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Estimation of volatility functionals: the case of a $\sqrt{n}$ window

Jean Jacod* Mathieu Rosenbaum†

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Abstract

We consider a multidimensional Itô semimartingale regularly sampled on $[0,t]$ at high frequency $1/\Delta_n$, with Δ_n going to zero. The goal of this paper is to provide an estimator for the integral over $[0,t]$ of a given function of the volatility matrix, with the optimal rate $1/\sqrt{\Delta_n}$ and minimal asymptotic variance. To achieve this we use spot volatility estimators based on observations within time intervals of length $k_n\Delta_n$. In [5] this was done with $k_n \to \infty$ and $k_n\sqrt{\Delta_n} \to 0$, and a central limit theorem was given after suitable de-biasing. Here we do the same with the choice $k_n \asymp 1/\sqrt{\Delta_n}$. This results in a smaller bias, although more difficult to eliminate.

Key words: semimartingale, high frequency data, volatility estimation, central limit theorem, efficient estimation

MSC2010: 60F05, 60G44, 62F12

1 Introduction

Consider an Itô semimartingale X_t, whose squared volatility c_t (a $d \times d$ matrices-valued process if X is d-dimensional) is itself another Itô semimartingale. The process X is observed at discrete times $i\Delta_n$ for $i = 0, 1, \cdots$, the time lag Δ_n being small (high-frequency setting) and eventually going to 0. The aim is to estimate integrated functionals of the volatility, that is $\int_0^t g(c_s)\,ds$ for arbitrary (smooth enough) functions g, on the basis of the observations at stage n and within the time interval $[0,t]$.

In [5], to which we refer for detailed motivations for this problem, we have exhibited estimators which are consistent, and asymptotically optimal, in the sense that they asymptotically achieve the best rate $1/\sqrt{\Delta_n}$, and also the minimal asymptotic variance in the cases where optimality is well-defined (namely, when X is continuous and has a Markov type structure, in the sense of [2]). These estimators have this rate and minimal asymptotic variance as soon as the jumps of X are summable, plus some mild technical conditions.

*Institut de Mathématiques de Jussieu, 4 Place Jussieu, 75 005 Paris, France (CNRS – UMR 7586, and Université Pierre et Marie Curie), Email: jean.jacod@upmc.fr

†Laboratoire de Probabilités et Modèles Aléatoires, 4 Place Jussieu, 75 005 Paris, France (CNRS – UMR 7599, and Université Pierre et Marie Curie), Email: mathieu.rosenbaum@upmc.fr

The aim of this report is to complement [5] with another estimator, of the same type, but using spot volatility estimators based on a different window size. In this introduction we explain the differences between the estimator in that paper and the one presented here.

For the sake of simplicity we consider the case when X is continuous and one-dimensional (the discontinuous and multi-dimensional case is considered later), that is of the form

$$X_t = X_0 + \int_0^t b_s\,ds + \int_0^t \sigma_s\,dW_s$$

and $c_t = \sigma_t^2$ is the squared volatility. Natural estimators for $\int_0^t g(c_s)\,ds$ are

$$V^n(g)_t = \Delta_n \sum_{i=1}^{[t/\Delta_n]-k_n+1} g(\widehat{c}_i^n), \quad \text{where } \widehat{c}_i^n = \frac{1}{k_n\Delta_n}\sum_{j=0}^{k_n-1}(X_{(i+j)\Delta_n} - X_{(i+j-1)\Delta_n})^2 \tag{1.1}$$

for an arbitrary sequence of integers such that $k_n \to \infty$ and $k_n\Delta_n \to 0$: One knows that $V^n(g)_t \xrightarrow{\mathbb{P}} V(g)_t$ (when g is continuous and of polynomial growth).

The variables $\widehat{c}_i^n$ are spot volatility estimators, and according to [4] we know that $\widehat{c}_{[t/\Delta_n]}^n$ estimate c_t, with a rate depending on the "window size" k_n. The optimal rate $1/\Delta_n^{1/4}$ is achieved by taking $k_n \asymp 1/\sqrt{\Delta_n}$. When k_n is smaller, the rate is $\sqrt{k_n}$ and the estimation error is a purely "statistical error"; when k_n is bigger, the rate is $1/\sqrt{k_n\Delta_n}$ and the estimation error is due to the variability of the volatility process c_t itself (its volatility and its jumps). With the optimal choice $k_n \asymp 1/\sqrt{\Delta_n}$ the estimation error is a mixture of the statistical error and the error due to the variability of c_t.

In [5] we have used a "small" window, that is $k_n \ll 1/\sqrt{\Delta_n}$. Somewhat surprisingly, this allows for optimality in the estimation of $\int_0^t g(c_s)\,ds$ (rate $1/\sqrt{\Delta_n}$ and minimal asymptotic variance). However, the price to pay is the need of a de-biasing term to be subtracted from $V^n(g)$, without which the rate is smaller and no Central Limit Theorem is available.

Here, we consider the window size $k_n \asymp 1/\sqrt{\Delta_n}$. This leads to a convergence rate $1/\sqrt{\Delta_n}$ for $V^n(g)$ itself, and the limit is again conditionally Gaussian with the "minimal" asymptotic variance, but with a bias that depends on the volatility of the volatility c_t, and on its jumps. It is however possible to subtract from $V^n(g)$ a de-biasing term again, so that the limit becomes (conditionally) centered.

Section 2 is devoted to presenting assumptions and results, and all proofs are gathered in Section 3. The reader is referred to [5] for motivation and various comments and a detailed discussion of optimality. However, in order to make this report readable, we basically give the full proofs, even though a number of partial results have already been proved in the above-mentioned paper, and with the exception of a few well designated lemmas.

2 The results

2.1 Setting and Assumptions

The underlying process X is d-dimensional, and observed at the times $i\Delta_n$ for $i = 0, 1, \cdots$, within a fixed interval of interest $[0, t]$. For any process we write $\Delta_i^n Y = Y_{i\Delta_n} - Y_{(i-1)\Delta_n}$

for the increment over the ith observation interval. We assume that the sequence Δ_n goes to 0. The precise assumptions on X are as follows:

First, X is an Itô semimartingale on a filtered space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$. It can be written in its Grigelionis form, as follows, using a d-dimensional Brownian motion W and a Poisson random measure μ on $\mathbb{R}_+ \times E$, with E is an auxiliary Polish space and with the (non-random) intensity measure $\nu(dt, dz) = dt \otimes \lambda(dz)$ for some σ-finite measure λ on E:

$$\begin{aligned} X_t &= X_0 + \int_0^t b_s\, ds + \int_0^t \sigma_s\, dW_s + \int_0^t \int_E \delta(s,z)\, 1_{\{\|\delta(s,z)\|\leq 1\}}\, (\mu-\nu)(ds,dz) \\ &\qquad + \int_0^t \int_E \delta(s,z)\, 1_{\{\|\delta(s,z)\|>1\}}\, \mu(ds,dz). \end{aligned} \tag{2.1}$$

This is a vector-type notation: the process b_t is $\mathbb{R}^d$-valued optional, the process σ_t is $\mathbb{R}^d \otimes \mathbb{R}^d$-valued optional, and $\delta = \delta(\omega, t, z)$ is a predictable $\mathbb{R}^d$-valued function on $\Omega \times \mathbb{R}_+ \times E$.

The spot volatility process $c_t = \sigma_t \sigma_t^*$ (* denotes transpose) takes its values in the set $\mathcal{M}_d^+$ of all nonnegative symmetric $d \times d$ matrices. We suppose that c_t is again an Itô semimartingale, which can be written as

$$\begin{aligned} c_t &= c_0 + \int_0^t \widetilde{b}_s\, ds + \int_0^t \widetilde{\sigma}_s\, dW_s + \int_0^t \int_E \widetilde{\delta}(s,z)\, 1_{\{\|\widetilde{\delta}(s,z)\|\leq 1\}}\, (\mu-\nu)(ds,dz) \\ &\qquad + \int_0^t \int_E \widetilde{\delta}(s,z)\, 1_{\{\|\widetilde{\delta}(s,z)\|>1\}}\, \mu(ds,dz). \end{aligned} \tag{2.2}$$

with the same W and μ as in (2.1). This is indeed *not a restriction*: if X and c are two Itô semimartingales, we have a representation as above for the pair (X, c) and, if the dimension of W exceeds the dimension of X one can always add fictitious component to X, arbitrarily set to 0, so that the dimensions of X and W agree.

In (2.2), $\widetilde{b}$ and $\widetilde{\sigma}$ are optional and $\widetilde{\delta}$ is as δ; moreover $\widetilde{b}$ and $\widetilde{\delta}$ are $\mathbb{R}^{d^2}$-valued. Finally, we need the spot volatility of the volatility and "spot covariation" of the continuous martingale parts of X and c, which are

$$\widetilde{c}_t^{ij,kl} = \sum_{m=1}^d \widetilde{\sigma}_t^{ij,m} \widetilde{\sigma}_t^{kl,m}, \qquad \widetilde{c}_t^{i,jk} = \sum_{l=1}^d \sigma_t^{il} \widetilde{\sigma}_t^{jk,l}.$$

The precise assumption on the coefficients are as follows, with r a real in $[0, 1)$:

Assumption (A'-r): There are a sequence (J_n) of nonnegative bounded λ-integrable functions on E and a sequence (τ_n) of stopping times increasing to ∞, such that

$$t \leq \tau_n(\omega) \implies \|\delta(\omega,t,z)\|^r \wedge 1 + \|\widetilde{\delta}(\omega,t,z)\|^2 \wedge 1 \leq J_n(z) \tag{2.3}$$

Moreover the processes $b'_t = b_t - \int \delta(t,z)\, 1_{\{\|\delta(t,z)\|\leq 1\}}\, \lambda(dz)$ (which is well defined), $\widetilde{c}_t$ and $\widetilde{c}'_t$ are càdlàg or càglàd, and the maps $t \mapsto \widetilde{\delta}(\omega,t,z)$ are càglàd (recall that $\widetilde{\delta}$ should be predictable), as well as the processes $\widetilde{b}_t + \int \widetilde{\delta}(t,z)\big(\kappa(\|\widetilde{\delta}(t,z)\|) - 1_{\{\|\widetilde{\delta}(t,z)\|\leq 1\}}\big)\, \lambda(dz)$ for one (hence for all) continuous function κ on $\mathbb{R}_+$ with compact support and equal to 1 on a neighborhood of 0. □

The bigger r is, the weakest Assumption (A-r) is, and when (A-0) holds the process X has finitely many jumps on each finite interval. The part of (A-r) concerning the jumps of X implies that $\sum_{s\leq t} \|\Delta X_s\|^r < \infty$ a.s. for all $t < \infty$, and it is in fact "almost" implied by this property. Since $r < 1$, this implies $\sum_{s\leq t} \|\Delta X_s\| < \infty$ a.s.

Remark 2.1 (A'-r) above is basically the same as Assumption (A-r) in [5], albeit (slightly) stronger (hence its name): some degree of regularity in time seems to be needed for $\widetilde{b}, \widetilde{c}, \widetilde{c}', \widetilde{\delta}$ in the present case.

2.2 A First Central Limit Theorem.

For defining the estimators of the spot volatility, we first choose a sequence k_n of integers which satisfies, as $n \to \infty$:

$$k_n \sim \frac{\theta}{\sqrt{\Delta_n}}, \qquad \theta \in (0, \infty), \tag{2.4}$$

and a sequence u_n in $(0, \infty]$. The $\mathcal{M}_d^+$-valued variables $\widehat{c}_i^n$ are defined, componentwise, as

$$\widehat{c}_i^{n,lm} = \frac{1}{k_n \Delta_n} \sum_{j=0}^{k_n-1} \Delta_{i+j}^n X^l \, \Delta_{i+j}^n X^m \, 1_{\{\|\Delta_{i+j}^n X\| \le u_n\}}, \tag{2.5}$$

and they implicitly depend on Δ_n, k_n, u_n.

One knows that $\widehat{c}_{[t/\Delta_n]}^n \xrightarrow{\mathbb{P}} c_t$ for any t, and there is an associated Central Limit Theorem under (A-2), with rate $1/\Delta_n^{1/4}$: the choice (2.4) is optimal, in the sense that it allows us to have the fastest possible rate by a balance between the involved "statistical error" which is of order $1/\sqrt{k_n}$, and the variation of c_t over the interval $[t, t + k_n \Delta_n]$, which is of order $\sqrt{k_n \Delta_n}$ because c_t is an Itô semimartingale (and even when it jumps), see [1, 4].

By Theorem 9.4.1 of [4], one also knows that under (A'-r) and if $u_n \asymp \Delta_n^{\varpi}$ for some $\varpi \in \left[\frac{p-1}{2p-r}, \frac{1}{2}\right)$ we have

$$V(g)_t^n := \Delta_n \sum_{i=1}^{[t/\Delta_n]-k_n+1} g(\widehat{c}_i^n) \overset{\text{u.c.p.}}{\Longrightarrow} V(g)_t := \int_0^t g(c_s)\, ds \tag{2.6}$$

(convergence in probability, uniform over each compact interval; by convention $\sum_{i=a}^b v_i = 0$ if $b < a$), as soon as the function g on $\mathcal{M}_d^+$ is continuous with $|g(x)| \le K(1 + \|x\|^p)$ for some constants K, p. Actually, for this to hold we need much weaker assumptions on X, but we do not need this below. Note also that when X is continuous, the truncation in (2.5) is useless: one may use (2.5) with $u_n \equiv \infty$, which reduces to (1.1) in the one-dimensional case.

Now, we want to determine at which rate the convergence (2.6) takes place. This amounts to proving an associated Central Limit Theorem. For an appropriate choice of the truncation levels, such a CLT is available for $V(g)^n$, with the rate $1/\sqrt{\Delta_n}$, but the limit exhibits a bias term. Below, g is a smooth function on $\mathcal{M}_d^+$, and the two first partial derivatives are denoted as $\partial_{jk} g$ and $\partial_{jk,lm}^2 g$, since any $x \in \mathcal{M}_d^+$ has d^2 components x^{jk}. The family of all partial derivatives of order j is simply denoted as $\partial^j g$.

