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From Navier-Stokes equations to the Oberbeck-Boussinesq approximation : a unified approach

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Abstract

We propose a unified asymptotic approach in order to derive the Oberbeck-Boussinesq approximation from the compressible Navier-Stokes equations coupled to a heat equation with an eventual source term. We point out, in the configuration of a horizontal infinite layer, the conditions for the density changes to be small, first for an ideal gas and then for a fluid with a divariant state law. We identify two small parameters. The original equations are then non-dimensionalized with different characteristic pressures and different linearized state laws for ideal gases and for a general fluid. We can either let the two small parameters go to zero and formally derive the asymptotic system of equations whereas in the gaseous case, we can directly use the Low Mach Number asymptotics. The coherence between the two approaches is provided and the link with the entropy production is established. It is emphasized that, in some situations, the work of the static pressure forces has to be retained in the final set of equation with both strategies since it involves the ratio of the two small parameters. It is related to the static pressure stratification of the fluid and can not be eliminated directly even if it is usually neglected in the Oberbeck-Boussinesq approximation. This original result proves the necessity to start from a divariant state law instead of the usual assumption that the density only depends on the temperature. Finally we prove, using a linear stability analysis and numerical simulations that this term has a stabilizing effect on the Rayleigh-Bénard problem and can even suppress the onset of natural convection for some values of the parameters.

Keywords : Boussinesq approximation/natural convection/asymptotic analysis

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1 Introduction

There has been several attempts at deriving the Oberbeck–Boussinesq approximation from the full Navier-Stokes equations. This approximation is often mentioned to be valid for a "mechanically incompressible but thermally compressible" fluid. It is generally used in the framework of the natural convection problems such as the Rayleigh–Bénard configuration, and provides a simplified set of equations which is much more tractable for both numerical and analytical purposes, since all the acoustic scales have been eliminated. There exists various studies devoted to the justification of this approximation from more general models, which we think are not entirely satisfactory, and this is the aim of the present paper to provide a general and unified approach.

As there are too numerous papers devoted to the present subject, we are going to present only a few of them, a cursory discussion of the quintessential contribution of which is going to both justify the re-examination of the subject in the present paper as well as the chosen framework and asymptotics. We have chosen to present the state of the art in the domain using four representative papers. The first paper to be published was written by Spiegel and Veronis [1]. They considered an ideal gas with the usual state law; an asymptotics was derived from the compressible Navier-Stokes equations. However, for the study of their thin layer of an ideal gas, they only introduced one small parameter related to the density variation. Consequently, in their discussion about the terms in the compressible equations to be retained, they had to use an adhoc argument in order to obtain the usual system of equations. The second one, often referred to as the "rigorous basis" for the derivation of the Oberbeck–Boussinesq equations, is the paper of Mihaljan [2]. Two small parameters were identified using the Buckingham Pi-theorem. Nevertheless, they assumed, from the starting point, that the density is a function of the only temperature of the fluid. It will be made clear by our study that such an assumption does not allow to take into account the work of the pressure forces in the temperature equations, an additional term which can change the stability limit in the Rayleigh-Bénard problem. It is worth to notice that the criticism expressed by Rajagopal et al. [3], that one of the parameter is exploding when the other is approaching zero is not correct since one of the two parameters is related to the temperature changes in the fluid layer and the other one to the acoustic scales; they are independent. A beautiful piece of work is to be found in Gray and Giorgini [4] where an extensive study of the validity of the Oberbeck–

Boussinesq approximation was provided. A general divariant fluid was considered and the final set of equations obtained. However, they did not conduct a rigorous asymptotic analysis and did not study the difference between ideal gases and general fluids. The relation of their approach to the Low Mach number asymptotics for ideal gases was not conducted and the influence of the work of the pressure force not studied. Part of our study can be considered as the rigorous exposition of the ideas present in this paper. Finally a more recent work by Rajagopal et al. [3] intends to provide a rigorous derivation of the Oberbeck–Boussinesq approximation in the framework of a full thermodynamical theory of the Navier-Stokes equations. They make the same assumption as Mihaljan; density is a function of the temperature only and the motion of fluid at constant temperature is isochoric. Besides, they pretend to relate the system of equations to the second law of thermodynamics. Since the work of the viscous forces is eliminated from the temperature equation in the limit of an incompressible fluid, the resulting system of equations is not compatible any more with the entropy production in the system. As a consequence of eliminating the pressure ab initio from the state law, they have to re-introduce it as the trace of the stress tensor; it is not the thermodynamical pressure any more but a "mechanical pressure". Since then it is difficult to follow the use of classical thermodynamics arguments when the functional dependencies of the various state function such as the free energy and the entropy are not explicitly given. Finally, they relate the choice of the free fall characteristic velocity in their asymptotics to the thickness of the layer which is then very large, and artificially let the reader think that the asymptotics depends on this characteristic velocity. The issue of the influence of the chosen characteristic velocity on the asymptotics will be discussed in the following.

In this paper we conduct two strategies, one for general divariant fluids which is based on two small parameters (close to the one used in [4]) and one based on the Low Mach Number asymptotics for ideal gases. In the present study, gases will always be assumed ideal, and for the sake of physical intuition, the first strategy will be considered to be well-suited for liquids, even if it is also valid for real divariant gases. For the sake of clear presentation, we focus on the generic infinite horizontal layer under gravity conditions.

The fundamental difference in terms of scales between liquids and gases is identified. The two strategies are then applied and yield asymptotic systems of equations. They are then proved to be compatible and provide the usual system of the Oberbeck-Boussinesq equations except that

there is a residual compressibility term in the heat equation. It is due to a static pressure gradient related to the stratification of the fluid and correspond to the work of the static pressure forces. It can only be eliminated under some precise assumption on the ratio of free fall velocity and velocity of sound.

In previous papers, this term is either neglected in the Oberbeck-Boussinesq approximation since the authors start from a simplified state law [2] [3] or it is mentioned but not derived in the framework of a rigorous asymptotics [1] [4]. The fact that the final system is not self consistent from the entropy conservation point of view is then discussed. Finally we make the link between the choice of various typical velocities as the characteristic velocity of the problem.

The influence of this residual compressibility term on the onset of natural convection for the Rayleigh–Bénard problem is studied through a linear stability analysis. Some numerical simulation confirm the theoretical analysis : the compressibility term can stabilize the layer and even prevent the onset of natural convection.

2 Original equations, characteristic pressure and density variations

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The purpose of this first section is to introduce the reference system of equations. It is constituted of the compressible Navier-Stokes equations coupled to the heat equation with an eventual source term and a general divariant state law. This system of equations is the basis in order to derive weakly compressible models. By weakly compressible models, we mean that the density variations are supposed to be small in the fluid around a reference value.

In this context, we consider the configuration of an infinite horizontal layer. We assume that the 1D heat equation in the vertical direction for the static fluid admits a stationary solution such that the temperature variations around one of the boundary conditions T_0, are small. We then precisely describe the pressure and density variation in the presence of gravity as well as the small parameters involved in the problem. We distinguish the treatment of ideal gases and divariant liquids and finally derive linearized state laws in both cases.

2. 1 The general framework of compressible Navier-Stokes Equations

The general system of equations we are going to start from is the system of the conservative Navier-Stokes equations with only one specific force acting on the fluid, the gravity. These equations of motion of the fluid are coupled to the enthalpy equation. The system reads:

$$\partial_t \rho + \quad \partial_x \cdot (\rho v) \quad = 0, \tag{2.1}$$

$$\partial_t \rho v + \partial_x \cdot (\rho v \otimes v) = -\partial_x \cdot \boldsymbol{P} - \rho\, g e_z, \tag{2.2}$$

$$\partial_t \rho h^{\mathrm{tot}} + \partial_x \cdot \left(\rho h^{\mathrm{tot}} v\right) = -\partial_x \cdot q - \partial_x \cdot (\boldsymbol{\Pi}.v) + \partial_t p - \rho\, g\, v \cdot e_z, \tag{2.3}$$

where x is the space coordinates vector, ρ denotes the density, v, the velocity, $h^{\mathrm{tot}} = h(p,T) + v \cdot v/2$ the total specific enthalpy and $h(p,T)$ the specific enthalpy of the fluid, p, the thermodynamic pressure, $\boldsymbol{\Pi}$ the shear stress tensor, $\boldsymbol{P} = p\boldsymbol{I} + \boldsymbol{\Pi}$, the pressure tensor, with $\boldsymbol{I}$ the unit second order tensor and q, the heat flux; g is the norm of the gravity force and e_z the unit vector in the vertical direction.

The dissipative fluxes $\boldsymbol{\Pi}$ and q are given by:

$$\boldsymbol{\Pi} = \left(\tfrac{2}{3}\eta - \eta_b\right)(\partial_x \cdot v)\boldsymbol{I} - \eta\left(\partial_x v + (\partial_x v)^t\right), \tag{2.4}$$

$$q = -\lambda\, \partial_x T, \tag{2.5}$$

where η denotes the shear viscosity, η_b, the bulk viscosity and λ, the heat conductivity.

For the system to be closed, we need to consider an equation of state relating the density to pressure and temperature:

$$\rho = \rho(p,T). \tag{2.6}$$

We assume that ρ is a strictly increasing function of pressure and strictly decreasing function of temperature (we are not too close to the $4°C$ point for water). We finally assume that the density is a smooth function. We also express the specific enthalpy :

$$h = h(p,T), \qquad h = e + p/\rho, \tag{2.7}$$

where $e(p,T)$ is the specific internal energy. We also assume the smoothness of h as a function of (p,T). It is important to notice here that we do not assume that the density is a function of

the temperature only as it is often done [2] [3]. We only suppose the fluid to be divariant, a fairly general basis. Let us then define:

$$\alpha(p,T) = -\frac{1}{\rho_0}\left(\frac{\partial \rho}{\partial T}\right)_P, \quad \alpha_0 = \alpha(p_0,T_0), \tag{2.8}$$

where α is the coefficient of thermal expansion at constant pressure and ρ_0, p_0 and T_0 are reference density, pressure and temperature. It is interesting to relate the isothermal compressibility coefficient to the sound velocity in the fluid through :

$$c(p,T) = \sqrt{\left(\frac{\partial p}{\partial \rho}\right)_S}, \qquad c_0 = c(p_0,T_0), \qquad \left(\frac{\partial \rho}{\partial p}\right)_T = \frac{\gamma}{c^2}, \qquad \gamma(p,T) = c_p(p,T)/c_v(p,T). \tag{2.9}$$

where c denotes the sound velocity, S, the entropy, γ the ratio of the heat capacity at constant pressure over the heat capacity at constant volume and $\gamma_0 = \gamma(p_0,T_0)$, its value at the reference temperature and pressure.

We can then derive a general equation for the temperature of the fluid :

$$\rho\, c_p\,(\partial_t T + v\cdot\partial_x T) = \partial_x(\lambda\,\partial_x T) - \boldsymbol{\Pi} : \partial_x v + \frac{\rho_0}{\rho}\,\alpha\, T\,(\partial_t p + v\cdot\partial_x p). \tag{2.10}$$

Before going into the core of the paper, let us make the following assumptions which can be easily relaxed, for the sake of clearness of the presentation.

[**S1**] The heat capacity at constant pressure of the fluid is assumed constant $c_p(p,T) = c_p(p_0,T_0) = c_{p0}$.

[**S2**] The shear viscosity as well as the thermal conductivity are supposed to be constant, $\lambda = \lambda_0$, $\mu = \mu_0$. We neglect the bulk viscosity η_b.

The purpose of the paper is to rigorously derive the Oberbeck–Boussinesq approximation from the previous system of equations for both liquids and gases, thus providing a unified approach. This approximation intends to retain in the final model the small density variations in the gravity terms without taking into account the acoustic scales. This model is essentially used for the determination of the onset of natural convection (which correspond to the stability limit of a given static configuration, by static we mean without fluid motion).

