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Factorization Method for Electromagnetic Inverse Scattering from Biperiodic Structures

Armin Lechleiter* Dinh-Liem Nguyen†

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Abstract

This paper is concerned with the inverse scattering problem of electromagnetic waves from penetrable biperiodic structures in three dimensions. We study the Factorization method as a tool for reconstructing the periodic media from measured data consisting of scattered electromagnetic waves for incident plane electromagnetic waves. We propose a rigorous analysis for the method. A simple criterion is provided to reconstruct the biperiodic structures. We also provide three-dimensional numerical experiments to indicate the performance of the method.

1 Introduction

We consider inverse scattering of electromagnetic waves from penetrable biperiodic structures in three dimensions. By biperiodic, we mean that the structure is periodic in the, say, x_1 - and x_2 -direction, while it is bounded in the x_3 direction. The inverse problem that we treat in this paper is the shape reconstruction of a biperiodic medium from measured data consisting of scattered electromagnetic waves. We consider plane electromagnetic waves as incident fields. The problem that we study here is motivated by the important applications of periodic structures in optics. Applications include diffractive optical filters and organic light-emitting diodes, and non-destructive testing is an important issue to guarantee the functioning of such devices.

Inverse scattering from periodic structures has been an active field of research in the last years. Uniqueness theorems for determining periodic scattering objects from the knowledge of scattered fields can be found in e.g. [1, 8, 9, 17, 32]. In the general context of acoustic and electromagnetic inverse scattering, qualitative methods has been received much considerable attentions, see Chapter 5 of [20]. Among those methods, the most developed is the linear sampling method which was first introduced in [13] for the scalar case of obstacle inverse scattering. It aims to compute a picture of the shape of the scattering object from measured data. Since the method is relatively rapid and does not need a-priori knowledge, it has attracted much research in recent years. One can find recent developments of the linear sampling method in [10, 11]. The linear sampling method has been extended to inverse scattering involving periodic media, in [15, 16, 33]. However, in spite of the advantages of the method, a full mathematical justification still remains open, see [10]. Some results on the justification of the linear sampling method have been recently obtained in [3, 6].

As an attempt to improve the linear sampling method, the so-called Factorization method has been developed in [18, 21]. The latter method has rigorous justification, keeps the previous advantages and of course is an interesting tool for reconstruction problems in inverse scattering. However, there is only a restricted class of scattering problems to which the Factorization method can be applied, see [20]. Recently this method has been extended to periodic inverse scattering problems. In [4, 5] the authors studied the Factorization method for the imaging problem of impenetrable periodic structures with Dirichlet and impedance boundary

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conditions. The paper [23] considered imaging of penetrable periodic interfaces between two dielectrics in two dimensions.

In the present work we aim to study the Factorization method as a tool for reconstructing three dimensional biperiodic structures for data consisting of scattered electromagnetic waves. More specific, the measured data that we consider here are the coefficients of evanescent and propagating modes of the scattered fields in the radiation condition. Given those coefficients of tangential components of the electromagnetic scattered fields, the inverse problem is then to determine the three-dimensional penetrable biperiodic scatterer. As presented in the rest of the paper, the Factorization method is shown to be an efficient tool to our imaging problem. From a full mathematical justification of the method, a simple criterion for imaging is shown to work accurately in the three-dimensional numerical experiments which, to the best of our knowledge, are the first numerical examples for this method in a biperiodic setting.

Our analysis extends approaches in [4, 20, 22] to Maxwell's equations in a biperiodic setting. We adapt the special plane incident fields introduced in [4] for the periodic scalar case to the vectorial problem, which allows us to suitably factorize the near field operator. Further, a modified version of the method studied in [22] treats the case that the imaginary part of the middle operator in the factorization is just semidefinite. Since this generalization is of some importance for the problem under consideration, we give a complete proof. Finally the necessary properties of the middle operator are obtained by the approach in [20] for obstacle inverse scattering of electromagnetic waves.

The paper is organized as follows: In Section 2 we introduce the direct problem and set up the corresponding inverse problem. Section 3 is dedicated to study the factorization of the near field operator. Section 4 contains an abstract result on range identity theorem with complete proof. We derive the necessary properties of the middle operator in the factorization in Section 5 while a characterization of the biperiodic structure is given in Section 6. Finally, section 7 is devoted to numerical experiments to examine the performance of the method

2 Problem Setting

We consider scattering of time-harmonic electromagnetic waves from a biperiodic structure. The electric field E and the magnetic field H are governed by the time-harmonic Maxwell equations at frequency $\omega > 0$ in \mathbb{R}^3 ,

$$\operatorname{curl} H + i\omega\varepsilon E = \sigma E \quad \text{in } \mathbb{R}^3, \tag{1}$$

$$\operatorname{curl} E - i\omega\mu_0 H = 0 \quad \text{in } \mathbb{R}^3. \tag{2}$$

Here the electric permittivity ε and the conductivity σ are real bounded measurable function which are 2π -periodic in x_1 and x_2 , and μ_0 is the positive constant magnetic permeability. Further, we assume that ε equals $\varepsilon_0 > 0$ and that σ vanishes outside the biperiodic structure. As usual, the problem (1)-(2) has to be completed by a radiation condition that we set up using Fourier series.

Let us denote the relative material parameter by

$$\varepsilon_r := \frac{\varepsilon + i\sigma}{\varepsilon_0}.$$

Note that ε_r equals 1 outside the biperiodic structure. Recall that the magnetic permeability μ_0 is constant which motivates us to work with the divergence-free magnetic field, that is, $\operatorname{div} H = 0$. Hence, introducing the wave number $k = \omega(\varepsilon_0\mu_0)^{1/2}$, and eliminating the electric field E from (1)-(2), we find that

$$\operatorname{curl} (\varepsilon_r^{-1} \operatorname{curl} H) - k^2 H = 0 \quad \text{in } \mathbb{R}^3. \tag{3}$$

Now we define that a function $u : \mathbb{R}^3 \rightarrow \mathbb{C}^3$ is called α -quasiperiodic if, for $\alpha := (\alpha_1, \alpha_2, 0)$ and $(x_1, x_2, x_3)^\top \in$

\mathbb{R}^3 ,

$$\begin{aligned} u(x_1 + 2\pi, x_2, x_3) &= e^{2\pi i \alpha_1} u(x_1, x_2, x_3), \\ u(x_1, x_2 + 2\pi, x_3) &= e^{2\pi i \alpha_2} u(x_1, x_2, x_3). \end{aligned}$$

Assume that the biperiodic structure is illuminated by α -quasiperiodic incident electric and magnetic fields E^i and H^i , respectively, satisfying

$$\operatorname{curl} H^i + i\omega \varepsilon_0 E^i = 0, \quad \operatorname{curl} E^i - i\omega \mu_0 H^i = 0 \quad \text{in } \mathbb{R}^3.$$

Simple examples for such α -quasiperiodic fields are certain plane waves that we introduce below. We wish to reformulate (3) in terms of the scattered field H^s , defined by $H^s := H - H^i$. Straightforward computations show that $\operatorname{curl} \operatorname{curl} H^i - k^2 H^i = 0$, and

$$\operatorname{curl} (\varepsilon_r^{-1} \operatorname{curl} H^s) - k^2 H^s = -\operatorname{curl} (q \operatorname{curl} H^i) \quad \text{in } \mathbb{R}^3, \quad (4)$$

where q is the contrast defined by

$$q := \varepsilon_r^{-1} - 1.$$

Since ε_r is 2π -periodic in x_1 and x_2 , and the right-hand side is α -quasiperiodic, we seek for an α -quasiperiodic solution H^s . Hence the problem is reduced to the domain $(0, 2\pi)^2 \times \mathbb{R}$. We complement this problem by a radiation condition that is set up using Fourier techniques. Since the scattered field H^s is α -quasiperiodic, the function $e^{-i\alpha \cdot x} H^s$ is 2π -periodic in x_1 and x_2 , and can hence be expanded as

$$e^{-i\alpha \cdot x} H^s(x) = \sum_{n \in \mathbb{Z}^2} \hat{H}_n(x_3) e^{i(n_1 x_1 + n_2 x_2)}, \quad x = (x_1, x_2, x_3)^\top \in \mathbb{R}^3. \quad (5)$$

Here the Fourier coefficients $\hat{H}_n(x_3) \in \mathbb{C}^3$ are defined by

$$\hat{H}_n(x_3) = \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} H^s(x_1, x_2, x_3) e^{-i\alpha_n \cdot x} dx_1 dx_2, \quad (6)$$

where

$$\alpha_n := (\alpha_{1,n}, \alpha_{2,n}, 0) = (\alpha_1 + n_1, \alpha_2 + n_2, 0).$$

We define, for $n \in \mathbb{Z}^2$,

$$\beta_n := \begin{cases} \sqrt{k^2 - |\alpha_n|^2}, & k^2 \geq |\alpha_n|^2, \\ i\sqrt{|\alpha_n|^2 - k^2}, & k^2 < |\alpha_n|^2, \end{cases}$$

and for some technical reason we assume in the following that

$$\beta_n \neq 0 \quad \text{for all } n \in \mathbb{Z}^2.$$

Recall that ε_r^{-1} equals one outside the structure. This means $\varepsilon_r^{-1} = 1$ and $q = 0$ for $|x_3| > h$ where $h > \sup\{|x_3| : (x_1, x_2, x_3)^\top \in \operatorname{supp}(q)\}$. Thus it holds that $\operatorname{div} H^s$ vanishes for $|x_3| > h$, and equation (4) becomes $(\Delta + k^2)H^s = 0$ in $\{|x_3| > h\}$. Using separation of variables, and choosing the upward propagating solution, we set up a radiation condition in form of a Rayleigh expansion condition, prescribing that H^s can be written as

$$H^s(x) = \sum_{n \in \mathbb{Z}^2} \hat{H}_n^\pm e^{i(\alpha_n \cdot x + \beta_n |x_3 - h|)} \quad \text{for } x_3 \gtrless \pm h, \quad (7)$$

where $(\hat{H}_n^\pm)_{n \in \mathbb{Z}^2}$ are the Rayleigh sequences given by

$$\hat{H}_n^\pm := \hat{H}_n(\pm h) = \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} H^s(x_1, x_2, \pm h) e^{-i\alpha_n \cdot x} dx_1 dx_2, \quad n \in \mathbb{Z}^2.$$

Note that we require that the series in (7) converges uniformly on compact subsets of $\{|x_3| > h\}$. Further, note that only a finite number of terms in (7) are propagating plane waves which are called propagating modes, the rest are evanescent modes which correspond to exponentially decaying terms.

