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# An easy upper bound for Ramsey numbers

Roland Bacher

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*Abstract: We prove easy upper bounds for Ramsey numbers.*<sup>1</sup>

## 1 Introduction

A theorem of Ramsey, see [4], implies the existence of a smallest natural integer  $R(n)$ , now called the *n-th Ramsey number*, such that every (simple unoriented) graph  $G$  with at least  $R(n)$  vertices contains either a complete graph with  $n$  vertices or  $n$  pairwise non-adjacent vertices (defining a complete graph in the complementary graph of  $G$ ).

The aim of this paper is to give a new simple proof of the following upper bound for Ramsey numbers:

**Theorem 1.1.** *We have*

$$R(n) \leq 2^{2n-3}$$

for  $n \geq 2$ .

The currently best asymptotic upper bound,

$$R(n+1) \leq \binom{2n}{n} n^{-C \log n / \log \log n},$$

(for a suitable constant  $C$ ) is due to Conlon, see [2].

The standard proof of Ramsey's theorem, due to Erdős and Szekeres (see [3] or Chapter 35 of [1]), uses a two parameter Ramsey number  $R(a, b)$  defined as the smallest integer such that every graph with  $R(a, b)$  vertices contains either a complete graph with  $a$  vertices or a subset of  $b$  non-adjacent vertices. It is slightly more involved than our proof and gives the upper bound  $R(n+1) \leq \binom{2n}{n}$  (based on the trivial values  $R(a, 1) = R(1, a) = 1$  and on the inequality  $R(a, b) \leq R(a-1, b) + R(a, b-1)$  for  $a, b > 1$ ).

Simple graphs are equivalent to complete graphs with edges of two colours (encoding edges, respectively nonedges of simple graphs). There is a generalization of Ramsey's theorem to an arbitrary finite number  $m$

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of colours as follows: There exists a smallest natural number  $R_m(n)$  such that every complete graph on  $R_m(n)$  vertices with edges of  $m$  colours contains  $n$  vertices belonging to a complete edge-monochromatic subgraph. The following result gives an upper bound for  $R_m(n)$ :

**Theorem 1.2.** *We have*

$$R_m(n) \leq 1 + \sum_{j=0}^{m(n-2)} m^j = 1 + \frac{m^{mn-2m+1} - 1}{m - 1}$$

for  $m, n \geq 2$ .

For  $m = 2$ , the upper bound  $1 + \frac{m^{mn-2m+1}-1}{m-1}$  of Theorem 1.2 coincides with the upper bound  $2^{2n-3} = 1 + \frac{2^{2n-4+1}-1}{2-1} = 2^{2n-3}$  for  $R_2(n) = R(n)$  given by Theorem 1.1.

This paper contains a simple proof of Theorem 1.1 (Section 2) and Theorem 1.2 (Section 3) which is a variation on the proofs usually found and is perhaps slightly simpler. In Section 4 we discuss a few generalizations of the numbers  $R'(n)$  and  $R'_m(n)$  playing a crucial role in the proofs.

## 2 Proof of Theorem 1.1

Given a finite graph  $G$ , we define  $\rho'(G)$  to be the largest natural number such that  $G$  contains two (not necessarily disjoint) subsets  $A$  and  $B$  of vertices satisfying the following two conditions:

1. All vertices of  $A$  are adjacent to each other and no vertices of  $B$  are adjacent.
2.  $\#(A) + \#(B) = \rho'(G)$ .

In this section, the letter  $A$  always denotes a set of pairwise adjacent vertices and  $B$  denotes a set of pairwise non-adjacent vertices. Two such subsets  $A, B$  of vertices in a graph  $G$  realize  $\rho'(G)$  if  $\rho'(G) = \#(A) + \#(B)$ .

We define  $R'(n)$  as the smallest natural integer such that  $\rho'(G) \geq n$  for every graph  $G$  with  $R'(n)$  vertices.