In order to derive the model and present the underlying ideas, we consider the usual configuration of a infinite horizontal layer as in the Rayleigh-Bénard problem ([5] [6] [7] [8]) and characterize, in the following sub-section, the characteristic density variations of the static stationary solution.

2. 2 Horizontal layer : Static stationary solution

We consider a horizontal layer of thickness L, the bottom line of which is located at the vertical coordinate $z = 0$ and the top line at $z = L$. We look for a static solution $(v_S = 0, p_S, T_S)$ of the system of equations (2.1), (2.2), (2.10).

The system of equations defining the static stationary solution is then given by :

$$-d_z\, p_S - \rho(p_S, T_S)\, g = 0 \tag{2.11}$$

$$d_z(\lambda\, d_z T_S) + Q(T_S) = 0, \tag{2.12}$$

with boundary conditions :

$$p_S(z = 0) = p_0, \qquad T_S(z = 0) = T_0, \quad T_S(z = L) = T_1. \tag{2.13}$$

For the sake of generality, we have added a heat source term $Q(T)$ in the temperature equation which can correspond to a chemical heat source term as in the thermal explosion problem treated in [9] [10] [11] [12]. For this system of two equations, we assume, firstly, that equation (2.12) with boundary conditions (2.13) on the temperature, admits a unique smooth solution $T_S(z)$; and secondly that equation (2.11) also admits a unique smooth solution $p_S(z)$.

The final Oberbeck–Boussinesq approximation is valid for a fluid density, the changes of which around a constant value $\rho_0 = \rho_0(p_0, T_0)$ are supposed to be small throughout the process and particularly for the static solution. It is then necessary to identify the conditions on the static temperature and static pressure such that the density changes inside the layer are small, as well as the corresponding small parameters.

2. 3 Small parameters

We identify in this paragraph, under the form of a proposition, the proof of which is given in Appendix I, the conditions for the density variations in the layer to be small both in the isothermal case and in the varying temperature case. The reason why the isothermal case is emphasized will be made clear in Remark 2.2 at the end of this subsection, and also in the following subsection.

Let us first treat the isothermal case. We assume that $T_1 = T_0$ and $Q = 0$ in such a way that the solution of problem (2.12) is $T_{SI}(z) = T_0$. Let $p_{SI}(z)$ denote the isothermal pressure field solution of (2.11) with $T_{SI}(z) = T_0$.

Let us now introduce two non-dimensional parameters:

$$\Upsilon^P(p,T) = \frac{\gamma(p,T)gL}{c^2(p,T)}, \qquad \Upsilon^T(p,T) = \alpha(p,T)\Delta T, \qquad \Delta T = \max_{z\in[0,L]} |T(z) - T_0|, \tag{2.14}$$

where ΔT is the maximal temperature difference in the layer. We also define, for convenience, $\Upsilon^P_0 = \Upsilon^P(p_0, T_0)$ and $\Upsilon^T_0 = \Upsilon^T(p_0, T_0)$. We then define the maximum of this quantities in both the isothermal and the non-isothermal cases :

$$\Upsilon^j_{\max} = \max_{z\in[0,L]} \{\Upsilon^j(p_{SI}(z), T_{SI}(z)), \Upsilon^j(p_S(z), T_S(z))\}, \qquad j = P, T. \tag{2.15}$$

PROPOSITION 2.1 ——

Let us assume that $\Upsilon^P_{\max} << 1$*, then for the isothermal case, we have the following estimate :*

$$|\frac{p_{SI} - (p_0 - \rho_0 gz)}{\rho_0 gL}| << 1, \quad |\frac{\rho_0 - \rho_{SI}(p_{SI}(z), T_{SI})}{\rho_0}| << 1, \tag{2.16}$$

and even, more precisely :

$$0 \le \frac{p_{SI} - (p_0 - \rho_0 gz)}{\rho_0 gL} \le \Upsilon^P_{\max} \frac{(z/L)^2}{2}, \quad 0 \le \frac{\rho_0 - \rho_{SI}}{\rho_0} \le \Upsilon^P_{\max}(z/L). \tag{2.17}$$

Let us then assume $\Upsilon^T_{\max} << \alpha_0 T_0 \le 1$*, then, for the non-isothermal case, we have :*

$$|\frac{p_S - (p_0 - \rho_0 gz)}{\rho_0 gL}| << 1, \quad |\frac{p_S - p_{SI}}{\rho_0 gL}| << 1, \quad |\frac{\rho_0 - \rho_S(p_S(z), T_S)}{\rho_0}| << 1. \tag{2.18}$$

However, these estimates can be made more precise :

$$-\Upsilon^T_{\max}(z/L) \le \frac{p_S - (p_0 - \rho_0 gz)}{\rho_0 gL} \le \Upsilon^T_{\max}(z/L) + \Upsilon^P_{\max} \frac{(z/L)^2}{2}, \tag{2.19}$$

$$-\Upsilon^T_{\max}(z/L) - \Upsilon^P_{\max} \frac{(z/L)^2}{2} \le \frac{p_S - p_{SI}}{\rho_0 gL} \le \Upsilon^T_{\max}(z/L) + \Upsilon^P_{\max} \frac{(z/L)^2}{2}, \tag{2.20}$$

$$-\Upsilon^T_{\max} + \Upsilon^P_{\max}(z/L)(1 - \Upsilon^T_{\max} - \Upsilon^P_{\max} \frac{(z/L)}{2}) \le \frac{\rho_0 - \rho_S(p_S(z), T_S)}{\rho_0} \le \Upsilon^T_{\max} + \Upsilon^P_{\max}(z/L)(1 + \Upsilon^T_{\max}). \tag{2.21}$$

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In the way the problem was formulated, for the sake of generality, we have made no assumptions on the variations of α and γ/c^2 as functions of (p,T). However, out of the two conditions, we mainly retain the associated assumptions on ΔT and L. Besides, $\Upsilon^T_{\max}$ and $\Upsilon^P_{\max}$ have been defined over the range of variation of p_S and T_S. In the following, we will consider the dynamical problem and linearize around the static solution so that the range of variations of pressure and temperature will change. Consequently, since we do not want to derive a priori estimates on p and T, we will consider a stronger assumption on α and γ/c^2 :

[**S3**] The coefficient of thermal expansion at constant pressure α as well as the ratio γ/c^2 are uniformly bounded as functions of p and T :

$$0 < \alpha < \alpha_{\max}, \qquad 0 < \frac{\gamma}{c^2} < \left(\frac{\gamma}{c^2}\right)_{\max},$$

with the assumption that :

$$\Upsilon^T_{\max} = \alpha_{\max}\,\Delta T << \alpha_0 T_0 \leq 1, \qquad \Upsilon^P_{\max} = \left(\frac{\gamma}{c^2}\right)_{\max} g\,L << 1, \tag{2.22}$$

as well as, $\alpha_{\max}/\alpha_0 = O(1)$ with respect to $\Upsilon^T_{\max}$ and $\left(\frac{\gamma}{c^2}\right)_{\max} / \left(\frac{\gamma_0}{c_0^2}\right) = O(1)$ with respect to $\Upsilon^P_{\max}$.

Remark 2.1.

This is the point where ideal gases and divariant fluids (and especially liquids) differ; for ideal gases, the inequality $\Upsilon^P_{\max} << 1$ implies $\frac{\rho_0 g L}{p_0} << 1$ since $c_0^2 = \gamma_0 p_0/\rho_0$, which means that the pressure changes are small compared to p_0, a condition which is going to allow the use of the Low Mach Number approximation for the dynamical set of equations as well as a linearization of the state law around (p_0, T_0). This is not the case for a liquid where the pressure changes can be of the order of p_0, even if $\frac{\rho_S - \rho_0}{\rho_0} << 1$ is satisfied.

Proposition 2.1 states that in both cases, the characteristic pressure drop due to gravity at the static level is $\rho_0 g L$, and it is almost linear as a function of the vertical coordinate.

Remark 2.2.

Similarly for a divariant fluid, if its thermal expansion coefficient is small, the temperature variations does not have to be small for the density variations to be small. However, for the sake of simplicity, we assume that we are in the framework of assumption **S3** so that the temperature variations are small compared to T_0. This is reasonable even in the configuration of thermal explosion where the temperature increase is of the order of the Frank-Kamenetskii temperature and is then small in comparison to T_0 in the large activation energy asymptotics [11]. The whole study can be extended to the case of a small thermal expansion coefficient to the cost of much more complicated notations since the linearization on the temperature are going to take place around the eventually non-linear stationary temperature profile T_S.

2. 4 Linearized state law, difference between gases and liquids

The state law can be linearized around (p_0, T_0):

$$\frac{\rho(p_S, T_S) - \rho_0}{\rho_0} \approx \frac{\gamma_0\, p_0}{c_0^2\, \rho_0}\, \frac{p_S - p_0}{p_0} - \alpha_0\, (T_S - T_0), \tag{2.23}$$

under the condition that $\frac{(T_S - T_0)}{T_0} << 1$, as well as $\frac{p_S - p_0}{p_0} << 1$. In the case of an ideal gas, the two conditions invoked in Proposition 2.1 are sufficient in order to insure the validity of the linearization, since $p_0 = \rho_0 r T_0$, with r the universal gas constant divided by the molar mass of the fluid, $\alpha_0 = 1/T_0$ and $c_0^2 = \gamma_0\, p_0/\rho_0$. We then retrieve that the density variations are small.

In the case of a liquid it is different in that $\frac{p_S - p_0}{p_0}$ does not have to be small for the density variations to be small; actually if $\rho_0\, g\, L/p_0$ is of order one then, the pressure variations are also of order one. On the other hand, from Remark 2.2, the temperature variations around T_0 are small. In this case the linearization has to take place around the profile p_{SI}. In the following we will use the following notations : $\bar\rho_S = \frac{\rho(p_S, T_S)}{\rho_0}$, $\hat\rho = \frac{\rho(p_{SI}, T_0)}{\rho_0}$, and $\bar p_S = \frac{p_S - p_{SI}}{\rho_0\, g\, L}$. We then obtain:

$$\bar\rho_S \approx \hat\rho + \Upsilon_0^P\, \bar p_S - \Upsilon_0^T \frac{T - T_0}{\Delta T}, \tag{2.24}$$

which is justified because $\bar p_S << 1$ from Proposition 2.1.

Finally just by examining the static mechanical equilibrium, we come to the conclusion that the state law can be linearized under the two conditions

$$\Upsilon^T_{\max} << \alpha_0 T_0 \le 1, \qquad \Upsilon^P_{\max} << 1, \tag{2.25}$$

so that the two small parameters are identified.

3 Non-dimensional form of the equations, characteristic quantities

In this section, we are going to obtain non-dimensional forms of the system of equation considered in the previous section. The plural is due to the fact that ideal gases and general divariant liquids can be treated using different approximations and also due to the fact that there is no natural characteristic velocity for the Rayleigh-Bénard problem.

3. 1 Non-dimensional quantities and state laws for gases and liquids

Let us first focus on the characteristic quantities which do not depend on the fact that the fluid is a liquid or a gas. The characteristic density is ρ_0, the characteristic temperature $T_0\,\alpha_0\,\Delta T$, and the characteristic velocity will be denoted by v_0. We will see that there exist several choices for v_0. The space characteristic length is L and the characteristic time scale is L/v_0, in such a way that :

$$\bar{x} = x/L, \quad \bar{t} = tv_0/L, \quad \bar{v} = v/v_0, \quad \bar{\rho} = \rho/\rho_0, \quad ,\theta = \frac{1}{\Upsilon_0^T}\frac{T-T_0}{T_0}. \tag{3.1}$$

Finally, we define $\bar{Q}(\theta) = Q(T)$.