Denote by \overline{D} the support of the contrast q in one period $\Omega := (0, 2\pi)^2 \times \mathbb{R}$. We make an assumption which is necessary for the subsequent factorization frame work.

Assumption 2.1. *We assume that the support $D \subset \Omega$ is open and bounded with Lipschitz boundary and that there exists a positive constant c such that $\operatorname{Re}(q) \geq c > 0$ and $\operatorname{Im}(q) \leq 0$ almost everywhere in Ω .*

Considering a more general source term on the right hand side of (4), we have the following direct problem: Given $f \in L^2(D)^3$, find $u : \Omega \rightarrow \mathbb{C}^3$ in a suitable function space such that

$$\operatorname{curl}(\varepsilon_r^{-1} \operatorname{curl} u) - k^2 u = -\operatorname{curl}(q/\sqrt{|q|}f) \quad \text{in } \Omega, \quad (8)$$

and u satisfies the Rayleigh expansion condition (7). In the following, a function which satisfies (7) is said to be radiating. It is also seen that if u is a solution of (4) then u solves (8) for the right hand side of $f = \operatorname{curl} H^i/\sqrt{|q|}$.

For a variational formulation of the problem, we define, for any Lipschitz domain \mathcal{O} ,

$$\begin{aligned} H(\operatorname{curl}, \mathcal{O}) &= \{v \in L^2(\mathcal{O})^3 : \operatorname{curl} v \in L^2(\mathcal{O})^3\}, \\ H_{\text{loc}}(\operatorname{curl}, \mathbb{R}^3) &= \{v : \mathbb{R}^3 \rightarrow \mathbb{C}^3 : v|_B \in H(\operatorname{curl}, B) \text{ for all balls } B \subset \mathbb{R}^3\}, \\ H_{\alpha, \text{loc}}(\operatorname{curl}, \Omega) &= \{u \in H_{\text{loc}}(\operatorname{curl}, \Omega) : u = U|_{\Omega} \text{ for some } \alpha\text{-quasiperiodic } U \in H_{\text{loc}}(\operatorname{curl}, \mathbb{R}^3)\}, \end{aligned}$$

and

$$\Omega_h = (0, 2\pi)^2 \times (-h, h) \quad \text{for } h > \sup\{|x_3| : (x_1, x_2, x_3)^\top \in \operatorname{supp}(q)\},$$

with boundaries $\Gamma_{\pm h} := (0, 2\pi)^2 \times \{\pm h\}$. The variational formulation to the problem (8) is to find a radiating solution $u \in H_{\alpha, \text{loc}}(\operatorname{curl}, \Omega)$ such that

$$\int_{\Omega} (\varepsilon_r^{-1} \operatorname{curl} u \cdot \operatorname{curl} \overline{\psi} - k^2 u \cdot \overline{\psi}) dx = - \int_{\Omega} q/\sqrt{|q|}f \cdot \operatorname{curl} \overline{\psi} dx, \quad (9)$$

for all $\psi \in H_{\alpha, \text{loc}}(\operatorname{curl}, \Omega)$ with compact support. Existence and uniqueness of this problem can be obtained for all but possibly a discrete set of wave numbers k , see e.g. [7, 14, 29]. In the sequel we assume that (9) is uniquely solvable for any $f \in L^2(D)^3$ and fixed $k > 0$. Then we define a solution operator $G : L^2(D)^3 \rightarrow \ell^2(\mathbb{Z}^2)^4$ which maps f to the Rayleigh sequences $(\hat{u}_{1,j}^+, \hat{u}_{1,j}^-, \hat{u}_{2,j}^+, \hat{u}_{2,j}^-)_{j \in \mathbb{Z}^2}$ of the first two components of $u \in H_{\alpha, \text{loc}}(\operatorname{curl}, \Omega)$, solution to (9). Note that the Rayleigh sequences $\hat{u}_{(1,2),j}^\pm$ are given by

$$\hat{u}_{(1,2),j}^\pm = \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} u_{(1,2)}(x_1, x_2, \pm h) e^{-i\alpha_j \cdot x} dx_1 dx_2, \quad j \in \mathbb{Z}^2. \quad (10)$$

Now we introduce the notation $\tilde{b} = (b_1, b_2, -b_3)^\top$ for $b = (b_1, b_2, b_3)^\top \in \mathbb{C}^3$. To obtain the data for the factorization method we consider the following α -quasiperiodic plane waves

$$\varphi_j^{(l)\pm} = p_j^{(l)} e^{i(\alpha_j \cdot x + \beta_j x_3)} \pm \tilde{p}_j^{(l)} e^{i(\alpha_j \cdot x - \beta_j x_3)}, \quad l = 1, 2, \quad j \in \mathbb{Z}^2, \quad (11)$$

where $p_j^{(l)} = (p_{1,j}^{(l)}, p_{2,j}^{(l)}, p_{3,j}^{(l)}) \in \mathbb{C}^3 \setminus \{0\}$ are polarizations chosen such that, for all $j \in \mathbb{Z}^2$,

$$i) \quad p_j^{(1)} \times p_j^{(2)} = c_j (\alpha_{1,j}, \alpha_{2,j}, \beta_j)^\top, \quad \text{for } c_j \in \mathbb{C} \setminus \{0\}. \quad (12)$$

$$ii) \quad |p_j^{(1)}| = |p_j^{(2)}| = 1. \quad (13)$$

Together with the assumption that $\beta_j \neq 0$ for all $j \in \mathbb{Z}^2$, such polarizations are linear independent. One possible choice is

$$p_j^{(1)} = (0, \beta_j, -\alpha_{2,j}) / (|\beta_j|^2 + \alpha_{2,j}^2)^{1/2}, \quad p_j^{(2)} = (-\beta_j, 0, \alpha_{1,j}) / (|\beta_j|^2 + \alpha_{1,j}^2)^{1/2}.$$

Note that $\varphi_j^{(l)\pm}$ are hence divergence-free functions for all $j \in \mathbb{Z}^2$, $l = 1, 2$. Due to the linearity of the problem, a linear combination of several incident fields will lead to a corresponding linear combination of the resulting scattered fields. We obtain such linear combination using sequences $(a_j)_{j \in \mathbb{Z}^2} = (a_j^{(1)+}, a_j^{(1)-}, a_j^{(2)+}, a_j^{(2)-})_{j \in \mathbb{Z}^2} \in \ell^2(\mathbb{Z}^2)^4$ and define the corresponding operator by

$$H(a_j) = \sqrt{|q|} \sum_{j \in \mathbb{Z}^2} \frac{1}{\beta_j w_j} \left[a_j^{(1)+} \operatorname{curl} \varphi_j^{(1)+} + a_j^{(2)+} \operatorname{curl} \varphi_j^{(2)+} + a_j^{(1)-} \operatorname{curl} \varphi_j^{(1)-} + a_j^{(2)-} \operatorname{curl} \varphi_j^{(2)-} \right], \quad (14)$$

where

$$w_j := \begin{cases} \mathbf{i}, & k^2 > \alpha_j^2, \\ \exp(-\mathbf{i}\beta_j h), & k^2 < \alpha_j^2. \end{cases}$$

Note that we divide by $\beta_j w_j$ to make later computations easier.

In our inverse problem the data that we measure are the Rayleigh sequences defined in (10). We know that only the propagating modes are measurable far away from the structure. However, it follows from [17] that we need all the modes to be able to uniquely determine the periodic structure. Hence the operator that models measurements from the periodic inhomogeneous medium of scattered fields caused by the incident fields (14) is referred to be the *near field* operator, denoted by N . We define $N : \ell^2(\mathbb{Z}^2)^4 \rightarrow \ell^2(\mathbb{Z}^2)^4$ to map a sequence $(a_j)_{j \in \mathbb{Z}^2}$ to the Rayleigh sequences of the first two components of the scattered field generated by the incident field $H(a_j)$ defined in (14), i.e.

$$[N(a_j)]_n := (\hat{u}_{1,n}^+, \hat{u}_{1,n}^-, \hat{u}_{2,n}^+, \hat{u}_{2,n}^-)_{n \in \mathbb{Z}^2},$$

where $u \in H_{\alpha, \text{loc}}(\operatorname{curl}, \Omega)$ is the radiating solution to (9) for the source $f = H(a_j)$. Then from the definition of the solution operator we have

$$N = GH. \quad (15)$$

The inverse scattering problem is now to reconstruct the support \overline{D} of the contrast $q = \varepsilon_r^{-1} - 1$ when the near field operator N is given. Note that it is not clear yet that N is a bounded linear operator, but we will prove this in the next section.

