_{k=1}^{\infty} \frac{1}{q^{k+1}} \sum_{j=0}^{k-1} \mathbb{E}_\nu \varphi(x_j, \cdots, x_{j+\ell-1}).$$

The function $\nu \mapsto M(\nu)$ is continuous, just because $\nu \mapsto \mathbb{E}_\nu \varphi(x_j, \cdots, x_{j+\ell-1})$ is continuous for all j.

What we have to show is that for $\mathbb{P}_{\mu_{-\infty}}$-a.e. $x \in \Sigma_m$ we have

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \varphi(x_k, \cdots, x_{kq^{\ell-1}}) = P'_\varphi(-\infty).$$

By Theorem 2.6, for $\mathbb{P}_{\mu_{-\infty}}$-a.e. $x \in \Sigma_m$ the limit in the left hand side of the above equation equals to $M(\mu_{-\infty})$. As M is continuous and μ_{s_n} converges to $\mu_{-\infty}$ when $n \to \infty$, we deduce that

$$\lim_{n\to\infty} M(\mu_{s_n}) = M(\mu_{-\infty}).$$

By Proposition 7.8, we know that

$$M(\mu_{s_n}) = P'_\varphi(s_n).$$

So

$$M(\mu_{-\infty}) = \lim_{n\to\infty} P'_\varphi(s_n).$$

By Theorem 4.7, the map $s \to P'_\varphi(s)$ is increasing, thus we deduce that the above limit exists and

$$M(\mu_{-\infty}) = P'_\varphi(-\infty).$$

This implies the desired result.

□

We have the following formula for $\dim_H \mathbb{P}_{\mu_{-\infty}}$.

Proposition 7.15. *We have*

$$\dim_H \mathbb{P}_{\mu_{-\infty}} = \lim_{s\to-\infty} \frac{[-sP'_\varphi(s) + P_\varphi(s)]}{q^{\ell-1}\log m} = \frac{P^*_\varphi(P'_\varphi(-\infty))}{q^{\ell-1}\log m}.$$

Proof. By Proposition 3.1, we know for any probability measure ν we have

$$\dim_H \mathbb{P}_\nu = \frac{(q-1)^2}{\log m}\sum_{k=1}^{\infty}\frac{H_k(\nu)}{q^{k+1}}.$$

As the series in the right hand side converges uniformly on ν, the map $\nu \to \dim_H \mathbb{P}_\nu$ is continuous. Since μ_{s_n} converges to $\mu_{-\infty}$ when $n\to\infty$, we deduce that

$$\lim_{n\to\infty} \dim_H \mathbb{P}_{\mu_{s_n}} = \dim_H \mathbb{P}_{\mu_{-\infty}}.$$

By Proposition 7.10, we have

$$\dim_H \mathbb{P}_{\mu_s} = \frac{[-sP'_\varphi(s) + P_\varphi(s)]}{q^{\ell-1}\log m}.$$

The derivative of the map $s \to \dim_H \mathbb{P}_{\mu_s}$ is

$$\frac{d}{ds}\dim_H \mathbb{P}_{\mu_s} = \frac{-sP''_\varphi(s)}{q^{\ell-1}\log m}.$$

As $P_\varphi(s)$ is convex on $\mathbb{R}$, $P''_\varphi(s)$ is non-negative, so for $s \le 0$ the map

$$s \to \dim_H \mathbb{P}_{\mu_s}$$

is increasing. Thus

$$\dim_H \mathbb{P}_{\mu_{-\infty}} = \lim_{n\to\infty} \dim_H \mathbb{P}_{\mu_{s_n}} = \lim_{s\to-\infty} \frac{[-sP'_\varphi(s) + P_\varphi(s)]}{q^{\ell-1}\log m}.$$

□

Proposition 7.16.

$$\dim_H E(P'_\varphi(-\infty)) = \frac{P^*_\varphi(P'_\varphi(-\infty))}{q^{\ell-1}\log m}.$$

Proof. By the last two propositions and Billingsley's lemma, we get

$$\dim_H E(P'_\varphi(-\infty)) \ge \frac{P^*_\varphi(P'_\varphi(-\infty))}{q^{\ell-1}\log m}.$$

We now show the reverse inequality. By the definition of $E^+(\alpha)$, we have

$$E(P'_\varphi(-\infty))) \subset \bigcap_{\alpha\in(P'_\varphi(-\infty),P'_\varphi(0)]} E^+(\alpha) = \bigcap_{s\le 0} E^+(P'_\varphi(s)).$$

So

$$\dim_H E(P'_\varphi(-\infty))) \le \dim_H E^+(P'_\varphi(s)) = \dim_H E(P'_\varphi(s)) = \dim_H \mathbb{P}_{\mu_s},\ \ \forall s \le 0.$$

Now as $s \to \dim_H \mathbb{P}_{\mu_s}$ is increasing we deduce that

$$\dim_H E(P'_\varphi(-\infty))) \leq \lim_{s\to-\infty} \dim_H \mathbb{P}_{\mu_s} = \frac{P^*_\varphi(P'_\varphi(-\infty))}{q^{\ell-1}\log m}.$$

□

8. The invariant part of $E(\alpha)$

From classical dynamical system point of view, the set $E(\alpha)$ is not invariant and its dimension can not be described by invariant measures supported on it, as we shall see. Let us first examine the largest dimension of ergodic measures supported on the set $E(\alpha)$.