In the case of an ideal gas, we simply consider the characteristic pressure p_0 and define

$$\widehat{p} = (p - p_0)/p_0. \tag{3.2}$$

The linearized state law then takes the expression :

$$\bar{\rho} \approx 1 + \widehat{p} - \Upsilon_0^T\,\theta, \tag{3.3}$$

In the case of a liquid, we have to distinguish the static pressure from the dynamic one and define :

$$\bar{p} = (p - p_{SI})/p_{DO}, \quad \widetilde{p} = (p_{SI} - p_0)/p_{SO}. \tag{3.4}$$

where the characteristic dynamic and static pressures are taken to be :

$$p_{DO} = \rho_0\,v_0^2, \qquad p_{SO} = \rho_0\,g\,L. \tag{3.5}$$

In these equations, p_{SO} stands for the static change of pressure due to gravity around p_0, P_{DO} for the dynamical change of pressure around p_{SI}. The linearized state law then reads :

$$\bar{\rho} \approx \hat{\rho} + \Upsilon_0^T\left(\Upsilon_0^P\,\frac{{\rm F_r}^2}{\Upsilon_0^T}\,\bar{p} - \,\alpha_0\,T_0\,\theta\right), \qquad \hat{\rho} = \rho(p_{SI}, T_0)/\rho_0, \tag{3.6}$$

where $\mathrm{F_r} = v_0/\sqrt{gL}$ is the Froude number. Let us notice that $\Upsilon_0^P\,\mathrm{F_r}^2 = \gamma_0\,\mathrm{M_a}^2$, where $\mathrm{M_a} = v_0/c_0$ is the Mach number. In the following, the non-dimensional numbers will always denote the ratio of quantities with a zero index.

3. 2 Non-dimensional equations for ideal gases

Using the non-dimensional variables introduced in the previous sub-section, we finally get for the non-dimensional form of the system of equations :

$$\partial_{\bar t}\bar\rho + \bar v\cdot\partial_{\bar x}\bar\rho = -\bar\rho\,\partial_{\bar x}\cdot\bar v, \tag{3.7}$$

$$\bar\rho\,(\partial_{\bar t}\bar v + \bar v\cdot\partial_{\bar x}\bar v) = -\frac{1}{\gamma_0\,\mathrm{M_a}^2}\partial_{\bar x}\widehat p + \frac{\mathrm{P_r}}{\mathrm{P_e}}(\partial_{\bar x\bar x}\bar v + \partial_{\bar x}(\partial_{\bar x}\cdot\bar v) - \tfrac{2}{3}\partial_{\bar x}\cdot(\partial_{\bar x}\cdot\bar v)\boldsymbol{I}) - \bar\rho\frac{1}{\mathrm{F_r}^2}e_z, \tag{3.8}$$

$$\begin{aligned}\bar\rho\,(\partial_{\bar t}\theta + \bar v\cdot\partial_{\bar x}\theta) &= \frac{1}{\mathrm{P_e}}\partial_{\bar x\bar x}\theta + \frac{(\gamma_0-1)\mathrm{M_a}^2}{\Upsilon_0^T}\frac{\mathrm{P_r}}{\mathrm{P_e}}\left(\partial_{\bar x}\bar v + (\partial_{\bar x}\bar v)^t - \tfrac{2}{3}\partial_{\bar x}\cdot\bar v\boldsymbol{I}\right) : \partial_{\bar x}\bar v\\ &\quad + \frac{\alpha_0\,T_0}{\bar\rho\,\Upsilon_0^T}\frac{\gamma_0-1}{\gamma_0}(\partial_{\bar t}\widehat p + \bar v\cdot\partial_{\bar x}\widehat p) + \frac{\bar Q(\theta)\,L^2}{q_0\,\kappa_0}\,\frac{1}{\mathrm{P_e}},\end{aligned} \tag{3.9}$$

where we have used

$$\frac{p_0}{\rho_0 T_0 c_{p0}} = \frac{r}{c_{p0}} = \frac{\gamma_0-1}{\gamma_0}.$$

and where $\mathrm{P_e} = Lv_0/\kappa_0$ is the Péclet number, $\mathrm{P_r} = \nu_0/\kappa_0$ the Prandtl number (with $\nu_0 = \eta_0/\rho_0$ the kinematic viscosity and $\kappa_0 = \lambda_0/\rho_0\,c_{p0}$ the thermal diffusivity) and $q_0 = \rho_0 c_{p0}\Delta T$ a reference enthalpy.

3. 3 Non-dimensional equations for divariant fluids

In the case of a divariant fluid, the system takes the form:

$$\partial_{\bar t}\bar\rho + \bar v\cdot\partial_{\bar x}\bar\rho = -\bar\rho\,\partial_{\bar x}\cdot\bar v, \tag{3.10}$$

$$\bar\rho\,(\partial_{\bar t}\bar v + \bar v\cdot\partial_{\bar x}\bar v) = -\frac{P_{DO}}{\rho_0 v_0^2}\,\partial_{\bar x}\bar p - \frac{p_{SO}}{\rho_0 v_0^2}\,\partial_{\bar x}\widetilde p + \frac{\mathrm{P_r}}{\mathrm{P_e}}(\partial_{\bar x\bar x}\bar v + \partial_{\bar x}(\partial_{\bar x}\cdot\bar v) - \tfrac{2}{3}\partial_{\bar x}\cdot(\partial_{\bar x}\cdot\bar v)\boldsymbol{I}) - \bar\rho\frac{1}{\mathrm{F_r}^2}e_z, \tag{3.11}$$

$$\begin{aligned}\bar\rho\,(\partial_{\bar t}\theta + \bar v\cdot\partial_{\bar x}\theta) &= \frac{1}{\mathrm{P_e}}\partial_{\bar x\bar x}\theta - \frac{P_{DO}}{q_0\,\alpha_0\,T_0}\frac{\mathrm{P_r}}{\mathrm{P_e}}\left(\partial_{\bar x}\bar v + (\partial_{\bar x}\bar v)^t - \tfrac{2}{3}\partial_{\bar x}\cdot\bar v\boldsymbol{I}\right) : \partial_{\bar x}\bar v\\ &\quad + \frac{P_{DO}}{\bar\rho\,q_0}\,(1+\Upsilon_0^T\,\theta)\,\frac{\Upsilon^T}{\Upsilon_0^T}(\partial_{\bar t}\bar p + \bar v\cdot\partial_{\bar x}\bar p) + \frac{p_{SO}}{\bar\rho\,q_0}\,(1+\Upsilon_0^T\,\theta)\,\frac{\Upsilon^T}{\Upsilon_0^T}(\bar v\cdot\partial_{\bar x}\widetilde p) + \frac{\bar Q(\theta)\,L^2}{q_0\,\kappa_0}\,\frac{1}{\mathrm{P_e}}.\end{aligned} \tag{3.12}$$

Remark 3.1.

The system for the divariant fluid is also valid for an ideal gas. Therefore, it can be shown that (3.11-3.12) is equivalent to (3.8-3.9) for an ideal gas.

3. 4 Characteristic velocity and associated non-dimensional equations

We have come to the point where a choice has to be made. The characteristic velocity v_0 is still to be defined. There are basically two choices which are more natural : either $v_0 = \kappa_0/L$ or $v_0 = \nu_0/L$, a diffusion velocity [6] [8] [2], or $v_0 = \sqrt{g\,L\,\alpha_0 \Delta T}$ the free fall velocity [4] [3].

It is interesting to note that this choice has no influence on the stability boundary defining the onset of convection from a linear stability analysis point of view. The influence of the choice of the characteristic velocity on the derivation of the asymptotics is discussed in section 7. In the present subsection, we use the free fall velocity. The non-dimensional equations for the choice of a diffusion velocity are provided in Appendix II.

$$-\partial_{\bar{x}}\cdot\bar{v} = \frac{1}{1+\widehat{p}}(\partial_{\bar{t}}\widehat{p} + \bar{v}\cdot\partial_{\bar{x}}\widehat{p}) - \frac{\Upsilon_0^T}{1+\Upsilon_0^T\,\theta}\,(\partial_{\bar{t}}\theta + \bar{v}\cdot\partial_{\bar{x}}\theta), \tag{3.13}$$

$$\bar{\rho}\,(\partial_{\bar{t}}\bar{v} + \bar{v}\cdot\partial_{\bar{x}}\bar{v}) = -\frac{1}{\gamma_0 \mathrm{Ma}^2}\partial_{\bar{x}}\widehat{p} + \frac{\mathrm{P_r}}{\mathrm{P_e}}(\partial_{\bar{x}\bar{x}}\bar{v} + \partial_{\bar{x}}(\partial_{\bar{x}}\cdot\bar{v}) - \tfrac{2}{3}\partial_{\bar{x}}\cdot(\partial_{\bar{x}}\cdot\bar{v})\boldsymbol{I}) - \frac{\bar{\rho}}{\Upsilon_0^T}e_z, \tag{3.14}$$

$$\begin{aligned}\bar{\rho}\,(\partial_{\bar{t}}\theta + \bar{v}\cdot\partial_{\bar{x}}\theta) = {} & \frac{1}{\mathrm{P_e}}\,\partial_{\bar{x}\bar{x}}\theta + \widetilde{\Upsilon}_0^P\,\frac{\mathrm{P_r}}{\mathrm{P_e}}\left(\partial_{\bar{x}}\bar{v} + (\partial_{\bar{x}}\bar{v})^t - \tfrac{2}{3}\partial_{\bar{x}}\cdot\bar{v}\boldsymbol{I}\right) : \partial_{\bar{x}}\bar{v} \\ & + \frac{\alpha_0\,T_0}{\Upsilon_0^T}\,\frac{\gamma_0 - 1}{\gamma_0}(\partial_{\bar{t}}\widehat{p} + \bar{v}\cdot\partial_{\bar{x}}\widehat{p}) + \frac{\bar{Q}(\theta)\,L^2}{q_0\,\kappa_0}\,\frac{1}{\mathrm{P_e}},\end{aligned} \tag{3.15}$$

where $\widetilde{\Upsilon}_0^P = \frac{\gamma_0 - 1}{\gamma_0}\Upsilon_0^P$ and with the linearized state law:

$$\bar{\rho} = 1 + \widehat{p} - \Upsilon_0^T\,\theta, \qquad \widehat{p} << 1, \qquad \Upsilon_0^T\,\theta << \alpha_0\,T_0 \leq 1.$$

In the case of a liquid, the system takes the form:

$$-\partial_{\bar{x}}\cdot\bar{v} = \frac{\Upsilon^P}{\bar{\rho}}\left(\Upsilon_0^T\,(\partial_{\bar{t}}\bar{p} + \bar{v}\cdot\partial_{\bar{x}}\bar{p}) + \bar{v}\cdot\partial_{\bar{x}}\widetilde{p}\right) - \alpha_0\,T_0\,\Upsilon^T(\partial_{\bar{t}}\theta + \bar{v}\cdot\partial_{\bar{x}}\theta), \tag{3.16}$$

$$\bar{\rho}\,(\partial_{\bar{t}}\bar{v} + \bar{v}\cdot\partial_{\bar{x}}\bar{v}) = -\partial_{\bar{x}}\bar{p} + \frac{\mathrm{P_r}}{\mathrm{P_e}}(\partial_{\bar{x}\bar{x}}\bar{v} + \partial_{\bar{x}}(\partial_{\bar{x}}\cdot\bar{v}) - \tfrac{2}{3}\partial_{\bar{x}}\cdot(\partial_{\bar{x}}\cdot\bar{v})\boldsymbol{I}) - \frac{1}{\Upsilon_0^T}\,(\partial_{\bar{x}}\widetilde{p} + \bar{\rho}\,e_z)\,, \tag{3.17}$$

$$\begin{aligned}\bar{\rho}\,(\partial_{\bar{t}}\theta + \bar{v}\cdot\partial_{\bar{x}}\theta) = {} & \frac{1}{\mathrm{P_e}}\,\partial_{\bar{x}\bar{x}}\theta + \frac{\widetilde{\Upsilon}_0^P}{(\alpha_0\,T_0)^2}\,\frac{\mathrm{P_r}}{\mathrm{P_e}}\left(\partial_{\bar{x}}\bar{v} + (\partial_{\bar{x}}\bar{v})^t - \tfrac{2}{3}\partial_{\bar{x}}\cdot\bar{v}\boldsymbol{I}\right) : \partial_{\bar{x}}\bar{v} + \frac{\bar{Q}(\theta)\,L^2}{q_0\,\kappa_0}\,\frac{1}{\mathrm{P_e}} \\ & + \frac{\widetilde{\Upsilon}_0^P}{\alpha_0\,T_0}\,(1 + \Upsilon_0^T\,\theta)\,\frac{1}{\bar{\rho}}\,\frac{\Upsilon^T}{\Upsilon_0^T}(\partial_{\bar{t}}\bar{p} + \bar{v}\cdot\partial_{\bar{x}}\bar{p}) + \frac{\widetilde{\Upsilon}_0^P}{\Upsilon_0^T}\,\frac{1}{\alpha_0\,T_0}\,(1 + \Upsilon_0^T\,\theta)\,\frac{1}{\bar{\rho}}\,\frac{\Upsilon^T}{\Upsilon_0^T}(\bar{v}\cdot\partial_{\bar{x}}\widetilde{p}),\end{aligned} \tag{3.18}$$

with the linearized state law:

$$\bar{\rho} = \hat{\rho} + \Upsilon_0^T\,\left(\Upsilon_0^P\,\bar{p} - \alpha_0 T_0\,\theta\right), \quad 0 \leq 1 - \hat{\rho} \leq \Upsilon_{\max}^P\,\bar{z} << 1, \quad \Upsilon_0^T\,\theta << 1, \quad \Upsilon_0^P\,\bar{p} << 1. \tag{3.19}$$