3 Factorization of the Near Field Operator

We study the inverse problem of the previous section using the factorization method. One of the important steps of the latter method that this section is devoted to is factorizing the near field operator. Before doing that, in the next lemma, we show some properties of the operator $H : \ell^2(\mathbb{Z}^2)^4 \rightarrow L^2(D)^3$ and its adjoint H^* . We need the sequence

$$w_j^* := \begin{cases} \exp(-\mathbf{i}\beta_j h), & k^2 > \alpha_j^2, \\ \mathbf{i}, & k^2 < \alpha_j^2. \end{cases}$$

Lemma 3.1. *For $p_j^{(l)} = (p_{1,j}^{(l)}, p_{2,j}^{(l)}, p_{3,j}^{(l)})$, $j \in \mathbb{Z}^2$, $l = 1, 2$, defined as in (12) and (13), the operator $H : \ell^2(\mathbb{Z}^2)^4 \rightarrow L^2(D)^3$ is compact and injective, and its adjoint $H^* : L^2(D)^3 \rightarrow \ell^2(\mathbb{Z}^2)^4$ satisfies*

$$(H^* f)_j = 8\pi^2 w_j^* \begin{pmatrix} \overline{p_{1,j}^{(1)}}(\hat{u}_{1,j}^+ + \hat{u}_{1,j}^-) + \overline{p_{2,j}^{(1)}}(\hat{u}_{2,j}^+ + \hat{u}_{2,j}^-) \\ \overline{p_{1,j}^{(2)}}(\hat{u}_{1,j}^+ + \hat{u}_{1,j}^-) + \overline{p_{2,j}^{(2)}}(\hat{u}_{2,j}^+ + \hat{u}_{2,j}^-) \\ \overline{p_{1,j}^{(1)}}(\hat{u}_{1,j}^+ - \hat{u}_{1,j}^-) + \overline{p_{2,j}^{(1)}}(\hat{u}_{2,j}^+ - \hat{u}_{2,j}^-) \\ \overline{p_{1,j}^{(2)}}(\hat{u}_{1,j}^+ - \hat{u}_{1,j}^-) + \overline{p_{2,j}^{(2)}}(\hat{u}_{2,j}^+ - \hat{u}_{2,j}^-) \end{pmatrix}^\top, \quad j \in \mathbb{Z}^2, \quad (16)$$

where $(\hat{u}_{1,j}^+, \hat{u}_{1,j}^-, \hat{u}_{2,j}^+, \hat{u}_{2,j}^-)_{j \in \mathbb{Z}^2}$ are the Rayleigh sequences of the first two components of $u \in H_{\alpha, \text{loc}}(\text{curl}, \Omega)$, the radiating variational solution to $\text{curl}^2 u - k^2 u = \text{curl}(\sqrt{|q|}f)$ in Ω .

Proof. For $l = 1, 2$ and $j \in \mathbb{Z}^2$, we have

$$\begin{aligned} \int_D H(a_j) \bar{f} \, dx &= \sum_{j \in \mathbb{Z}^2} \left[\sum_{l=1,2} \frac{a_j^{(l)+}}{\beta_j w_j} \int_D \sqrt{|q|} \bar{f} \cdot \text{curl} \varphi_j^{(l)+} \, dx + \sum_{l=1,2} \frac{a_j^{(l)-}}{\beta_j w_j} \int_D \sqrt{|q|} \bar{f} \cdot \text{curl} \varphi_j^{(l)-} \, dx \right] \\ &= \left\langle (a_j), \left(\int_D \sqrt{|q|} f \cdot \overline{\left(\frac{\text{curl} \varphi_j^{(l)+}}{\beta_j w_j} \right)} \, dx, \int_D \sqrt{|q|} f \cdot \overline{\left(\frac{\text{curl} \varphi_j^{(l)-}}{\beta_j w_j} \right)} \, dx \right) \right\rangle_{\ell^2(\mathbb{Z}^2)^4}. \end{aligned}$$

Note that the equation $\text{curl}^2 u - k^2 u = \text{curl}(\sqrt{|q|}f)$ in Ω with Rayleigh expansion condition is uniquely solvable for all wave number $k > 0$. The Fredholm property can be obtained as in [7, 14, 29], and using integral representation formulas from Theorem 3.1 in [28] one shows the uniqueness. Now we define $v_j^{(l)\pm} = \varphi_j^{(l)\pm} / (\beta_j w_j)$ and consider a smooth function $\phi \in C^\infty(\mathbb{R})$ such that $\phi = 1$ in $(-h, h)$, $\phi = 0$ in $\mathbb{R} \setminus \overline{(-2h, 2h)}$. Then $\phi v_j^{(l)\pm}$ belongs to $H_\alpha(\text{curl}, \Omega)$ with compact support in $\{|x_3| < 2h\}$. Assume that $u \in H_{\alpha, \text{loc}}(\text{curl}, \Omega)$ is the variational radiating solution to $\text{curl}^2 u - k^2 u = -\text{curl}(\sqrt{|q|}f)$ in Ω . We have

$$\begin{aligned} \int_D \sqrt{|q|} f \cdot \text{curl} \overline{v_j^{(l)\pm}} \, dx &= \int_{\Omega_h} (\text{curl} u \cdot \text{curl} \overline{v_j^{(l)\pm}} - k^2 u \cdot \overline{v_j^{(l)\pm}}) \, dx \\ &\quad + \int_{\Omega_{2h} \setminus \Omega_h} (\text{curl} u \cdot \text{curl}(\overline{\phi v_j^{(l)\pm}}) - k^2 u \cdot \overline{\phi v_j^{(l)\pm}}) \, dx. \end{aligned}$$

Now using Green's theorems and exploiting the fact that $v_j^{(l)\pm}$ and u are divergence-free solutions to the Hemholtz equation in \mathbb{R}^3 and $\Omega \setminus \Omega_h$, respectively, we obtain that

$$\begin{aligned} \int_D \sqrt{|q|} f \cdot \text{curl} \overline{v_j^{(l)\pm}} \, dx &= \int_{\Gamma_h} (e_3 \times \text{curl} u \cdot \overline{v_j^{(l)\pm}} - e_3 \times \text{curl} \overline{v_j^{(l)\pm}} \cdot u) \, ds \\ &\quad + \int_{\Gamma_{-h}} (e_3 \times \text{curl} \overline{v_j^{(l)\pm}} \cdot u - e_3 \times \text{curl} u \cdot \overline{v_j^{(l)\pm}}) \, ds \\ &= \left(\int_{\Gamma_h} - \int_{\Gamma_{-h}} \right) \left[\partial_3 \overline{v_{2,j}^{(l)\pm}} u_2 - \partial_3 u_2 \overline{v_{2,j}^{(l)\pm}} + \partial_3 \overline{v_{1,j}^{(l)\pm}} u_1 - \partial_3 u_1 \overline{v_{1,j}^{(l)\pm}} \right] \, ds. \end{aligned} \tag{17}$$

Note that we have

$$\begin{aligned} \overline{v_{1,j}^{(l)+}} &= \overline{\left(\frac{p_{1,j}^{(l)}}{\beta_j w_j} \right)} (e^{i\beta_j x_3} + e^{-i\beta_j x_3}) e^{-i\alpha_j \cdot x}, \\ \partial_3 \overline{v_{1,j}^{(l)+}} &= i\beta_j \overline{\left(\frac{p_{1,j}^{(l)}}{\beta_j w_j} \right)} (e^{i\beta_j x_3} - e^{-i\beta_j x_3}) e^{-i\alpha_j \cdot x}. \end{aligned}$$

Then by straightforward computation we obtain

$$\int_{\Gamma_h} (\partial_3 \overline{v_{1,j}^{(l)+}} u_1 - \partial_3 u_1 \overline{v_{1,j}^{(l)+}}) \, ds = \sum_{n \in \mathbb{Z}^2} \hat{u}_{1,n}^+ \int_{\Gamma_h} e^{i\alpha_n \cdot x} [\partial_3 \overline{v_{1,j}^{(l)+}} - i\beta_n \overline{v_{1,j}^{(l)+}}] \, ds = 8\pi^2 w_j^* \overline{p_{1,j}^{(l)}} \hat{u}_{1,j}^+.$$

Similarly we also have

$$\begin{aligned} \int_{\Gamma_h} (\partial_3 \overline{v_{2,j}^{(l)+}} u_2 - \partial_3 u_2 \overline{v_{2,j}^{(l)+}}) \, ds &= 8\pi^2 w_j^* \overline{p_{2,j}^{(l)}} \hat{u}_{2,j}^+, \\ \int_{\Gamma_{-h}} (\partial_3 \overline{v_{2,j}^{(l)+}} u_2 - \partial_3 u_2 \overline{v_{2,j}^{(l)+}} + \partial_3 \overline{v_{1,j}^{(l)+}} u_1 - \partial_3 u_1 \overline{v_{1,j}^{(l)+}}) \, ds &= -8\pi^2 w_j^* (\overline{p_{1,j}^{(l)}} \hat{u}_{1,j}^- + \overline{p_{2,j}^{(l)}} \hat{u}_{2,j}^-). \end{aligned}$$

Now substituting the last two equations into (17) we derive

$$\int_D \sqrt{|q|} f \cdot \operatorname{curl} \overline{v_j^{(l)+}} \, dx = 8\pi^2 w_j^* (\overline{p_{1,j}^{(l)}} \hat{u}_{1,j}^- + \overline{p_{2,j}^{(l)}} \hat{u}_{2,j}^- + \overline{p_{1,j}^{(l)}} \hat{u}_{1,j}^+ + \overline{p_{2,j}^{(l)}} \hat{u}_{2,j}^+).$$