Here we can consider a more general setting. Let $f_1, f_2, \cdots, f_\ell$ be real functions defined on Σ_m. Let

$$M_{f_1,\cdots,f_\ell}(x) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} f_1(T^k x) f_2(T^{2k} x) \cdots f_\ell(T^{\ell k} x) \tag{53}$$

if the limit exists. In this section, for a real number α, we define

$$E(\alpha) = \{x \in \Sigma_m : M_{f_1,\cdots,f_\ell}(x) = \alpha\}.$$

In order to describe the invariant part of $E(\alpha)$, we introducing the so-called invariant spectrum:

$$F_{\text{inv}}(\alpha) = \sup\{\dim \mu : \mu \text{ ergodic}, \mu(E(\alpha)) = 1\}.$$

In general, $F_{\text{inv}}(\alpha)$ is smaller than $\dim E(\alpha)$. It is even possible that no ergodic measure is supported on $E(\alpha)$.

Theorem 8.1. *Let $\ell = 2$. Let f_1 and f_2 be two Hölder continuous functions on Σ_m. If $E(\alpha)$ supports an ergodic measure, then*

$$F_{\text{inv}}(\alpha) = \sup\left\{\dim \mu : \mu \text{ ergodic}, \int f_1 d\mu \int f_2 d\mu = \alpha\right\}.$$

Proof. Let μ be an ergodic measure such that $\mu(E(\alpha)) = 1$. Then

$$\begin{aligned}
\alpha &= \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \mathbb{E}_\mu[f_1(T^k x) f_2(T^{2k} x)] \\
&= \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \mathbb{E}_\mu[f_1(x) f_2(T^k x)] \\
&= \mathbb{E}_\mu[f_1(x) M_{f_2}(x)]
\end{aligned}$$

where the first and third equalities are due to Lebesgue convergence theorem and the second one is due to the invariance of μ. Since μ is ergodic, $M_{f_2}(x) = \mathbb{E}_\mu f_2$ for μ-a.e. x. So, $\alpha = \mathbb{E}_\mu f_1 \mathbb{E}_\mu f_2$. It follows that

$$F_{\text{inv}}(\alpha) \leq \sup\{\dim \mu : \mu \text{ ergodic}, \mathbb{E}_\mu f_1 \mathbb{E}_\mu f_2 = \alpha\}.$$

To obtain the inverse inequality, it suffices to observe from standard higher–dimensional multifractal analysis for Hölder continuous functions that the above supremum is attained by a Gibbs measure ν which is mixing and that the mixing property implies $M_{f_1,f_2}(x) = \mathbb{E}_\nu f_1 \mathbb{E}_\nu f_2$ ν-a.e.. □

Remark 8.2. *In the above theorem, the assumption that μ is ergodic can be relaxed to μ is invariant. In fact, if ν is an invariant measure such that $\nu(E(\alpha)) = 1$. Then, by the ergodic decomposition theorem and the corresponding decomposition of entropy (a theorem due to Jacobs), there is an ergodic measure μ such that $\mu(E(\alpha)) = 1$ and $h_\nu \leq h_\mu$. When $\ell \geq 3$, the result in above theorem remains true if we replace "ergodic" by "multiple mixing", i.e.*

$$F_{\text{mix}}(\alpha) = \sup\{\dim \mu : \mu \text{ multiple mixing}, \ \mathbb{E}_\mu f_1 \cdots \mathbb{E}_\mu f_\ell = \alpha \ \},$$

where

$$F_{\text{mix}}(\alpha) = \sup\{\dim \mu : \mu \text{ multiple mixing}, \mu(E(\alpha)) = 1 \ \}.$$

Here is a remarkable corollary of the above theorem. Assume that $f_1 = f_2 = f$. If $\mu(E(\alpha) = 1$ for some ergodic measure μ, then we must have

$$\alpha = \left(\int f d\mu\right)^2 \geq 0.$$

There are examples of f taking negative value such that for some $\alpha < 0$ we have $\dim E(\alpha) > 0$. However, the theorem together with the remark shows that there is no invariant measure with positive dimension supported by $E(\alpha)$. See Example 2 below.

In the proof of the theorem, the fact that M_{f_1} is almost constant plays an important role. It is not the case for M_{f_1,f_2}. So we can not generalize the theorem to $\ell = 3$.

For $f_1, f_2 \in L^2(\mu)$ where μ is an ergodic measure, Bourgain proved that $M_{f_1,f_2}(x)$ exists for μ-almost all x. The limit is in general not constant, but can be written by the Kronecker factor (Z, m, S), which is considered as a rotation on a compact abelian group Z. Let π be the factor map. Let

$$\tilde{f}_i = \mathbb{E}(f_i|Z).$$

Then μ-almost surely

$$M_{f_1,f_2}(x) = \int_Z \tilde{f}_1(\pi(x) + z)\tilde{f}_1(\pi(x) + 2z)dm(z).$$

Then it is easy to deduce that $M_{f_1,f_2}(x)$ is μ-almost surely constant if and only if

$$\forall \gamma \in \widehat{Z} \text{ with } \gamma \neq 1, \widehat{\tilde{f}}_1(\gamma)\widehat{\tilde{f}}_2(\gamma^2) = 0.$$

This condition is extremely strong if μ is not weakly mixing. In other words, when $\ell = 3$, it would be exceptional that $E(\alpha)$ carries an ergodic

measure which is not weakly mixing. When μ is mixing, we have $M_{f_1,f_2}(x) = \int f_1 d\mu \int f_2 d\mu$ for μ-almost all x.