We have used the fact that

$$\frac{\alpha_0}{c_{p0}} = \frac{\gamma_0 - 1}{c_0^2\,\alpha_0 T_0},$$

so that

$$\frac{P_{DO}}{q_0} = \frac{\widetilde{\Upsilon}_0^P}{\alpha_0\,T_0}, \qquad \frac{p_{SO}}{q_0} = \frac{\widetilde{\Upsilon}_0^P}{\alpha_0\,T_0}\,\frac{1}{\Upsilon_0^T}.$$

3. 5 First approximation, link with the entropy production

At this level of the study, it is clear that a first approximation can be conducted using assumption **S3** on the two parameters Υ_0^P and Υ_0^T on system (3.16-3.18). It yields a divergence free velocity field, and allows to neglect the work of the dynamical pressure and viscous forces in the temperature equation.

The first idea is to express the particular derivative of the density as in (3.16) and to see that the divergence of the velocity is constituted of two terms, the first one multiplied by Υ_0^P and the second one by Υ_0^T. This yields :

$$\partial_{\bar{x}} \cdot \bar{v} \approx 0.$$

Besides, the source term in the momentum equation reads :

$$\frac{\partial_{\bar{x}}\widetilde{p} + \bar{\rho}\,e_z}{\Upsilon_0^T} \approx (\Upsilon_0^P\,\bar{p} - \alpha_0\,T_0\,\theta)e_z, \tag{3.20}$$

so that we can conclude on this point, with the complementary fundamental assumption :

[**S4**] The Péclet number and the Prandtl number are of order zero with respect to both small parameters Υ_0^P and Υ_0^T :

$$\mathrm{P_e} = O(1), \qquad \mathrm{P_r} = O(1). \tag{3.21}$$

The meaning of the previous statement is that the three velocities, the diffusion velocities and the free fall velocity have to be at the same order in the asymptotics. The same conclusion could be drawn from the study by Rajagopal et al. [3] where they choose the free fall velocity and assume the Reynolds number to be order one in their asymptotics. We will come back on this matter in the discussion section at the end of the paper.

Assuming $\frac{\bar{Q}(\theta)\,L^2}{q_0\,\kappa_0} = O(1)$ with respect to both Υ_0^P and Υ_0^T, the last term to be examined takes into account the ratio of the two small quantities. Either $\frac{\Upsilon_0^P}{\Upsilon_0^T} \approx 1$ then the term $\bar{v}_z$ is to be

retained :

$$\partial_{\bar{x}} \cdot \bar{v} \;\; = 0, \tag{3.22}$$

$$\partial_{\bar{t}} \bar{v} \; + \; \bar{v} \cdot \partial_{\bar{x}} \bar{v} \;\; = -\partial_{\bar{x}} \bar{p} + \frac{\mathrm{P_r}}{\mathrm{P_e}} \, \partial_{\bar{x}\bar{x}} \bar{v} + \alpha_0 T_0 \, \theta \, e_z, \tag{3.23}$$

$$\partial_{\bar{t}} \theta + \; \bar{v} \cdot \partial_{\bar{x}} \cdot \theta \;\; = \qquad \frac{1}{\mathrm{P_e}} \, \partial_{\bar{x}\bar{x}} \theta - \frac{\widetilde{\Upsilon}_0^P}{\Upsilon_0^T} \, \frac{\Upsilon^T}{\Upsilon_0^T} \, \frac{1}{\alpha_0 \, T_0} \, \bar{v}_z + \frac{\bar{Q} \, L^2}{q_0 \, \kappa_0} \, \frac{1}{\mathrm{P_e}}, \tag{3.24}$$

or $\frac{\Upsilon_0^P}{\Upsilon_0^T} << 1$ so that this terms does not play any role at the zeroth order and the final equations read :

$$\partial_{\bar{x}} \cdot \bar{v} \;\; = 0, \tag{3.25}$$

$$\partial_{\bar{t}} \bar{v} \; + \; \bar{v} \cdot \partial_{\bar{x}} \bar{v} \;\; = -\partial_{\bar{x}} \bar{p} + \frac{\mathrm{P_r}}{\mathrm{P_e}} \partial_{\bar{x}\bar{x}} \bar{v} + \alpha_0 \, T_0 \, \theta e_z, \tag{3.26}$$

$$\partial_{\bar{t}} \theta + \; \bar{v} \cdot \partial_{\bar{x}} \cdot \theta \;\; = \qquad \frac{1}{\mathrm{P_e}} \partial_{\bar{x}\bar{x}} \theta + \frac{\bar{Q} \, L^2}{q_0 \, \kappa_0} \, \frac{1}{\mathrm{P_e}}, \tag{3.27}$$

a system of equations which corresponds to the usual Oberbeck-Boussinesq approximation.

Remark 3.2.

It is especially important here to emphasize that the model of a fluid "mechanically incompressible but thermally compressible" can be obtained from (3.16-3.18) by letting Υ_0^P go to zero. We then rigorously obtain the model presented by Rajagopal et al. as the limit of the full compressible model and justify it. However, we do not proceed with this part since we realize that, doing so, the possible interaction between the two scales of the problem disappears. By performing successively $\Upsilon_0^P \to 0$ and then $\Upsilon_0^T \to 0$, the only possible result of the asymptotics is the usual Oberbeck-Boussinesq approximation and the term $\frac{\widetilde{\Upsilon}_0^P}{\Upsilon_0^T} \, \frac{\Upsilon^T}{\Upsilon_0^T} \, \frac{1}{(\alpha_0 \, T_0)^2} \, \bar{v}_z$ can not be attained. This is the reason why we conduct an asymptotics with both scales involved and qualify the obtained model of "weakly compressible".

Before going into the link with the Low Mach Number asymptotics, let us come back on the problem of the entropy production for the previous system of equations in reference to system (3.16-3.18). It has to be noticed that, neglecting the work of the viscous forces, the entropic structure of the compressible and conservative original equation is destroyed. In order to visualize it, let us recall the entropy conservation equation related to system (2.1-2.3) :

$$\partial_t \rho \, s + \partial_x \cdot (\rho \, s \, v) = -\partial_x \cdot q + \lambda \, \frac{\partial_x T \cdot \partial_x T}{T^2} - \frac{\boldsymbol{\Pi} : \partial_x v}{T}, \tag{3.28}$$

where s denotes the specific entropy and where $-\frac{\boldsymbol{\Pi}:\partial_x v}{T}$ can be shown to be non-negative. The work of the viscous forces can then be interpreted as the entropy production due to the viscous dissipation phenomenon. Neglecting this terms yields the decoupling of the hydrodynamic part of the equations from the thermal one. However, even if the equations are simpler, the resulting system of equation has lost, for example, the symmetrizability property as a system of mixed hyperbolic-parabolic equations. We refer to [13] [14] for precise results on the entropic structure and symmetrizability of the compressible Navier-Stokes equations.

4 Low Mach Number asymptotics for ideal gases

We have seen that the pressure variations at the scale L of the layer can not be considered as small in comparison to p_0 for a liquid. For this reason, we have introduced two separate pressure scales in sub-section 3.1. However, for an ideal gas, the pressure variation, under the assumptions of Proposition 2.1 can be considered as small in comparison to p_0. In this context, we can use the low Mach number asymptotics [15]. To that purpose, we introduce a small parameter and derive in a first sub-section the system of equations. We prove, in a second sub-section that it can be seen as equivalent to the systems derived in the previous section.

4. 1 Principle and Asymptotics

In the case of a gaseous phase, it can be shown from the previous developments that :

$$p = p_0(1+\widehat{p}), \quad \widehat{p} << 1, \quad \widehat{p} = O(\Upsilon_0^P). \tag{4.1}$$

The idea is then to put the two small parameters into a unique scale, and to distinguish between two cases.

We consider system (3.13-3.15) and define the small parameter $\varepsilon = \Upsilon_0^T$; we assume :

$$\Upsilon_0^P = O(\varepsilon^{j+1}), \quad j \geq 0. \tag{4.2}$$

This yields $\gamma_0\,\mathrm{Ma}^2 = \Upsilon_0^T\Upsilon_0^P = O(\varepsilon^{j+2})$. We further assume

$$\frac{1}{\mathrm{Pe}} = O(1), \quad \mathrm{Pr} = O(1), \quad \frac{\bar{Q}\,L^2}{q_0\,\kappa_0} = O(1). \tag{4.3}$$

It can be noticed that j can not be negative from equation (3.15). Besides, the arguments presented in Remark 3.2 are also valid in this section where both scales are considered at the same time and their possible interaction described through the integer j.

From there, the pressure and gravity terms in the momentum equation can be rewritten $-\frac{1}{\gamma_0\,\mathrm{Ma}^2}\partial_{\bar{x}}\widehat{p} - \frac{\bar{\rho}}{\varepsilon}e_z$, which means that the perturbation of the pressure is going to be $O(\varepsilon^{j+1})$.