Theorem 2.2 *Assume (A'-r) for some $r < 1$. Let g be a C^3 function on $\mathcal{M}_d^+$ such that*

$$\|\partial^j g(x)\| \le K(1 + \|x\|^{p-j}), \qquad j = 0, 1, 2, 3 \tag{2.7}$$

for some constants $K > 0$, $p \geq 3$. Either suppose that X is continuous and $u_n/\Delta_n^{\varepsilon} \to \infty$ for some $\varepsilon < 1/2$ (for example, $u_n \equiv \infty$, so there is no truncation at all), or suppose that

$$u_n \asymp \Delta_n^{\varpi}, \qquad \frac{2p-1}{2(2p-r)} \leq \varpi < \frac{1}{2}. \tag{2.8}$$

Then we have the finite-dimensional (in time) stable convergence in law

$$\frac{1}{\sqrt{\Delta_n}}\left(V(g)_t^n - V(g)_t\right) \stackrel{\mathcal{L}_{f-s}}{\longrightarrow} A_t^1 + A_t^2 + A_t^3 + A_t^4 + Z_t, \tag{2.9}$$

where Z is a process defined on an extension $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, (\widetilde{\mathcal{F}}_t)_{t\geq 0}, \widetilde{\mathbb{P}})$ of $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$, which conditionally on $\mathcal{F}$ is a continuous centered Gaussian martingale with variance

$$\widetilde{\mathbb{E}}\big((Z_t)^2 \mid \mathcal{F}\big) \;=\; \sum_{j,k,l,m=1}^{d} \int_0^t \partial_{jk} g(c_s)\, \partial_{lm} g(c_s)\, \big(c_s^{jl} c_s^{km} + c_s^{jm} c_s^{kl}\big)\, ds, \tag{2.10}$$

and where, with the notation

$$G(x,y) = \int_0^1 \big(g(x+wy) - wg(x+y) - (1-w)g(x)\big)\, dw, \tag{2.11}$$

we have

$$\begin{aligned}
A_t^1 &= -\tfrac{\theta}{2}\big(g(c_0) + g(c_t)\big)\\
A_t^2 &= \tfrac{1}{2\theta} \sum_{j,k,l,m=1}^{d} \textstyle\int_0^t \partial^2_{jk,lm} g(c_s)\, \big(c_s^{jl} c_s^{km} + c_s^{jm} c_s^{kl}\big)\, ds\\
A_t^3 &= -\tfrac{\theta}{12} \sum_{j,k,l,m=1}^{d} \textstyle\int_0^t \partial^2_{jk,lm} g(c_s)\, \widetilde{c}_s^{jk,lm}\, ds\\
A_t^4 &= \theta \sum_{s\leq t} G(c_{s-}, \Delta c_s).
\end{aligned} \tag{2.12}$$

Note that $|G(x,y)| \leq K(1+\|x\|)^p\, \|y\|^2$, so the sum defining A_t^4 is absolutely convergent, and vanishes when c_t is continuous.

Remark 2.3 The bias has four parts:

1) The first one is due to a border effect: the formula giving $V^n(g)_t$ contains $[t/\Delta_n] - k_n + 1$ summands only, whereas the natural (unfeasible) approximation $\Delta_n \sum_{i=1}^{[t/\Delta_n]} g(c_{(i-1)\Delta_n})$ contains $[t/\Delta_n]$ summands. The sum of the lacking k_n summands is of order of magnitude $(k_n - 1)\Delta_n$, which goes to 0 and thus does not impair consistency, but it creates an obvious bias after normalization by $1/\sqrt{\Delta_n}$. Removing this source of bias is straightforward: since $g(c_s)$ is "under-represented" when s is close to 0 or to t, we add to $V^n(g)_t$ the variable

$$\frac{(k_n-1)\Delta_n}{2}\big(g(\widehat{c}_1^n) + g(\widehat{c}^n_{[t/\Delta_n]-k_n+1}\big). \tag{2.13}$$

Of course, other weighted averages of $g(\widehat{c}_i^n)$ for i close to 0 or to $[t/\Delta_n] - k_n + 1$ would be possible.

2) The second part A^2 is continuous in time and is present even for the toy model $X_t = \sqrt{c}\,W_t$ with c a constant and $\Delta_n = \frac{1}{n}$ and $T = 1$. In this simple case it can be interpreted as follows: instead of taking the "optimal" $g(\widehat{c}_n)$ for estimating $g(c)$, with $\widehat{c}_n = \sum_{i=1}^n (\Delta_i^n X)^2$, one takes $\frac{1}{n}\sum_{i=1}^n g(\widehat{c}_i^n)$ with $\widehat{c}_i^n$ a "local" estimator of c. This adds a statistical error which results in a bias. Note that, even in the general case, this bias would disappear, were we taking in (2.4) the (forbidden) value $\theta = \infty$ (with still $k_n\Delta_n \to 0$, at the expense of a slower rate of convergence.

3) The third and fourth parts A^3 and A^4 are respectively continuous and purely discontinuous, and due to the continuous part and to the jumps of the volatility process c_t itself. These two biases disappear if we take $\theta = 0$ in (2.4) (with still $k_n \to \infty$), again a forbidden value, and again at the expense of a slower rate of convergence.

The only test function g for which the last three biases disappear is the identity $g(x) = x$. This is because, in this case, and up to the border terms, $V(g)_t^n$ is nothing but the realized quadratic variation itself and the spot estimators $\widehat{c}_i^n$ actually merge together and disappear as such.

Remark 2.4 Observe that (2.8) implies $r < 1$. This restriction is not a surprise, since one needs $r \leq 1$ in order to estimate the integrated volatility by the (truncated) realized volatility, with a rate of convergence $1/\sqrt{\Delta_n}$. When $r = 1$ it is likely that the CLT still holds fore an appropriate choice of the sequence u_n, and with another additional bias, see e.g. [6] for a slightly different context. Here we let this borderline case aside.

2.3 Estimation of the Bias.

Now we proceed to "remove" the bias, which means subtracting consistent estimators for the bias from $V'(g)_t^n$. As written before, we have

$$A_t^{n,1} = -\frac{k_n\sqrt{\Delta_n}}{2}\left(g(\widehat{c}_1^n) + g(\widehat{c}_{[t/\Delta_n]-k_n+1}^n\right) \stackrel{\mathbb{P}}{\longrightarrow} A_t^1 \tag{2.14}$$

(this comes from $\widehat{c}_1^n \stackrel{\mathbb{P}}{\longrightarrow} c_0$ and $\widehat{c}_{[t/\Delta_n]-k_n+1}^n \stackrel{\mathbb{P}}{\longrightarrow} c_{t-}$, plus $c_{t-} = c_t$ a.s.). Next, observe that $A^2 = \frac{1}{\theta}V(h)$ for the test function h defined on $\mathcal{M}_d^+$ by

$$h(x) = \frac{1}{2}\sum_{j,k,l,m=1}^d \partial^2_{jk,lm}g(x)\left(x^{jl}x^{km} + x^{jm}x^{kl}\right). \tag{2.15}$$

Therefore

$$A_t^{n,2} = \frac{1}{k_n\sqrt{\Delta_n}}V(h)_t^n \stackrel{\mathbb{P}}{\longrightarrow} A_t^2. \tag{2.16}$$

The term A_t^3 involves the volatility of the volatility, for which estimators have been provided in the one-dimensional case by M. Vetter in [7]; namely, if $d = 1$ and under suitable technical assumptions (slightly stronger than here), *plus* the continuity of X_t and c_t, he proves that

$$\frac{3}{2k_n}\sum_{i=1}^{[t/\Delta_n]-2k_n+1}(\widehat{c}_{i+k_n}^n - \widehat{c}_i^n)^2$$

converges to $\int_0^t \left(\widetilde{c}_s + \frac{6}{\theta^2}\,(c_s)^2\right) ds$. Of course, we need to modify this estimator here, in order to include the function $\partial^2 g$ in the limit and account for the possibilities of having $d \geq 2$ and having jumps in X. We propose to take

$$A_t^{n,3} = -\frac{\sqrt{\Delta_n}}{8} \sum_{i=1}^{[t/\Delta_n]-2k_n+1} \sum_{j,k,l,m=1}^{d} \partial^2_{jk,lm} g(\widehat{c}_i^n)\,(\widehat{c}_{i+k_n}^{n,jk} - \widehat{c}_i^{n,jk})\,(\widehat{c}_{i+k_n}^{n,lm} - \widehat{c}_i^{n,lm}). \tag{2.17}$$

When X and c are continuous one may expect the convergence to $A_t^3 - \frac{1}{2}\,A_t^2$ (observe that $\frac{\sqrt{\Delta_n}}{4} \sim \frac{3}{2k_n}\,\frac{\theta}{12}$), and one may expect the same when X jumps and c is still continuous, because in (2.5) the truncation basically eliminates the jumps of X. In contrast, when c jumps, the limit should rather be related to the "full" quadratic variation of c, and indeed we have:

Theorem 2.5 *Under the assumptions of Theorem 2.2, for all $t \geq 0$ we have*

$$A_t^{n,3} \stackrel{\mathbb{P}}{\longrightarrow} -\frac{1}{2}\,A_t^2 + A_t^3 + A_t'^4, \tag{2.18}$$

where

$$A_t'^4 = \theta \sum_{s \leq t} G'(c_{s-}, \Delta c_s) \tag{2.19}$$

and

$$G'(x,y) = -\frac{1}{8} \sum_{j,k,l,m} \int_0^1 \left(\partial^2_{jk,lm}\, g(x) + \partial^2_{jk,lm}\, g(x + (1-w)y)\right) w^2\, y^{jk}\, y^{lm}\, dw. \tag{2.20}$$

At this stage, it remains to find consistent estimators for $A_t^4 - A_t'^4$, which has the form

$$A_t^4 - A_t'^4 = \theta \sum_{s \leq t} G''(c_{s-}, \Delta c_s), \quad \text{where } \; G'' = G - G'.$$

More generally, we aim at estimating

$$\mathcal{V}(F)_t = \sum_{s \leq t} F(c_{s-}, \Delta c_s), \tag{2.21}$$

at least when the function F on $\mathcal{M}_d^+ \times \mathcal{M}_d$, where $\mathcal{M}_d$ is the set of all $d \times d$ matrices, is C^1 and $|F(x,y)| \leq K\|y\|^2$ uniformly in x within any compact set, as is the function G'' above.

The solution to this problem is not as simple as it might appear at first glance. We first truncate from below, taking any sequence u_n' of truncation levels satisfying

$$u_n' \to 0, \qquad \frac{u_n'}{\Delta_n^{\varpi'}} \to \infty \quad \text{for some } \; \varpi' \in \left(0, \frac{1}{8}\right) \tag{2.22}$$

Second, we resort on the following trick. Since $\widehat{c}_i^n$ is "close" to the average of c_t over the interval $(i\Delta_n, (i+k_n)\Delta_n]$, we (somehow wrongly) pretend that, for all j:

$$\begin{gathered} \exists s \in ((j-1)k_n\Delta_n, jk_n\Delta_n] \;\text{ with }\; \|\Delta c_s\| > u_n' \;\Leftrightarrow\; \|\widehat{c}_{jk_n}^n - \widehat{c}_{(j-2)k_n}^n\| > u_n' \\ \Delta c_s \sim \widehat{c}_{jk_n}^n - \widehat{c}_{(j-2)k_n}^n, \quad \|\widehat{c}_{(j-1)k_n}^n - \widehat{c}_{(j-3)k_n}^n\| \bigvee \|\widehat{c}_{(j+1)k_n}^n - \widehat{c}_{(j-1)k_n}^n\| < \|\widehat{c}_{jk_n}^n - \widehat{c}_{(j-2)k_n}^n\|. \end{gathered} \tag{2.23}$$

The condition (2.22) implies that for n large enough there is at most one jump of size bigger than u'_n in each interval $(i-1)\Delta_n, (i-1+k_n)\Delta_n]$ within $[0,t]$, and no two consecutive intervals of this form contain such jumps. Despite this, the statement above is of course not true, the main reason being that $\widehat{c}^n_i$ and c^n_i do not exactly agree. However it is "true enough" to allow for the next estimators to be consistent for $\mathcal{V}(F)_t$:

$$\begin{array}{c}\mathcal{V}(F)^n_t = \sum_{j=3}^{[t/k_n\Delta_n]-3} F(\widehat{c}^n_{(j-3)k_n+1}, \delta^n_j\widehat{c})\, 1_{\{\|\delta^n_{j-1}\widehat{c}\|\vee\|\delta^n_{j+1}\widehat{c}\|\vee u'_n<\|\delta^n_j\widehat{c}\|\}}, \\ \text{where } \ \delta^n_j\widehat{c} = \widehat{c}^n_{jk_n+1} - \widehat{c}^n_{(j-2)k_n+1}.\end{array} \tag{2.24}$$

Since this is a sum of approximately $[t/k_n\Delta_n]$ terms, the rate of convergence of $\mathcal{V}(F)^n_t$ toward $\mathcal{V}(F)_t$ is law, probably $1/\Delta_n^{1/4}$ only. However, here we are looking for consistent estimators, and the rate is not of concern to us. Note that, again, the upper limit in the sum above is chosen in such a way that $\mathcal{V}(F)^n_t$ is computable on the basis of the observations within the interval $[0,t]$.

Theorem 2.6 *Assume all hypotheses of Theorem 2.2, and let F be a continuous function on $\mathbb{R}_+\times\mathbb{R}$ satisfying, with the same $p\geq 3$ as in (2.8),*

$$|F(x,y)| \leq K(1+\|x\|+\|y\|)^{p-2}\,\|y\|^2. \tag{2.25}$$

Then for all $t\geq 0$ we have

$$\mathcal{V}(F)^n_t \ \stackrel{\mathbb{P}}{\longrightarrow}\ \mathcal{V}(F)_t. \tag{2.26}$$

2.4 An Unbiased Central Limit Theorem.

At this stage, we can set, with the notation (2.15), (2.16), (2.17) and (2.24), and also (2.12) and (2.20) for G and G':

$$\overline{V}(g)^n_t = V(g)^n_t + \frac{k_n\Delta_n}{2}\left(g(\widehat{c}^n_1)+g(\widehat{c}^n_{[t/\Delta_n]-k_n+1}\right) - \sqrt{\Delta_n}\left(\frac{3}{2}\,A^{n,2}_t + A^{n,3}_t\right) - k_n\Delta_n\,\mathcal{V}(G-G')^n_t. \tag{2.27}$$

We then have the following, which is a straightforward consequence of the three previous theorems and of $k_n\sqrt{\Delta_n}\to\theta$, plus (2.14) and (2.16) and the fact that the function $G-G'$ satisfies (2.25) when g satisfies (2.7):

Theorem 2.7 *Under the assumptions of Theorem 2.2, and with Z as in this theorem, for all $t\geq 0$ we have the finite-dimensional stable convergence in law*

$$\frac{1}{\sqrt{\Delta_n}}\,(\overline{V}(g)^n_t - V(g)_t) \ \stackrel{\mathcal{L}_{f-s}}{\longrightarrow}\ Z_t. \tag{2.28}$$

Note that θ no longer explicitly appears in this statement, so one can replace (2.4) by the weaker statement

$$k_n \ \asymp\ \frac{1}{\sqrt{\Delta_n}} \tag{2.29}$$

(this is easily seen by taking subsequences n_l such that $k_{n_l}\sqrt{\Delta_{n_l}}$ converge to an arbitrary limit in $(0,\infty)$).