For three or more functions, the existence of the almost everywhere limit $M_{f_1,f_2,\cdots,f_\ell}$ is not yet proved. But the L^2-convergence is proved by Host and Kra [17]. The limit can be written as a similar integral, but the integral is taken over a nilmanifold of order 2 [5].

Let us also remark that the supremum in the theorem is also equal to the dimension of the α-level set of

$$\lim_{n\to\infty} \frac{1}{n^2} \sum_{1\le j,k\le n} f_1(T^j x) f_2(T^k x).$$

See [11]. Also see [14], where general V-statistics are studied.

9. Examples

The motivation of the subject initiated in [10] is the following example. The Riesz product method used in [10] doesn't work for this case. However Theorem 1.1 does.

Example 1. *Let $q = 2$, $m = 2$, $\ell = 2$ and φ the potential given by $\varphi(x, y) = x_1 y_1$ with $x = (x_i)_{i=1}^\infty, y = (y_i)_{i=1}^\infty \in \Sigma_2$. So*

$$[\varphi(i,j)]_{(i,j)\in\{0,1\}^2} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

The system of equations (5) in this case becomes

$$\begin{aligned} \psi_s(0)^2 &= \psi_s(0) + \psi_s(1), \\ \psi_s(1)^2 &= \psi_s(0) + e^s \psi_s(1). \end{aligned}$$

Fix $s \in \mathbb{R}$. By solving an fourth order algebraic equation, we get the unique positive solution of the above system:

$$\begin{aligned} \psi_s(0) &= \frac{1}{6} a(s) + \frac{2/3 - 2e^s}{a(s)} + \frac{2}{3}, \\ \psi_s(1) &= \psi_s(0)^2 - \psi_s(0), \end{aligned}$$

where

$$a(s) = \left(100 - 36e^s + 12\sqrt{69 - 54e^s - 3e^{2s} - 12e^{3s}}\right)^{\frac{1}{3}}.$$

Recall that the pressure function is equal to

$$P_\varphi(s) = \log(\psi_s(0) + \psi_s(1)).$$

The minimal and maximal values of φ are 0 and 1, which are respectively attained by the sequences $(x_j)_{j=0}^\infty = (0)^\infty$ and $(y_j)_{j=0}^\infty = (1)^\infty$ in the sense of

$$\varphi(x_j, x_{j+1}) = 0, \quad \varphi(y_j, y_{j+1}) = 1, \quad \forall j \ge 0.$$

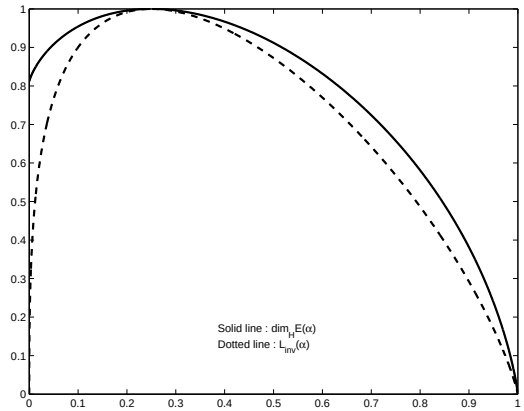

FIGURE 1. The graphs of the spectrum $\alpha \mapsto \dim_H E(\alpha)$ and $\alpha \mapsto F_{\text{inv}}(\alpha)$ (Example 1).

Then by Theorem 1.2, we have

$$P'_\varphi(-\infty) = 0, \quad P'_\varphi(+\infty) = 1.$$

Therefore, according to Theorem 1.1, for any $\alpha \in [0,1]$ we have

$$\dim_H E(\alpha) = \frac{-\alpha s_\alpha + P_\varphi(s_\alpha)}{2\log 2},$$

where s_α is the unique real such that $P'_\varphi(s_\alpha) = \alpha$.

We now consider the invariant spectrum of $E(\alpha)$. As $\varphi(x,y) = f(x)f(y)$ with $f(x) = x_1$, by Theorem 8.1, we have

$$F_{\text{inv}}(\alpha) = \sup\left\{\frac{h_\mu}{\log 2} \ : \ \mu \in \mathcal{M}_{\text{inv}}(\Sigma_2), \ \int x_1 d\mu = \sqrt{\alpha}\right\}.$$

It is well known (see [9]) that the right hand side, which is attained by a Bernoulli measure, is equal to

$$H(\sqrt{\alpha}) = -\sqrt{\alpha}\log_2\sqrt{\alpha} - (1-\sqrt{\alpha})\log_2(1-\sqrt{\alpha}).$$

So

$$F_{\text{inv}}(\alpha) \ = \ H(\sqrt{\alpha}).$$

See Figure 1 for the graphs of the spectra $\alpha \mapsto \dim_H E(\alpha)$ and $\alpha \mapsto L_{\text{inv}}(\alpha)$. We remark that, except at the extremal points ($\alpha = 1/4$ or 1), we have a strict inequality $F_{\text{inv}}(\alpha) < \dim_H E(\alpha)$. This shows that the invariant part of $E(\alpha)$ is much smaller than $E(\alpha)$ itself. This is different of the classical ergodic theory ($\ell = 1$) where in general we have $F_{\text{inv}}(\alpha) = \dim_H E(\alpha)$ for all α and actually $E(\alpha)$ is invariant.

The following example is a special case of a situation studied in [10]. So, the result is not new. Applying Theorem 1.1 only provides a second way to get it. But when we compare its invariant spectrum with its multifractal spectrum we will discover a new phenomenon–there is "no" invariant part in $E(\alpha)$ for some α.