If we then expand $\bar{p}$ and $\bar{\rho}$ in ε, we get:

$$\widehat{p} = \varepsilon^{j+1}\widehat{p}_{j+1} + \varepsilon^{j+2}\widehat{p}_{j+2} + O(\varepsilon^{j+3}), \tag{4.4}$$

$$\bar{\rho} = 1 + \varepsilon^1\bar{\rho}_1 + O(\varepsilon^2), \tag{4.5}$$

We then have to consider the linearized equation of state which states that :

$$\bar{\rho}_1 = -\varepsilon^j\widehat{p}_{j+1} - \theta + O(\varepsilon), \tag{4.6}$$

and recalling that $\partial_{\bar{x}}\,\widehat{p}_{j+1} = -\frac{\Upsilon_0^P}{\varepsilon^{j+1}}e_z$, we finally get :

$$\bar{\rho} = 1 - \frac{\Upsilon_0^P}{\varepsilon}\,\bar{z} - \varepsilon\,\theta + O(\varepsilon^2). \tag{4.7}$$

Let us consider the first case when $j = 0$; we denote $\tilde{\chi}_0 = \frac{\Upsilon_0^P}{\Upsilon_0^T}$, so that $\tilde{\chi}_0 = O(1)$. In this case, the system at the zeroth order reads :

$$\partial_{\bar{x}}\cdot\bar{v} \;= 0, \tag{4.8}$$

$$\partial_{\bar{t}}\bar{v} \,+\, \bar{v}\cdot\partial_{\bar{x}}\bar{v} \;= -\frac{1}{\tilde{\chi}}\,\partial_{\bar{x}}\widehat{p}_2 + \frac{\mathrm{Pr}}{\mathrm{Pe}}\partial_{\bar{x}\bar{x}}\bar{v} + \theta e_z + \tilde{\chi}\,\bar{z}e_z, \tag{4.9}$$

$$\partial_{\bar{t}}\theta \,+\, \bar{v}\cdot\partial_{\bar{x}}\cdot\theta \;= \qquad \frac{1}{\mathrm{Pe}}\,\partial_{\bar{x}\bar{x}}\theta - \frac{\gamma_0 - 1}{\gamma}\,\tilde{\chi}\,\bar{v}_z + \frac{\bar{Q}(\theta)\,L^2}{q_0\,\kappa_0}\,\frac{1}{\mathrm{Pe}}, \tag{4.10}$$

If $j > 0$, we denote $\tilde{\chi}_j = \frac{\Upsilon_0^P}{\Upsilon_0^T}$, so that $\tilde{\chi}_j = O(\varepsilon^j)$. In this case, the system at the zeroth order reads

$$\partial_{\bar{x}}\cdot\bar{v} \;= 0, \tag{4.11}$$

$$\partial_{\bar{t}}\bar{v} \,+\, \bar{v}\cdot\partial_{\bar{x}}\bar{v} \;= -\frac{1}{\tilde{\chi}_j}\,\partial_{\bar{x}}\widehat{p}_{j+2} + \frac{\mathrm{Pr}}{\mathrm{Pe}}\partial_{\bar{x}\bar{x}}\bar{v} + \theta e_z, \tag{4.12}$$

$$\partial_{\bar{t}}\theta \,+\, \bar{v}\cdot\partial_{\bar{x}}\cdot\theta \;= \qquad \frac{1}{\mathrm{Pe}}\,\partial_{\bar{x}\bar{x}}\theta + \frac{\bar{Q}(\theta)\,L^2}{q_0\,\kappa_0}\,\frac{1}{\mathrm{Pe}}, \tag{4.13}$$

which is the usual form of the Oberbeck–Boussinesq approximation [8] [9] [10] [11] [12].

4. 2 Coherence between the two approaches

It is interesting to note that both approaches yield the same systems of equations (3.24-3.26) and (4.12-4.14), if $\Upsilon_0^P << \Upsilon_0^T$. However, in the situation where $\Upsilon_0^P \approx \Upsilon_0^T$, equations (3.22) and (4.9) differ.

The two equation can be related by considering the two expressions of the pressure :

$$\frac{p - p_0}{p_0} = -\varepsilon\, \tilde{\chi}_0\, \bar{z} + \varepsilon^2\, \widehat{p}_2 + O(\varepsilon^3) = \varepsilon\, \tilde{\chi}_0\, (\varepsilon\, \bar{p} + \widetilde{p}), \tag{4.14}$$

so that, at the leading order,

$$-\bar{z} + \frac{\varepsilon\, \widehat{p}_2}{\tilde{\chi}_0} = \varepsilon\, \bar{p} + \widetilde{p}$$

Besides $|\widetilde{p} + \bar{z}| \leq \Upsilon^P_{\max}\, \dfrac{\bar{z}^2}{2}$, so that finally :

$$\widehat{p}_2 = \tilde{\chi}_0\, (\bar{p} + \frac{\widetilde{p} + \bar{z}}{\varepsilon}), \quad |\frac{\widetilde{p} + \bar{z}}{\varepsilon}| \leq \frac{\Upsilon^P_{\max}\, \bar{z}^2}{2\, \varepsilon} = O(1), \tag{4.15}$$

thanks to assumption **S3**. Finally, we can relate the pressure $\widehat{p}_2$ to its equivalent in the other derivation $\bar{p}$ through a constant coefficient of order one modulo an other part which can be calculated easily :

$$\tilde{\chi}_0\, \frac{\widetilde{p} + \bar{z}}{\varepsilon} = \tilde{\chi}_0\, \frac{\bar{z}^2}{2}. \tag{4.16}$$

The conclusion to be drawn from this is the fact that both forms are equivalent through a change of pressure and that the low Mach number asymptotics is a little more precise since it provides, at the second order, the exact pressure corrections.

The models have been completely identified; we are now going to evaluate the influence of the work of the pressure forces when $j = 0$. We first consider the linear stability analysis and then compare to numerical simulations.

5 Rayleigh–Bénard problem for an ideal gas : Linear stability analysis

The system of equations we are going to use for the linear stability analysis is the one used usually, which means the one with $v_0 = \kappa_0/L$ given in Appendix II, without the heat source term. There has been a vast literature on the linear stability analysis of this type of equations for the Rayleigh-Bénard problem (See for example [5] [6] [7] [16] [8]) and we shortly reproduce the leading ideas in this section.

The aim is the computation of Ra_c (the critical value of the Rayleigh number Ra for the onset of convection in the layer) as a function of the coefficient related to the work of the static pressure forces $\chi = \frac{\widetilde{\Upsilon}_0^P}{\Upsilon_0^T} = \frac{\gamma_0 - 1}{\gamma_0}\,\tilde{\chi}_0$. We should retrieve the usual values for the critical Rayleigh number as $\chi \to 0$.

5. 1 System of equations

The system with $v_0 = \kappa_0/L$ for a gas reads :

$$\partial_{\bar{x}} \cdot \bar{v} \; = 0, \tag{5.1}$$

$$\partial_{\bar{t}} \bar{v} \; + \; \bar{v} \cdot \partial_{\bar{x}} \bar{v} \; = -\partial_{\bar{x}} \, \bar{\bar{p}} - \mathrm{Pr}\, \partial_{\bar{x}\bar{x}} \bar{v} + \mathrm{Ra}\, \mathrm{Pr}\, \theta \, e_z, \tag{5.2}$$

$$\partial_{\bar{t}} \theta + \; \bar{v} \cdot \partial_{\bar{x}} \theta \; = \partial_{\bar{x}\bar{x}} \theta - \chi \, w, \tag{5.3}$$

where $\bar{\bar{p}}$ can be easily related to $\widehat{p}_2$ and where we omit, in the following, the double bar on the pressure. Problem (5.1-5.3) is considered in a 2D square domain ($\bar{x} = (y, z)^t$, $0 \leq y \leq 1$, $0 \leq z \leq 1$), where u and w represent the horizontal and vertical coordinates of the velocity and with the boundary conditions:

$$y = 0, 1 : \; \partial_y \theta = 0, u = 0, \partial_y w = 0; \;\; z = 0 : \; \theta = 1, w = 0, \partial_z u = 0; \;\; z = 1 : \; \theta = 0, w = 0, \partial_z u = 0. \tag{5.4}$$

5. 2 Linear stability analysis

Problem (5.1-5.4) is linearized about a static solution $\theta_S(z)$, $u = w = 0$, (in the configuration considered in the present paper, static solutions depend only on the vertical coordinate). In the following we still use the notation θ for the linearized temperature around θ_S. Using simple algebra, pressure can be eliminated to yield:

$$\partial_{\bar{t}} \theta = \partial_{\bar{x}\bar{x}} \theta - (\theta_S^{'} + \chi) w, \tag{5.5}$$

$$\partial_{\bar{t}}\partial_{\bar{x}\bar{x}}w = \mathrm{P_r}\ \partial_{\bar{x}\bar{x}}\partial_{\bar{x}\bar{x}}w + \mathrm{P_r}\,\mathrm{R_a}\ \partial_{yy}\theta. \tag{5.6}$$

with boundary conditions:

$$y = 0, 1:\ \partial_y\theta = 0, \partial_y w = 0;\ \ z = 0, 1:\ \theta = 0, w = 0, \partial_{zz}w = 0. \tag{5.7}$$

We look for the solution of (5.5-5.7) of the form $\theta(y,z,t) = \widetilde{\theta}(z)e^{-\lambda t}\cos(ky)$, $w(y,z,t) = \widetilde{w}(z)e^{-\lambda t}\cos(ky)$ where $k = \pi m$, $m = 1, 2, ...$ which yields the eigenvalue problem:

$$-\lambda\widetilde{\theta} = \widetilde{\theta}^{''} - k^2\widetilde{\theta} - (\theta_S^{'} + \chi)\widetilde{w}, \tag{5.8}$$

$$-\lambda(\widetilde{w}^{''} - k^2\widetilde{w}) = \mathrm{P_r}\ (\widetilde{w}^{(4)} - 2k^2\widetilde{w}^{''} + k^4\widetilde{w}) - \mathrm{P_r}\,\mathrm{R_a}\ k^2\widetilde{\theta}, \tag{5.9}$$

with the boundary conditions

$$z = 0, 1 : \widetilde{\theta} = 0, \widetilde{w} = \widetilde{w}^{''} = 0. \tag{5.10}$$

The convective instability boundary can be found from the condition that the eigenvalue λ with the maximal real part is zero. Eliminating $\widetilde{\theta}$ from equations (5.8-5.9) and setting $\lambda = 0$ yields

$$\widetilde{w}^{(6)} - 3\ k^2\ \widetilde{w}^{(4)} + 3\ k^4\ \widetilde{w}^{''} -\ k^6\ \widetilde{w} = \mathrm{R_a}\,k^2(\theta_S^{'} + \chi)\widetilde{w}. \tag{5.11}$$

with boundary conditions

$$z = 0, 1 : \widetilde{w} = \widetilde{w}^{''} = \widetilde{w}^{(4)} = 0. \tag{5.12}$$

In the present configuration the static solution $\theta_S(z)$ is linear $\theta_S(z) = 1 - z$. Setting $\widetilde{w}(z) = \sin(\pi n z)$, we get $((\pi n)^2 + k^2)^3 = \mathrm{R_a}\,k^2(\chi - 1)$, which defines

$$(\mathrm{R_a})_j = \frac{((\pi j)^2 + k^2)^3}{k^2(1-\chi)}. \tag{5.13}$$

The minimum of $(\mathrm{R_a})_1(k^2)$ is attained for $k^2 = \pi^2/2$ and is equal to $\frac{27\pi^4}{4(1-\chi)} \approx \frac{657.5}{1-\chi}$. In the square domain where $k = \pi m$ the minimum of $(\mathrm{R_a})_1(k^2)$ is attained for $m = 1$ and is equal to $\frac{8\pi^4}{(1-\chi)} \approx \frac{780}{1-\chi}$.

6 Comparison with numerical simulations

A 2D vorticity-stream function formulation of system (5.1-5.3) is used:

$$\partial_{\bar{x}\bar{x}}\psi = -\omega, \tag{6.1}$$

$$\partial_{\bar{t}}\omega + u\,\partial_y\omega + w\,\partial_z\omega = \mathrm{Pr}\partial_{\bar{x}\bar{x}}\omega - \mathrm{Pr}\,\mathrm{Ra}\,\partial_z\theta \tag{6.2}$$

$$\partial_{\bar{t}}\theta + u\,\partial_y\theta + w\,\partial_z\theta = \partial_{\bar{x}\bar{x}}\theta - \chi w \tag{6.3}$$

where ψ is the stream function, ω the vorticity, $u = \partial_z\psi$, $w = -\partial_y\psi$. The Prandtl number has no influence on the critical value of the Rayleigh number and is taken to be 1. System (6.1-6.3) has been considered in a square domain with boundary conditions: $\psi = \omega = 0$, $\partial_y\theta = 0$ on the vertical boundaries and $\theta = 0$ on the horizontal ones. It is solved by using finite differences and an alternate directions method: the discretization of equation (6.3) on the first and on the second time half steps is, respectively,

$$2(\theta_{ij}^{n+1/2} - \theta_{ij}^{n})/\tau + u_{ij}^{n}\,(\theta_{i+1,j}^{n+1/2} - \theta_{i-1,j}^{n+1/2})/2h + v_{ij}^{n}\,\left((\theta_{i,j+1}^{n} - \theta_{i,j-1}^{n})/2h + \chi\right) =$$

$$(\theta_{i+1,j}^{n+1/2} - 2\theta_{i,j}^{n+1/2} + \theta_{i-1,j}^{n+1/2})/h^2 + (\theta_{i,j+1}^{n} - 2\theta_{i,j}^{n} + \theta_{i,j-1}^{n})/h^2$$

and

$$2(\theta_{ij}^{n+1} - \theta_{ij}^{n+1/2})/\tau + u_{ij}^{n}\,(\theta_{i+1,j}^{n+1/2} - \theta_{i-1,j}^{n+1/2})/2h + v_{ij}^{n}\,\left((\theta_{i,j+1}^{n+1} - \theta_{i,j-1}^{n+1})/2h + \chi\right) =$$

$$(\theta_{i+1,j}^{n+1/2} - 2\theta_{i,j}^{n+1/2} + \theta_{i-1,j}^{n+1/2})/h^2 + (\theta_{i,j+1}^{n+1} - 2\theta_{i,j}^{n+1} + \theta_{i,j-1}^{n+1})h^2.$$

Here τ is the time step and h the space step. Other equations are discretized similarly.