It is simple to make this CLT "feasible", that is usable in practice for determining a confidence interval for $V(g)_t$ at any time $t > 0$. Indeed, we can define the following function on $\mathcal{M}_d^+$:

$$\overline{h}(x) = \sum_{j,k,l,m=1}^{d} \partial_{jk}\, g(x)\, \partial_{lm}\, g(x) \left(x^{jl} x^{km} + x^{jm} x^{kl}\right). \tag{2.30}$$

We then have $V(\overline{h})^n \stackrel{\text{u.c.p.}}{\Longrightarrow} V(\overline{h})$, whereas $V(\overline{h})_t$ is the right hand side of (2.9). Then we readily deduce:

Corollary 2.8 *Under the assumptions of the previous theorem, for any $t > 0$ we have the following stable convergence in law, where Y is an $\mathcal{N}(0,1)$ variable:*

$$\frac{\overline{V}(g)_t^n - V(g)_t}{\sqrt{\Delta_n\, V(\overline{h})_t^n}} \stackrel{\mathcal{L}-s}{\longrightarrow} Y, \quad \text{in restriction to the set } \{V(\overline{h})_t > 0\}, \tag{2.31}$$

Finally, let us mention that the estimators $\overline{V}(g)_t^n$ enjoy excatly the same asymptotic efficiency properties as the estimators in [5], and we refer to this paper for a discussion of this topic.

Example 2.9 (Quarticity) Suppose $d = 1$ and take $g(x) = x^2$, so we want tho estimate the quarticity $\int_0^t c_s^2\, ds$. In this case we have

$$h(x) = 2x^2, \qquad G(x,y) - G'(x,y) = -\frac{y^2}{6}.$$

Then the "optimal" estimator for the quarticity is

$$\Delta_n\big(1-\frac{3}{k_n}\big) \sum_{i=1}^{[t/\Delta_n]-k_n+1} (\widehat{c}_i^n)^2 + \frac{\Delta_n}{4} \sum_{i=1}^{[t/\Delta_n]-2k_n+1} (\widehat{c}_{i+k_n}^n - \widehat{c}_i^n)^2 + \frac{(k_n-1)\Delta_n}{2} \left((\widehat{c}_1^n)^2 + (\widehat{c}_{[t/\Delta_n]-k_n+1}^n)^2\right).$$

The asymptotic variance is $8\int_0^t c_s^4\, ds$, to be compared with the asymptotic variance of the more usual estimators $\frac{1}{3\Delta_n}\sum_{i=1}^{[t/\Delta_n]} (\Delta_i^n X)^4$, which is $\frac{32}{3}\int_0^t c_s^4\, ds$.

3 Proofs

3.1 Preliminaries.

According to the localization lemma 4.4.9 of [4] (for the assumption (K) in that lemma), it is enough to show all four Theorems 2.2, 2.5, 2.6, 2.7 under the following stronger assumption:

Assumption (SA'-r): We have (A'-r). Moreover we have, for a λ-integrable function J on E and a constant A:

$$\|b\|,\ \|b'\|,\ \|\widetilde{b}\|,\ \|c\|,\ \|\widetilde{c}\|,\ \|\widetilde{c}'\|,\ J \ \le\ A, \quad \|\delta(\omega,t,z)\|^r \le J(z), \quad \|\widetilde{\delta}(\omega,t,z)\|^2 \le J(z), \tag{3.1}$$

In the sequel we thus suppose that X satisfies (SA'-r), and also that (2.4) holds: these assumptions are typically not recalled. Below, all constants are denoted by K, and they vary from line to line. They may implicitly depend on the process X (usually through A in (3.1)). When they depend on an additional parameter p, we write K_p.

We will usually replace the discontinuous process X by the continuous process

$$X'_t = \int_0^t b'_s\, ds + \int_0^t \sigma_s\, dW_s, \tag{3.2}$$

connected with X by $X_t = X_0 + X'_t + \sum_{s\le t} \Delta X_s$. Note that b' is bounded, and without loss of generality we will use below its càdlàg version. Note also that, since the jumps of c are bounded, one can rewrite (2.2) as

$$c_t = c_0 + \int_0^t \widetilde{b}_s\, ds + \int_0^t \widetilde{\sigma}_s\, dW_s + \int_0^t \int_E \widetilde{\delta}(s,z)\,(\mu-\nu)(ds,dz). \tag{3.3}$$

This amounts to replacing $\widetilde{b}$ in (2.2) by $\widetilde{b}_{t+} + \int_E \delta(t+,z)(\kappa(\|\widetilde{\delta}(t+,z)\|) - 1_{\{\|\widetilde{\delta}(t+,z)\|\le 1\}})\,\lambda(dz)$, where κ is a continuous function with compact support, equal to 1 on the set $[0,A]$. Note that the new process $\widetilde{b}$ is bounded càdlàg.

With any process Z we associate the variables

$$\eta(Z)_{t,s} = \sqrt{\mathbb{E}\big(\sup_{v\in(t,t+s]}\ \|Z_{t+v} - Z_t\|^2 \mid \mathcal{F}_t\big)}, \tag{3.4}$$

and we recall Lemma 4.2 of [5]:

Lemma 3.1 *For all $t>0$, all bounded càdlàg processes Z, and all sequences $v_n \ge 0$ of reals tending to 0, we have $\Delta_n \mathbb{E}\big(\sum_{i=1}^{[t/\Delta_n]} \eta(Z)_{(i-1)\Delta_n, v_n}\big) \to 0$, and for all $0 \le v \le s$ we have $\mathbb{E}(\eta(Z)^n_{t+v,s} \mid \mathcal{F}_t) \le \eta(Z)_{t,s}$.*

3.2 An Auxiliary Result on Itô Semimartingales

In this subsection we give some simple estimates for a d-dimensional semimartingale

$$Y_t = \int_0^t b^Y_s\, ds + \int_0^t \sigma^Y_s\, dW_s + \int_0^t \int_E \delta^Y(s,z)\,(\mu-\nu)(ds,dz)$$

on some space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\ge 0}, \mathbb{P})$, which may be different from the one on which X is defined, as well as W and μ, but we still suppose that the intensity measure ν is the same. Note that $Y_0 = 0$ here. We assume that for some constant A and function J^Y we have, with $c^Y = \sigma^Y \sigma^{Y,*}$:

$$\|b^Y\| \le A,\ \|c^Y\| \le A^2, \quad \|\delta^Y(\omega,t,z)\|^2 \le J^Y(z) \le A^2, \quad \int_E J^Y(z)\,\lambda(dz) \le A^2. \tag{3.5}$$

The compensator of the quadratic variation of Y is of the form $\int_0^t \overline{c}^Y_s\, ds$, where $\overline{c}^Y_t = c^Y_t + \int_E \delta^Y(t,z)\,\delta^Y(t,z)^*\,\lambda(dz)$. Moreover, if the process c^Y is itself an Itô semimartingale, the quadratic covariation of the continuous martingale parts of Y and c^Y is also of the form $\int_0^t \widehat{c}'^Y_s\, ds$ for some process $\widehat{c}'^Y$, necessarily bounded if both Y and c^Y satisfy (3.5) (and, if $Y = X$, we have $c^Y = c$ and $\widehat{c}'^Y = \widehat{c}'$).

Lemma 3.2 *Below we assume (3.5), and the constant K only depends on A.*

a) We have for $t \in [0,1]$:

$$\begin{aligned} &\big\|\mathbb{E}(Y_t \mid \mathcal{F}_0) - tb_0^Y\big\| \le t\,\eta(b^Y)_{0,t} \le Kt \\ &\big|\mathbb{E}(Y_t^j\,Y_t^m \mid \mathcal{F}_0) - t\overline{c}_0^{Y,jm}\big| \le Kt(t+\sqrt{t}\,\eta(b^Y)_{0,t} + \eta(\overline{c}^Y)_{0,t}) \le Kt, \end{aligned} \tag{3.6}$$

and if further $\|\mathbb{E}(\overline{c}_t^Y - \overline{c}_0^Y \mid \mathcal{F}_0)\| \le A^2 t$ for all t, we also have

$$\big|\mathbb{E}(Y_t^j\,Y_t^m \mid \mathcal{F}_0) - t\overline{c}_0^{Y,jm}\big| \le 2\,t^{3/2}(2A^2\sqrt{t} + A\eta(b^Y)_{0,t}) \le Kt^{3/2}. \tag{3.7}$$

b) When Y is continuous, and if $\mathbb{E}(\|\overline{c}_t^Y - \overline{c}_0^Y\|^2 \mid \mathcal{F}_0) \le A^4 t$ for all t, we have

$$\big|\mathbb{E}\big(Y_t^j\,Y_t^k\,Y_t^l\,Y_t^m \mid \mathcal{F}_0\big) - t^2(c_0^{Y,jk}c_0^{Y,lm} + c_0^{Y,jl}c_0^{Y,km} + c_0^{Y,jm}c_0^{Y,kl})\big| \le Kt^{5/2}. \tag{3.8}$$

c) When c^Y is a (possibly discontinuous) semimartingale satisfying the same conditions (3.5) as Y, and if Y itself is continuous, we have

$$\big|\mathbb{E}\big((Y_t^j\,Y_t^k - tc_0^{Y,jk})(c_t^{Y,lm} - c_0^{Y,lm}) \mid \mathcal{F}_0\big) \le Kt^{3/2}(\sqrt{t} + \eta(\widehat{c}'^Y)_{0,t}). \tag{3.9}$$

Proof. The first part of (3.6) follows by taking the $\mathcal{F}_0$-conditional expectation in the decomposition $Y_t = M_t + tb_0^Y + \int_0^t (b_s^Y - b_0^Y)\,ds$, where M is a d-dimensional martingale with $M_0 = 0$. For the second part, we deduce from Itô's formula that Y^jY^m is the sum of a martingale vanishing at 0 and of

$$b_0^j\int_0^t Y_s^m\,ds + b_0^m\int_0^t Y_s^j\,ds + \int_0^t Y_s^m(b_s^j - b_0^j)\,ds + \int_0^t Y_s^j(b_s^m - b_0^m)\,ds + \overline{c}_0^{Y,jm}t + \int_0^t (\overline{c}_s^{Y,jm} - \overline{c}_0^{Y,jm})\,ds.$$

Since $\mathbb{E}(\|Y_t\| \mid \mathcal{F}_0) \le KA\sqrt{t}$, as in (3.10), we deduce the second part of (3.6) and also (3.7) by taking again the conditional expectation and by using the Cauchy-Schwarz inequality and the first part.

(3.8) is a part of Lemma 5.1 of [5]. For (3.9), we first observe that $Y_t^jY_t^k - tc_0^{Y,jk} = B_t + M_t$ and $c_t^{Y,lm} - c_0^{Y,lm} = B'_t + M'_t$, with M and M' martingales (M is continuous). The processes B, B', $\langle M,M\rangle$, $\langle M',M'\rangle$ and $\langle M,M'\rangle$ are absolutely continuous, with densities $\overline{b}_s$, $\overline{b}'_s$, h_s, h'_s and h''_s satisfying, by (3.5) for Y and c^Y:

$$|\overline{b}_s| \le 2\|Y_s\|\,\|b_s^Y\| + \|c_s^Y - c_0^Y\|, \quad |\overline{b}'_s| \le K, \quad |h_s| \le K\|Y_s\|^2, \quad |h'_s| \le K,$$

whereas $h''_s = Y_s^j\widehat{c}'^{Y,k,lm} + Y_s^k\widehat{c}'^{Y,j;lm}$. As seen before, $\mathbb{E}(\|Y_t\|^q \mid \mathcal{F}_0) \le K_q t^{q/2}$ for all q, and $\mathbb{E}(\|c_t^Y - c_0^Y\|^2 \mid \mathcal{F}_0) \le Kt$. This yields $\mathbb{E}(B_t^2 \mid \mathcal{F}_0) \le Kt^3$ and $\mathbb{E}(M_t^2 \mid \mathcal{F}_0) \le Kt^2$. Since $|B'_t| \le Kt$ and $\mathbb{E}(M_t'^2 \mid \mathcal{F}_0) \le Kt$, we deduce that the $\mathcal{F}_0$- conditional expectations of $B_tB'_t$ and $B_tM'_t$ and $M_tB'_t$ are smaller than Kt^2.

Finally $\mathbb{E}(M_tM'_t \mid \mathcal{F}_0) = \mathbb{E}(\langle M,M'\rangle_t \mid \mathcal{F}_0)$, and $\langle M,M'\rangle_t$ is the sum of $\widehat{c}_0'^{Y,k,lm}\int_0^t Y_s^j\,ds + \int_0^t Y_s^j(\widehat{c}_s'^{Y,k,lm} - \widehat{c}_0'^{Y,k,lm})\,ds$ and a similar term with k and j exchanged. Then using again $\mathbb{E}(\|Y_t\|^2 \mid \mathcal{F}_0) \le Kt$, plus $\|\mathbb{E}(Y_t \mid \mathcal{F}_0)\| \le Kt$ and Cauchy-Schwarz inequality, we obtain that the above conditional expectation is smaller than $K(t^2 + t^{3/2}\eta(\widehat{c}'^Y)_t)$. This completes the proof of (3.9). □

3.3 Some Estimates.

1) We begin with well known estimates for X' and c, under (3.1) and for $s, t \geq 0$ and $q \geq 0$:

$$\begin{array}{ll} \mathbb{E}\big(\sup_{w\in[0,s]} \|X'_{t+w} - X'_t\|^q \mid \mathcal{F}_t\big) \leq K_q\, s^{q/2}, & \|\,\mathbb{E}(X'_{t+s} - X'_t \mid \mathcal{F}_s)\| \leq Ks \\ \mathbb{E}\big(\sup_{w\in[0,s]} \|c_{t+w} - c_t\|^q \mid \mathcal{F}_t\big) \leq K_q\, s^{1\wedge(q/2)}, & \|\,\mathbb{E}(c_{t+s} - c_t \mid \mathcal{F}_s)\| \leq Ks. \end{array} \tag{3.10}$$

Next, it is much easier (although unfeasible in practice) to replace $\widehat{c}_i^n$ in (2.6) by the estimators based on the process X' given by (3.2). Namely, we will replace $\widehat{c}_i^n$ by the following:

$$\widehat{c}_i'^n = \frac{1}{k_n\Delta_n} \sum_{j=0}^{k_n-1} \Delta_{i+j}^n X'\, \Delta_{i+j}^n X'^*.$$

The difference between $\widehat{c}_i^n$ and $\widehat{c}_i'^n$ is estimated by the following inequality, valid when $u_n \asymp \Delta_n^{\varpi}$ and $q \geq 1$, and where a_n denotes a sequence of numbers (depending on u_n), going to 0 as $n \to \infty$ (this is Equation 4.8 of [5]):

$$\mathbb{E}\big(\|\widehat{c}_i^n - \widehat{c}_i'^n\|^q\big) \leq K_q a_n\, \Delta_n^{(2q-r)\varpi+1-q}. \tag{3.11}$$

2) The jumps of c also potentially cause troubles. So we will eliminate the "big" jumps as follows. For any $\rho > 0$ we consider the subset $E_\rho = \{z : J(z) > \rho\}$, which satisfies $\lambda(E_\rho) < \infty$, and we denote by $\mathcal{G}^\rho$ the σ-field generated by the variables $\mu([0,t] \times A)$, where $t \geq 0$ and A runs through all Borel subsets of E_ρ. The process

$$N_t^\rho = \mu((0,t] \times E_\rho) \tag{3.12}$$

is a Poisson process and we let $S_1^\rho, S_2^\rho, \cdots$ be its successive jump times, and $\Omega_{n,t,\rho}$ be the set on which $S_j^\rho \notin \{i\Delta_n : i \geq 1\}$ for all $j \geq 1$ such that $S_j^\rho < t$, and $S_{j+1}^\rho > t \wedge S_j^\rho + (6k_n+1)\Delta_n$ for all $j \geq 0$ (with the convention $S_0^\rho = 0$; taking $6k_n$ here instead of the more natural k_n will be needed in the proof of Theorem 2.6, and makes no difference here). All these objects are $\mathcal{G}^\rho$-measurable, and $\mathbb{P}(\Omega_{n,t,\rho}) \to 1$ as $n \to \infty$, for all $t, \rho > 0$.