Example 2. *Let $q = 2$, $m = 2$, $\ell = 2$ and φ be the potential given by $\varphi(x, y) = (2x_1 - 1)(2x_2 - 1)$. So*

$$[\varphi(i,j)]_{(i,j)\in\{0,1\}^2} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}.$$

The system of equations (5) in this case reduces to

$$\begin{aligned} \psi_s(0)^2 &= e^s\psi_s(0) + e^{-s}\psi_s(1), \\ \psi_s(1)^2 &= e^{-s}\psi_s(0) + e^s\psi_s(1). \end{aligned}$$

Because of the symmetry of φ, it is easy to find the unique positive solution of the system:

$$\psi_s(0) \;=\; \psi_s(1) \;=\; e^s + e^{-s}.$$

Thus we get the pressure function

$$P_\varphi(s) = \log(\psi_s(0) + \psi_s(1)) = \log 2 + \log(e^s + e^{-s}).$$

It is evident that

$$P'_\varphi(s) \;=\; \frac{e^s - e^{-s}}{e^s + e^{-s}}.$$

and

$$P'_\varphi(-\infty) = -1, \qquad P'_\varphi(+\infty) = 1.$$

So, by Theorem 1.1, we have $L_\varphi = [-1, 1]$, and for any $\alpha \in [-1, 1]$ we have

$$\dim_H E(\alpha) = \frac{-\alpha s_\alpha + P_\varphi(s_\alpha)}{2\log 2},$$

where s_α is such that

$$\frac{e^{s_\alpha} - e^{-s_\alpha}}{e^{s_\alpha} + e^{-s_\alpha}} = \alpha.$$

We now consider the invariant spectrum of $E(\alpha)$. We have $\varphi(x, y) = f(x)f(y)$ with $f(x) = 2x_1 - 1$, then by Theorem 8.1, we have

$$F_{\text{inv}}(\alpha) = \sup\left\{\frac{h_\mu}{\log 2} \;:\; \mu \in \mathcal{M}_{\text{inv}}(\Sigma_2),\ \left(\int (2x_1 - 1)d\mu\right)^2 = \alpha\right\}.$$

We see that we must assume $\alpha \geq 0$. As $\int(2x_1 - 1)d\mu = 2\int x_1 d\mu - 1$, the condition $\left(\int(2x_1 - 1)d\mu\right)^2 = \alpha$ means $\int x_1 d\mu = \frac{1}{2}(1 \pm \sqrt{\alpha})$. The above supremum is attained by a Bernoulli measure determined by the probability vector $((1 + \sqrt{\alpha})/2, (1 - \sqrt{\alpha})/2)$. In other word,

$$F_{\text{inv}}(\alpha) \;=\; H\left(\frac{1 + \sqrt{\alpha}}{2}\right)$$

where $H(x) = -x\log_2 x - (1 - x)\log_2(1 - x)$.

See Figure 2 for the graphs of the spectra $\alpha \mapsto \dim_H E(\alpha)$ and $\alpha \mapsto F_{\text{inv}}(\alpha)$. We see that, except at the extremal point $\alpha = 0$, we have $F_{\text{inv}}(\alpha) < \dim_H E(\alpha)$. Moreover, for $-1 \leq \alpha < 0$, we have $F_{\text{inv}}(\alpha) = 0$. That is to say, there is no invariant measure with positive dimension sitting on $E(\alpha)$ for $-1 \leq \alpha < 0$. But $\dim_H E(\alpha) > 0$.

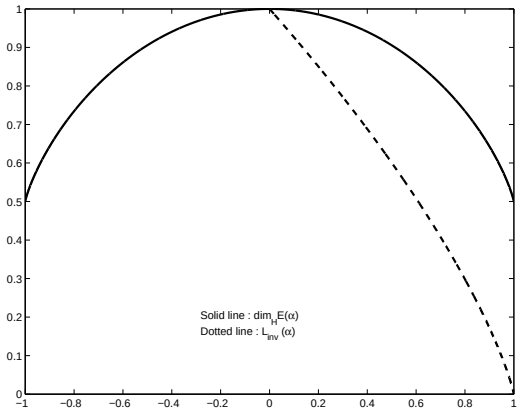

FIGURE 2. The graph of the spectrum $\alpha \mapsto \dim_H E(\alpha)$ (Example 2).

The following example presents a case where the L_φ is strictly contained in the interval $[\alpha_{\min}, \alpha_{\max}]$.

Example 3. *Let $q = 2$, $m = 2$, $\ell = 2$ and φ be the potential given by $\varphi(x, y) = y_1 - x_1$. In other words,*

$$[\varphi(i,j)]_{(i,j)\in\{0,1\}^2} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

The system of equations (5) in this case reduces to

$$\begin{aligned} \psi_s(0)^2 &= \psi_s(0) + e^s\psi_s(1), \\ \psi_s(1)^2 &= e^{-s}\psi_s(0) + \psi_s(1). \end{aligned}$$

It is easy to find the unique positive solution of the system:

$$\psi_s(0) = 1 + e^{\frac{s}{2}}, \qquad \psi_s(1) = 1 + e^{-\frac{s}{2}}.$$

The pressure function is then given by

$$P_\varphi(s) = \log(\psi_s(0) + \psi_s(1)) = \log(2 + e^{\frac{s}{2}} + e^{-\frac{s}{2}}).$$

