



**HAL**  
open science

## An easy upper bound for Ramsey numbers

Roland Bacher

► **To cite this version:**

| Roland Bacher. An easy upper bound for Ramsey numbers. 2012. hal-00763927

**HAL Id: hal-00763927**

**<https://hal.science/hal-00763927v1>**

Preprint submitted on 11 Dec 2012

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# An easy upper bound for Ramsey numbers

Roland Bacher

March 15, 2016

*Abstract:* We prove easy upper bounds for Ramsey numbers.<sup>1</sup>

## 1 Introduction

A theorem of Ramsey, see [4], implies the existence of a smallest natural integer  $R(n)$ , now called the  $n$ -th Ramsey number, such that every (simple unoriented) graph  $G$  with at least  $R(n)$  vertices contains either a complete graph with  $n$  vertices or  $n$  pairwise non-adjacent vertices (defining a complete graph in the complementary graph of  $G$ ).

The aim of this paper is to give a new simple proof of the following upper bound for Ramsey numbers:

**Theorem 1.1.** *We have*

$$R(n) \leq 2^{2n-3}$$

for  $n \geq 2$ .

The currently best asymptotic upper bound,

$$R(n+1) \leq \binom{2n}{n} n^{-C \log n / \log \log n},$$

(for a suitable constant  $C$ ) is due to Conlon, see [2].

The standard proof of Ramsey's theorem, due to Erdős and Szekeres (see [3] or Chapter 35 of [1]), uses a two parameter Ramsey number  $R(a, b)$  defined as the smallest integer such that every graph with  $R(a, b)$  vertices contains either a complete graph with  $a$  vertices or a subset of  $b$  non-adjacent vertices. It is slightly more involved than our proof and gives the upper bound  $R(n+1) \leq \binom{2n}{n}$  (based on the trivial values  $R(a, 1) = R(1, a) = 1$  and on the inequality  $R(a, b) \leq R(a-1, b) + R(a, b-1)$  for  $a, b > 1$ ).

Simple graphs are equivalent to complete graphs with edges of two colours (encoding edges, respectively nonedges of simple graphs). There is a generalization of Ramsey's theorem to an arbitrary finite number  $m$

---

<sup>1</sup>Keywords: Ramsey theory. Math. class: 5D10, 5C55

of colours as follows: There exists a smallest natural number  $R_m(n)$  such that every complete graph on  $R_m(n)$  vertices with edges of  $m$  colours contains  $n$  vertices belonging to a complete edge-monochromatic subgraph. The following result gives an upper bound for  $R_m(n)$ :

**Theorem 1.2.** *We have*

$$R_m(n) \leq 1 + \sum_{j=0}^{m(n-2)} m^j = 1 + \frac{m^{mn-2m+1} - 1}{m - 1}$$

for  $m, n \geq 2$ .

For  $m = 2$ , the upper bound  $1 + \frac{m^{mn-2m+1}-1}{m-1}$  of Theorem 1.2 coincides with the upper bound  $2^{2n-3} = 1 + \frac{2^{2n-4+1}-1}{2-1} = 2^{2n-3}$  for  $R_2(n) = R(n)$  given by Theorem 1.1.

This paper contains a simple proof of Theorem 1.1 (Section 2) and Theorem 1.2 (Section 3) which is a variation on the proofs usually found and is perhaps slightly simpler. In Section 4 we discuss a few generalizations of the numbers  $R'(n)$  and  $R'_m(n)$  playing a crucial role in the proofs.

## 2 Proof of Theorem 1.1

Given a finite graph  $G$ , we define  $\rho'(G)$  to be the largest natural number such that  $G$  contains two (not necessarily disjoint) subsets  $A$  and  $B$  of vertices satisfying the following two conditions:

1. All vertices of  $A$  are adjacent to each other and no vertices of  $B$  are adjacent.
2.  $\#(A) + \#(B) = \rho'(G)$ .

In this section, the letter  $A$  always denotes a set of pairwise adjacent vertices and  $B$  denotes a set of pairwise non-adjacent vertices. Two such subsets  $A, B$  of vertices in a graph  $G$  realize  $\rho'(G)$  if  $\rho'(G) = \#(A) + \#(B)$ .

We define  $R'(n)$  as the smallest natural integer such that  $\rho'(G) \geq n$  for every graph  $G$  with  $R'(n)$  vertices.