Similarly we have

$$\int_D \sqrt{|q|} f \cdot \operatorname{curl} \overline{v_j^{(l)-}} \, dx = 8\pi^2 w_j^* (-\overline{p_{1,j}^{(l)}} \hat{u}_{1,j}^- - \overline{p_{2,j}^{(l)}} \hat{u}_{2,j}^- + \overline{p_{1,j}^{(l)}} \hat{u}_{1,j}^+ + \overline{p_{2,j}^{(l)}} \hat{u}_{2,j}^+)$$

which shows that H^* satisfies (16). Next we show the compactness of H^* . This relies on the operator $W : \ell^2(\mathbb{Z}^2)^4 \rightarrow \ell^2(\mathbb{Z}^2)^4$ defined by

$$W((a_l)_{l \in \mathbb{Z}^2}) = -8\pi^2 w_j^* \begin{pmatrix} \overline{p_{1,j}^{(1)}}(a_j^{(1)+} + a_j^{(1)-}) + \overline{p_{2,j}^{(1)}}(a_j^{(2)+} + a_j^{(2)-}) \\ \overline{p_{1,j}^{(2)}}(a_j^{(1)+} + a_j^{(1)-}) + \overline{p_{2,j}^{(2)}}(a_j^{(2)+} + a_j^{(2)-}) \\ \overline{p_{1,j}^{(1)}}(a_j^{(1)+} - a_j^{(1)-}) + \overline{p_{2,j}^{(1)}}(a_j^{(2)+} - a_j^{(2)-}) \\ \overline{p_{1,j}^{(2)}}(a_j^{(1)+} - a_j^{(1)-}) + \overline{p_{2,j}^{(2)}}(a_j^{(2)+} - a_j^{(2)-}) \end{pmatrix}^\top, \quad j \in \mathbb{Z}^2. \quad (18)$$

Since $(w_j^*)_{j \in \mathbb{Z}^2}$ is a bounded sequence, and since the sequences $(\overline{p_j^{(l)}})_{j \in \mathbb{Z}^2}$ are bounded for $l = 1, 2$ due to (13), the operator W is bounded. Now we define the operator

$$Q : L^2(D)^3 \rightarrow \ell^2(\mathbb{Z}^2)^4 \quad (19)$$

which maps f to $(\hat{u}_{1,j}^+, \hat{u}_{1,j}^-, \hat{u}_{2,j}^+, \hat{u}_{2,j}^-)$ where u is the radiating variational solution to $\operatorname{curl}^2 u - k^2 u = \operatorname{curl}(\sqrt{|q|}f)$ in Ω . Then we have

$$H^* = -WQ. \quad (20)$$

The following trace spaces are necessary for our proof: We define

$$Y(\Gamma_{\pm h}) = \{f \in H^{-1/2}(\Gamma_{\pm h})^3 \mid \text{there exists } u \in H_\alpha(\operatorname{curl}, \Omega_h) \text{ with } \pm e_3^\top \times u|_{\Gamma_{\pm h}} = f\}$$

with norm

$$\|f\|_{Y(\Gamma_{\pm h})} = \inf_{u \in H_\alpha(\operatorname{curl}, \Omega_h), \pm e_3^\top \times u|_{\Gamma_{\pm h}} = f} \|u\|_{H_\alpha(\operatorname{curl}, \Omega_h)}.$$

The trace spaces $Y(\Gamma_{\pm h})$ are Banach spaces with this norm, see [26]. In the latter reference one also shows that the operation $u \mapsto ((0, 0, \pm 1) \times u|_{\Gamma_{\pm h}}) \times (0, 0, \pm 1)$ is bounded from $H_\alpha(\operatorname{curl}, \Omega_h)$ into $Y'(\Gamma_{\pm h})$ which is the dual space of $Y(\Gamma_{\pm h})$.

Now we know that the operation which maps $f \in L^2(D)^3$ into $u \in H_{\alpha, \text{loc}}(\operatorname{curl}, \Omega)$, radiating variational solution to $\operatorname{curl}^2 u - k^2 u = \operatorname{curl}(\sqrt{|q|}f)$, is bounded. Note that $((0, 0, \pm 1) \times u|_{\Gamma_{\pm h}}) \times (0, 0, \pm 1) = (u_1, u_2, 0)$. We obtain that the operations $f \mapsto (u_1, u_2, 0)|_{\Gamma_h}$ and $(u_1, u_2, 0)|_{\Gamma_h} \mapsto (\hat{u}_{1,j}^+, \hat{u}_{2,j}^+)$ are bounded from $L^2(D)^3$ into $Y'(\Gamma_h)$ and from $Y'(\Gamma_h)$ into $\ell^2(\mathbb{Z}^2)^2$, respectively. Similarly for Γ_{-h} we obtain that $f \mapsto (\hat{u}_{1,j}^-, \hat{u}_{2,j}^-)$ are bounded from $L^2(D)^3$ into $\ell^2(\mathbb{Z}^2)^2$. Together with the boundedness of the sequence $(w_j^*)_{j \in \mathbb{Z}^2}$, Q is a bounded operator. We know that in a neighborhood of $\Gamma_{\pm h}$ u solves the Helmholtz equation. Hence elliptic regularity results [25] imply that u is H^2 -regular in a neighborhood of $\Gamma_{\pm h}$, thus, $f \mapsto (u_1, u_2, 0)|_{\Gamma_{\pm h}}$ is compact operations from $L^2(D)^3$ into $Y'(\Gamma_{\pm h})$. Then Q is a compact operator and H^* is compact. Therefore H is compact as well.

To obtain the injectivity of H , we prove that H^* has dense range. It is sufficient to prove that W has dense range and all sequences $((\delta_{jl})_{l \in \mathbb{Z}^2}, 0, 0, 0)$, $(0, (\delta_{jl})_{l \in \mathbb{Z}^2}, 0, 0)$, $(0, 0, (\delta_{jl})_{l \in \mathbb{Z}^2}, 0)$ and $(0, 0, 0, (\delta_{jl})_{l \in \mathbb{Z}^2})$ belong to the range of Q (by definition, the Kronecker symbol δ_{jl} equals one for $j = l$ and zero otherwise).

The operator W has dense range due to the fact that

$$\det \begin{pmatrix} \overline{p_{1,j}^{(1)}} & \overline{p_{1,j}^{(1)}} & \overline{p_{2,j}^{(1)}} & \overline{p_{2,j}^{(1)}} \\ \overline{p_{1,j}^{(2)}} & \overline{p_{1,j}^{(2)}} & \overline{p_{2,j}^{(2)}} & \overline{p_{2,j}^{(2)}} \\ \overline{p_{1,j}^{(1)}} & \overline{p_{1,j}^{(1)}} & \overline{p_{2,j}^{(1)}} & \overline{p_{2,j}^{(1)}} \\ \overline{p_{1,j}^{(2)}} & \overline{p_{1,j}^{(2)}} & \overline{p_{2,j}^{(2)}} & \overline{p_{2,j}^{(2)}} \end{pmatrix} = -4 \left(\overline{p_{1,j}^{(2)} p_{2,j}^{(1)}} - \overline{p_{2,j}^{(2)} p_{1,j}^{(1)}} \right)^2 = 4(\overline{c_j \beta_j})^2 \neq 0,$$

due to the property (12) of the polarizations. Now we show that $((\delta_{jl})_{l \in \mathbb{Z}^2}, 0, 0, 0)$ belongs to the range of Q , and the other cases can be done in a similar way. We choose a cut-off function $\chi_{1,j} \in C^\infty(\mathbb{R})$ such that $\chi_{1,j}(t) = 0$ for $t < 0$ and $\chi(t) = 1$ for $t > h/2$. Then $(x_1, x_2, x_3) \mapsto \chi_{1,j}(x_3) \exp(i(\alpha_j \cdot x + \beta_j(x_3 - h)))$ has Rayleigh sequence $((\delta_{jl})_{l \in \mathbb{Z}^2}, 0)$. For all $j \in \mathbb{Z}^2$, we define

$$\varphi_j(x) = (\chi_{1,j}(x_3), 0, \chi_{3,j}(x_3))^\top \exp(i(\alpha_j \cdot x + \beta_j(x_3 - h))),$$

where

$$\chi_{3,j}(x_3) = -i\alpha_{1,j} e^{-i\beta_j x_3} \int_0^{x_3} e^{i\beta_j t} \chi_{1,j}(t) dt.$$

Then $\operatorname{div} \varphi_j = 0$ in Ω and the Rayleigh sequences of the first two components of φ_j are $((\delta_{jl})_{l \in \mathbb{Z}^2}, 0, 0, 0)$. Next we show that there exists $f_j \in L^2(D)^3$ such that $\operatorname{curl}^2 \varphi_j - k^2 \varphi_j = \operatorname{curl}(\sqrt{|q|} f_j)$ in Ω holds in the variational sense. Set

$$g_j(x) := \operatorname{curl}^2 \varphi_j(x) - k^2 \varphi_j(x), \quad x \in \Omega,$$

then we have $\operatorname{div}(g_j) = 0$ in Ω which also implies that

$$\int_{\partial\Omega_h} g_j \cdot \nu ds = 0.$$

Therefore, due to Theorem 3.38 in [26], there exists $\psi_j \in H^1(\Omega_h)^3$ such that

$$g_j = \operatorname{curl} \psi_j \quad \text{in } \Omega_h.$$

Define $f_j = \sqrt{|q|}^{-1} \psi_j$, then $f_j \in L^2(D)^3$ and we have, in the weak sense,

$$\operatorname{curl}^2 \varphi_j - k^2 \varphi_j = \operatorname{curl}(\sqrt{|q|} f_j) \quad \text{in } \Omega_h.$$

Together with $\operatorname{curl}^2 \varphi_j - k^2 \varphi_j = 0$ in $\Omega \setminus \Omega_h$, we complete the proof. \square