We define the processes

$$\widetilde{b}(\rho)_t = \widetilde{b}_t - \int_{E_\rho} \widetilde{\delta}(t+,z)\, \lambda(dz), \qquad \overline{c}(\rho)_t = \widetilde{\sigma}_t\, \widetilde{\sigma}_t^* + \int_{(E_\rho)^c} \widetilde{\delta}(t+,z)\, \widetilde{\delta}(t+,z)^*\, \lambda(dz)$$

$$\begin{array}{rcl} c(\rho)_t & = & c_t - \int_0^t \int_{E_\rho} \widetilde{\delta}(s,z)\, \mu(ds,dz) \;=\; c^{(1)}(\rho)_t + c^{(2)}(\rho)_t, \quad \text{where} \\ & & c^{(1)}(\rho)_t = c_0 + \int_0^t \widetilde{b}(\rho)_s\, ds + \int_0^t \widetilde{\sigma}_s\, dW_s \\ & & c^{(2)}(\rho)_t = \int_0^t \int_{(E_\rho)^c} \widetilde{\delta}(t-,z)\, (\mu-\nu)(ds,dz), \end{array} \tag{3.13}$$

so $\overline{c}(\rho)$, which is $\mathbb{R}^{d^2} \otimes \mathbb{R}^{d^2}$-valued, is the càdlàg version of the density of the predictable quadratic variation of $c(\rho)$. Moreover $\mathcal{G}^\rho = \{\emptyset, \Omega\}$ and $(\widetilde{b}(\rho), c(\rho)) = (\widetilde{b}, c)$ when ρ exceeds the bound of the function J. Note also that $\widetilde{b}(\rho)$ and $\overline{c}(\rho)$ are càdlàg.

By Lemma 2.1.5 and Proposition 2.1.10 in [4] applied to each components of X' and $c^{(2)}(\rho)$, plus the property $\|\widetilde{b}(\rho)\| \leq K/\rho$, for all $t \geq 0$, $s \in [0,1]$, $\rho \in (0,1]$, $q \geq 2$, we have

$$\begin{array}{l} \mathbb{E}\big(\sup_{w\in[0,s]} \|X'_{t+w} - X'_t\|^q \mid \mathcal{F}_t \vee \mathcal{G}^\rho\big) \leq K_q\, s^{q/2} \\ \|\,\mathbb{E}(X'_{t+s} - X'_t \mid \mathcal{F}_s \vee \mathcal{G}^\rho)\| + \|\,\mathbb{E}(c(\rho)_{t+s} - c(\rho)_t \mid \mathcal{F}_s \vee \mathcal{G}^\rho)\| \leq Ks \\ \mathbb{E}\big(\sup_{w\in[0,s]} \|c^{(2)}(\rho)_{t+w} - c^{(2)}(\rho)_t\|^q \mid \mathcal{F}_t \vee \mathcal{G}^\rho\big) \leq K_q\, \phi_\rho\, (s + s^{q/2}) \\ \mathbb{E}\big(\sup_{w\in[0,s]} \|c(\rho)_{t+w} - c(\rho)_t\|^q \mid \mathcal{F}_t \vee \mathcal{G}^\rho\big) \leq K_q\big(\phi_\rho\, s + s^{q/2} + \frac{s^q}{\rho^q}\big) \leq K_{q,\rho}\, s. \end{array} \tag{3.14}$$

where $\phi_\rho = \int_{(E_\rho)^c} J(z)\,\lambda(dz) \to 0$ as $\rho \to 0$. Note also that $\|\widetilde{b}(\rho)_t\| \leq K/\rho$.

3) For convenience, we put

$$\begin{array}{lll} b_i^n = b_{(i-1)\Delta_n}, & c_i^n = c_{(i-1)\Delta_n} & \\ \widetilde{b}(\rho)_i^n = \widetilde{b}(\rho)_{(i-1)\Delta_n}, & \overline{c}(\rho)_i^n = \overline{c}(\rho)_{(i-1)\Delta_n}, & c(\rho)_i^n = c(\rho)_{(i-1)\Delta_n} \\ \mathcal{F}_i^n = \mathcal{F}_{(i-1)\Delta_n}, & \mathcal{F}_i^{n,\rho} = \mathcal{F}_i^n \vee \mathcal{G}^\rho. & \end{array} \tag{3.15}$$

All the above variables are $\mathcal{F}_i^{n,\rho}$-measurable. Recalling (3.4), and writing $\eta(Z,(\mathcal{H}_t))_{t,s}$ if we use the filtration $(\mathcal{H}_t)$ instead of $(\mathcal{F}_t)$, we also set

$$\eta(\rho)_{i,j}^n = \max(\eta(Y,(\mathcal{G}^\rho \bigvee \mathcal{F}_t))_{(i-1)\Delta_n, j\Delta_n} :\ Y = b', \widetilde{b}(\rho), c, \overline{c}(\rho), \widehat{c}'\big), \quad \eta(\rho)_i^n = \eta(\rho)_{i,i+2k_n}^n.$$

Therefore, Lemma 3.1 yields for all $t, \rho > 0$ and j, k such that $j + k \leq 2k_n$:

$$\Delta_n \mathbb{E}\big(\sum_{i=1}^{[t/\Delta_n]} \eta(\rho)_i^n\big) \to 0, \qquad \mathbb{E}(\eta(\rho)_{i+j,k}^n \mid \mathcal{F}_i^{n,\rho}) \leq \eta(\rho)_i^n. \tag{3.16}$$

We still need some additional notation. First, define $\mathcal{G}^\rho$-measurable (random) set of integers:

$$L(n,\rho) = \{i = 1, 2, \cdots :\ N^\rho_{(i+2k_n)\Delta_n} - N^\rho_{(i-1)\Delta_n} = 0\} \tag{3.17}$$

(taking above, $2k_n$ instead of k_n, is necessary for the proof of Theorem 2.5). Observe that

$$i \in L(n,\rho),\ 0 \leq j \leq 2k_n + 1 \ \Rightarrow\ c_{i+j}^n - c_i^n = c(\rho)_{i+j}^n - c(\rho)_i^n. \tag{3.18}$$

Second, we define the following $\mathbb{R}^d \otimes \mathbb{R}^d$-valued variables

$$\begin{array}{l} \alpha_i^n = \Delta_i^n X'\, \Delta_i^n X'^* - c_i^n\, \Delta_n \\ \beta_i^n = \widehat{c}_i'^n - c_i^n = \frac{1}{k_n\Delta_n} \sum_{j=0}^{k_n-1} \big(\alpha_{i+j}^n + (c_{i+j}^n - c_i^n)\Delta_n\big) \\ \gamma_i^n = \widehat{c}_{i+k_n}'^n - \widehat{c}_i'^n = \beta_{i+k_n}^n - \beta_i^n + c_{i+k_n}^n - c_i^n. \end{array} \tag{3.19}$$

4) Now we proceed with estimates. (3.14) yields, for all $q \geq 0$:

$$\begin{array}{ll} \mathbb{E}(\|\alpha_i^n\|^q \mid \mathcal{F}_i^{n,\rho}) \leq K_q \Delta_n^q, & \|\mathbb{E}(\alpha_i^n \mid \mathcal{F}_i^{n,\rho})\| \leq K\Delta_n^{3/2} \\ \mathbb{E}\big(\|\sum_{j=0}^{k_n-1} \alpha_{i+j}^n\|^q \mid \mathcal{F}_i^{n,\rho}\big) \leq K_q \Delta_n^{3q/4}, & \mathbb{E}(\|\widehat{c}_i'^n\|^q \mid \mathcal{F}_i^{n,\rho}) \leq K_q \end{array} \tag{3.20}$$

the third inequality following from the two first one, plus Burkholder-Gundy and Hölder inequalities, and the last inequality form the third one and the boundedness of c_t. Moreover,

since the set $\{i \in L(n,\rho)\}$ is $\mathcal{G}^\rho$-measurable, the last part of (3.14), (3.18), and Hölder's inequality, readily yield

$$q \geq 2,\ i \in L(n,\rho) \quad \Rightarrow \quad \mathbb{E}\big(\|\beta_i^n\|^q \mid \mathcal{F}_i^{n,\rho}\big)\big| \leq K_q\Big(\sqrt{\Delta_n}\,\phi_\rho + \Delta_n^{q/4} + \frac{\Delta_n^{q/2}}{\rho^q}\Big). \tag{3.21}$$

5) The previous estimates are not enough for us. We will apply the estimates of Lemma 3.2 with $Y_t = X'_{(i-1)\Delta_n+t} - X'_{(i-1)\Delta_n}$ for any given pair n,i, and with the filtration $(\mathcal{F}_{(i-1)\Delta_n+t} \vee \mathcal{G}^\rho)_{t\geq 0}$. We observe that on the set $A(\rho,n,i) = \{\exists j \leq 2k_n : i-j \in L(n,\rho)\}$, which is $\mathcal{G}^\rho$-measurable, and because of (3.18), the process c^Y coincide with $c(\rho)_{(i-1)\Delta_n+t} - c(\rho)_{(i-1)\Delta_n}$ if $t \in [0,\Delta_n]$. Then in restriction to this set, by (3.7) and (3.8) and by the definition of $\eta(\rho)_{i,1}^n$, we have

$$\begin{aligned}
&\big|\mathbb{E}(\Delta_i^n X'^j\, \Delta_i^n X'^m \mid \mathcal{F}_i^{n,\rho}) - c_i^{n,jm}\Delta_n\big| \leq K_\rho \Delta_n^{3/2}(\sqrt{\Delta_n} + \eta(\rho)_{i,1}^n)\\
&\big|\mathbb{E}\big(\Delta_i^n X'^j\, \Delta_i^n X'^k\, \Delta_i^n X'^l\, \Delta_i^n X'^m \mid \mathcal{F}_i^{n,\rho}\big) - (c_i^{n,jk}c_i^{n,lm} + c_i^{n,jl}c_i^{n,km} + c_i^{n,jm}c_i^{n,kl})\Delta_n^2\big| \leq K_\rho \Delta_n^{5/2}
\end{aligned}$$

(the constant above depends on ρ, through the bound K/ρ for the drift of $c(\rho)$). Then a simple calculation gives us

$$\left.\begin{aligned}
&\big\|\,\mathbb{E}(\alpha_i^n \mid \mathcal{F}_i^{n,\rho})\big\| \leq K_\rho \Delta_n^{3/2}(\sqrt{\Delta_n} + \eta(\rho)_{i,1}^n)\\
&\big|\mathbb{E}\big(\alpha_i^{n,jk}\alpha_i^{n,lm} \mid \mathcal{F}_i^{n,\rho}\big) - (c_i^{n,jl}c_i^{n,km} + c_i^{n,jm}c_i^{n,kl})\Delta_n^2\big| \leq K_\rho \Delta_n^{5/2}
\end{aligned}\right\} \quad \text{on} \quad A(\rho,n,i) \tag{3.22}$$

Next, we apply Lemma 3.2 to the process $Y_t = c(\rho)_{(i-1)\Delta_n+t} - c(\rho)_{(i-1)\Delta_n}$ for any given pair n,i, and with the filtration $(\mathcal{F}_{(i-1)\Delta_n+t} \vee \mathcal{G}^\rho)_{t\geq 0}$. We then deduce from (3.6), plus again (3.18), that

$$\begin{aligned}
&i \in L(n,\rho),\ 0 \leq t \leq k_n\Delta_n \Rightarrow\\
&\big|\mathbb{E}((c^{jk}_{(i-1)\Delta_n+t} - c^{jk}_{(i-1)\Delta_n})(c^{lm}_{(i-1)\Delta_n+t} - c^{lm}_{(i-1)\Delta_n}) \mid \mathcal{F}_i^{n,\rho}) - t\overline{c}(\rho)_i^{n,jklm}\big| \leq K_\rho t\,\eta(\rho)_{i,k_n}^n\\
&\big\|\mathbb{E}(c_{(i-1)\Delta_n+t} - c_{(i-1)\Delta_n} \mid \mathcal{F}_i^{n,\rho}) - t\widetilde{b}(\rho)_i^n\big\| \leq K_\rho t\,\eta(\rho)_{i,k_n}^n \leq K_p\, t.
\end{aligned} \tag{3.23}$$

Moreover, the Cauchy-Schwarz inequality and (3.20) on the one hand, and (3.9) applied with the process $Y_t = X'_{(i-1)\Delta_n+t} - X'_{(i-1)\Delta_n}$ on the other hand, give us

$$i \in L(n,\rho) \ \Rightarrow \left\{\begin{aligned}
&\big|\mathbb{E}\big(\alpha_i^{n,kl}\,\Delta_i^n\widetilde{b}(\rho)^{ms} \mid \mathcal{F}_i^{n,\rho}\big)\big| \leq K\Delta_n\eta(\rho)_{i,1}^n\\
&\big|\mathbb{E}\big(\alpha_i^{n,kl}\,\Delta_i^n c^{ms} \mid \mathcal{F}_i^{n,\rho}\big)\big| \leq K_\rho\Delta_n^{3/2}(\sqrt{\Delta_n} + \eta(\rho)_{i,1}^n).
\end{aligned}\right. \tag{3.24}$$

6) We now proceed to estimates on β_i^n:

Lemma 3.3 *We have on the set where i belongs to $L(n,\rho)$:*

$$\begin{aligned}
&\big|\mathbb{E}(\beta_i^{n,jk}\,\beta_i^{n,lm} \mid \mathcal{F}_i^{n,\rho}) - \tfrac{1}{k_n}\,(c_i^{n,jl}c_i^{n,km} + c_i^{n,jm}c_i^{n,kl}) - \tfrac{k_n\Delta_n}{3}\,\overline{c}(\rho)_i^{n,jklm}\big|\\
&\qquad\qquad \leq K_\rho\sqrt{\Delta_n}\,(\Delta_n^{1/4} + \eta(\rho)_i^n)\\
&\big|\mathbb{E}(\beta_i^{n,jk}(c_{i+k_n}^{n,lm} - c_i^{n,lm}) \mid \mathcal{F}_i^{n,\rho}) - \tfrac{k_n\Delta_n}{2}\,\overline{c}(\rho)_i^{n,jklm}\big| \leq K_\rho\sqrt{\Delta_n}\,(\sqrt{\Delta_n} + \eta(\rho)_i^n).
\end{aligned}$$

Proof. We set $\zeta_{i,j}^n = \alpha_{i+j}^n + (c_{i+j}^n - c_i^n)\Delta_n$ and write $\beta_i^{n,jk}\beta_i^{n,lm}$ as

$$\frac{1}{k_n^2\Delta_n^2}\sum_{u=0}^{k_n-1}\zeta_{i,u}^{n,jk}\zeta_{i,u}^{n,lm} + \frac{1}{k_n^2\Delta_n^2}\sum_{u=0}^{k_n-2}\sum_{v=u+1}^{k_n-1}\zeta_{i,u}^{n,jk}\zeta_{i,v}^{n,lm} + \frac{1}{k_n^2\Delta_n^2}\sum_{u=0}^{k_n-2}\sum_{v=u+1}^{k_n-1}\zeta_{i,u}^{n,lm}\zeta_{i,v}^{n,jk}. \quad (3.25)$$

For the estimates below, we implicitly assume $i \in L(n,\rho)$ and $u, v \in \{0, \cdots, k_n - 1\}$.