**Lemma 2.1.** *We have  $R(n) \leq R'(2n - 1)$ .*

**Proof** A graph  $G$  with  $R'(2n - 1)$  vertices contains subsets  $A$  and  $B$  of vertices realizing  $\rho'(G) \geq 2n - 1$ . One of the subsets  $A, B$  thus contains at least  $n$  vertices. If  $\#(A) \geq n$ , the graph  $G$  contains a complete subgraph of  $n$  vertices, if  $\#(B) \geq n$ , the graph  $G$  contains  $n$  pairwise non-adjacent vertices.  $\square$

**Lemma 2.2.** *We have  $R'(n + 1) \leq 2R'(n)$ .*

**Proof** We choose a vertex  $v$  in a graph  $G$  with  $2R'(n)$  vertices. We denote by  $G_v$  the subgraph of  $G \setminus \{v\}$  defined by all neighbours of  $v$ . Up to replacing  $G$  by its complementary graph (and exchanging the roles of the sets  $A$  and  $B$ ), we can suppose that  $G_v$  has at least  $\lceil (2R'(n) - 1)/2 \rceil = R'(n)$  vertices. Hence we have  $\rho'(G) \geq n$  and we can find subsets  $A, B$  of vertices in  $G_v$  which realize  $\rho'(G)$ . The subset  $A \cup \{v\}$  contains thus  $\#(A) + 1$  pairwise adjacent vertices of  $G$  and we have  $\rho'(G) \geq \#(A \cup \{v\}) + \#(B) = \rho'(G_v) + 1 \geq n + 1$ .  $\square$

**Proposition 2.3.** *We have  $R'(n) \leq 2^{n-2}$  for  $n \geq 2$ .*

**Proof** If  $n = 2$  we take  $A = B = \{v\}$  where  $v$  is the unique vertex of the trivial graph  $G = \{v\}$  on one vertex  $v$ .

Induction on  $n$  using Lemma 2.2 ends the proof.  $\square$

**Proof of Theorem 1.1** The proof follows from the inequalities

$$R(n) \leq R'(2n - 1) \leq 2^{2n-1-2} = 2^{2n-3}$$

given by Lemma 2.2 and Proposition 2.3.  $\square$

**Remark 2.4.** *The proof of Proposition 2.3 can easily be rewritten algorithmically: Given a graph  $G$  with at least  $2^{n-2} \geq 2$  vertices, set  $A = B = \emptyset$ . While  $G$  has at least 4 vertices, choose a vertex  $v$ . Set  $A = A \cup \{v\}$  and replace  $G$  by the subgraph induced by all neighbours of  $v$  if  $v$  has more neighbours than non-adjacent vertices in  $G$ . Otherwise set  $B = B \cup \{v\}$  and replace  $G$  by the subgraph induced by all non-neighbours ( $\neq v$ ) of  $v$ .*

*If  $G$  has 2 or 3 vertices, choose two vertices  $v, w$  and replace  $A$  by  $A \cup \{v, w\}$  if  $v$  and  $w$  are adjacent. Otherwise replace  $B$  by  $B \cup \{v, w\}$ .*

## 2.1 Value of $R(3)$

The inequality  $R'(4) \leq 2^{4-2} = 4$  given by the case  $n = 2$  of Proposition 2.3 is not sharp: Indeed, we have  $R'(4) = 3$  as can be seen by inspecting all four possible graphs on three vertices. (The set  $A$  has respectively 1, 2, 2, 3 elements for a 3-vertex graph with 0, 1, 2, 3 edges.) Lemma 2.2 shows now  $R'(5) \leq 6$  and we get  $R(3) \leq R'(5) \leq 6$  by Lemma 2.1. Since a cycle with 5 vertices contains no triangle and no triplet of pairwise non-adjacent vertices, both inequalities are sharp and we have  $R(3) = R'(5) = 6$ .

**Remark 2.5.** *The value  $R'(4) = 3$  can of course be used for improving the upper bound  $R(n) \leq 2^{2n-3}$  in Theorem 1.1 to  $3 \cdot 2^{2n-5}$  for  $n \geq 3$ . More generally, any interesting upper bound on  $R'(n)$  for  $n > 4$  easily yields an improvement of Theorem 1.1.*

### 3 Proof of Theorem 1.2

We define  $R'_m(n)$  to be the smallest integer such that every complete graph with  $R'_m(n)$  vertices and edges of  $m$  colours contains  $m$  (not necessarily disjoint) subsets of vertices  $A_1, \dots, A_m$  with  $\#(A_1) + \#(A_2) + \dots + \#(A_m) = n$  and with  $A_1, \dots, A_m$  defining  $m$  complete edge-monochromatic graphs of different edge-colours.

Given a complete graph  $G$  with  $m$ -coloured edges, we denote by  $\rho'(G)$  the largest integer such that  $G$  contains  $m$  (not necessarily disjoint) subsets  $A_1, \dots, A_m$  of vertices defining complete edge-monochromatic subgraphs of different colours and  $\rho'(G) = \#(A_1) + \dots + \#(A_m)$ . We say that  $m$  such subsets  $A_1, \dots, A_m$  realize  $\rho'(G)$ .

We have of course  $\rho'(G) \geq n$  if  $G$  contains at least  $R'(n)$  vertices.