Now we show a factorization of the near field operator N in the following theorem. To this end, we define the sign of q by

$$\operatorname{sign}(q) := \frac{q}{|q|}.$$

Theorem 3.2. *Assume that q satisfies the Assumption 2.1. The operator W is defined as in (18). Let $T : L^2(D)^3 \rightarrow L^2(D)^3$ be defined by $Tf = \operatorname{sign}(q)(f + \sqrt{|q|} \operatorname{curl} v)$, where $v \in H_{\alpha, \operatorname{loc}}(\operatorname{curl}, \Omega)$ is the radiating solution to (9). Then the near field operator satisfies*

$$WN = H^*TH.$$

Proof. We recall the operator Q in (19) that maps $f \in L^2(D)^3$ to the Rayleigh sequences $(\hat{u}_{1,j}^+, \hat{u}_{1,j}^-, \hat{u}_{2,j}^+, \hat{u}_{2,j}^-)$ where u is the radiating variational solution to $\operatorname{curl}^2 u - k^2 u = \operatorname{curl}(\sqrt{|q|} f)$ in Ω . By definition of the solution operator G we have $Gf = (\hat{u}_{1,j}^+, \hat{u}_{1,j}^-, \hat{u}_{2,j}^+, \hat{u}_{2,j}^-)$ where $u \in H_{\alpha, \operatorname{loc}}(\operatorname{curl}, \Omega)$ is a radiating weak solution to $\operatorname{curl}(\varepsilon_r^{-1} \operatorname{curl} u) - k^2 u = -\operatorname{curl}(q/\sqrt{|q|} f)$. This means that $\operatorname{curl}^2 u - k^2 u = -\operatorname{curl}(\sqrt{|q|} \operatorname{sign}(q)(f + \sqrt{|q|} \operatorname{curl} v))$, thus, $Gf = -(QT)f$. Now due to the fact that $N = GH$ we have

$$WN = WGH = -WQTH.$$

Additionally we know from (20) that $H^* = -WQ$ which completes the proof. \square

4 The Range Identity Theorem

This section presents an abstract result on range identities which is necessary to characterize the support D of the contrast q . For the convenience of the reader, we give a rather complete proof, see also in [20, 22]. First, we introduce real and imaginary part of a bounded linear operator. Let $X \subset U \subset X^*$ be a Gelfand triple, that is, U is a Hilbert space, X is a reflexive Banach space with dual X^* for the inner product of U , and the embeddings are injective and dense. Then the real and imaginary part of a bounded operator $T : X^* \rightarrow X$ are defined in accordance with the corresponding definition for complex numbers,

$$\operatorname{Re}(T) := \frac{1}{2}(T + T^*), \quad \operatorname{Im}(T) := \frac{1}{2i}(T - T^*).$$

Theorem 4.1. *Let $X \subset U \subset X^*$ be a Gelfand triple with Hilbert space U and reflexive Banach space X . Furthermore, let V be a second Hilbert space and $F : V \rightarrow V$, $H : V \rightarrow X$ and $T : X \rightarrow X^*$ be linear and bounded operators with*

$$F = H^*TH$$

We make the following assumptions:

a) H is compact and injective.

b) There exists $t \in [0, 2\pi]$ such that $\operatorname{Re}(e^{it}T)$ has the form $\operatorname{Re}(e^{it}T) = T_0 + T_1$ with some positive definite selfadjoint operator T_0 and some compact operator $T_1 : X \rightarrow X^*$.

c) $\operatorname{Im} T$ is non positive on X , i.e., $\langle \operatorname{Im} T\phi, \phi \rangle \leq 0$ for all $\phi \in X$.

Moreover, we assume that one of the two following conditions is fulfilled

d) T is injective and t from b) does not equal $\pi/2$ or $3\pi/2$.

e) $\operatorname{Im} T$ is negative on the (finite dimensional) null space of $\operatorname{Re}(e^{it}T)$, i.e., for all $\phi \neq 0$ such that $\operatorname{Re}(e^{it}T)\phi = 0$ it holds $\langle \operatorname{Im} T\phi, \phi \rangle < 0$.

Then the operator $F_{\sharp} := |\operatorname{Re}(e^{it}F)| - \operatorname{Im} F$ is positive definite and the ranges of $H^* : X^* \rightarrow V$ and $F_{\sharp}^{\frac{1}{2}} : V \rightarrow V$ coincide.

Proof. We know that from Theorem 2.15 in [19] it is sufficient to assume that $X = U$ is a Hilbert space and that H has dense range in U . The factorization of F implies that $\operatorname{Re}(e^{it}F) = H^*\operatorname{Re}(e^{it}T)H$ is compact and selfadjoint. By the spectral theorem for such operators, there exists a complete orthonormal eigensystem $(\lambda_j, \psi_j)_{j \in \mathbb{N}}$ of $\operatorname{Re}(e^{it}F)$. In consequence, the spaces

$$V^+ = \operatorname{span}\{\psi_j : \lambda_j > 0\} \quad \text{and} \quad V^- = \operatorname{span}\{\psi_j : \lambda_j \leq 0\}$$

are invariant under $\operatorname{Re}(e^{it}F)$ and satisfy $V = V^+ \oplus V^-$. We set $U^- = HV^-$.

In the next step we show that U^- is finite dimensional. The operator $T_1 = \operatorname{Re}(e^{it}T) - T_0$ is a selfadjoint and compact operator, we denote by $(\mu_j, \phi_j)_{j \in \mathbb{N}}$ an eigensystem of T_1 . By assumption of T_0 , there exists $\alpha > 0$ such that $\langle T_0\varphi, \varphi \rangle \geq \alpha\|\varphi\|^2$ for all $\varphi \in U$. We set $W^+ = \operatorname{span}\{\phi_j : \mu_j > -\alpha\}$, $W^- = \operatorname{span}\{\phi_j : \mu_j \leq -\alpha\}$ and note that W^- is finite dimensional since $\mu_j \rightarrow 0$. Let now $\phi = H\psi \in U^-$ with (unique) decomposition $\phi = \phi^+ + \phi^-$, $\phi^{\pm} \in W^{\pm}$. Since $\psi \in V^-$,

$$\begin{aligned} 0 &\geq \langle \operatorname{Re}(e^{it}F)\psi, \psi \rangle = \langle \operatorname{Re}(e^{it}T)H\psi, H\psi \rangle = \langle \operatorname{Re}(e^{it}T)(\phi^+ + \phi^-), \phi^+ + \phi^- \rangle \\ &= \langle \operatorname{Re}(e^{it}T)\phi^+, \phi^+ \rangle + \langle \operatorname{Re}(e^{it}T)\phi^-, \phi^- \rangle \geq c\|\phi^+\|^2 - \|\operatorname{Re}(e^{it}T)\|\|\phi^-\|^2, \end{aligned}$$

thus, $\|\phi\|^2 = \|\phi^+\|^2 + \|\phi^-\|^2 \leq C\|\phi^-\|^2$. This shows that the mapping $\phi \mapsto \phi^-$ is boundedly invertible from U^- into W^- . Consequently, U^- is finite dimensional.

Denseness of the range of H implies that the sum $\overline{HV^+} + U^-$ is dense in U . Since U^- is a finite dimensional and therefore complemented subspace, we can choose a closed subspace U^+ of $\overline{HV^+}$ such that the (non-orthogonal) sum $U = U^+ \oplus U^-$ is direct. Let moreover $U^0 := \overline{HV^+} \cap U^-$ be the intersection of $\overline{HV^+}$ and U^- , we will show that U^0 is contained in the kernel of $\operatorname{Re}(e^{it}T)$. We denote $P_{U^{\pm}} : U \rightarrow U^{\pm}$ the

canonical projections, that is, every $\phi \in U$ has the unique decomposition $\phi = P_{U^+}\phi + P_{U^-}\phi$. Both operators P_{U^\pm} are bounded and $P_{U^+} - P_{U^-}$ is an isomorphism, since

$$(P_{U^+} - P_{U^-})^2 = P_{U^+}^2 + P_{U^-}^2 - P_{U^+}P_{U^-} - P_{U^-}P_{U^+} = P_{U^+} + P_{U^-} = \text{Id}.$$

From the factorization $\text{Re}(e^{it}F) = H^*\text{Re}(e^{it}T)H$ and the definition of U^\pm we obtain that $H^*\text{Re}(e^{it}T)(U^-) = \text{Re}(e^{it}F)(V^-) \subset V^-$. Note also that, by definition we have $U^+ \subset \overline{HV^+}$. In consequence, for $\phi^- \in U^-$ and $\psi^+ \in V^+$ we have

$$0 = \langle H^*(\text{Re}(e^{it}T))\phi^-, \psi^+ \rangle = \langle \text{Re}(e^{it}T)\phi^-, H\psi^+ \rangle = \langle \phi^-, (\text{Re}(e^{it}T))H\psi^+ \rangle. \quad (21)$$

We conclude that $\text{Re}(e^{it}T)U^- \subset (HV^+)^\perp = (U^+ \oplus U^0)^\perp \subset (U^+)^\perp$ and, $\text{Re}(e^{it}T)U^+ \subset \text{Re}(e^{it}T)\overline{HV^+} \subset (U^-)^\perp$. Indeed, for $\phi^+ \in \overline{HV^+}$ there is a sequence $\psi_n^+ \in V^+$ such that $H\psi_n^+ \rightarrow \phi^+$ and $\text{Re}(e^{it}T)H\psi_n^+ \subset (U^-)^\perp$ by (21), thus, $\text{Re}(e^{it}T)\phi^+ \subset (U^-)^\perp$. For $\phi^0 \in \overline{HV^+} \cap U^-$, these mapping properties of $\text{Re}(e^{it}T)$ imply that $\text{Re}(e^{it}T)\phi^0$ is orthogonal both to U^- and U^+ . Therefore $\text{Re}(e^{it}T)\phi^0 = 0$ and we conclude that $U^0 = \overline{HV^+} \cap U^-$ is contained in the kernel of $\text{Re}(e^{it}T)$. This inclusion allows to show a factorization of F_\sharp in the next step.