We examine the effect of the term χw. Recall that this term has to be retained in the equation if $\frac{\gamma_0\, g\, L}{c_0^2\,\alpha_0\,\Delta T} \approx 1$ which means that the density changes due to temperature variations and the density changes due to the static pressure variations have the same order of magnitude.

This term is seen to have a stabilizing effect which confirms the linear stability analysis. On Figure 1, we have plotted the maximum of the stationary stream function as a function of the Rayleigh number, for various values of χ. For each curve a continuation based on the Rayleigh number has been performed. This bifurcation diagram is classical for $\chi = 0$. On Figure 2, we have plotted Ra_c as a function of χ, as well as the curve $\mathrm{Ra}_c = \frac{8\pi^4}{(1-\chi)}$ predicted by the stability analysis, for comparison. It is shown to perfectly match.

We have also performed the numerical simulations in the configuration of no-slip boundary conditions. In this case, the domain was taken rectangular ($0 \leq y \leq 3.117$ and $0 \leq z \leq 1$. The value 3.117 corresponds to the wave length leading to the minimal critical Rayleigh number on the marginal stability curve : $\mathrm{Ra}_c = 1708$ [7] [8]). Figure 3 shows the behavior of the stationary

maximal stream function with the Rayleigh number as a bifurcation parameter for various values of the compressibility parameter. Finally, Figure 4 presents Ra_c as a function of χ as well as the curve $\mathrm{Ra}_c = \frac{1708}{(1-\chi)}$. As in the case of the free surface boundary conditions, the agreement is very good.

7 Discussion and Conclusion

Three main directions of discussion can be drawn from the previous study. The first point to be underlined is the necessity to start from general enough equations, i.e. from the full compressible system of equation with a divariant state law in order to rigorously justify the Oberbeck-Boussinesq approximation. The mathematical representation of a mechanically incompressible but thermally compressible fluid, should only be obtained as the limit $\Upsilon_0^P \to 0$ for the systems of equation derived in sub-section 3.4. This leads in some cases to the additional term involving the compressibility parameter χ which can not be attained by just assuming the density to depend only on the temperature as in [2] [3]. As stated in the previous two sections, the influence of this term can be of importance since it can completely stabilize the layer. A physical explanation for this phenomenon can be given. In the asymptotic limit where the compressibility of the medium is negligible at the scale L, but where the work of the static pressure forces due to the pressure stratification can bring a temperature difference of the order of the temperature difference imposed on the fluid layer, convection is prevented.

The second point is related to the typical order of magnitude of the compressibility parameter χ. In the case of a gaseous layer of air at usual temperature and pressure, the thickness of the layer has to be between a hundred and a thousand meters for χ to be in the range where is stabilizes the layer, whereas for water, is is of the order of 10^4 m. We have assumed in the previous evaluation that $\delta T \approx 5$ K. We then refer at this level to the study of Gray and Giorgini [4] where the regions of validity of the approximation are provided. It is clear that for such thick layers, the associated Rayleigh numbers will be sufficient for turbulence to appear. However, even if the equations are then not valid any more, the problem of the stability of the stationary solution is still relevant and the stabilizing effect of the work of the static pressure forces should be kept in the governing equations.

The third point to be discussed is related to the choice of the characteristic velocity in order to non-dimensionalize the equations. We have presented the case $v_0 = \sqrt{g\,L\,\Upsilon_0^T}$ of the free fall velocity. In this case, the bifurcation parameter can still be taken as the Rayleigh number, or more precisely as the square root of it since $\mathrm{P_e}/\sqrt{\mathrm{P_r}} = \sqrt{\mathrm{R_a}}$. Instead of having the bifurcation parameter in front of the source term in the momentum equation as usual when $v_0 = \kappa_0/L$, it is in front of the viscous second order term as the denominator. From a theoretical point of view, it yields of course the same results. From the asymptotics point of view, let us underline that it has no implication since $\mathrm{R_a} = O(1)$ for both small parameters. In order to check the validity of such a result, we have represented on Fig. 5 three characteristic velocities involved in the Rayleigh-Bénard problem with $\chi = 0$. The first one is the ratio of the free fall velocity to the diffusion velocity. In our case, since $\mathrm{P_r} = 1$, this ratio is equal to the square root of the Rayleigh number. The second velocity is the diffusion one, $v_0 = \kappa_0/L$ which corresponds in our study to the non-dimensionalizing velocity and is represented on Fig. 5 with the horizontal line $|\bar{v}| = 1$. Finally, we have plotted the maximum of the norm of the stationary velocity as a function of the Rayleigh number. It is particularly interesting to note that this last velocity is always in between the previous two velocities, that the diffusion velocity is adapted to the numerical detection of the critical value of the Rayleigh number but gets especially small as the Rayleigh number is increased whereas the maximum of the velocity is closer to the free fall velocity. In this case, the Péclet number is linearly increasing which is known to yield the appearance of a thermal boundary layer.

Finally we have provided a rigorous framework for the derivation of the Oberbeck-Boussinesq approximation from a full compressible model. The difference between general divariant liquids and ideal gases has been pointed out and two asymptotics derived. The asymptotics is not dependent on the choice of the characteristic velocity. The presence of a residual term in the temperature equation due to the work of the static pressure forces due to stratification has been discussed and justified; besides the influence of this term on the stability of the layer has been studied both theoretically and numerically. We think that our work brings a unified approach to this problem and provides the basis for further mathematical investigation and justification of the presented asymptotics.

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References

[1] E. A. Spiegel, G. Veronis, *On the Boussinesq approximation for a compressible fluid*, *Astrophys. J.*, Vol. 131(1960) 442–447

[2] J. M. Mihaljan, *A rigorous exposition of the Boussinesq approximations applicable to a thin layer of fluid*, *Astrophys. J.*, Vol. 136(1962) 1126–1133

[3] K. R. Rajagopal, M. Ruzicka, A. R. Srinivasa, *On the Oberbeck-Boussinesq approximation*, *Math. Models and Meth. in Appl. Sci.*, Vol. 6, n. 8(1996) 1157–1167

[4] D. D. Gray, A. Giorgini, *The validity of the Boussinesq approximation for liquids and gases*, *Int. J. Heat Mass transfer*, Vol. 19(1976) 545–551

[5] S. Chandrasekhar, Hydrodynamic and Hydromagnetic Stability, Clarendon Press, Oxford (1961)

[6] G.Z. Gershuni and E.M. Zhukovitskii, Convection Stability of Incompressible Fluids, Ketter Publications (1976)

[7] P.G. Drazin and W.H. Reid, Hydrodynamic Stability, Cambridge University Press (1981)

[8] A.V. Getling, Rayleigh-Bénard Convection, Structure and Dynamics, Advanced Series in Nonlinear Dynamics, Vol. 11, World Scientific (1998)

[9] A. G. Merzhanov, E. A. Shtessel, *Free Convection and Thermal Explosion in Reactive Systems*, *Astronautica Acta*, Vol. 18(1973) 191–199

[10] A.G. Merzhanov, V.V. Barzykin, V.G. Abramov, *The theory of heat explosion: from N.N. Semenov to our days*, *Himicheskaya Fizika*, Vol. 15, No. 6(1996) 3–44

[11] T. Dumont, S. Genieys, M. Massot, V. Volpert, *Interaction of thermal explosion and natural convection : critical conditions and new oscillating regimes*, *Internal Report of MAPLY, Submitted* (2001)

[12] V. Volpert, T. Dumont, S. Genieys, M. Massot, *Influence of Natural Convection on Heat Explosion (Third prize for the best paper) Proceedings of the 3rd International Seminar on Fire and Explosion Hazards, 10th-14th April, 2000, University of Lancashire, Preston, UK* (2000)

[13] V. Giovangigli, M. Massot, *The Local Cauchy Problem for Multicomponent Reactive Flows in Full Vibrational Nonequilibrium*, *Math. Meth. Appl. Sci.*, 21(1998) 1415–1439

[14] V. Giovangigli, M. Massot, *Asymptotic Stability of Equilibrium States for Multicomponent Reactive Flows*, *Mathematical Models and Methods in Applied Sciences,* 2(1998) 251–297

[15] V. Giovangigli, Multicomponent flow modeling, Modeling and Simulation in Science, Engineering, & Technology. Birkhuser Boston, Inc., Boston, MA (1999)

[16] S. Fauve, *Pattern forming Instabilities*, In Hydrodynamics and Nonlinear Instabilities, C. Godrèche, P. Mannevile Eds., Cambridge University Press (1998) 387–492

8 Appendix I

In this Appendix, we give the proof of Proposition 2.1. Let us first recall the result. Under the assumption that $\Upsilon^P_{\max} << 1$, then for the isothermal case, we have the following estimate :

$$0 \leq \frac{p_{SI} - (p_0 - \rho_0 g z)}{\rho_0 g L} \leq \tfrac{1}{2} \Upsilon^P_{\max} \bar{z}^2 << 1, \quad 0 \leq \frac{\rho_0 - \rho_{SI}}{\rho_0} \leq \Upsilon^P_{\max} \bar{z} << 1,$$

where $\bar{z} = z/L$, L is the characteristic length.