First, we deduce from (3.22) and (3.23), plus (3.24) and successive conditioning, that

$$\big|\mathbb{E}(\zeta_{i,u}^{n,jk}\zeta_{i,u}^{n,lm} \mid \mathcal{F}_i^{n,\rho}) - (c_i^{n,jl}c_i^{n,km} + c_i^{n,jm}c_i^{n,kl})\Delta_n^2\big| \le K\Delta_n^{5/2}. \quad (3.26)$$

Second, if $u < v$, the same type of arguments and the boundedness of $\widetilde{b}(\rho))_t$ and c_t yield

$$\begin{aligned}
&|\mathbb{E}(\zeta_{i,v}^{n,jk} \mid \mathcal{F}_{i+u+1}^{n,\rho}) - (c_{i+u+1}^{n,jk} - c_i^{n,jk})\Delta_n - \widetilde{b}(\rho)_{i+u+1}^{n,jk}\Delta_n^2(v-u-1)| \\
&\qquad\qquad\qquad\qquad \le K\Delta_n^{3/2}(k_n\sqrt{\Delta_n} + \eta(\rho)_{i+v,1}^n) \\
&|\mathbb{E}(\alpha_{i+u}^{n,lm}\,(c_{i+u+1}^{n,jk} - c_{i+u}^{n,jk}) \mid \mathcal{F}_{i+u}^{n,\rho})| \le K_\rho\Delta_n^{3/2}(\sqrt{\Delta_n} + \eta(\rho)_{i+u,1}^n) \\
&|\mathbb{E}(\alpha_{i+u}^{n,lm}\,(c_{i+u}^{n,jk} - c_i^{n,jk}) \mid \mathcal{F}_{i+u}^{n,\rho})| \le K\Delta_n^{3/2}(\sqrt{\Delta_n} + \eta_{i+u,1}^n) \\
&|\mathbb{E}(\alpha_{i+u}^{n,lm}\,(\widetilde{b}(\rho)_{i+u+1}^{n,jk} - \widetilde{b}(\rho)_{i+u}^{n,jk}) \mid \mathcal{F}_{i+u}^{n,\rho})| \le K_\rho\Delta_n^{3/2}(\sqrt{\Delta_n} + \eta(\rho)_{i+u,1}^n) \\
&|\mathbb{E}(\alpha_{i+u}^{n,lm}\,\widetilde{b}(\rho)_{i+u}^{n,jk} \mid \mathcal{F}_{i+u}^{n,\rho})| \le K_\rho\Delta_n^{3/2}(\sqrt{\Delta_n} + \eta(\rho)_{i+u,1}^n) \\
&|\mathbb{E}((c_{i+u}^{n,lm} - c_i^{n,lm})\,(c_{i+u+1}^{n,jk} - c_i^{n,jk}) \mid \mathcal{F}_i^{n,\rho}) - \overline{c}(\rho)_i^{n,jklm}\Delta_n u| \le K_\rho\Delta_n\,\eta(\rho)_i^n \\
&|\mathbb{E}((c_{i+u}^{n,lm} - c_i^{n,lm})\,\widetilde{b}(\rho)_{i+u+1}^{n,jk} \mid \mathcal{F}_i^{n,\rho})| \le K_\rho\Delta_n^{1/4}.
\end{aligned}$$

Since $\sum_{u=0}^{k_n-2}\sum_{v=u+1}^{k_n-1} u = k_n^3/6 + \mathrm{O}(k_n^2)$, we easily deduce that the $\mathcal{F}_i^{n,\rho}$-conditional expectation of the last term in (3.25) is $\frac{1}{6}\,\overline{c}(\rho)_i^{n,jklm}k_n\Delta_n$, up to a remainder term which is $\mathrm{O}(\sqrt{\Delta_n}\,(\Delta_n^{1/4} + \eta(\rho)_i^n))$, and the same is obviously true of the second term. The first claim of the lemma readily follows from this and (3.25) and (3.26).

The proof of the second claim is similar. Indeed, we have

$$\beta_i^{n,jk}(c_{i+k_n}^{n,lm} - c_i^{n,lm}) = \frac{1}{k_n\Delta_n}\sum_{u=0}^{k_n-1}\big(\alpha_{i,u}^{n,jk} + (c_{i+u}^{n,jk} - c_i^{n,jk})\Delta_n\big)\,\big(c_{i+k_n}^{n,lm} - c_i^{n,lm})\big)$$

and

$$\big|\mathbb{E}(c_{i+k_n}^{n,lm} - c_i^{n,lm} \mid \mathcal{F}_{i+u+1}^{n,\rho}) - c_{i+u+1}^{n,lm} - c_i^{n,lm} - \widetilde{b}(\rho)_{i+u+1}^{n,lm}\Delta_n(k_n - u - 1)\big| \le K\Delta_n\eta(\rho)_{i+u+1,k_n-u}^n.$$

Using the previous estimates, we conclude as for the first claim. □

Finally, we deduce the following two estimates on the variables γ_i^n of (3.19), for any $q \ge 2$:

$$i \in L(n,\rho) \;\Rightarrow\; \begin{cases} \big|\mathbb{E}\big(\gamma_i^{n,jk}\,\gamma_i^{n,lm} \mid \mathcal{F}_i^{n,\rho}\big) - \frac{2}{k_n}\,(c_i^{n,jl}c_i^{n,km} + c_i^{n,jm}c_i^{n,kl}) \\ \qquad\qquad - \frac{2k_n\Delta_n}{3}\,\overline{c}(\rho)_i^{n,jklm}\big| \le K_\rho\,\sqrt{\Delta_n}\,\big(\Delta_n^{1/8} + \eta(\rho)_i^n\big) \\ \mathbb{E}(\|\gamma_i^n\|^q \mid \mathcal{F}_i^{n,\rho}) \le K_q\big(\sqrt{\Delta_n}\,\phi_\rho + \Delta_n^{q/4} + \frac{\Delta_n^{q/2}}{\rho^q}\big). \end{cases} \quad (3.27)$$

To see that the first claim holds, one expands the product $\gamma_i^{n,jk}\,\gamma_i^{n,lm}$ and use successive conditioning, the Cauchy-Schwarz inequality and (3.14), (3.18) and(3.23), and Lemma 3.3; the contributing terms are

$$\begin{array}{c}\beta_i^{n,jk}\,\beta_i^{n,lm}+\beta_{i+k_n}^{n,jk}\,\beta_{i+k_n}^{n,lm}+(c_{i+k_n}^{n,jk}-c_i^{n,jk})(c_{i+k_n}^{n,lm}-c_i^{n,lm})\\ -\beta_i^{n,jk}(c_{i+k_n}^{n,lm}-c_i^{n,lm})-\beta_i^{n,lm}(c_{i+k_n}^{n,jk}-c_i^{n,jk}).\end{array}$$

For the second claim we use (3.14), (3.18) and (3.21), and it holds for all $q\ge 2$.

3.4 The Behavior of Some Functionals of $c(\rho)$.

For $\rho>0$ we set

$$\begin{array}{c}U(\rho)_t^n=\sum_{j=3}^{[t/k_n\Delta_n]-3}\|\mu(\rho)_j^n\|^2\,1_{\{\|\mu(\rho)_j^n\|>u_n'/4\}},\quad\text{where}\\ \mu(\rho)_j^n=\frac{1}{k_n}\sum_{w=0}^{k_n-1}(c(\rho)_{jk_n+w}^n-c(\rho)_{(j-2)k_n+w}^n).\end{array}\tag{3.28}$$

The aim of this subsection is to prove the following lemma:

Lemma 3.4 *Under (SA'-r) and (2.22) we have*

$$\lim_{\rho\to 0}\ \limsup_{n\to\infty}\ \mathbb{E}\big(U(\rho)_t^n\big)\ =\ 0.$$

Assumption (SA'-r) is of course not fully used. What is needed is the assumptions concerning the process c_t only.

Proof. With the notation (3.13), and for $l=1,2$ we define $\mu^{(l)}(\rho)_j^n$ and $U^{(l)}(\rho)_t^n$ as above, upon substituting $c(\rho)$ and $u_n'/4$ with $c^{(l)}(\rho)$ and $u_n'/8$. Since $U(\rho)_t^n\le 4U^{(1)}(\rho)_t^n+4U^{(2)}(\rho)_t^n$, it suffices to prove the result for each $U^{(l)}(\rho)_t^n$.

First, $\|\mu^{(1)}(\rho)_j^n\|^2\,1_{\{\|\mu^{(1)}(\rho)_i^n\|>u_n'/8\}}$ is smaller than $K\|\mu^{(1)}(\rho)_j^n\|^4/u_n'^2$, whereas (recalling $\|\widetilde{b}(\rho)\|\le K/\rho$) classical estimates yield $\mathbb{E}\big(\|\mu^{(1)}(\rho)_j^n\|^4\big)\le K\Delta_n(1+\Delta_n/\rho)$. Thus the expectation of $U^{(1)}(\rho)_t^n$ is less than $K\Delta_n^{1/2-2\varpi'}(1+\Delta_n/\rho)$, yielding the result for $U^{(1)}(\rho)_t^n$.

Secondly, we have $U^{(2)}(\rho)_t^n\le\sum_{j=3}^{[t/k_n\Delta_n]}\|\mu^{(2)}(\rho)_i^n\|^2$ and the first part of (3.14) yields $\mathbb{E}\big(\|\mu^{(2)}(\rho)_i^n\|^2\big)\le K\phi_\rho\sqrt{\Delta_n}$. Since $\phi_\rho\to 0$ as $\rho\to 0$, the result for $U^{(1)}(\rho)_t^n$ follows. □

3.5 A Basic Decomposition.

We start the proof of Theorem 2.2 by giving a decomposition of $V(g)^n-V(g)$, with quite a few terms. It is based on the key property $\widehat{c}_i^n=c_i^n+\beta_i^n$ and on the definition (3.19) of α_i^n and β_i^n. A simple calculation shows that $\frac{1}{\sqrt{\Delta_n}}\,(V'(g)_t^n-V(g)_t)=\sum_{j=1}^5 V_t^{n,j}$, as soon as

$t > k_n\Delta_n$, where (the sums on components below always extend from 1 to d):

$$
\begin{aligned}
V_t^{n,1} &= \sqrt{\Delta_n}\,\textstyle\sum_{i=1}^{[t/\Delta_n]-k_n+1}\big(g(\widehat{c}_i^n) - g(\widehat{c}_i'^n)\big)\\
V_t^{n,2} &= \frac{1}{\sqrt{\Delta_n}}\sum_{i=1}^{[t/\Delta_n]-k_n+1}\int_{(i-1)\Delta_n}^{i\Delta_n}(g(c_i^n) - g(c_s))\,ds\\
V_t^{n,3} &= \frac{1}{k_n\sqrt{\Delta_n}}\sum_{i=1}^{[t/\Delta_n]-k_n+1}\sum_{l,m}\partial_{lm}g(c_i^n)\sum_{u=0}^{k_n-1}\alpha_{i+u}^{n,lm}\\
V_t^{n,4} &= \frac{\sqrt{\Delta_n}}{k_n}\sum_{i=1}^{[t/\Delta_n]-k_n+1}\sum_{l,m}\partial_{lm}g(c_i^n)\sum_{u=1}^{k_n-1}(c_{i+u}^{n,lm} - c_i^{n,lm}) - \frac{1}{\sqrt{\Delta_n}}\int_{\Delta_n([t/\Delta_n]-k_n+1}^{t} g(c_s)\,ds\\
V_t^{n,5} &= \sqrt{\Delta_n}\sum_{i=1}^{[t/\Delta_n]-k_n+1}\Big(g(c_i^n + \beta_i^n) - g(c_i^n) - \sum_{l,m}\partial_{lm}g(c_i^n)\,\beta_i^{n,lm}\Big).
\end{aligned}
$$

The leading term is $V^{n,3}$, the bias comes from the terms $V^{n,4}$ and $V^{n,5}$, and the first two terms are negligible, in the sense that they satisfy

$$
j = 1,2 \quad\Rightarrow\quad V_t^{n,j} \xrightarrow{\mathbb{P}} 0 \qquad \text{for all } t > 0. \tag{3.29}
$$

We end this subsection with the proof of (3.29).

The case $j = 1$: (2.7) implies

$$
|g(\widehat{c}_i^n) - g(\widehat{c}_i'^n)| \le K(1+\|\widehat{c}_i^n\|+\|\widehat{c}_i'^n\|)^{p-1}\,\|\widehat{c}_i^n - \widehat{c}_i'^n\| \le K(1+\|\widehat{c}_i'^n\|)^{p-1}\,\|\widehat{c}_i^n - \widehat{c}_i'^n\| + K\|\widehat{c}_i^n - \widehat{c}_i'^n\|^p.
$$

Recalling the last part of (3.20), we deduce from (3.11), from the fact that $1 - r\varpi - p(1 - 2\varpi) < \frac{(2-r)\varpi}{2q}$ for all $q > 1$ small enough, and from Hölder's inequality, that $\mathbb{E}(|g(\widehat{c}_i^n) - g(\widehat{c}_i'^n)|) \le Ka_n\Delta_n^{(2p-r)\varpi+1-p}$. Therefore

$$
\mathbb{E}\Big(\sup_{s\le t}|V_s^{n,1}|\Big) \le Kta_n\Delta_n^{(2p-r)\varpi+1/2-p}
$$

and (3.29) for $j = 1$ follows.