**Lemma 2.1.** *We have  $R(n) \leq R'(2n - 1)$ .*

**Proof** A graph  $G$  with  $R'(2n - 1)$  vertices contains subsets  $A$  and  $B$  of vertices realizing  $\rho'(G) \geq 2n - 1$ . One of the subsets  $A, B$  thus contains at least  $n$  vertices. If  $\#(A) \geq n$ , the graph  $G$  contains a complete subgraph of  $n$  vertices, if  $\#(B) \geq n$ , the graph  $G$  contains  $n$  pairwise non-adjacent vertices.  $\square$

**Lemma 2.2.** *We have  $R'(n + 1) \leq 2R'(n)$ .*

**Proof** We choose a vertex  $v$  in a graph  $G$  with  $2R'(n)$  vertices. We denote by  $G_v$  the subgraph of  $G \setminus \{v\}$  defined by all neighbours of  $v$ . Up to replacing  $G$  by its complementary graph (and exchanging the roles of the sets  $A$  and  $B$ ), we can suppose that  $G_v$  has at least  $\lceil (2R'(n) - 1)/2 \rceil = R'(n)$  vertices. Hence we have  $\rho'(G) \geq n$  and we can find subsets  $A, B$  of vertices in  $G_v$  which realize  $\rho'(G)$ . The subset  $A \cup \{v\}$  contains thus  $\#(A) + 1$  pairwise adjacent vertices of  $G$  and we have  $\rho'(G) \geq \#(A \cup \{v\}) + \#(B) = \rho'(G_v) + 1 \geq n + 1$ .  $\square$

**Proposition 2.3.** *We have  $R'(n) \leq 2^{n-2}$  for  $n \geq 2$ .*

**Proof** If  $n = 2$  we take  $A = B = \{v\}$  where  $v$  is the unique vertex of the trivial graph  $G = \{v\}$  on one vertex  $v$ .

Induction on  $n$  using Lemma 2.2 ends the proof.  $\square$

**Proof of Theorem 1.1** The proof follows from the inequalities

$$R(n) \leq R'(2n - 1) \leq 2^{2n-1-2} = 2^{2n-3}$$

given by Lemma 2.2 and Proposition 2.3.  $\square$

**Remark 2.4.** *The proof of Proposition 2.3 can easily be rewritten algorithmically: Given a graph  $G$  with at least  $2^{n-2} \geq 2$  vertices, set  $A = B = \emptyset$ . While  $G$  has at least 4 vertices, choose a vertex  $v$ . Set  $A = A \cup \{v\}$  and replace  $G$  by the subgraph induced by all neighbours of  $v$  if  $v$  has more neighbours than non-adjacent vertices in  $G$ . Otherwise set  $B = B \cup \{v\}$  and replace  $G$  by the subgraph induced by all non-neighbours ( $\neq v$ ) of  $v$ .*

*If  $G$  has 2 or 3 vertices, choose two vertices  $v, w$  and replace  $A$  by  $A \cup \{v, w\}$  if  $v$  and  $w$  are adjacent. Otherwise replace  $B$  by  $B \cup \{v, w\}$ .*

## 2.1 Value of $R(3)$

The inequality  $R'(4) \leq 2^{4-2} = 4$  given by the case  $n = 2$  of Proposition 2.3 is not sharp: Indeed, we have  $R'(4) = 3$  as can be seen by inspecting all four possible graphs on three vertices. (The set  $A$  has respectively 1, 2, 2, 3 elements for a 3-vertex graph with 0, 1, 2, 3 edges.) Lemma 2.2 shows now  $R'(5) \leq 6$  and we get  $R(3) \leq R'(5) \leq 6$  by Lemma 2.1. Since a cycle with 5 vertices contains no triangle and no triplet of pairwise non-adjacent vertices, both inequalities are sharp and we have  $R(3) = R'(5) = 6$ .

**Remark 2.5.** *The value  $R'(4) = 3$  can of course be used for improving the upper bound  $R(n) \leq 2^{2n-3}$  in Theorem 1.1 to  $3 \cdot 2^{2n-5}$  for  $n \geq 3$ . More generally, any interesting upper bound on  $R'(n)$  for  $n > 4$  easily yields an improvement of Theorem 1.1.*

### 3 Proof of Theorem 1.2

We define  $R'_m(n)$  to be the smallest integer such that every complete graph with  $R'_m(n)$  vertices and edges of  $m$  colours contains  $m$  (not necessarily disjoint) subsets of vertices  $A_1, \dots, A_m$  with  $\#(A_1) + \#(A_2) + \dots + \#(A_m) = n$  and with  $A_1, \dots, A_m$  defining  $m$  complete edge-monochromatic graphs of different edge-colours.

Given a complete graph  $G$  with  $m$ -coloured edges, we denote by  $\rho'(G)$  the largest integer such that  $G$  contains  $m$  (not necessarily disjoint) subsets  $A_1, \dots, A_m$  of vertices defining complete edge-monochromatic subgraphs of different colours and  $\rho'(G) = \#(A_1) + \dots + \#(A_m)$ . We say that  $m$  such subsets  $A_1, \dots, A_m$  realize  $\rho'(G)$ .