Let us then assume $\Upsilon^T_{\max} << \alpha_0 T_0 \leq 1$, then

$$-\Upsilon^T_{\max} \bar{z} \leq \frac{p_S - (p_0 - \rho_0 g z)}{\rho_0 g L} \leq \Upsilon^T_{\max} \bar{z} + \tfrac{1}{2} \Upsilon^P_{\max} \bar{z}^2 << 1,$$

$$-\Upsilon^T_{\max} \bar{z} - \Upsilon^P_{\max} \frac{\bar{z}^2}{2} \leq \frac{p_S - p_{SI}}{\rho_0 g L} \leq \Upsilon^T_{\max} \bar{z} + \Upsilon^P_{\max} \frac{\bar{z}^2}{2},$$

$$-\Upsilon^T_{\max} + \Upsilon^P_{\max} \bar{z} (1 - \Upsilon^T_{\max} - \Upsilon^P_{\max} \frac{\bar{z}}{2}) \leq \frac{\rho_0 - \rho_S(p_S(z), T_S)}{\rho_0} \leq \Upsilon^T_{\max} + \Upsilon^P_{\max} \bar{z} (1 + \Upsilon^T_{\max}),$$

We first consider a non dimensional form of the pressure equation, the characteristic pressure drop is taken to be $\rho_0 gL$, and the non dimensional pressure will then be denoted $\widetilde{p} = (p_{SI} - p_0)/(\rho_0 gL)$. The static equation at constant temperature on the pressure $\widetilde{p}$ reads:

$$d_{\bar{z}}\widetilde{p} + 1 = -\frac{\rho(p_0 + \widetilde{p}\rho_0 gL, T_0) - \rho_0}{\rho_0}.$$

Since ρ is a strictly increasing function of p at constant temperature, it is clear that $\widetilde{p}$ is then a decreasing and convex function of $\bar{z}$ which belongs to the interval $[-1, 0]$; besides, from Rolle's theorem applied to the application $p \to \rho(p, T_0)$, there exists $p(z = L) < p^* < p_0$ such that, denoting $\hat{\rho} = \rho/\rho_0$, we get :

$$1 - \hat{\rho} = \frac{\gamma(p^*, T_0)}{c^2(p^*, T_0)}\, g\, L(-\widetilde{p}(z)) < \Upsilon^P_{\max}\,(-\widetilde{p}(z))$$

but

$$-\widetilde{p} = \bar{z} - \int_0^{\bar{z}} (1 - \hat{\rho}) dz$$

so that

$$0 \leq -\widetilde{p} \leq \bar{z}, \qquad 0 \leq 1 - \hat{\rho} \leq \Upsilon^P_{\max}\, \bar{z}.$$

this also yield

$$0 \leq \widetilde{p} + \bar{z} \leq \tfrac{1}{2}\Upsilon^P_{\max}\, \bar{z}^2.$$

Let us now turn to the non-isothermal case and denote $\widetilde{p}_S = \frac{p_S(z) - p_0}{\rho_0 gL}$ and $\hat{\rho}_S = \frac{\rho_S(p_S(z), T_0)}{\rho_0}$.

$$\frac{\rho_0 - \rho_S(p_S(z), T_S(z))}{\rho_0} = \hat{\rho}_S - \frac{\rho_S(p_S(z), T_S(z))}{\rho_0} + 1 - \hat{\rho}_S;$$

however, at constant pressure, ρ is a smooth decreasing function of T so that there exists $T_0 - \Delta T \leq T^* \leq T_0 + \Delta T$, such that

$$\hat{\rho}_S - \frac{\rho_S(p_S(z), T_S(z))}{\rho_0} = \alpha(p_S(z), T^*)(T_0 - T_S(z)), \qquad |\hat{\rho}_S - \frac{\rho_S(p_S(z), T_S(z))}{\rho_0}| \leq \Upsilon^T_{\max};$$

but

$$-\widetilde{p}_S = \bar{z} - \int_0^{\bar{z}} (1 - \hat{\rho}_S)\, dz + \int_0^{\bar{z}} (\hat{\rho}_S - \frac{\rho_S(p_S(z), T_S)}{\rho_0}) dz.$$

Besides, there exists p^{**} such that

$$1 - \hat{\rho}_S = \frac{\gamma(p^{**}, T_0)}{c^2(p^{**}, T_0)}\, g\, L\,(-\widetilde{p}_S),$$

so that

$$-\Upsilon^T_{\max}\,\bar z \le \widetilde{p}_S + \bar z \le \Upsilon^T_{\max}\,\bar z + \Upsilon^P_{\max}\,\frac{\bar z^2}{2},$$

and finally

$$-\Upsilon^T_{\max} + \Upsilon^P_{\max}\,\bar z\,(1-\Upsilon^T_{\max} - \Upsilon^P_{\max}\,\frac{\bar z}{2}) \le \frac{\rho_0 - \rho_S(p_S(z),T_S)}{\rho_0} \le \Upsilon^T_{\max} + \Upsilon^P_{\max}\,\bar z\,(1+\Upsilon^T_{\max}).$$

9 Appendix II

We give the system of non-dimensional equations in the case when we chose $v_0 = \frac{\kappa_0}{L}$ as the characteristic velocity of the problem. The final system then reads, for the gaseous case:

$$\begin{aligned}
-\partial_{\bar x}\cdot\bar v &= \frac{1}{1+\widehat{p}}(\partial_{\bar t}\widehat{p} + \bar v\cdot\partial_{\bar x}\widehat{p}) - \frac{\Upsilon^T_0}{1+\Upsilon^T_0\,\theta}\,(\partial_{\bar t}\theta + \bar v\cdot\partial_{\bar x}\theta),\\
\bar\rho\,(\partial_t\bar v + \bar v\cdot\partial_{\bar x}\bar v) &= -\frac{\mathrm{Ra}\,\mathrm{Pr}}{\Upsilon^P_0\,\Upsilon^T_0}\partial_{\bar x}\widehat{p} - \mathrm{Pr}(\partial_{\bar x\bar x}\bar v + \partial_{\bar x}(\partial_{\bar x}\cdot\bar v) - \tfrac{2}{3}\partial_{\bar x}\cdot(\partial_{\bar x}\cdot\bar v)\boldsymbol{I}) - \frac{\mathrm{Ra}\,\mathrm{Pr}}{\Upsilon^T_0}\,\bar\rho e_z,\\
\bar\rho\,(\partial_{\bar t}\theta + \bar v\cdot\partial_{\bar x}\theta) &= \partial_{\bar x\bar x}\theta - \frac{\widetilde{\Upsilon}^P_0}{\mathrm{Ra}}\left(\partial_{\bar x}\bar v + (\partial_{\bar x}\bar v)^t - \tfrac{2}{3}\partial_{\bar x}\cdot\bar v\boldsymbol{I}\right) : \partial_{\bar x}\bar v\\
&\quad + \frac{1}{\Upsilon^T_0}\,\frac{\gamma_0-1}{\gamma_0}(\partial_{\bar t}\widehat{p} + \bar v\cdot\partial_{\bar x}\widehat{p}) + \frac{\bar Q(\theta)\,L^2}{q_0\,\kappa_0},
\end{aligned}$$

where $Ra = gL^3\alpha_0\Delta T/\kappa_0\,\nu_0$ is the Rayleigh number, and with the linearized state law:

$$\bar\rho = 1 + \widehat{p} - \Upsilon^T_0\,\theta, \qquad \widehat{p} << 1, \qquad \Upsilon^T_0\,\theta << 1.$$

In the case of a liquid, the system takes the form:

$$\begin{aligned}
-\partial_{\bar x}\cdot\bar v &= \frac{\Upsilon^P}{\bar\rho}\left(\Upsilon^T_0\,(\partial_{\bar t}\bar p + \bar v\cdot\partial_{\bar x}\bar p) + \bar v\cdot\partial_{\bar x}\widetilde{p}\right) - \alpha_0\,T_0\,\Upsilon^T\,(\partial_{\bar t}\theta + \bar v\cdot\partial_{\bar x}\theta),\\
\bar\rho\,(\partial_{\bar t}\bar v + \bar v\cdot\partial_{\bar x}\bar v) &= -\partial_{\bar x}\bar p + \mathrm{Pr}\,(\partial_{\bar x\bar x}\bar v + \partial_{\bar x}(\partial_{\bar x}\cdot\bar v) - \tfrac{2}{3}\partial_{\bar x}\cdot(\partial_{\bar x}\cdot\bar v)\boldsymbol{I}) - \frac{\mathrm{Ra}\,\mathrm{Pr}}{\Upsilon^T_0}\,(\partial_{\bar x}\widetilde{p} + \bar\rho\,e_z)\,,\\
\bar\rho\,(\partial_{\bar t}\theta + \bar v\cdot\partial_{\bar x}\theta) &= \partial_{\bar x\bar x}\theta - \widetilde{\Upsilon}^P_0\,\frac{1}{(\alpha_0\,T_0)^2}\,\frac{1}{\mathrm{Ra}}\left(\partial_{\bar x}\bar v + (\partial_{\bar x}\bar v)^t - \tfrac{2}{3}\partial_{\bar x}\cdot\bar v\boldsymbol{I}\right) : \partial_{\bar x}\bar v + \frac{\bar Q(\theta)\,L^2}{q_0\,\kappa_0}\\
&\quad + \frac{\widetilde{\Upsilon}^P_0}{\alpha_0\,T_0}\,\frac{(1+\Upsilon^T_0\,\theta)}{\bar\rho\,\mathrm{Ra}\,\mathrm{Pr}}\,\frac{\Upsilon^T}{\Upsilon^T_0}\,(\partial_{\bar t}\bar p + \bar v\cdot\partial_{\bar x}\bar p) + \frac{\widetilde{\Upsilon}^P_0}{\Upsilon^T_0\,\alpha_0\,T_0}\,\frac{(1+\Upsilon^T_0\,\theta)}{\bar\rho\,\mathrm{Ra}\,\mathrm{Pr}}\,\frac{\Upsilon^T}{\Upsilon^T_0}\,(\bar v\cdot\partial_{\bar x}\widetilde{p}),
\end{aligned}$$

with the linearized state law:

$$\bar\rho = \hat\rho + \Upsilon^T_0\,\left(\Upsilon^P_0\,\frac{\mathrm{Fr}^2}{\Upsilon^T_0}\,\bar p - \alpha_0\,T_0\,\theta\right), \quad \Upsilon^T_0\,\alpha_0\,T_0\,\theta << 1, \quad \mathrm{Fr}^2\,\Upsilon^P_0\,\bar p << 1.$$

Captions

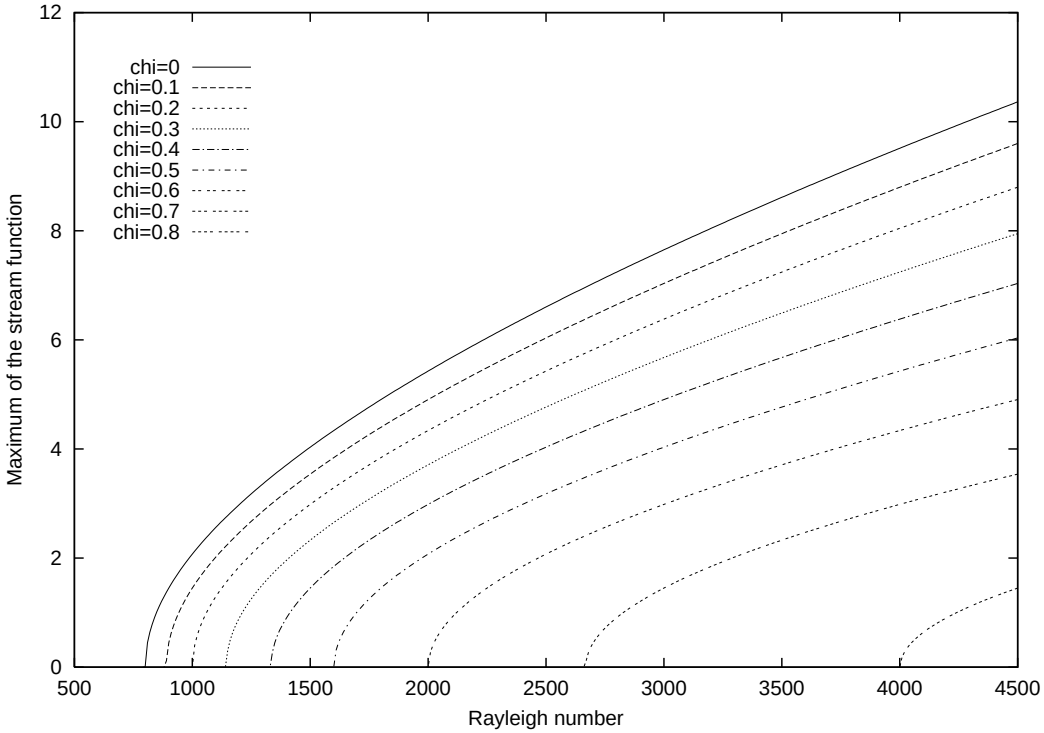
Figure 1. : Behavior of the stationary maximal stream function depending on the Rayleigh number for various values of the compressibility parameter χ with free surface boundary conditions.