Let $\psi \in V$ and ψ^\pm be its orthogonal projection on V^\pm . Then

$$\begin{aligned} |\text{Re}(e^{it}F)|\psi &= H^*\text{Re}(e^{it}T)H(\psi^+ - \psi^-) \\ &= H^*\text{Re}(e^{it}T)(P_{U^+}H\psi^+ + P_{U^-}H\psi^+ - P_{U^+}H\psi^- - P_{U^-}H\psi^-) \\ &= H^*\text{Re}(e^{it}T)(P_{U^+}H\psi + 2 \underbrace{P_{U^-}H\psi^+}_{\in U^0 \subset \ker(\text{Re}(e^{it}T))} - P_{U^+}H\psi) \\ &= H^*\text{Re}(e^{it}T)(P_{U^+} - P_{U^-})H\psi \end{aligned}$$

This factorization of $|\text{Re}(e^{it}F)|$ yields a factorization of F_\sharp ,

$$F_\sharp = |\text{Re}(e^{it}F)| - \text{Im}F = H^*(\text{Re}(e^{it}T)(P_{U^+} - P_{U^-}) - \text{Im}T)H = H^*T_\sharp H,$$

where $T_\sharp = \text{Re}(e^{it}T)(P_{U^+} - P_{U^-}) - \text{Im}T$. Due to the fact that

$$\langle \text{Re}(e^{it}T)(P_{U^+} - P_{U^-})H\phi, H\phi \rangle = \langle |\text{Re}(e^{it}F)|\phi, \phi \rangle \geq 0$$

for all $\phi \in V$ and denseness of the range of H in U we conclude that $\text{Re}(e^{it}T)(P_{U^+} - P_{U^-})$ is nonnegative on U . Since T_\sharp is therefore a nonnegative operator, we can apply the inequality [19, Estimate (4.5)] for bounded nonnegative operators,

$$\langle T_\sharp\psi, \psi \rangle \geq \frac{1}{\|T_\sharp\|} \|T_\sharp\psi\|^2, \quad \psi \in U \quad (22)$$

Now, we show that assumption d) implies assumption e). Under the assumption d), let ϕ belong to the null space of $\text{Re}(e^{it}T)$ and suppose that $\langle \text{Im}T\phi, \phi \rangle = 0$. We need to show that this implies that $\phi = 0$. By definition of the real part of an operator,

$$e^{it}T\phi + e^{-it}T^*\phi = 0 \quad (23)$$

Furthermore, $-\text{Im}T$ is a bounded nonnegative operator so the application of (22) to $-\text{Im}T$ yields

$$0 = \langle -\text{Im}T\phi, \phi \rangle \geq \frac{1}{\|\text{Im}T\|} \|\text{Im}T\phi\|^2, \quad \phi \in U,$$

hence $\|\text{Im}T\phi\| = 0$ and $\text{Im}\phi = 0$. By definition of the imaginary part, this is to say that $T\phi - T^*\phi = 0$. Combine this equation with (23) yields that $(1 + e^{i2t})T\phi = 0$. Since $t \in [0, 2\pi] \setminus \{\frac{\pi}{2}, \frac{3\pi}{2}\}$, this implies $T\phi = 0$ and $\phi = 0$ by assumption d). We have hence proven that $\langle \text{Im}T\phi, \phi \rangle < 0$ for all $0 \neq \phi \in \ker(\text{Re}(e^{it}T))$. This is precisely assumption e) which is considered next.

Assuming e), we will show that T_{\sharp} is injective. Suppose that $T_{\sharp}\phi = 0$, then we have $\langle \operatorname{Re}(e^{it}T)(P_{U^+} - P_{U^-})\phi, \phi \rangle - \langle \operatorname{Im}T\phi, \phi \rangle = 0$. Both terms on the left are nonnegative so we have

$$\begin{cases} \langle \operatorname{Re}(e^{it}T)(P_{U^+} - P_{U^-})\phi, \phi \rangle = 0 \\ \langle \operatorname{Im}T\phi, \phi \rangle = 0 \end{cases} \quad (24)$$

From this and application of (22) to $\operatorname{Re}(e^{it}T)(P_{U^+} - P_{U^-})$ yield $\operatorname{Re}(e^{it}T)(P_{U^+} - P_{U^-})\phi = 0$. Moreover, due to the selfadjointness we obtain

$$\operatorname{Re}(e^{it}T)(P_{U^+} - P_{U^-}) = (P_{U^+} - P_{U^-})^* \operatorname{Re}(e^{it}T)$$

and since $P_{U^+} - P_{U^-}$ is an isomorphism so is $(P_{U^+} - P_{U^-})^*$. Consequently, $\operatorname{Re}(e^{it}T)\phi = 0$. Assumption e) now implies that $\langle \operatorname{Im}T\phi, \phi \rangle < 0$ if $\phi \neq 0$. However, we showed, in (24), that $\langle -\operatorname{Im}T\phi, \phi \rangle = 0$, that is, $\phi = 0$ and therefore T_{\sharp} is injective.

Hence, by assumption d) or e), T_{\sharp} is an injective Fredholm operator on index 0 (Fredholmness is due to assumption b)) and hence boundedly invertible. By (22) we obtain

$$\langle T_{\sharp}\psi, \psi \rangle \geq \frac{1}{\|T_{\sharp}\|} \|T_{\sharp}\psi\|^2 \geq C\|\psi\|^2 \quad \text{for all } \psi \in U$$

Now, as T_{\sharp} has been show to be positive definite, the square root $T_{\sharp}^{1/2}$ of T_{\sharp} is also positive definite on U , see, e.g., [27], hence the inverse $T_{\sharp}^{-1/2}$ is bounded and we can write

$$F_{\sharp} = F_{\sharp}^{1/2}(F_{\sharp}^{1/2})^* = H^*T_{\sharp}H = (H^*T_{\sharp}^{1/2})(H^*T_{\sharp}^{1/2})^*$$

However, if two positive operators agree, then the ranges of their square root agree, as the following well known lemma shows.

Lemma 4.2. (Lemma 2.4 in [19]). *Let V , U_1 and U_2 be Hilbert spaces and $A_j : U_j \rightarrow V$, $j = 1, 2$, bounded and injective such that $A_1A_1^* = A_2A_2^*$. Then the ranges of A_1 and A_2 coincide and $A_1^{-1}A_2$ is an isomorphism from U_2 onto U_1 .*

Setting $A_1 = F_{\sharp}^{1/2}$ and $A_2 = H^*T_{\sharp}^{1/2}$, the last lemma states that the ranges of $F_{\sharp}^{1/2}$ and $H^*T_{\sharp}^{1/2}$ agree and that $F_{\sharp}^{-1/2}H^*T_{\sharp}^{1/2}$ is an isomorphism from U to V . Since $T_{\sharp}^{1/2}$ is an isomorphism on U , we conclude that the range of $H^*T_{\sharp}^{1/2}$ equals the range of H^* and that $F_{\sharp}^{-1/2}H^* : U \rightarrow V$ is bounded with bounded inverse. \square

5 Study of the Middle Operator

In this section we analyze the middle operator T in the factorization of Theorem 3.2 and derive its necessary properties for the application of the Theorem 4.1. This is seen in the following lemma.

Lemma 5.1. *Suppose that the contrast q satisfies the Assumption 2.1 and that the direct scattering problem (9) is uniquely solvable for any $f \in L^2(D)^3$. Let $T : L^2(D)^3 \rightarrow L^2(D)^3$ be the operator defined as in Theorem 3.2, i.e.*

$$Tf = \operatorname{sign}(q)(f + \sqrt{|q|} \operatorname{curl} v),$$

where $v \in H_{\alpha, \text{loc}}(\operatorname{curl}, \Omega)$ is the radiating variational solution to

$$\operatorname{curl}(\varepsilon_r^{-1} \operatorname{curl} u) - k^2 u = -\operatorname{curl}(q/\sqrt{|q|}f). \quad (25)$$

Then we have

(a) T is injective and $\langle \operatorname{Im}Tf, f \rangle \leq 0$ for all $f \in L^2(D)^3$.

(b) Define the operator $T_0 : L^2(D)^3 \rightarrow L^2(D)^3$ by $T_0 f = \text{sign}(q)(f + \sqrt{|q|} \text{curl } \tilde{v})$ where $\tilde{v} \in H_{\alpha, \text{loc}}(\text{curl}, \Omega)$ solves (25) for $k = i$, $f \in L^2(D)^3$, in the variational sense. Then we have that $T - T_0$ is compact in $L^2(D)^3$.

(c) For T_0 defined as in (b), if $\text{Re}(q) > 0$ on $L^2(D)^3$ then $\text{Re}(T_0)$ is coercive in $L^2(D)^3$, i.e., there exists a constant $\gamma > 0$ such that

$$\langle \text{Re}(T_0)f, f \rangle_{L^2(D)^3} \geq \gamma \|f\|_{L^2(D)^3}.$$

Note that the proofs of (b) and (c) can be found in Theorem 4.9 [28] or Theorem 5.12 [20]. Here, for convenience, we repeat the proof of (b) in [20] with slight adaptations.

Proof. (a) We show the injectivity of T by assuming that $Tf = \text{sign}(q)(f + \sqrt{|q|} \text{curl } v) = 0$, then v is a radiating variational solution to the homogeneous problem $\text{curl}^2 v - k^2 v = 0$. However, we showed in the proof of Lemma 3.1 that the latter problem has only the trivial solution which implies that $v = 0$ in Ω . Thus, $f = 0$ or T is injective.