We have of course  $\rho'(G) \geq n$  if  $G$  contains at least  $R'(n)$  vertices.

**Examples:**

1. We have  $R'_m(m) = 1$  by setting  $A_1 = A_2 = \dots = A_m = \{v\}$  where  $v$  is the unique vertex of the trivial graph with one vertex (the empty sets of edges in  $A_1, \dots, A_m$  have different colours by convention).

The value  $R'_m(m) = 1$  also follows from  $R_m(1) = 1$  applied to the trivial inequality  $R'_m(n + m - 1) \leq R_m(n)$  obtained by completing a complete edge-monochromatic subgraph on  $n$  vertices with  $m - 1$  singletons representing complete edge-monochromatic subgraphs of the  $m - 1$  remaining colours.

2.  $R'_m(m + 1) = 2$  since an edge-coloured complete graph on 2 vertices is always monochromatic.
3.  $R'_m(m + 2) = 3$  since every edge-coloured triangle is either edge-monochromatic or contains two edges of different colours.

**Lemma 3.1.** *We have  $R_m(n) \leq R'(m(n - 1) + 1)$ .*

**Proof** A set of  $m$  integers summing up to  $m(n - 1) + 1$  contains an element at least equal to  $n$ . For every realization  $A_1, \dots, A_m$  of  $\rho'(G) \geq m(n - 1) + 1$  of a graph  $G$  with  $R'(m(n - 1) + 1)$  vertices there thus exists an index  $i$  such that  $A_i$  defines an edge-monochromatic complete graph on at least  $n$  vertices.  $\square$

**Lemma 3.2.** *We have  $R'_m(n + 1) \leq 2 + m(R'_m(n) - 1)$ .*

**Proof** Fixing a vertex  $v$  in a complete graph  $G$  with  $2 + m(R'_m(n) - 1)$  vertices and edges of  $m$  colours, we get a partition  $V \setminus \{v\} = V_1 \cup \dots \cup V_m$  of all vertices different from  $v$  by considering the set  $V_i$  of vertices joined by an edge of colour  $i$  to  $v$ . Since  $V \setminus \{v\}$  has  $1 + m(R'_m(n) - 1)$  elements, there exists a set  $V_i$  containing at least  $R'_m(n)$  vertices. The subgraph  $G_i$  with vertices  $V_i$  thus contains a realization  $A_1, \dots, A_m$  of  $\rho'_m(G_i) \geq n$ . Since  $v$  is

joined by edges of colour  $i$  to all elements of  $A_i$ , the set of vertices  $A_i \cup \{v\}$  defines a complete edge-monochromatic subgraph of colour  $i$  in  $G$ . This proves  $\rho'(G) \geq \sharp(A_1) + \dots + \sharp(A_i \cup \{v\}) + \dots + \sharp(A_m) = \rho'G_i) + 1 \geq n + 1$ .  $\square$

**Proposition 3.3.** *We have*

$$R'_m(m+k) \leq 1 + \sum_{j=0}^{k-1} m^j = 1 + \frac{m^k - 1}{m - 1}$$

for every natural integer  $k$  (using the convention  $\sum_{j=0}^{-1} m^j = 0$  if  $k = 0$ ).

**Proof** The formula holds for  $k = 0$  with  $A_1 = A_2 = \dots = A_m = \{v\}$  the unique vertex of the trivial graph  $\{v\}$  reduced to one vertex.

Using Lemma 3.2 and induction on  $k$  we have

$$\begin{aligned} R'(m+k+1) &\leq 2 + m(R'_m(m+k) - 1) \\ &\leq 2 + m \left( \left( 1 + \sum_{j=0}^{k-1} m^j \right) - 1 \right) \\ &= 1 + \sum_{j=0}^k m^j \end{aligned}$$

which ends the proof.  $\square$

**Proof of Theorem 1.2** We have

$$R_m(n) \leq R'(m(n-1) + 1) \leq 1 + \frac{m^{m(n-1)+1-m} - 1}{m - 1} = 1 + \frac{m^{mn-2m+1} - 1}{m - 1}$$

where the first inequality is Lemma 3.1 and the second inequality is Proposition 3.3.  $\square$

## 4 Generalizations of the number $R'_m(n)$

The number  $R'_m(n)$  has two obvious generalizations.