**Examples:**

1. We have  $R'_m(m) = 1$  by setting  $A_1 = A_2 = \dots = A_m = \{v\}$  where  $v$  is the unique vertex of the trivial graph with one vertex (the empty sets of edges in  $A_1, \dots, A_m$  have different colours by convention).

The value  $R'_m(m) = 1$  also follows from  $R_m(1) = 1$  applied to the the trivial inequality  $R'_m(n + m - 1) \leq R_m(n)$  obtained by completing a complete edge-monochromatic subgraph on  $n$  vertices with  $m - 1$  singletons representing complete edge-monochromatic subgraphs of the  $m - 1$  remaining colours.

2.  $R'_m(m + 1) = 2$  since an edge-coloured complete graph on 2 vertices is always monochromatic.
3.  $R'_m(m + 2) = 3$  since every edge-coloured triangle is either edge-monochromatic or contains two edges of different colours.

**Lemma 3.1.** *We have  $R_m(n) \leq R'(m(n - 1) + 1)$ .*

**Proof** A set of  $m$  integers summing up to  $m(n - 1) + 1$  contains an element at least equal to  $n$ . For every realization  $A_1, \dots, A_m$  of  $\rho'(G) \geq m(n - 1) + 1$  of a graph  $G$  with  $R'(m(n - 1) + 1)$  vertices there thus exists an index  $i$  such that  $A_i$  defines an edge-monochromatic complete graph on at least  $n$  vertices.  $\square$

**Lemma 3.2.** *We have  $R'_m(n + 1) \leq 2 + m(R'_m(n) - 1)$ .*

**Proof** Fixing a vertex  $v$  in a complete graph  $G$  with  $2 + m(R'_m(n) - 1)$  vertices and edges of  $m$  colours, we get a partition  $V \setminus \{v\} = V_1 \cup \dots \cup V_m$  of all vertices different from  $v$  by considering the set  $V_i$  of vertices joined by an edge of colour  $i$  to  $v$ . Since  $V \setminus \{v\}$  has  $1 + m(R'_m(n) - 1)$  elements, there exists a set  $V_i$  containing at least  $R'_m(n)$  vertices. The subgraph  $G_i$  with vertices  $V_i$  thus contains a realization  $A_1, \dots, A_m$  of  $\rho'_m(G_i) \geq n$ . Since  $v$  is

joined by edges of colour  $i$  to all elements of  $A_i$ , the set of vertices  $A_i \cup \{v\}$  defines a complete edge-monochromatic subgraph of colour  $i$  in  $G$ . This proves  $\rho'(G) \geq \sharp(A_1) + \dots + \sharp(A_i \cup \{v\}) + \dots + \sharp(A_m) = \rho'(G_i) + 1 \geq n + 1$ .  $\square$

**Proposition 3.3.** *We have*

$$R'_m(m+k) \leq 1 + \sum_{j=0}^{k-1} m^j = 1 + \frac{m^k - 1}{m - 1}$$

for every natural integer  $k$  (using the convention  $\sum_{j=0}^{-1} m^j = 0$  if  $k = 0$ ).

**Proof** The formula holds for  $k = 0$  with  $A_1 = A_2 = \dots = A_m = \{v\}$  the unique vertex of the trivial graph  $\{v\}$  reduced to one vertex.

Using Lemma 3.2 and induction on  $k$  we have

$$\begin{aligned} R'(m+k+1) &\leq 2 + m(R'_m(m+k) - 1) \\ &\leq 2 + m \left( \left( 1 + \sum_{j=0}^{k-1} m^j \right) - 1 \right) \\ &= 1 + \sum_{j=0}^k m^j \end{aligned}$$

which ends the proof.  $\square$

**Proof of Theorem 1.2** We have

$$R_m(n) \leq R'(m(n-1) + 1) \leq 1 + \frac{m^{m(n-1)+1-m} - 1}{m - 1} = 1 + \frac{m^{mn-2m+1} - 1}{m - 1}$$

where the first inequality is Lemma 3.1 and the second inequality is Proposition 3.3.  $\square$

## 4 Generalizations of the number $R'_m(n)$

The number  $R'_m(n)$  has two obvious generalizations.

The first one is given by considering  $R'_{m,j}(n)$  with  $j \in \{1, \dots, m\}$  defined as the smallest integer such that every complete graph with  $R'_{m,j}(n)$  vertices and edges of  $m$  colours contains  $j$  edge-monochromatic complete subgraphs of different edge-colours and of size  $\alpha_1, \dots, \alpha_j$  such that  $\alpha_1 + \dots + \alpha_j = n$ . Therefore we consider only the  $j$  colours corresponding to the  $j$  largest edge-monochromatic complete subgraphs. For  $j = 1$ , we recover the usual Ramsey numbers  $R_m(n)$ , for  $j = m$  we get the numbers  $R'_m(n)$  introduced previously.