Now we set $w = f + \sqrt{|q|} \text{curl } v$, then $Tf = \text{sign}(q)w$ and

$$\begin{aligned} \langle Tf, f \rangle_{L^2(D)^3} &= \int_D \text{sign}(q)w \cdot (\bar{w} - \sqrt{|q|} \text{curl } \bar{v}) \, dx \\ &= \int_D (\text{sign}(q)|w|^2 - q/\sqrt{|q|}w \cdot \text{curl } \bar{v}) \, dx \end{aligned}$$

For $r > \sup\{|x_3| : (x_1, x_2, x_3)^\top \in D\}$, we consider a smooth function $\chi \in C^\infty(\mathbb{R})$ such that $\chi = 1$ in Ω_r , $\chi = 0$ in $\Omega \setminus \Omega_{2r}$. Then χv belongs to $H_{\alpha, \text{loc}}(\text{curl}, \Omega)$ with compact support in Ω_{3r} . Since $v \in H_{\alpha, \text{loc}}(\text{curl}, \Omega)$ is the radiating solution to (25), we have

$$\begin{aligned} - \int_D q/\sqrt{|q|}w \cdot \text{curl } \bar{v} \, dx &= \int_{\Omega_r} (|\text{curl } v|^2 - k^2|v|^2) \, dx \\ &\quad + \int_{\Omega_{2r} \setminus \Omega_r} (\text{curl } v \cdot \text{curl}(\bar{\chi}v) - k^2v \cdot \bar{\chi}v) \, dx \end{aligned}$$

Now using Green's theorems and exploiting the fact that v solve the Helmholtz equation in $\Omega \setminus \Omega_h$, we obtain that

$$\begin{aligned} - \int_D q/\sqrt{|q|}w \cdot \text{curl } \bar{v} \, dx &= \int_{\Omega_r} (|\text{curl } v|^2 - k^2|v|^2) \, dx + \left(\int_{\Gamma_r} - \int_{\Gamma_{-r}} \right) (e_3 \times \text{curl } v \cdot \bar{v}) \, ds \\ &= \int_{\Omega_r} (|\text{curl } v|^2 - k^2|v|^2) \, dx + \left(\int_{\Gamma_r} - \int_{\Gamma_{-r}} \right) (-\bar{v}_1 \partial_3 v_1 - \bar{v}_2 \partial_3 v_2 + v_3 \partial_3 \bar{v}_3) \, ds. \end{aligned} \quad (26)$$

Taking the imaginary part of the latter equation we have

$$-\text{Im} \int_D q/\sqrt{|q|}w \cdot \text{curl } \bar{v} \, dx = \text{Im} \left(\int_{\Gamma_r} - \int_{\Gamma_{-r}} \right) (-\bar{v}_1 \partial_3 v_1 - \bar{v}_2 \partial_3 v_2 + v_3 \partial_3 \bar{v}_3) \, ds.$$

Recall that v satisfies the radiating Rayleigh condition for $|x_3| > r$. Thus all the terms corresponding to evanescent modes tend to zero as r tends to infinity. Then due to a straightforward computation we derive

$$\begin{aligned} -\text{Im} \int_D q/\sqrt{|q|}w \cdot \text{curl } \bar{v} \, dx &= \lim_{r \rightarrow \infty} \text{Im} \left(\int_{\Gamma_r} - \int_{\Gamma_{-r}} \right) (-\bar{v}_1 \partial_3 v_1 - \bar{v}_2 \partial_3 v_2 + v_3 \partial_3 \bar{v}_3) \, ds \\ &= -4\pi^2 \sum_{j: k^2 > \alpha_j^2} \beta_j (|\hat{v}_j^+|^2 + |\hat{v}_j^-|^2), \end{aligned}$$

which implies that

$$\begin{aligned} (\operatorname{Im} T f, f)_{L^2(D)^3} &= \int_D \operatorname{Im} q/|q| |w|^2 dx - \operatorname{Im} \int_D q/\sqrt{|q|} w \cdot \operatorname{curl} \bar{v} dx \\ &= \int_D \operatorname{Im} q/|q| |w|^2 dx - 4\pi^2 \sum_{j:k^2 > \alpha_j^2} \beta_j (|\hat{v}_j^+|^2 + |\hat{v}_j^-|^2) \leq 0, \end{aligned}$$

since $\operatorname{Im}(q) \leq 0$ in D .

(b) From the definitions of T and T_0 we note that $T f - T_0 f = q/\sqrt{|q|} \operatorname{curl}(v - \tilde{v})$ where $v, \tilde{v} \in H_{\alpha, \text{loc}}(\operatorname{curl}, \Omega)$ are the radiating solutions, for k and $k = i$, of

$$\int_{\Omega} (\varepsilon_r^{-1} \operatorname{curl} v \cdot \operatorname{curl} \bar{\psi} - k^2 v \cdot \bar{\psi}) dx = - \int_{\Omega} q/\sqrt{|q|} f \cdot \operatorname{curl} \bar{\psi} dx, \quad (27)$$

$$\int_{\Omega} (\varepsilon_r^{-1} \operatorname{curl} \tilde{v} \cdot \operatorname{curl} \bar{\psi} + \tilde{v} \cdot \bar{\psi}) dx = - \int_{\Omega} q/\sqrt{|q|} f \cdot \operatorname{curl} \bar{\psi} dx, \quad (28)$$

respectively, for all $\psi \in H_{\alpha}(\operatorname{curl}, \Omega)$ with compact support. By substituting $\psi = \nabla \varphi$ for scalar functions $\varphi \in C^{\infty}(\Omega)$ with compact support we obtain that $\int_{\Omega} v \cdot \nabla \bar{\varphi} dx = 0$ for all $\varphi \in C^{\infty}(\Omega)$ with compact support which means that $\operatorname{div} v = 0$, and analogously, $\operatorname{div} \tilde{v} = 0$ in Ω . The difference $w = v - \tilde{v}$ solves

$$\int_{\Omega} (\varepsilon_r^{-1} \operatorname{curl} w \cdot \operatorname{curl} \bar{\psi} - k^2 w \cdot \bar{\psi}) dx = (k^2 + 1) \int_{\Omega} \tilde{v} \cdot \bar{\psi} dx,$$

for all $\psi \in H_{\alpha}(\operatorname{curl}, \Omega)$ with compact support.

Let now the sequence f_j converge weakly to zero in $L^2(D)^3$ and denote by $v_j, \tilde{v}_j \in H_{\alpha, \text{loc}}(\operatorname{curl}, \Omega)$ the corresponding radiating solutions of (27) and (28), respectively. Define $w_j \in H_{\alpha, \text{loc}}(\operatorname{curl}, \Omega)$ again by the difference $w_j = v_j - \tilde{v}_j$. Set $R > \supp\{|x_3| : (x_1, x_2, x_3)^{\top} \in D\}$, then $\bar{D} \subset \Omega_R$. By the boundedness of the solution operator we conclude that v_j and \tilde{v}_j converge weakly to zero in $H_{\alpha}(\operatorname{curl}, \Omega_R)$. Furthermore, v_j and \tilde{v}_j are smooth outside of \bar{D} and converges uniformly (with all of its derivatives) to zero on $\Gamma_{\pm h}$. In consequence, w_j converges to zero in $C(\partial\Omega_R)$. We determine $p_j \in H_{\alpha, \diamond}^1(\Omega_R)$ as the solution of

$$\int_{\Omega_R} \nabla p_j \cdot \nabla \varphi dx = \int_{\partial\Omega_R} (\nu \cdot w_j) \bar{\varphi} ds \quad (29)$$

for all $\varphi \in H_{\alpha, \diamond}^1(\Omega_R)$. Here the subspace $H_{\alpha, \diamond}^1(\Omega_R)$ of $H_{\alpha}^1(\Omega_R)$ is defined as $H_{\alpha, \diamond}^1(\Omega_R) = \{\varphi \in H_{\alpha}^1(\Omega_R) : \int_{\Omega_R} \varphi ds = 0\}$. The solution of (29) exists and is unique since the form $(p, \varphi) \mapsto \int_{\Omega_R} \nabla p \cdot \nabla \varphi dx$ is bounded and coercive on $H_{\alpha, \diamond}^1(\Omega_R)$ by the inequality of Poincaré (cf. [30]). The latter states that there exists a constant $c > 0$ with

$$\int_{\Omega_R} |\nabla \varphi|^2 dx \geq c \|\varphi\|_{H_{\alpha}^1(\Omega_R)}^2 \text{ for all } \varphi \in H_{\alpha, \diamond}^1(\Omega_R). \quad (30)$$

Problem (29) is the variational form of the Neumann boundary value problem

$$\Delta p_j = \operatorname{div} w_j = 0 \text{ in } \Omega_R, \quad \partial_{\nu} p_j = \nu \cdot w_j \text{ on } \partial\Omega_R.$$

We observe that (29) holds even for all $\varphi \in H_{\alpha}^1(\Omega_R)$ since $\int_{\partial\Omega_R} (\nu \cdot w_j) ds$ vanishes by the divergence theorem and the fact that $\operatorname{div} w_j = 0$. Substituting $\varphi = p_j$ into (29) yields, using (30) and the trace theorem,

$$c \|p_j\|_{H_{\alpha}^1(\Omega_R)}^2 \leq \int_{\Omega_R} |\nabla p_j|^2 dx = \int_{\partial\Omega_R} (\nu \cdot w_j) \bar{p}_j ds \leq \tilde{c} \|w_j\|_{C(\partial\Omega_R)} \|p_j\|_{H_{\alpha}^1(\Omega_R)},$$

i.e. $\|p_j\|_{H_{\alpha}^1(\Omega_R)} \leq (\tilde{c}/c) \|w_j\|_{C(\partial\Omega_R)}$ which converges to zero. Therefore, the functions $\tilde{w}_j := w_j - \nabla p_j \in H_{\alpha}(\operatorname{curl}, \Omega_R)$ satisfy

- $\tilde{w}_j \in H_{\alpha, \text{div}}(\text{curl}, \Omega_R) := \{u \in H_\alpha(\text{curl}, \Omega_R) : \int_{\Omega_R} \nabla \varphi \cdot u \, dx = 0 \text{ for all } \varphi \in H_\alpha^1(\Omega_R)\}$
- $\tilde{w}_j \rightharpoonup 0$ weakly in $L^2(\Omega_R)^3$,
- $\text{curl } \tilde{w}_j = \text{curl } w_j \rightharpoonup 0$ weakly in $L^2(\Omega_R)^3$.