The case $j = 2$: Since g is C^2 and c_t is an Itô semimartingale with bounded characteristics, the convergence $V^{n,2} \overset{\text{u.c.p.}}{\Longrightarrow} 0$ is well known: see for example the proof of (5.3.24) in [4], in which one replaces $\rho_{c_s}(f)$ by $g(c_s)$.

3.6 The Leading Term $V^{n,3}$.

Our aim here is to prove that

$$
V^{n,3} \overset{\mathcal{L}-\text{s}}{\Longrightarrow} Z \tag{3.30}
$$

(functional stable convergence in law), where Z is the process defined in Theorem 2.2.

A change of order of summation allows us to rewrite $V^{n,3}$ as

$$
V_t^{n,3} = \frac{1}{\sqrt{\Delta_n}}\sum_{i=1}^{[t/\Delta_n]}\sum_{l,m} w_i^{n,lm}\,\alpha_i^{n,lm}, \quad \text{where } w_i^{n,lm} = \frac{1}{k_n}\sum_{j=(i-[t/\Delta_n]+k_n-1)^+}^{(i-1)\wedge(k_n-1)}\partial_{lm}g(c_{i-j}^n).
$$

Observe that w_i^n and α_i^n are measurable with respect to $\mathcal{F}_i^n$ and $\mathcal{F}_{i+1}^n$, respectively, so by Theorem IX.7.28 of [3] (with $G = 0$ and $Z = 0$ in the notation of that theorem) it suffices to prove the following four convergences in probability, for all $t > 0$ and all component indices:

$$\frac{1}{\sqrt{\Delta_n}} \sum_{i=1}^{[t/\Delta_n]-k_n+1} w_i^{n,lm}\, \mathbb{E}(\alpha_i^{n,lm} \mid \mathcal{F}_i^n) \;\stackrel{\mathbb{P}}{\longrightarrow}\; 0 \tag{3.31}$$

$$\frac{1}{\Delta_n} \sum_{i=1}^{[t/\Delta_n]-k_n+1} w_i^{n,jk}\, w_i^{n,lm}\, \mathbb{E}(\alpha_i^{n,jk}\, \alpha_i^{n,lm} \mid \mathcal{F}_i^n) \stackrel{\mathbb{P}}{\longrightarrow} \int_0^t \partial_{jk} g(c_s)\, \partial_{lm} g(c_s)\, \big(c_s^{jl} c_s^{km} + c_s^{jm} c_s^{kl}\big)\, ds \tag{3.32}$$

$$\frac{1}{\Delta_n^2} \sum_{i=1}^{[t/\Delta_n]-k_n+1} \|w_i^n\|^4\, \mathbb{E}(\|\alpha_i^n\|^4 \mid \mathcal{F}_i^n) \;\stackrel{\mathbb{P}}{\longrightarrow}\; 0 \tag{3.33}$$

$$\frac{1}{\sqrt{\Delta_n}} \sum_{i=1}^{[t/\Delta_n]-k_n+1} w_i^{n,lm}\, \mathbb{E}(\alpha_i^{n,lm}\, \Delta_i^n N \mid \mathcal{F}_i^n) \;\stackrel{\mathbb{P}}{\longrightarrow}\; 0, \tag{3.34}$$

where $N = W^j$ for some j, or is an arbitrary bounded martingale, orthogonal to W.

For proving these properties, we pick a ρ bigger than the upper bound of the function J, so $\mathcal{G}^\rho$ becomes the trivial σ-field and $\mathcal{F}_i^n = \mathcal{F}_i^{n,\rho}$ and $L(n,\rho) = \mathbb{N}$. In such a way, we can apply all estimates of the previous subsections with the conditioning σ-fields $\mathcal{F}_i^n$. Therefore (3.20) and the property $\|w_i^n\| \le K$ readily imply (3.31) and (3.33). In view of the form of α_i^n, a usual argument (see e.g. [4]) shows that in fact $\mathbb{E}(\alpha_i^{n,lm}\, \Delta_i^n N \mid \mathcal{F}_i^n) = 0$ for all N as above, hence (3.34) holds.

For (3.32), by (3.22) it suffices to prove that

$$\Delta_n \sum_{i=1}^{[t/\Delta_n]-k_n+1} w_i^{n,jk}\, w_i^{n,lm}\, (c_i^{n,jl} c_i^{n,km} + c_i^{n,jm} c_i^{n,kl}) \stackrel{\mathbb{P}}{\longrightarrow} \int_0^t \partial_{jk} g(c_s)\, \partial_{lm} g(c_s)\, \big(c_s^{jl} c_s^{km} + c_s^{jm} c_s^{kl}\big)\, ds.$$

In view of the definition of w_i^n, for each t we have $w_{i(n,t)}^{n,jk} \to \partial_{jk} g(c_t)$ and $c_{i(n,t)}^{n,jk} \to c_t^{jk}$ almost surely if $|i(n,t)\Delta_n - t| \le k_n \Delta_n$ (recall that c is almost surely continuous at t, for any fixed t), and the above convergence follows by the dominated convergence theorem, thus ending the proof of (3.30).

3.7 The Term $V^{n,4}$.

In this subsection we prove that, for all t,

$$V_t^{n,4} \stackrel{\mathbb{P}}{\longrightarrow} \frac{\theta}{2} \sum_{l,m} \int_0^t \partial_{lm} g(c_{s-})\, dc_s^{lm} - \theta\, g(c_t). \tag{3.35}$$

We call $V_t'^{n,4}$ and $V_t''^{n,4}$, respectively, the first sum, and the last integral, in the definition of $V_t^{n,4}$. Since $k_n\sqrt{\Delta_n} \to \theta$ and c is a.s. continuous at t, it is obvious that $V_t''^{n,4}$ converges almost surely to $-\theta\, g(c_t)$, and it remains to prove the convergence of $V_t'^{n,4}$ to the first term in the right side of (3.35).

We first observe that $c^n_{i+u} - c^n_i = \sum_{v=0}^{u-1} \Delta^n_{i+v} c$. Then, upon changing the order of summation, we can rewrite $V'^{n,4}_t$ as

$$V^{n,4}_t = \sum_{i=1}^{[t/\Delta_n]-1} \sum_{l,m} w^{n,lm}_i \, \Delta^n_i c^{lm}, \qquad w^{n,lm}_i = \frac{\sqrt{\Delta_n}}{k_n} \sum_{u=0\vee(i+k_n-1-[t/\Delta_n]}^{i-1)\wedge(k_n-2)} (k_n - 1 - u) \partial_{lm} g(c^n_{i-u}).$$

In other words, recalling $k_n\sqrt{\Delta_n} \le K$ and $\|\partial g(c_s)\| \le K$, we see that

$$V^{n,4}_t = \sum_{l,m} \int_0^t H(n,t)^{lm}_s \, dc^{lm}_s,$$

where $H(n,t)_s$ is a $d \times d$-dimensional predictable process, bounded uniformly (in n, s, ω) and given on the set $[k_n\Delta_n, t - k_n\Delta_n]$ by

$$(i-1)\Delta_n < s \le i\Delta_n \quad \Rightarrow \quad H(n,t)^{lm}_s = \frac{\sqrt{\Delta_n}}{k_n} \sum_{u=0}^{k_n-2} (k_n - 1 - u) \partial_{lm} g(c^n_{i-u})$$

(its expression on $[0, k_n\Delta_n)$ and on $(t - k_n\Delta_n, t]$ is more complicated, but not needed, apart from the fact that it is uniformly bounded). Now, since $\sum_{u=0}^{k_n-2}(k_n - 1 - u) = k_n^2/2 + \mathrm{O}(k_n)$ as $n \to \infty$, we observe that $H(n,t)^{lm}_s$ converges to $\frac{\theta}{2}\, \partial_{lm} g(c_{s-})$ for all $s \in (0,t)$. Since c is a.s. continuous at t, we deduce from the dominated convergence theorem for stochastic integrals that $V'^{n,4}_t$ indeed converges in probability to the first term in the right side of (3.35).

3.8 The Term $V^{n,5}$.

The aim of this subsection is to prove the convergence

$$V^{n,5}_t \xrightarrow{\mathbb{P}} A^2_t - 2A^3_t + \theta \sum_{s \le t} \int_0^1 \big(g(c_{s-} + w\Delta c_s) - g(c_{s-}) - w \sum_{l,m} \partial_{lm} g(c_{s-}) \, \Delta c^{lm}_s \big) \, dw \quad (3.36)$$

We have $V^{n,5}_t = \sum_{i=1}^{[t/\Delta_n]-k_n+1} v^n_i$, where

$$v^n_i = \sqrt{\Delta_n} \big(g(c^n_i + \beta^n_i) - g(c^n_i) - \sum_{l,m} \partial_{lm} g(c^n_i) \, \beta^{n,lm}_i \big).$$

We also set

$$\begin{array}{ll} \overline{\alpha}^n_i = \frac{1}{k_n\Delta_n} \sum_{u=0}^{k_n-1} \alpha^n_{i+u}, & \overline{\beta}^n_i = \beta^n_i - \overline{\alpha}^n_i = \frac{1}{k_n} \sum_{u=1}^{k_n-1} (c^n_{i+u} - c^n_i) \\ v'^n_i = \sqrt{\Delta_n} \big(g(c^n_i + \overline{\beta}^n_i) - g(c^n_i) - \sum_{l,m} \partial_{lm} g(c^n_i) \, \overline{\beta}^{n,lm}_i \big), & v''^n_i = v^n_i - v'^n_i. \end{array} \quad (3.37)$$

We take $\rho \in (0,1]$, and will eventually let it go to 0. With the sets $L(n,\rho)$ of (3.17), we associate

$$\begin{array}{l} L(n,\rho,t) = \{1, \cdots, [t/\Delta_n] - k_n + 1\} \cap L(n,\rho) \\ \overline{L}(n,\rho,t) = \{1, \cdots, [t/\Delta_n] - k_n + 1\} \backslash L(n,\rho). \end{array}$$

We split the sum giving $V_t^{n,5}$ into three terms:

$$U_t^{n,\rho} = \sum_{i\in L(n,\rho,t)} v_i^n, \qquad U_t^{\prime n,\rho} = \sum_{i\in \overline{L}(n,\rho,t)} v_i^{\prime n}, \qquad U_t^{\prime\prime n,\rho} = \sum_{i\in \overline{L}(n,\rho,t)} v_i^{\prime\prime n}. \tag{3.38}$$

A) The processes $U^{n,\rho}$. A Taylor expansion and (2.7) give us

$$v_i^n = v(1)_i^n + v(2)_i^n + v(3)_i^n, \text{ where } \begin{cases} v(1)_i^n = \frac{\sqrt{\Delta_n}}{2} \sum_{j,k,l,m} \partial^2_{jk,lm} g(c_i^n)\, \mathbb{E}(\beta_i^{n,jk}\, \beta_i^{n,lm} \mid \mathcal{F}_i^{n,\rho}) \\ v(2)_i^n = \frac{\sqrt{\Delta_n}}{2} \sum_{j,k,l,m} \partial^2_{jk,lm} g(c_i^n)\, \beta_i^{n,jk}\, \beta_i^{n,lm} - v(1)_i^n \\ |v(3)_i^n| \le K\sqrt{\Delta_n}\, (1+\|\beta_i^n\|)^{p-3}\, \|\beta_i^n\|^3. \end{cases}$$

Therefore

$$U^{n,\rho} = \sum_{j=1}^{3} U(j)^{n,\rho}, \quad \text{where} \quad U(j)_t^n = \sum_{i\in L(n,\rho,t)} v(j)_i^n. \tag{3.39}$$

On the one hand, and letting

$$w(\rho)_i^n = \sum_{j,k,l,m} \partial^2_{jk,lm} g(c_i^n) \Big(\frac{1}{2k_n\sqrt{\Delta_n}} \, (c_i^{n,jl} c_i^{n,km} + c_i^{n,jm} c_i^{n,kl}) + \frac{k_n\sqrt{\Delta_n}}{6}\, \overline{c}(\rho)_i^{n,jklm} \Big),$$

the càdlàg property of c and $\overline{c}(\rho)$ and $k_n\sqrt{\Delta_n} \to \theta$ imply

$$W(\rho)_t^n := \Delta_n \sum_{i=1}^{[t/\Delta_n]-k_n+1} w(\rho)_i^n \xrightarrow{\mathbb{P}} U(1)_t^\rho := A_t^2 + \frac{\theta}{6} \sum_{j,k,l,m} \int_0^t \partial^2_{jk,lm} g(c_s)\, \overline{c}(\rho)_s^{jklm}\, ds.$$

On the other hand, Lemma 3.3 yields $[v(1)_i^n - \Delta_n w(\rho)_i^n| \le K_\rho \Delta_n\, (\Delta_n^{1/4} + \eta(\rho)_i^n)$ when $i \in L(n,\rho)$, whereas $|w(\rho)_i^n| \le K$ always. Therefore

$$\mathbb{E}\big(|U(1)_t^{n,\rho} - W(\rho)_t^n|\big) \le K_\rho \Delta_n\, \mathbb{E}\Big(\sum_{i=1}^{[t/\Delta_n]} (\sqrt{\Delta_n} + \eta(\rho)_i^n) \Big) + K\Delta_n \mathbb{E}\big(\#(\overline{L}(n,\rho,t))\big).$$

Now, $\#(\overline{L}(n,\rho,t))$ is not bigger than $(2k_n+1)N_t^\rho$, implying that $\Delta_n \mathbb{E}(\#(L'(n,\rho,t))) \le K_\rho\sqrt{\Delta_n}$. Taking advantage of (3.16), we deduce that the above expectation goes to 0 as $n \to \infty$, and thus

$$U(1)_t^{n,\rho} \xrightarrow{\mathbb{P}} U(1)_t^\rho. \tag{3.40}$$

Next, $v(2)_i^n$ is $\mathcal{F}_{i+k_n}^{n,\rho}$-measurable, with vanishing $\mathcal{F}_i^{n,\rho}$-conditional expectation, and each set $\{i \in L(n,\rho)\}$ is $\mathcal{F}_0^{n,\rho}$-measurable. It follows that

$$\begin{aligned} \mathbb{E}\big((U(2)_t^{n,\rho})^2\big) &\le 2k_n\, E\Big(\textstyle\sum_{i\in L(n,\rho,t)} \mathbb{E}\big(|v(2)_i^n|^2 \mid \mathcal{F}_i^{n,\rho}\big)\Big) \\ &\le K k_n \Delta_n\, E\Big(\textstyle\sum_{i\in L(n,\rho,t)} \mathbb{E}\big(|\beta_i^n|^4 \mid \mathcal{F}_i^{n,\rho}\big)\Big) \;\le\; Kt\phi_\rho + K_\rho t\sqrt{\Delta_n}, \end{aligned}$$

where we have applied (3.21) for the last inequality. Another application of the same estimate gives us

$$\mathbb{E}\big(|U(3)_t^n|\big) \;\le\; Kt\phi_\rho + K_\rho t \Delta_n^{1/4}.$$