So

$$P'_\varphi(s) = \frac{1}{2}\frac{e^{s/2} - e^{-s/2}}{2 + e^{s/2} + e^{-s/2}},$$

and

$$P'_\varphi(-\infty) = -\frac{1}{2}, \qquad P'_\varphi(+\infty) = \frac{1}{2}.$$

Remark that in this case we have

$$\alpha_{\min} < P'_\varphi(-\infty) < P'_\varphi(+\infty) < \alpha_{\max}.$$

By Theorem 1.1, we have $L_\varphi = [-1/2, 1/2]$, and for any $\alpha \in [-1/2, 1/2]$ we have

$$\dim_H E(\alpha) = \frac{-\alpha s_\alpha + P_\varphi(s_\alpha)}{2\log 2},$$

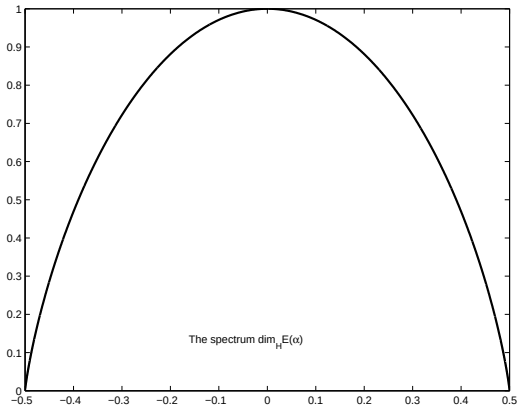

FIGURE 3. The graph of the spectrum $\alpha \mapsto \dim_H E(\alpha)$ (Example 3).

where s_α is the solution of

$$\frac{e^{s_\alpha/2} - e^{-s_\alpha/2}}{2 + e^{s_\alpha/2} + e^{-s_\alpha/2}} = 2\alpha.$$

We now consider the invariant spectrum of $E(\alpha)$. We have $\varphi(x, y) = f(y) - f(x)$ with $f(x) = x_1$. By Lebesgue convergence theorem, for any $\alpha \in \mathbb{R}$ such that there exists an invariant measure μ with $\mu(E(\alpha)) = 1$ we have

$$\alpha = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \mathbb{E}_\mu(x_{2k} - x_k) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} (\mathbb{E}_\mu(x_{2k}) - \mathbb{E}_\mu(x_k)) = 0.$$

(The last equality is due to the invariance of μ). This means that the only α such that there is an invariant measure with positive dimension sitting on $E(\alpha)$ is $\alpha = 0$. The invariant spectrum then degenerates to one point. We have $F_{\text{inv}}(0) = 1$.

See Figure 3 for the graph of the spectrum $\alpha \mapsto \dim_H E(\alpha)$.

We can easily solve the system (5) for a class of symmetric functions described in the following example. The example 2 is a special case.

Example 4. *Let $\ell = 2$, $q \geq 2$ and $m \geq 2$. Let $\varphi = [\varphi(i, j)]_{(i,j)\in\{0,\cdots,m-1\}^2}$ be a potential considered as a matrix. Suppose that each row of the matrix is a permutation of the first row.*

Recall the system of equations (5):

$$\psi_s(i)^q = \sum_{j=0}^{m-1} e^{s\varphi(i,j)} \psi_s(j), \quad i \in \{0, \cdots, m-1\}.$$

It is straightforward to verify that the constant vector $(a, \cdots, a)$, with

$$a = \left(\sum_{j=0}^{m-1} e^{s\varphi(1,j)} \right)^{\frac{1}{q-1}},$$

is the unique positive solution of the above system (see Theorem 4.1). The pressure function is then given by

$$P_\varphi(s) = \log \sum_{j=0}^{m-1} e^{s\varphi(1,j)} + (q-1)\log m.$$

We have

$$P'_\varphi(s) = \frac{\sum_{j=0}^{m-1} e^{s\varphi(1,j)}\varphi(1,j)}{\sum_{j=0}^{m-1} e^{s\varphi(1,j)}}.$$

Then

$$\lim_{s\to-\infty} P'_\varphi(s) = \lim_{s\to-\infty} \frac{\sum_{j=0}^{m-1} e^{s(\varphi(1,j)-\alpha_{\min})}\varphi(1,j)}{\sum_{j=0}^{m-1} e^{s(\varphi(1,j)-\alpha_{\min})}} = \alpha_{\min} = \min_j \varphi(1,j).$$

Similarly, we have

$$\lim_{s\to+\infty} P'_\varphi(s) = \alpha_{\max} = \max_j \varphi(1,j).$$

By the hypothesis of symmetry on φ, it is easy to see that there exist sequences $(x_j)_{j=0}^\infty$ and $(y_j)_{j=0}^\infty \in \Sigma_m$ such that

$$\varphi(x_j, x_{j+1}) = \alpha_{\min}, \qquad \varphi(y_j, y_{j+1}) = \alpha_{\max}, \ \forall j \geq 0.$$

Therefore, by Theorem 1.1, $L_\varphi = [\alpha_{\min}, \alpha_{\max}]$, and for any $\alpha \in [\alpha_{\min}, \alpha_{\max}]$ we have

$$\dim_H E(\alpha) = \frac{-\alpha s_\alpha + P_\varphi(s_\alpha)}{2\log m},$$

where s_α is the solution of

$$\frac{\sum_{j=0}^{m-1} e^{s_\alpha\varphi(1,j)}\varphi(1,j)}{\sum_{j=0}^{m-1} e^{s_\alpha\varphi(1,j)}} = \alpha.$$

The invariant spectrum: For $\alpha \in [\alpha_{\min}, \alpha_{\max}]$, the invariant spectrum is attained by a Markov measure. That is to say

$$F_{\text{inv}}(\alpha) = \sup\left\{ -\sum_{0\leq i,j\leq m-1} \pi_i p_{i,j} \log_m p_{i,j} : \sum_{0\leq i,j\leq m-1} \varphi(i,j)\pi_i p_{i,j} = \alpha \right\}$$

where $P = (p_{i,j})$ is a stochastic matrix and $\pi = (\pi_0, \cdots, \pi_{m-1})$ is an invariant probability vector of P, i.e. $\pi P = \pi$.