The first one is given by considering  $R'_{m,j}(n)$  with  $j \in \{1, \dots, m\}$  defined as the smallest integer such that every complete graph with  $R'_{m,j}(n)$  vertices and edges of  $m$  colours contains  $j$  edge-monochromatic complete subgraphs of different edge-colours and of size  $\alpha_1, \dots, \alpha_j$  such that  $\alpha_1 + \dots + \alpha_j = n$ . Therefore we consider only the  $j$  colours corresponding to the  $j$  largest edge-monochromatic complete subgraphs. For  $j = 1$ , we recover the usual Ramsey numbers  $R_m(n)$ , for  $j = m$  we get the numbers  $R'_m(n)$  introduced previously.

The second generalization depends on an unbounded function  $s : \mathcal{G} \rightarrow \mathbb{N}$  (one can also work with  $m$  different unbounded functions  $s_c : \mathcal{G} \rightarrow \mathbb{N}$

indexed by colours or replace the target-set of natural integers by the set of non-negative real numbers) on the set  $\mathcal{G}$  of all finite simple graphs.

For  $n \geq 1$  we define  $R'_{m,s}(n)$  as the smallest integer such that every complete graph on  $R'_{m,s}(n)$  vertices contains  $m$  (not necessarily complete) edge-monochromatic subgraphs  $G_1, \dots, G_m$  of colour  $1, \dots, m$  satisfying  $s(G_1) + s(G_2) + \dots + s(G_m) \geq n$  (respectively  $s_1(G_1) + \dots + s_m(G_m) \geq n$ ).

The numbers  $R'_m(n)$  correspond to the choice  $s(G) = n$  if  $G$  is the complete graph on  $n$  vertices and  $s(G) = 0$  otherwise.

Other perhaps interesting choices are  $s(G) = n$  if  $G$  is an  $n$ -cycle and  $s(G) = 0$  otherwise, or  $s(G) = n$  if  $G$  is a simple path (two endpoints of degree 1 and all other vertices of degree 2) with  $n$  vertices.

It is of course possible to combine both generalizations by defining  $R'_{m,j,s}(n)$  in the obvious way considering only the  $j$  colours giving the largest contributions to the sum  $s(G_1) + \dots + s(G_m)$ .

#### 4.1 Analogues of $R'$ for van der Waerden numbers

Van der Waerden's Theorem gives the existence of a function  $W : \{2, 3, 4, \dots\} \times \{2, 3, 4, \dots\} \rightarrow \mathbb{N}$  associating to two integers  $m, n \geq 2$  the smallest natural integer  $W(m, n)$  such that every colouring of the  $W(m, n)$  consecutive natural integers  $1, 2, \dots, W(m, n)$  with  $m$  colours contains a monochromatic arithmetic progression with  $n$  elements.

We define  $W'(m, n)$  in the obvious way as the smallest natural integer such that every colouring of  $1, 2, \dots, W'(m, n)$  with  $m$  colours contains  $m$  (perhaps empty) monochromatic progressions of different colours and of lengths  $\alpha_1, \dots, \alpha_m$  summing up to  $n = \alpha_1 + \dots + \alpha_m$ .

We have  $W(m, n) \leq W'(m, m(n-1) + 1)$  since a set of  $m$  integers strictly smaller than  $n$  sums up at most to  $m(n-1)$ . It is easy to check that  $W'(2, 1) = 1, W'(2, 2) = 2$  and  $W'(2, 3) = 3$ .

For  $W'(2, 4)$  we get  $W'(2, 4) = 6$  as can be seen as follows:  $W'(2, 4) > 5$  by inspection of the black-white colouring  $bbwbb$  of  $1, 2, 3, 4, 5$ . Consider a black-white colouring of  $1, \dots, 6$  not containing a black progression of size  $\alpha$  and a white progression of size  $\beta$  such that  $\alpha + \beta \geq 4$ . Such a colouring cannot use only one colour (otherwise we can take  $\alpha = 6$  or  $\beta = 6$ ). It cannot use both colours twice (otherwise we can take  $\alpha = 2$  and  $\beta = 2$ ). It uses thus one colour, say white, only once and we have necessarily  $\alpha \geq 3$  and  $\beta = 1$  since either  $1, 3, 5$  (for an even white element) or  $2, 4, 6$  (for an odd white element) are all black.

It is of course also possible to consider the numbers  $W'_j(m, n)$  defined by considering only the  $j$  largest arithmetical progressions. For  $j = 1$  we get the classical van der Waerden number  $W(m, n)$ .

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