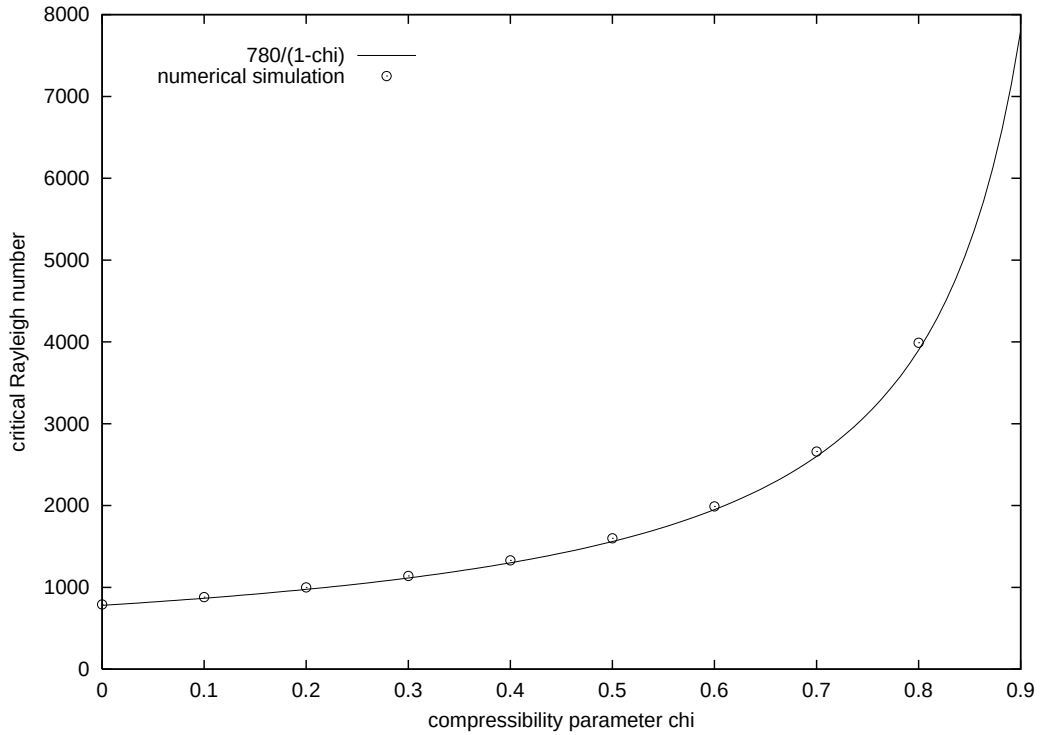
Figure 2. : Symbols : Critical Rayleigh number as a function of the compressibility parameter χ, solid line : theoretical value of the critical Rayleigh number with free surface boundary conditions.

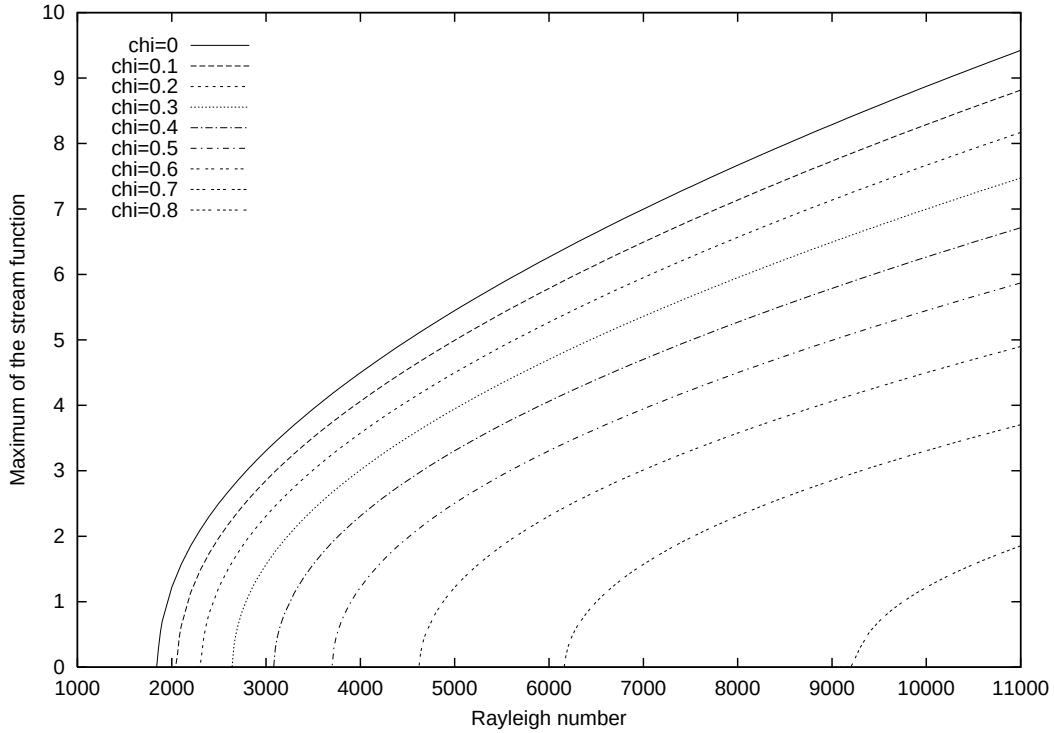
Figure 3. : Behavior of the stationary maximal stream function depending on the Rayleigh number for various values of the compressibility parameter χ with no-slip velocity boundary conditions.

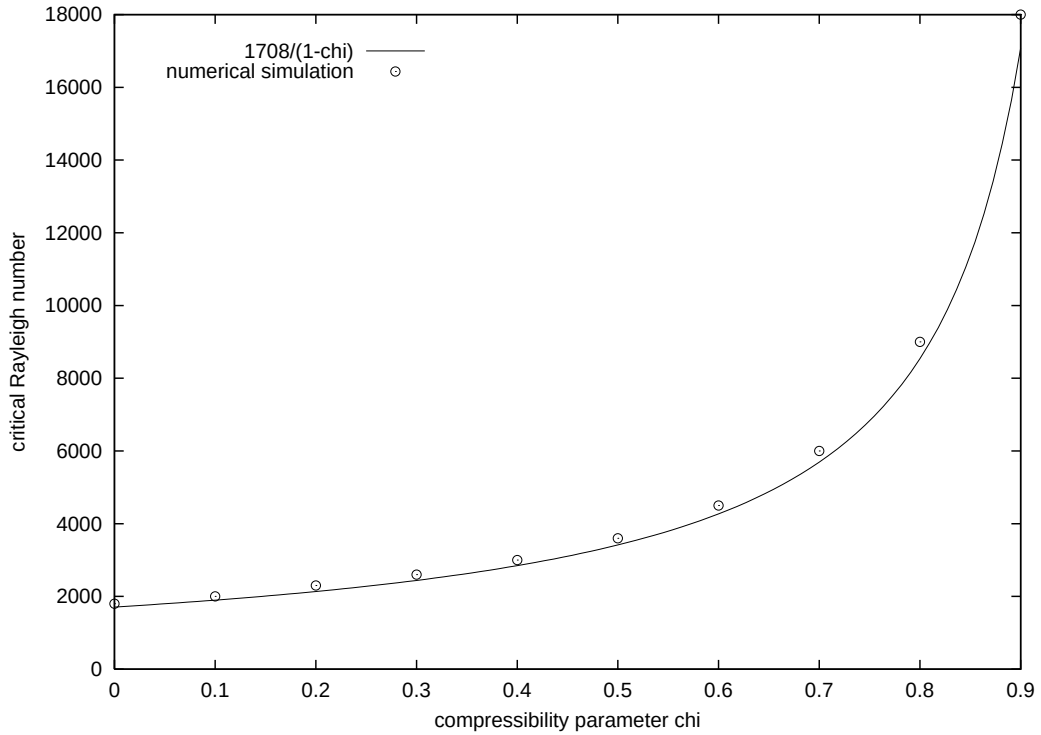
Figure 4. : Symbols : Critical Rayleigh number as a function of the compressibility parameter χ with no-slip velocity boundary conditions, solid line : curve $\mathrm{Ra}_c = \frac{1708}{1-\chi}$, where 1708 is the theoretical value of the critical Rayleigh number for $\chi = 0$.

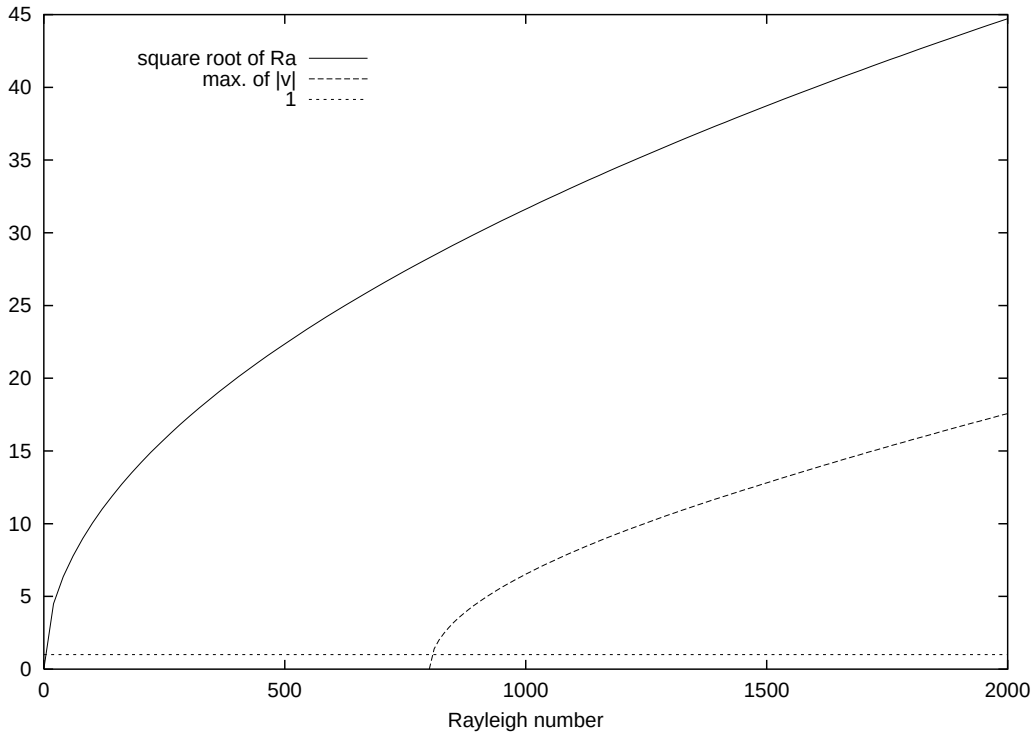
Figure 5. : Comparison of the various characteristic velocities in the Rayleigh-Bénard problem with $\chi = 0$: solid line is the square root of the Rayleigh number $\sqrt{\mathrm{Ra}} = \mathrm{Pe}$ since $\mathrm{Pr} = 1$, dashed line is the maximum of the velocity in the layer, and the dotted line corresponds to the diffusion velocity $v_0 = \kappa_0/L = \nu_0/L$.


chi=0
chi=0.1
chi=0.2
chi=0.3
chi=0.4
chi=0.5
chi=0.6
chi=0.7
chi=0.8
Maximum of the stream function
Rayleigh number
0
2
4
6
8
10
12
500
1000
1500
2000
2500
3000
3500
4000
4500


780/(1-chi)
numerical simulation
critical Rayleigh number
compressibility parameter chi
8000
7000
6000
5000
4000
3000
2000
1000
0
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9


chi=0
chi=0.1
chi=0.2
chi=0.3
chi=0.4
chi=0.5
chi=0.6
chi=0.7
chi=0.8
Maximum of the stream function
Rayleigh number
0
1
2
3
4
5
6
7
8
9
10
1000
2000
3000
4000
5000
6000
7000
8000
9000
10000
11000


1708/(1-chi)
numerical simulation
critical Rayleigh number
compressibility parameter chi
18000
16000
14000
12000
10000
8000
6000
4000
2000
0
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9


45
40
35
30
25
20
15
10
5
0
0
500
1000
1500
2000
Rayleigh number
square root of Ra
max. of |v|
1