The second generalization depends on an unbounded function  $s : \mathcal{G} \rightarrow \mathbb{N}$  (one can also work with  $m$  different unbounded functions  $s_c : \mathcal{G} \rightarrow \mathbb{N}$

indexed by colours or replace the target-set of natural integers by the set of non-negative real numbers) on the set  $\mathcal{G}$  of all finite simple graphs.

For  $n \geq 1$  we define  $R'_{m,s}(n)$  as the smallest integer such that every complete graph on  $R'_{m,s}(n)$  vertices contains  $m$  (not necessarily complete) edge-monochromatic subgraphs  $G_1, \dots, G_m$  of colour  $1, \dots, m$  satisfying  $s(G_1) + s(G_2) + \dots + s(G_m) \geq n$  (respectively  $s_1(G_1) + \dots + s_m(G_m) \geq n$ ).

The numbers  $R'_m(n)$  correspond to the choice  $s(G) = n$  if  $G$  is the complete graph on  $n$  vertices and  $s(G) = 0$  otherwise.

Other perhaps interesting choices are  $s(G) = n$  if  $G$  is an  $n$ -cycle and  $s(G) = 0$  otherwise, or  $s(G) = n$  if  $G$  is a simple path (two endpoints of degree 1 and all other vertices of degree 2) with  $n$  vertices.

It is of course possible to combine both generalizations by defining  $R'_{m,j,s}(n)$  in the obvious way considering only the  $j$  colours giving the largest contributions to the sum  $s(G_1) + \dots + s(G_m)$ .

#### 4.1 Analogues of $R'$ for van der Waerden numbers

Van der Waerden's Theorem gives the existence of a function  $W : \{2, 3, 4, \dots\} \times \{2, 3, 4, \dots\} \rightarrow \mathbb{N}$  associating to two integers  $m, n \geq 2$  the smallest natural integer  $W(m, n)$  such that every colouring of the  $W(m, n)$  consecutive natural integers  $1, 2, \dots, W(m, n)$  with  $m$  colours contains a monochromatic arithmetic progression with  $n$  elements.

We define  $W'(m, n)$  in the obvious way as the smallest natural integer such that every colouring of  $1, 2, \dots, W'(m, n)$  with  $m$  colours contains  $m$  (perhaps empty) monochromatic progressions of different colours and of lengths  $\alpha_1, \dots, \alpha_m$  summing up to  $n = \alpha_1 + \dots + \alpha_m$ .

We have  $W(m, n) \leq W'(m, m(n-1) + 1)$  since a set of  $m$  integers strictly smaller than  $n$  sums up at most to  $m(n-1)$ . It is easy to check that  $W'(2, 1) = 1, W'(2, 2) = 2$  and  $W'(2, 3) = 3$ .

For  $W'(2, 4)$  we get  $W'(2, 4) = 6$  as can be seen as follows:  $W'(2, 4) > 5$  by inspection of the black-white colouring  $bbwbb$  of  $1, 2, 3, 4, 5$ . Consider a black-white colouring of  $1, \dots, 6$  not containing a black progression of size  $\alpha$  and a white progression of size  $\beta$  such that  $\alpha + \beta \geq 4$ . Such a colouring cannot use only one colour (otherwise we can take  $\alpha = 6$  or  $\beta = 6$ ). It cannot use both colours twice (otherwise we can take  $\alpha = 2$  and  $\beta = 2$ ). It uses thus one colour, say white, only once and we have necessarily  $\alpha \geq 3$  and  $\beta = 1$  since either  $1, 3, 5$  (for an even white element) or  $2, 4, 6$  (for an odd white element) are all black.

It is of course also possible to consider the numbers  $W'_j(m, n)$  defined by considering only the  $j$  largest arithmetical progressions. For  $j = 1$  we get the classical van der Waerden number  $W(m, n)$ .

**Acknowledgements.** I thank S. Eliahou and G. McShane for comments.

## References

- [1] M. Aigner, G.M. Ziegler, *Proofs from The Book*. Third edition. Springer-Verlag, Berlin, 2004.
- [2] D. Conlon, *A new upper bound for diagonal Ramsey numbers.*, *Ann. of Math. (2)* 170 (2009), no. 2, 941–960.
- [3] P. Erdős, G. Szekeres, *A combinatorial problem in geometry.* *Comp. Math.* 2 (1935), 463–470.
- [4] F.P. Ramsey, *On a Problem of Formal Logic.* *Proc. London Math. Soc.* 30 (1928), 264–286.

Roland BACHER, Université Grenoble I, CNRS UMR 5582, Institut Fourier,  
100 rue des maths, BP 74, F-38402 St. Martin d'Hères, France.  
e-mail: Roland.Bacher@ujf-grenoble.fr