These two results and the property $\phi_\rho \to 0$ as $\rho \to 0$ clearly imply

$$\lim_{\rho\to 0} \limsup_{n\to\infty} \mathbb{E}(|U(2)_t^{n,\rho}| + |U(3)_t^{n,\rho}|) \;=\; 0. \tag{3.41}$$

B) The processes $U'^{n,\rho}$. We will use here the jump times $S_1^\rho, S_2^\rho, \cdots$ of the Poisson process N^ρ, and will restrict our attention to the set $\Omega_{n,t,\rho}$ defined before (3.13), whose probability goes to 1 as $n \to \infty$. On this set, $\overline{L}(n,\rho,t)$ is the collection of all integers i which are between $[S_q^\rho/\Delta_n] - 2k_n + 2$ and $[S_q^\rho/\Delta_n] + 1$, for some q between 1 and N_t^ρ. Thus

$$U_t'^{n,\rho} = \sum_{q=1}^{N_t^\rho} H(n,\rho,q), \quad \text{where } \; H(n,\rho,q) = \sum_{i=[S_q^\rho/\Delta_n]-2k_n+1}^{[S_q^\rho/\Delta_n]+1} v_i'^n. \tag{3.42}$$

The behavior of each $H(n,\rho,q)$ is a pathwise question. We fix q and set $S = S_q^\rho$ and $a_n = [S/\Delta_n]$, so $S > a_n\Delta_n$ because S is not a multiple of Δ_n. For further reference we consider a case slightly more general than strictly needed here. We have $c_i^n \to c_{S-}$ when $a_n - 6k_n + 1 \le i \le a_n + 1$ and $c_i^n \to c_S$ when $a_n + 2 \le i \le a_n + 6k_n$, uniformly in i (for each given outcome ω). Hence

$$\overline{\beta}_i^n - \frac{(k_n - a_n + i - 2)^+ \wedge (k_n - 1)}{k_n}\,\Delta c_S \to 0 \quad \text{uniformly in } i \in \{a_n - 6k_n + 2, \cdots, a_n + 5k_n\}. \tag{3.43}$$

Thus, the following convergence holds, uniform in $i \in \{a_n - 2k_n + 1, \cdots, a_n + 1\}$:

$$\tfrac{1}{\sqrt{\Delta_n}}\, v_i'^n - \Big(g\big(c_{S-} + \tfrac{k_n - a_n + i - 2}{k_n}\,\Delta c_S\big) - g(c_{S-}) \\ - \textstyle\sum_{l,m} \partial_{lm} g(c_{S-})\,\big(c_{S-}^{jk} + \tfrac{k_n - a_n + i - 2}{k_n}\,\Delta c_S^{lm}\big)\Big) \to 0,$$

which implies

$$H(n,\rho,q) - \sqrt{\Delta_n} \sum_{u=1}^{k_n-3} \Big(g\big(c_{S_q-} + \frac{u}{k_n}\,\Delta c_{S_q}\big) - g(c_{S_q-}) - \sum_{l,m} \partial_{lm} g(c_{S_q-})\,\frac{u}{k_n}\,\Delta c_{S_q}^{lm}\Big) \to 0$$

and by Riemann integration this yields

$$H(n,\rho,q) \to \theta \int_0^1 \big(g(c_{S_q-} + w\Delta c_{S_q}) - g(c_{S_q-}) - w \sum_{l,m} \partial_{lm} g(c_{S_q-})\,\Delta c_S^{lm}\big)\, dw.$$

Henceforth, we have

$$U_t'^{n,\rho} \;\xrightarrow{\mathbb{P}}\; U_t'^\rho := \theta \sum_{q=1}^{N_t^\rho} \int_0^1 \big(g(c_{S_q-} + w\Delta c_{S_q}) - g(c_{S_q-}) - w \sum_{l,m} \partial_{lm} g(c_{S_q-})\,\Delta c_{S_q}^{lm}\big)\, dw. \tag{3.44}$$

C) The processes $U''^{n,\rho}$. Since $|\overline{\beta}_i^n| \le K$ we deduce from (2.7) that $|v_i''^n| \le K\sqrt{\Delta_n}\,(\|\overline{\alpha}_i^n\| + \|\overline{\alpha}_i^n\|^p)$. (3.20) yields $\mathbb{E}(\|\overline{\alpha}_i^n\|^q \mid \mathcal{F}_i^{n,\rho}) \le K_q \Delta_n^{q/4}$ for all $q > 0$. Therefore

$$\mathbb{E}\big(|U_t''^{n,\rho}|\big) \;\le\; K\Delta_n^{3/4}\,\mathbb{E}(\#(\overline{L}(,n,\rho,t))) \;\le\; K_\rho \Delta_n^{1/4},$$

by virtue of what precedes (3.40). We then deduce

$$U_t''^{n,\rho} \stackrel{\mathbb{P}}{\longrightarrow} 0. \tag{3.45}$$

D) Proof of (3.36)**.** On the one hand, $V^{n,5} = U(1)^{n,\rho} + U(2)^{n,\rho} + U(3)^{n,\rho} + U'^{n,\rho} + U''^{n,\rho}$; on the other hand, the dominated convergence theorem (observe that $\overline{c}(\rho)_t \to \widetilde{\sigma}_t^2$ for all t) yields that $U(1)_t^\rho \stackrel{\mathbb{P}}{\longrightarrow} A^2 - \frac{1}{2}\, A_t^3$ and

$$U_t'^{\rho} \stackrel{\mathbb{P}}{\longrightarrow} \theta \sum_{s\le t} \int_0^1 \big(g(c_{s-} + w\Delta c_s) - g(c_{s-}) - w \sum_{l,m} \partial_{lm} g(c_{s-})\, \Delta c_s^{lm}\big)\, dw$$

as $\rho \to 0$ (for the latter convergence, note that $|g(x+y) - g(x) - \sum_{l,m} \partial_{lm} g(x) y^{lm}| \le K\|y\|^2$ when x, y stay in a compact set). Then the property (3.36) follows from (3.40), (3.41), (3.44) and (3.45).

E) Proof of Theorem 2.2. We are now ready to prove Theorem 2.2. Recall that $\frac{1}{\sqrt{\Delta_n}}\,(V(g)n_t - V(g)) = \sum_{j=1}^5 V^{n,j}$. By virtue of (3.29), (3.30), (3.35), (3.36), it is enough to check that

$$\begin{aligned} A_t^1 + A_t^3 + A_t^4 + A_t^5 &= \tfrac{\theta}{2} \textstyle\sum_{l,m} \int_0^t \partial_{lm} g(c_{s-})\, dc_s^{lm} - \theta\, g(c_t) \\ &\quad -2A_t^3 + \theta \textstyle\sum_{s\le t} \int_0^1 \big(g(c_{s-} + w\Delta c_s) - g(c_{s-}) - w \sum_{l,m} \partial_{lm} g(c_{s-})\, \Delta c_s^{lm}\big)\, dw. \end{aligned}$$

To this aim, we observe that Itô's formula gives us

$$g(c_t) = g(c_0) + \sum_{l,m} \int_0^t \partial_{lm} g(c_{s-})\, dc_s^{lm} - \frac{6}{\theta} A_t^3 + \sum_{s\le t} \big(g(c_{s-} + \Delta c_s) - g(c_{s-}) - \sum_{l,m} \partial_{lm} g(c_{s-})\, \Delta c_s^{lm}\big),$$

so the desired equality is immediate (use also $\int_0^1 w\, dw = \frac{1}{2}$), and the proof of Theorem 2.2 is complete.

3.9 Proof of Theorem 2.5.

The proof of Theorem 2.5 follows the same line as in Subsection 3.8, and we begin with an auxiliary step.

Step 1) Replacing $\widehat{c}_i^n$ by $\widehat{c}_i'^n$. The summands in the definition (2.17) of $A_t^{n,3}$ are $R(\widehat{c}_i^n, \widehat{c}_{i+k_n}^n)$, where $R(x,y) = \sum_{j,k,l,m} \partial^2_{jk,lm} g(x)(y^{jk} - x^{jk})(y^{lm} - x^{lm})$, and we set

$$A_t'^{n,3} = -\frac{\sqrt{\Delta_n}}{8} \sum_{i=1}^{[t/\Delta_n]-2k_n+1} R(\widehat{c}_i'^n, \widehat{c}_{i+k_n}'^n).$$

We prove here that

$$A_t^{n,3} - A_t'^{n,3} \stackrel{\mathbb{P}}{\longrightarrow} 0 \tag{3.46}$$

for all t, and this is done as in to the step $j=1$ in Subsection 3.5. The function R is C^1 on $\mathbb{R}^2_+$ with $\|\partial^j R(x,y)\| \leq K(1+\|x\|+\|y\|)^{p-j}$ for $j=0,1$, by (2.7). Thus

$$\begin{aligned}|R(\widehat{c}_i^n,\widehat{c}_{i+k_n}^n)-R(\widehat{c}_i'^n,\widehat{c}_{i+k_n}'^n)| \;&\leq\; K(1+\|\widehat{c}_i'^n\|+\|\widehat{c}_{i+k_n}'^n)\|)^{p-1}(\|\widehat{c}_i^n-\widehat{c}_i'^n\|+\|\widehat{c}_{i+k_n}^n-\widehat{c}_{i+k_n}'^n\|)\\ &\quad +K\|\widehat{c}_i^n-\widehat{c}_i'^n\|^p+K\|\widehat{c}_{i+k_n}^n-\widehat{c}_{i+k_n}'^n\|^p.\end{aligned}$$

Then, exactly as in the case afore-mentioned, we conclude (3.46), and it remains to prove that, for all t, we have

$$A_t'^{n,3} \;\stackrel{\mathbb{P}}{\longrightarrow}\; -\frac{1}{2}A_t^2+A_t^3+A_t'^4.$$

Step 2) From now on we use the same notation as in Subsection 3.8, although they denote different variables or processes. For any $\rho\in(0,1]$ we have $A'^{n,3}=U^{n,\rho}+U'^{n,\rho}+U''^{n,\rho}$, as defined in (3.38), but with

$$\begin{aligned}v_i^n &= -\tfrac{\sqrt{\Delta_n}}{8}\,R(c_i^n+\beta_i^n,c_{i+k_n}^n+\beta_{i+k_n}^n)\\ v_i'^n &= -\tfrac{\sqrt{\Delta_n}}{8}\,R(c_i^n+\overline{\beta}_i^n,c_{i+k_n}^n+\overline{\beta}_{i+k_n}^n),\quad v_i''^n=v_i^n-v_i'^n.\end{aligned}$$

Recalling γ_i^n in (3.19), the decomposition (3.39) holds with

$$\begin{aligned}v(1)_i^n &= -\tfrac{\sqrt{\Delta_n}}{8}\textstyle\sum_{j,l,k,m}\partial^2_{jl,km}\,g(c_i^n)\;\mathbb{E}\big(\gamma_i^{n,jk}\,\gamma_i^{n,lm}\mid\mathcal{F}_i^{n,\rho}\big)\\ v(2)_i^n &= -\tfrac{\sqrt{\Delta_n}}{8}\textstyle\sum_{j,l,k,m}\partial^2_{jl,km}\,g(c_i^n)\,\gamma_i^{n,jk}\,\gamma_i^{n,lm}-v(1)_i^n\\ v(3)_i^n &= v_i^n-v(1)_i^n-v(2)_i^n.\end{aligned}$$

Use $\widehat{c}_i'^n-c_i^n=\beta_i^n$ and (2.7) and a Taylor expansion to check that

$$|v(3)_i^n|\leq K\sqrt{\Delta_n}\,\|\gamma_i^n\|^2\,\|\beta_i^n\|\,(1+\|\beta_i^n\|)^{p-3}.$$

We also have $|v(2)_i^n|\leq K\sqrt{\Delta_n}\,\|\gamma_i^n\|^2$, hence (3.21) and (3.27) yield

$$\mathbb{E}(|v(3)_i^n|\mid\mathcal{G}^\rho)+\mathbb{E}(|v(2)_i^n|^2\mid\mathcal{G}^\rho)\leq K\Delta_n\Big(\phi_\rho+\Delta_n^{1/4}+\frac{\Delta_n}{\rho^p}\Big),$$

and thus (3.41) holds here as well, by the same argument. Moreover, (3.27) again yields (3.40), with now

$$U_t^\rho=-\sum_{j,k,l,m}\int_0^t\partial^2_{jk,lm}\,g(c_s)\,\Big(\frac{\theta}{12}\,\overline{c}(\rho)_s^{jklm}+\frac{1}{4\theta}\,(c_s^{jl}c_s^{km}+c_s^{jm}c_s^{kl})\Big)\,ds.$$

This goes to $A_t^3-\frac{1}{2}A_t^2$ as $\rho\to0$.

Another application of (2.7) gives us

$$|v_i''^n|\leq K\sqrt{\Delta_n}\,\big(1+\|\gamma_i^n\|^2\big)\big(\|\overline{\alpha}_i^n\|+\|\overline{\alpha}_{i+k_n}^n\|+\|\overline{\alpha}_i^n\|^p+\|\overline{\alpha}_{i+k_n}^n\|^p\big).$$

Then another application of (3.20), (3.21) and (3.27) yields $\mathbb{E}(|v_i''^n|\mid\mathcal{G}^\rho)\leq K\Delta_n^{3/4}$ and we conclude (3.45) as previously. We are thus left to prove that

$$\rho>0\;\Rightarrow\;U_t'^{n,\rho}\stackrel{\mathbb{P}}{\longrightarrow}U_t'^\rho,\quad\text{with, as }\rho\to0,\;U_t'^\rho\stackrel{\mathbb{P}}{\longrightarrow}A_t'^4.\tag{3.47}$$

Step 3) On the set $\Omega_{n,t,\rho}$ we have (3.42) and we study $H(n,\rho,q)$, in the same way as before, on the set $\Omega_{n,t,\rho}$. We fix q and set $S = S_q$ and $a_n = [S/\Delta_n]$. We then apply (3.43) and also $c_i^n \to c_{S-}$ or $c_i^n \to c_S$, according to whether $a_n - 2k_n + 1 \le i \le a_n + 1$ or $a_n + 2 \le i \le a_n + k_n$, to obtain $v_i'^n - \overline{v}_i'^n \to 0$, uniformly in i between $a_n - 2k_n + 1$ and $a_n + 1$, where

$$\overline{v}_i'^n = \begin{cases} 0 & \text{if } a_n - 2k_n + 1 \le i \le a_n - 2k_n + 2 \\ -\frac{(2k_n - a_n + i - 2)^2 \sqrt{\Delta_n}}{8k_n^2} \sum_{j,k,l,m} \partial^2_{jk,lm}\, g(c_{S-})\, \Delta c_S^{jk}\, \Delta c_S^{lm} & \\ & \text{if } a_n - 2k_n + 3 \le i \le a_n - k_n + 1 \\ \frac{(a_n - i + 2)^2 \sqrt{\Delta_n}}{8k_n^2} \sum_{j,k,l,m} \partial^2_{jk,lm}\, g\Big(c_{S-} + \frac{k_n - a_n + i + 2}{k_n}\, \Delta c_S\Big)\, \Delta c_S^{jk}\, \Delta c_S^{lm} & \\ & \text{if } a_n - k_n + 2 \le i \le a_n + 1. \end{cases}$$

We then deduce, by Riemann integration, that

$$H(n,\rho,q) \to -\frac{\theta}{8} \sum_{j,k,l,m} \int_0^1 \big(\partial^2_{jk,lm}\, g(c_{S_q-}) + \partial^2_{jk,lm}\, g(c_{S_q-} + (1-w)\Delta c_{S_q})\big)\, w^2\, \Delta c_{S_q}^{jk}\, \Delta c_{S_q}^{lm}\, dw,$$

which is $\theta G'(c_{S_q-}, \Delta c_{S_q})$, hence the first part of (3.47), with $U_t'^\rho = \theta \sum_{q=1}^{N_t^\rho} G'(c_{S_q^\rho -}, \Delta c_{S_q^\rho})$. The second part of (3.47) follows from the dominated convergence theorem, and the proof of Theorem 2.5 is complete.