In the next example we show that in general the invariant spectrum can be strictly larger than the mixing spectrum for some level set $E(\alpha)$.

Example 5. *Let $m \geq 2$. Consider two functions f and h on Σ_m defined by*

$$f(i) = \begin{cases} 1 & 0 \leq i < m-1 \\ 2 & i = m-1 \end{cases} \qquad h(i) = \begin{cases} -2 & 0 \leq i < m-1 \\ 1 & i = m-1 \end{cases}.$$

Consider the level set

$$E(0) = \left\{ x \in \Sigma_m \,:\, \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} f(x_k) f(x_{2k}) h(x_{3k}) = 0 \right\}.$$

(That means $\phi(x,y,z) = f(x)f(y)h(z)$). We claim that $F_{\text{mix}}(0) < F_{\text{inv}}(0)$ for $m \geq 49$.

Let δ_j denotes the Dirac measure at $j \in \{0, 1, \cdots, m-1\}$. Let

$$\nu = \frac{1}{m-1} \sum_{j=0}^{m-2} \delta_j.$$

We note that ν restricted on Σ_{m-1} gives rise to the measure of maximal dimension on Σ_{m-1}. We consider a probability measure on Σ_m defined by

$$\mu = \frac{1}{2}\mu_1 + \frac{1}{2}\mu_2,$$

where

$$\mu_1([x_1 x_2 \cdots x_n]) = \prod_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \delta_{m-1}(x_{2k+1}) \cdot \prod_{k=1}^{\lfloor \frac{n}{2} \rfloor} \nu(x_{2k}),$$

and

$$\mu_2([x_1 x_2 \cdots x_n]) = \prod_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \nu(x_{2k+1}) \cdot \prod_{k=1}^{\lfloor \frac{n}{2} \rfloor} \delta_{m-1}(x_{2k}).$$

Note that $T^{-1} \circ \mu_1 = \mu_2$ and $T^{-1} \circ \mu_2 = \mu_1$. So μ is shift invariant. The measure μ sits on the set $A = A_1 \bigcup A_2$ where

$$A_1 = \{x \in \Sigma_m \,:\, x_{2k+1} = m-1,\, x_{2k} \neq m-1,\, k \in \mathbb{N}\},$$

$$A_2 = \{x \in \Sigma_m \,:\, x_{2k} = m-1,\, x_{2k+1} \neq m-1,\, k \in \mathbb{N}\}.$$

Actually $\mu_1(A_1) = 1$ and $\mu_2(A_2) = 1$ and the sets A_1 and A_2 are disjoint.

We claim that μ is ergodic but not mixing. To see that μ is not mixing, we only need to observe that $T^{-1}A_1 = A_2$ and $T^{-1}A_2 = A_1$. From this and that A_1 and A_2 are disjoint we deduce that

$$\mu\left(T^{-2k}A_1 \cap A_2\right) = 0, \quad \forall k \in \mathbb{N}.$$

This implies that μ is not mixing. The ergodicity of μ with respect to T is due to the fact that μ_1 and μ_2 are ergodic with respect to $T^2 = T \circ T$ and that they are supported by disjoint sets.

For every $x \in A_1$ we have

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} f(x_k) f(x_{2k}) h(x_{3k}) = \lim_{n\to\infty} \frac{1}{n} \left(\sum_{k-\text{even}} + \sum_{k-\text{odd}} \right) = \frac{1}{2}(-2+2) = 0$$

and for every $x \in A_2$ we have

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} f(x_k) f(x_{2k}) h(x_{3k}) = \lim_{n\to\infty} \frac{1}{n} \left(\sum_{k-\text{even}} + \sum_{k-\text{odd}} \right) = \frac{1}{2}(4-4) = 0.$$

Hence, $\mu(E(\alpha) = 1$. We note that

$$\int_{\Sigma_m} f\, d\mu \cdot \int_{\Sigma_m} f\, d\mu \cdot \int_{\Sigma_m} h\, d\mu = \left(\frac{3}{2}\right)^2 \cdot \left(-\frac{1}{2}\right) = -\frac{9}{8} < 0.$$

Let us compute the dimension of μ by computing the local entropy at typical points. If $x \in A$ then

$$\mu([x_1 \cdots x_{2n}]) = (m-1)^{-n}.$$

Since $\mu(A) = 1$ this implies that $\dim_H \mu = \frac{1}{2}\log(m-1)$. So that $F_{\text{inv}}(0) \geq \frac{1}{2}\log(m-1)$. On the other hand, by Theorem 8.1 and Remark 8.2, we have

$$F_{\text{mix}}(0) = \sup \left\{ h_\mu \,:\, \mu - \text{multiple mixing}, \quad \int_{\Sigma_m} h\, d\mu = 0 \right\}$$

since f is strictly positive. From standard multifractal analysis we know that the supremum is attained by a Bernoulli measure and

$$\begin{aligned} F_{\text{mix}}(0) &= \max_{p_i \geq 0} \left\{ -\sum_{i=0}^{m-1} p_i \log p_i \,:\, p_0 + \cdots + p_{m-2} = \frac{1}{3}, p_{m-1} = \frac{2}{3} \right\} \\ &= \frac{1}{3}\log(m-1) + \frac{1}{3}\log 3 + \frac{2}{3}\log\frac{3}{2}. \end{aligned}$$

If $m > 48$ we conclude $F_{\text{inv}}(0) > F_{\text{mix}}(0)$.

10. Remarks and Problems

Multiplicatively invariant sets. The first basic example (**Example 1** above) which motivated our study leads to the set

$$X_2 = \{(x_k)_{k\geq 1} \in \Sigma_2 : \forall k \geq 1, x_k x_{2k} = 0\}$$

which was introduced in [10]. It is known to Fürstenberg [15] that any shift-invariant closed set has its Hausdorff dimension equal to its Minkowski (box-counting) dimension. Unfortunately the closed set X_2 is not shift-invariant. Its Minkowski dimension was computed by Fan, Liao and Ma [10] and its Hausdorff dimension was computed by Kenyon, Peres and Solomyak [18]. The results show that the Hausdorff dimension is smaller than the Minkowski dimension. Recall that

$$\dim_M X_2 = 0.82429..., \quad \dim_H X_2 = 0.81137...$$

As observed by Kenyon, Peres and Solomyak, the set X_2 is invariant under the action of the semigroup $\mathbb{N}$ in the sense that $T_r X_2 \subset X_2$ for all $r \in \mathbb{N}$ where T_r is defined by

$$x = (x_k)_{k\geq 1} \mapsto T_r x = (x_{rk})_{k\geq 1}.$$

As observed by Fan, Liao and Ma, we have the decomposition

$$\mathbb{N} = \bigsqcup_{i:\text{odd}} i\Lambda$$

where $\Lambda = \{1, 2, 2^2, 2^3, \cdots\}$ is the (multiplicative) sub-semigroup generated by 2. This is one of the key point in the present study. A similar decomposition holds for semigroups generated by a finite number of prime numbers. Using this decomposition, Peres, Schmeling, Solomyak and Seuret [23] computed the Hausdorff dimension and the Minkowski dimension of sets like

$$X_{2,3} = \{(x_k)_{k\geq 1} \in \Sigma_2 : \forall k \geq 1, x_k x_{2k} x_{3k} = 0\}.$$

This is an important step.

A generalization. Combining the ideas in [23] and those in the present paper, we can study the following limit

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \varphi(x_k, x_{2k}, x_{3k}).$$

See [25]. Notice that the computation in this case are more involved. Also notice that, by chance, the Riesz product method used in [10] is well adapted to the study of the special limit

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} (2x_k - 1)(2x_{2k} - 1) \cdots (2x_{\ell k} - 1)$$

where $\ell \geq 2$ is any integer.

Vector valued potential. We indicate here how to extend our results to vector valued potentials. First, let φ, γ be 2 functions defined on S^ℓ taking real values. Instead of considering the transfer operator $\mathcal{L}_s$ as defined in (4), we consider the following one.

$$\mathcal{L}_s \psi(a) = \sum_{j\in S} e^{s\varphi(a,j)+\gamma(a,j)} \psi(Ta, j),\ a \in S^{\ell-1},\ s \in \mathbb{R}.$$

Still by Theorem 4.1, there exists a unique solution to the equation

$$(\mathcal{L}_s \psi)^{\frac{1}{q}} = \psi.$$

Then, we can similarly define the pressure function as indicated in (6) and (7). We denote this pressure function by $P_{\varphi,\gamma}(s)$. The arguments with which we proved the analyticity and convexity of $s \mapsto P_\varphi(s)$ can be also used to prove the same results for $s \mapsto P_{\varphi,\gamma}(s)$.

Let $\underline{\varphi} = (\varphi_1, \cdots, \varphi_d)$ be a function defined on S^ℓ taking values in $\mathbb{R}^d$. For $\underline{s} = (s_1, \cdots, s_d) \in \mathbb{R}^d$, we consider the following transfer operator.

$$\mathcal{L}_{\underline{s}} \psi(a) = \sum_{j\in S} e^{\langle \underline{s}, \underline{\varphi} \rangle} \psi(Ta, j),\ a \in S^{\ell-1},$$

where $\langle \cdot, \cdot \rangle$ denotes the scalar product in $\mathbb{R}^d$. We denote the associated pressure function by $P(\underline{\varphi})(\underline{s})$. Then, by the above discussion, for any vectors $u, v \in \mathbb{R}^d$ the function

$$\mathbb{R} \ni s \longmapsto P(\underline{\varphi})(us + v)$$

is analytical and convex. We deduce from this that the function

$$\underline{s} \longmapsto P(\underline{\varphi})(\underline{s})$$

is infinitely differentiable and convex on $\mathbb{R}^d$. We can prove that $P(\underline{\varphi})(\underline{s})$ is indeed analytical by the same argument used to prove the analyticity of $P_\varphi(s)$.

Similarly, we define the level sets $E(\underline{\alpha})$ ($\underline{\alpha} \in \mathbb{R}^d$) of $\underline{\varphi}$. A vector version of Theorem 1.1 is stated by just replacing the derivative of the pressure function by gradient.

We finish the paper with two problems.

Subshifts of finite type. Our study is strictly restricted to the full shift dynamics. It is a challenging problem to study the dynamics of subshift of finite type.