3.10 Proof of Theorem 2.6.

The proof is once more somewhat similar to the proof of Subsection 3.8, although the way we replace $\widehat{c}_i^n$ by $\widehat{c}_i'^n$ and further by $\overline{\alpha}_i^n + \beta_i^n$ is different.

A) Preliminaries. The jth summand in (2.24) involves several estimators $\widehat{c}_i^n$, spanning the time interval $((j-3)k_n\Delta_n, (j+2)k_n\Delta_n]$. It is thus convenient to replace the sets $L(n,\rho)$, $L(n,\rho,t)$ and $\overline{L}(n,\rho,t)$, for $\rho, t > 0$, by the following ones:

$$\begin{aligned} &L'(n,\rho) = \{j = 3,4,\cdots :\ N^\rho_{(j+2)k_n\Delta_n} - N^\rho_{(j-3)k_n\Delta_n} = 0\} \\ &L'(n,\rho,t) = \{3,\cdots,[t/k_n\Delta_n] - 3\} \cap L'(n,\rho) \\ &\overline{L}'(n,\rho,t) = \{3,\cdots,[t/k_n\Delta_n] - 3\} \cap (\mathbb{N} \backslash L'(n,\rho)). \end{aligned}$$

For any $\rho \in (0,1]$ we write $\mathcal{V}(F)_t^n = \mathcal{V}_t^{n,\rho} + \overline{\mathcal{V}}_t^{n,\rho}$, where

$$\begin{aligned} &v_j^n = F(\widehat{c}^n_{(j-3)k_n+1}, \delta_j^n \widehat{c})\, 1_{\{\|\delta^n_{j-1}\widehat{c}\| \vee \|\delta^n_{j+1}\widehat{c}\| \vee u_n' < \|\delta_j^n \widehat{c}\|\}} \\ &\mathcal{V}_t^{n,\rho} = \textstyle\sum_{j \in L'(n,\rho,t)} v_j^n, \qquad \overline{\mathcal{V}}_t^{n,\rho} = \sum_{j \in \overline{L}'(n,\rho,t)} v_j^n. \end{aligned}$$

We also set

$$\begin{aligned} &\delta_j^n \widehat{c}' = \widehat{c}'^n_{jk_n+1} - \widehat{c}'^n_{(j-2)k_n+1}, && \delta_j^n \beta = \beta^n_{jk_n+1} - \beta^n_{(j-2)k_n+1} \\ &w_j^n = \textstyle\sum_{m=-3}^{2} \|\widehat{c}^n_{(j+m)k_n+1} - \widehat{c}'^n_{(j+m)k_n+1}\|, && w_j'^n = (1 + \|\widehat{c}'^n_{(j-3)k_n+1}\|)^{p-1}\, (1 + \|\delta_j^n \widehat{c}\|)^2. \end{aligned}$$

(3.11) and the last part of (3.20) yield

$$q \ge 1 \quad \Rightarrow \quad \mathbb{E}\big((w_j^n)^q\big) \le K_q\, \Delta_n^{(2q-r)\varpi + 1 - q}, \qquad \mathbb{E}((w_i'^n)^q) \le K_q. \tag{3.48}$$

Observe that $\delta_j^n \widehat{c}'$ is analogous to γ_i^n, with a doubled time lag, so it satisfies a version of (3.27) and, for $q \geq 2$, we have

$$i \in L'(n,\rho) \;\Rightarrow\; \mathbb{E}\big(\|\delta_j^n \widehat{c}'\|^q \mid \mathcal{F}^{n,\rho}_{(j-2)k_n+1}\big)\big| \leq K_q\big(\sqrt{\Delta_n}\,\phi_\rho + \Delta_n^{q/4} + \frac{\Delta_n^{q/2}}{\rho^q}\big). \tag{3.49}$$

B) The processes $\mathcal{V}^{n,\rho}$. (2.25) yields

$$|v_j^n| \leq K(1 + \|\widehat{c}^n_{(j-3)k_n+1}\|)^{p-2}\, \|\delta_j^n \widehat{c}\|^2\, 1_{\{\|\delta_j^n \widehat{c}\| > u'_n\}} + K\|\delta_j^n \widehat{c}\|^p.$$

Thus a (tedious) computation shows that, with the notation

$$a_j^n = (1 + \|\widehat{c}'^n_{(j-3)k_n+1}\|)^{p-2}\, \|\delta_j^n \widehat{c}'\|^2\, 1_{\{\|\delta_j^n \widehat{c}'\| > u'_n/2\}}, \quad a_j'^n = w_j'^n\Big(w_i^n + (w_i^n)^p + \frac{(w_i^n)^v}{u_n'^v}\Big),$$

with $v > 0$ arbitrary, we have $|v_j^n| \leq K(a_j^n + \|\delta_j^n \widehat{c}'\|^p + a_j'^n)$ (with K depending on v). Therefore we have $|\mathcal{V}_t^{n,\rho}| \leq K(B_t^{n,\rho} + C_t^{n,\rho} + D_t^n)$, where

$$B_t^{n,\rho} = \sum_{j \in L'(n,\rho,t)} a_j^n, \qquad C_t^{n,\rho} = \sum_{j \in L'(n,\rho,t)} \|\delta_j^n \widehat{c}'\|^p, \qquad D_t^n = \sum_{j=3}^{[t/k_n\Delta_n]} a_j'^n.$$

First, (3.48) and Hólder's inequality give us $\mathbb{E}(a_j'^n) \leq K_{q,v}\Delta_n^{l(q,v)}$ for any $q > 1$ and $v > 0$, where (recalling (2.8) and (2.22) for ϖ and ϖ') we have set $l(q,v) = \frac{1-r\varpi}{q} - \big(p(1-2\varpi) \vee v(1-2\varpi+\varpi')\big)$. Upon choosing v small enough and q close enough to 1, and in view of (2.8), we see that $l(q,v) > \frac{1}{2}$, thus implying

$$\mathbb{E}(D_t^n) \;\to\; 0. \tag{3.50}$$

Next, we deduce from (3.49) that

$$\mathbb{E}\big(C_t^{n,\rho}\big) \leq K\mathbb{E}\Big(\mathbb{E}\Big(\sum_{i \in L'(n,\rho,t)} \|\delta_j^n \widehat{c}'\|^p \mid \mathcal{G}^\rho\Big)\Big) \leq Kt\Big(\phi_\rho + \Delta_n^{p/4} + \frac{\Delta_n^{p/2}}{\rho^p}\Big),$$

and thus, since $p \geq 3$,

$$\lim_{\rho \to 0} \limsup_{n \to \infty} \mathbb{E}(|C_t^{n,\rho}|) \;=\; 0. \tag{3.51}$$

The analysis of $B_t^{n,\rho}$ is more complicated. We have $\delta_j^n \widehat{c}' = z_j^n + z_j'^n$, where

$$z_j^n = \overline{\alpha}^n_{jk_n+1} - \overline{\alpha}^n_{(j-2)k_n+1}, \quad z_j'^n = \frac{1}{k_n}\sum_{m=1}^{k_n} (c^n_{jk_n+m} - c^n_{(j-2)k_n+m})$$

(recall (3.37) for $\overline{\alpha}_i^n$), hence

$$a_j^n \leq 4(1 + \|\widehat{c}'^n_{(j-3)k_n+1}\|)^{p-2}\, \Big(\|z_j^n\|^2\, 1_{\{\|z_i^n\| > u'_n/4\}} + \|z_j'^n\|^2\, 1_{\{\|z_i'^n\| > u'_n/4\}}\Big).$$

It easily follows that for all $A > 1$,

$$B_t^{n,\rho} \;\le\; 16\, B_t^{n,\rho,1} + 4A^{p-2}\, B_t^{n,\rho,2} + \frac{2^p}{A}\, B_t^{n,\rho,3}, \tag{3.52}$$

where

$$B_t^{n,\rho,m} = \textstyle\sum_{j\in L'(n,\rho,t)} a(m)_j^n, \qquad a(1)_j^n = (1 + \|\widehat{c}'^n_{(j-3)k_n+1}\|)^{p-2}\, \frac{\|z_j^n\|^3}{u'_n}$$
$$a(2)_j^n = \|z_j'^n\|^2\, 1_{\{\|z_i'^n\| > u'_n/4\}}, \quad a(3)_j^n = \|\widehat{c}'^n_{(j-3)k_n+1}\|^{p-1}\, \|z_j'^n\|^2.$$

On the one hand, (3.20) and Hölder's inequality yield $\mathbb{E}(a(1)_j^n \mid \mathcal{G}^\rho) \le K\Delta_n^{3/4-\varpi'}$ and, since $\varpi' < \frac{1}{4}$, we deduce

$$\mathbb{E}\big(B_t^{n,\rho,1}\big) \;\to\; 0. \tag{3.53}$$

On the other hand, observe that $z_j'^n = \mu(\rho)_j^n$, with the notation (3.28), and as soon as $j \in L'(n,\rho)$, so Lemma 3.4 gives us

$$\lim_{\rho\to 0}\; \limsup_{n\to\infty}\; \mathbb{E}\big(B_t^{n,\rho,2}\big) \;=\; 0. \tag{3.54}$$

Finally, (3.14) shows that $\mathbb{E}(\|z_j'^n\|^q| \mid \mathcal{F}^{n,\rho}_{(j-2)k_n+1}) \le K_{q,\rho}\sqrt{\Delta_n}$ for all $q \ge 2$ and $j \in L'(n,\rho)$, whereas $\widehat{c}'^n_{(j-3)k_n+1}$ is $\mathcal{F}^n_{(j-2)k_n+1}$-measurable, so (3.14), (3.20) and successive conditioning yield $\mathbb{E}(a(3)_j^n| \mid \mathcal{G}^\rho) \le K_{q,\rho}\sqrt{\Delta_n}$. Then, again as for (3.53), one obtains

$$\mathbb{E}\big(B_t^{n,\rho,3}\big) \;\le\; K_\rho\, t. \tag{3.55}$$

At this stage, we gather (3.50)–(3.55) and obtain, by letting first $n \to \infty$, then $\rho \to 0$, then $A \to \infty$, that

$$\lim_{\rho\to 0}\; \limsup_{n\to\infty}\; \mathbb{E}\big(|\mathcal{V}_t^{n,\rho}|\big) \;=\; 0. \tag{3.56}$$

C) The processes $\overline{\mathcal{V}}^{n,\rho}$. With the previous notation S_j^ρ and N_t^ρ, and on the set $\Omega_{n,\rho,t}$, we have

$$\overline{\mathcal{V}}_t^{n,\rho} = \sum_{m=1}^{N_t^\rho}\; \sum_{j=-2}^{2} v^n_{[S_m^\rho/k_n\Delta_n]+j}\,. \tag{3.57}$$

This is a finite sum (bounded in n for each ω). Letting $S = S_m^\rho$ for m and ρ fixed and $w_n = \frac{S}{k_n\Delta_n} - \big[\frac{S}{k_n\Delta_n}\big]$, we know that for any given $j \in \mathbb{Z}$ the variable $\widehat{c}^n_{([S/k_n\Delta_n]+j)k_n+1}$ converge in probability to c_{S-} if $j < 0$ and to c_S if $j > 0$, whereas for $j = 0$ we have $\widehat{c}^n_{[S/k_n\Delta_n]k_n+1} - w_n c_S - (1-w_n)c_S \xrightarrow{\mathbb{P}} 0$. This in turn implies

$$j < 0 \text{ or } j > 2 \;\Rightarrow\; \delta^n_{[S/k_n\Delta_n]+j}\widehat{c} \xrightarrow{\mathbb{P}} 0$$
$$\delta^n_{[S/k_n\Delta_n]}\widehat{c} - (1-w_n)\Delta c_S \xrightarrow{\mathbb{P}} 0, \quad \delta^n_{[S/k_n\Delta_n]+1}\widehat{c} \xrightarrow{\mathbb{P}} \Delta c_S, \quad \delta^n_{[S/k_n\Delta_n]+2}\widehat{c} - w_n\Delta c_S \xrightarrow{\mathbb{P}} 0.$$

By virtue of the definition of v_j^n, and since $u'_n \to 0$ and also since w_n is almost surely in $(0,1)$ and F is continuous and $F(x,0) = 0$, one readily deduces that

$$v^n_{[S/k_n\Delta_n]+j} \;\xrightarrow{\mathbb{P}}\; \begin{cases} F(c_{S-}, \Delta c_S) & \text{if } j = 1 \\ 0 & \text{if } j \ne 1. \end{cases}$$

Coming back to (3.57), we deduce that

$$\overline{\mathcal{V}}_t^{n,\rho} \xrightarrow{\mathbb{P}} \overline{\mathcal{V}}_t^{\rho} := \sum_{m=1}^{N_t^\rho} F(c_{S_m^\rho -}, \Delta c_{S_m^\rho}). \tag{3.58}$$

In view of (2.25), an application of the dominated convergence theorem gives $\overline{\mathcal{V}}_t^{\rho} \to \mathcal{V}(F)_t$. Then (2.26) follows from $\mathcal{V}(F)_t^n = \mathcal{V}_t^{n,\rho} + \overline{\mathcal{V}}_t^{n,\rho}$ and (3.56) and (3.58), and the proof of Theorem 2.6 is complete.

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