More general are dynamics with Markov property. More efforts are needed to deal with β-shift which are not Markovian. New ideas are needed to deal with these dynamics.

Nonlinear cookie cutter. The full shift is essentially the doubling dynamics $Tx = 2x \mod 1$ on the interval $[0, 1)$. Cookie cutters are the first interval maps coming into the mind after the doubling map. If the cookie cutter maps are not linear, it is a difficult problem.

Based on the computation made in [22], Liao and Rams [21] considered a special piecewise linear map of two branches defined on two intervals I_0 and I_1 and studied the following limit

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} 1_{I_1}(T^k x) 1_{I_1}(T^{2k} x).$$

The techniques presented in the present paper can be used to treat the problem for general piecewise linear cookie cutter dynamics [12, 25].

References

[1] I. Assani, *Multiple recurrence and almost sure convergence for weakly mixing dynamical systems*, Israel. J. Math. **1-3** (1987), 111–124.

[2] L. Barreira, "Dimension and recurrence in hyperbolic dynamics," Progress in Mathematics. Soc. **272**. Birkhäuser Verlag, Basel, 2008.

[3] L. Barreira, B. Saussol, J. Schmeling, *Higher-dimensional multifractal analysis*, J. Math. Pures Appl. **81** (2002), 67–91.

[4] V. Bergelson, *Weakly mixing PET*, Ergodic Theory Dynam. Systems **3** (1987), 337–349.

[5] V. Bergelson, B. Host and B. Kra, *Multiple recurrence and nilsequences* (with an appendix by I. Rusza), Inventiones Math. 160, 2 (2005), 261-303.

[6] J. Bourgain, *Double recurrence and almost sure convergence*, J. Reine Angew. Math. **404** (1990), 140-161.
[7] K. Falconer, "Fractal geometry. Mathematical foundations and applications," John Wiley Sons, Chichester, 1990.
[8] A.H. Fan, *Sur les dimensions de mesures*, Studia Math. **111** (1994), 1-17.
[9] A.H. Fan, D. J. Feng and J. Wu, *Recurrence, entropy and dimension*, J. London Math. Soc. **64** (2001), 229–244.
[10] A.H. Fan, L. M. Liao and J. H. Ma, *Level sets of multiple ergodic averages*, Monatsh. Math. **168** (2012), 17-26.
[11] A.H. Fan, L. M. Liao and J. Peyrière, *Generic points in systems of specification and Banach valued Birkhoff ergodic average*, Discrete Contin. Dyn. Syst. **21** (2008), 1103-1128.
[12] A.H. Fan, L. M. Liao and M. Wu, *Multifractal analysis of some multiple ergodic averages in linear Cookie-Cutter dynamical systems*, preprint.
[13] A.H. Fan, J. Schmeling and M. Wu, *Multifractal analysis of multiple ergodic averages*, Comptes Rendus Mathématique, Volume **349**, numéro 17–18 (2011), 961–964.
[14] A.H. Fan, J. Schmeling and M. Wu, *Multifractal analysis of V-statistics*, to appear in *Further Developments in Fractals and Related Fields*, Eds. J. Barral and S. Seuret, 2013.
[15] H. Fürstenberg, *Disjointness in ergodic theory, minimal sets, and a problem in Diophantine approximation*, Math. Systems Theory **1** (1967), 1–49.
[16] H. Fürstenberg, *Ergodic behavior of diagonal measures and a theorem of Szemerédi on arithmetic progressions*, J. d'Analyse Math. **31** (1977), 204–256.
[17] B. Host and B. Kra, *Nonconventional ergodic averages and nilmanifolds*, Ann. Math. **161** (2005), 397-488.
[18] R. Kenyon, Y. Peres and B. Solomyak, *Hausdorff dimension of the multiplicative golden mean shift*, Comptes Rendus Mathematique, volume **349**, numéro 11–12 (2011), 625–628.
[19] R. Kenyon, Y. Peres and B. Solomyak, *Hausdorff dimension for fractals invariant under the multiplicative integers*, to appear in Ergodic Theory Dynam. Systems.
[20] Yu. Kifer, *A nonconventional strong law of large numbers and fractal dimensions of some multiple recurrence sets*, preprint.
[21] L. M. Liao and M. Rams, *Multifractal analysis of some multiple ergodic averages for the systems with non-constant Lyapunov exponents*, preprint.
[22] Y. Peres and B. Solomyak, *Dimension spectrum for a nonconventional ergodic average*, to appear in Real Analysis Exchange.
[23] Y. Peres, J. Schmeling, B. Solomyak and S. Seuret, *Dimensions of some fractals defined via the semigroup denerated by 2 and 3*, preprint.
[24] R.S. Varga, "Geršgorin and his circles," Springer series in computational mathematics **36**, Springer-Verlag, Berlin, 2004.
[25] M. Wu, Ph D Thesis, in preparation.

A. F. Fan: LAMFA, UMR 7352 (ex 6140) CNRS, Université de Picardie, 33 rue Saint Leu, 80039 Amiens, France. E-mail: ai-hua.fan@u-picardie.fr

J. Scheming: MCMS, Lund Institute of Technology, Lund University Box 118 SE-221 00 Lund, Sweden. E-mail: joerg@maths.lth.se

M. Wu: LAMFA, UMR 7352 (ex 6140) CNRS, Université de Picardie, 33 rue Saint Leu, 80039 Amiens, France. E-mail: meng.wu@u-picardie.fr