These three conditions assure that \tilde{w}_j converges to zero in the norm of $L^2(\Omega_R)^3$ since the closed subspace $H_{\alpha, \text{div}}(\text{curl}, \Omega_R)$ of $H_\alpha(\text{curl}, \Omega_R)$ is compactly embedded in $L^2(\Omega)^3$. We refer to [31], see also [26], Theorem 4.7. Since also $\|\nabla p_j\|_{L^2(\Omega_R)^3} \rightarrow 0$ this yields $\|w_j\|_{L^2(\Omega_R)} \rightarrow 0$ as j tends to infinity. Now we return to the variational equation for w_j and substitute $\psi = \phi \overline{w}_j$ where $\phi \in C^\infty(\Omega)$ is some function with compact support such that $\phi = 1$ on Ω_R . This yields

$$\begin{aligned} \int_{\Omega_R} (\varepsilon_r^{-1} |\text{curl } w_j|^2 - k^2 |w_j|^2) \, dx &= \int_{\Omega \setminus \Omega_R} (\varepsilon_r^{-1} \text{curl } w_j \cdot \text{curl}(\phi \overline{w}_j) - k^2 \phi |w_j|^2) \, dx \\ &+ (k^2 + 1) \int_{\Omega} \phi \tilde{v}_j \cdot \overline{w}_j \, dx. \end{aligned}$$

We note that w_j is smooth in $\Omega \setminus \Omega_R$. Green's theorem in $\Omega_{mR} \setminus \Omega_R$ (for a sufficiently large value of m) and application of $\text{curl}^2 w_j - k^2 w_j = (k^2 + 1) \tilde{v}_j$ in this region yields

$$\int_{\Omega_R} (\varepsilon_r^{-1} |\text{curl } w_j|^2 - k^2 |w_j|^2) \, dx = \int_{\partial \Omega_R} (\nu \times \text{curl } w_j) \cdot \overline{w}_j \, ds + (k^2 + 1) \int_{\Omega_R} \tilde{v}_j \cdot \overline{w}_j \, dx$$

which tends to zero as j tends to infinity since \tilde{v}_j and $\text{curl } w_j$ are bounded sequences and $\|w_j\|_{L^2(\Omega_R)}$, $\|w_j\|_{C(\partial \Omega_R)}$ tend to zero. Therefore, also $\text{curl } w_j$ tends to zeros in $L^2(\Omega_R)^3$ which complete the proof. \square

6 Characterization of the Biperiodic Support

In this section, we give a characterization for a point z belonging to the support of the contrast q by exploiting special test sequences. A simple criterion for imaging the periodic support is also proposed.

First we introduce some basic facts about α -quasiperiodic Green functions. It is well known that the function $G_k(x, y)$ given by

$$G_k(x, y) = \frac{i}{8\pi^2} \sum_{j \in \mathbb{Z}} \frac{1}{\beta_j} e^{i\alpha_j \cdot (x-y) + i\beta_j |x_3 - y_3|}, \quad x, y \in \Omega, \quad x_3 \neq y_3, \quad (31)$$

is the α -quasiperiodic Green's function of the Helmholtz operator in three dimensions. That means, for fixed $y \in \Omega$,

$$\Delta_x G_k(x, y) + k^2 G_k(x, y) = -\delta_y(x), \quad x \in \Omega.$$

Also, another form of $G_k(x, y)$ can be given, see e.g. [2],

$$G_k(x, y) = \frac{e^{ik|x-y|}}{4\pi|x-y|} + \Psi_k(x-y), \quad (32)$$

where Ψ_k is the analytic solution to the Helmholtz equation in $(-2\pi, 2\pi)^2 \times \mathbb{R}$.

The α -quasiperiodic Green's tensor $\mathbb{G}_k(x, y) \in \mathbb{C}^{3 \times 3}$ defined by

$$\mathbb{G}_k(x, y) = G_k(x, y) \mathbf{I}_{3 \times 3} + k^{-2} \nabla_x \text{div}_x (G_k(x, y) \mathbf{I}_{3 \times 3}), \quad x, y \in \Omega, \quad x_3 \neq y_3,$$

solves

$$\text{curl}_x^2 \mathbb{G}_k(x, y) - k^2 \mathbb{G}_k(x, y) = \delta_y(x) \mathbf{I}_{3 \times 3}, \quad x \in \Omega,$$

where $\mathbf{I}_{3 \times 3}$ is the identity matrix. Here, the curl of a matrix is taken columnwise, the div of a matrix and the ∇ are meant to be taken columnwise and componentwise, respectively. Note that \mathbb{G}_k satisfies the Rayleigh expansion condition and has a strong singularity due to the representation of G_k in (32).

Lemma 6.1. *Let the operator W be defined as in (18). For any $z \in \Omega$ and fixed nonzero $p = (p_1, p_2, p_3) \in \mathbb{C}^3$ we denote by $(\hat{\Psi}_{z,j}^\pm)_{j \in \mathbb{Z}^2} \in \ell^2(\mathbb{Z}^2)^4$ the Rayleigh coefficients of the first two components of*

$$\begin{aligned} \Psi_z(x) &:= k^2 \mathbb{G}_k(x, z)p \\ &= \begin{pmatrix} \left[k^2 G_k(x, z) + \frac{\partial^2 G_k(x, z)}{\partial x_1^2} \right] p_1 + \frac{\partial^2 G_k(x, z)}{\partial x_1 \partial x_2} p_2 + \frac{\partial^2 G_k(x, z)}{\partial x_1 \partial x_3} p_3 \\ \frac{\partial^2 G_k(x, z)}{\partial x_2 \partial x_1} p_1 + \left[k^2 G_k(x, z) + \frac{\partial^2 G_k(x, z)}{\partial x_2^2} \right] p_2 + \frac{\partial^2 G_k(x, z)}{\partial x_2 \partial x_3} p_3 \\ \frac{\partial^2 G_k(x, z)}{\partial x_3 \partial x_1} p_1 + \frac{\partial^2 G_k(x, z)}{\partial x_3 \partial x_2} p_2 + \left[k^2 G_k(x, z) + \frac{\partial^2 G_k(x, z)}{\partial x_3^2} \right] p_3 \end{pmatrix}, \end{aligned}$$

for $x \in \Omega$, $x \neq z$. Then z belongs to D if and only if $W(\hat{\Psi}_{z,j}^\pm) \in \text{Rg}(H^*)$.

Remark 6.2. *Note that the Rayleigh sequences $\hat{G}_{k,j}^\pm(z)$ of the α -quasiperiodic Green's function $G_k(\cdot, z)$ can be obtained from the representation of $G_k(\cdot, z)$ in (31)*

$$\hat{G}_{k,j}^\pm(z) = \frac{i}{8\pi^2 \beta_j} e^{-i[\alpha_{1,j} z_1 + \alpha_{2,j} z_2 \pm \beta_j(z_3 \mp h)]}.$$

Then the Rayleigh sequences $(\hat{\Psi}_{z,j}^\pm)_{j \in \mathbb{Z}^2} \in \ell^2(\mathbb{Z}^2)^4$ of the first two components of Ψ_z can be given as

$$\hat{\Psi}_{z,j}^\pm = \begin{pmatrix} (k^2 - \alpha_{1,j}^2) \hat{G}_{k,j}^\pm(z) p_1 - \alpha_{1,j} \alpha_{2,j} \hat{G}_{k,j}^\pm(z) p_2 \mp \alpha_{1,j} \beta_j \hat{G}_{k,j}^\pm(z) p_3 \\ -\alpha_{2,j} \alpha_{1,j} \hat{G}_{k,j}^\pm(z) p_1 + (k^2 - \alpha_{2,j}^2) \hat{G}_{k,j}^\pm(z) p_2 \mp \alpha_{2,j} \beta_j \hat{G}_{k,j}^\pm(z) p_3 \end{pmatrix}.$$

Proof. First, let $z \in D$. Recall the operator Q defined in (19). Due to the fact that $H^* = -WQ$, it is sufficient to show that $(\hat{\Psi}_{z,j}^\pm)_{j \in \mathbb{Z}^2} \in \text{Rg}(Q)$. Choose $r > 0$ such that $B(z, r) \in D$ and consider a cut-off function $\varphi \in C^\infty(\mathbb{R}^3)$ with $\varphi(x) = 0$ for $|x - z| \leq r/2$ and $\varphi(x) = 1$ for $|x - z| \geq r$. We define

$$w(x) = \text{curl}^2(\varphi(x) \mathbb{G}_k(x, z)p), \quad x \in \Omega.$$

Note that, for $|x - z| \geq r$, we have

$$w(x) = \text{curl}^2(\varphi(x) \mathbb{G}_k(x, z)p) = k^2 \mathbb{G}_k(x, z)p,$$

and further $(\hat{w}_j)_{j \in \mathbb{Z}^2} = (\hat{\Psi}_{z,j}^\pm)_{j \in \mathbb{Z}^2}$. Using Green's theorem we obtain

$$\begin{aligned} \int_{\Omega} (\text{curl } w \cdot \text{curl } \bar{\psi} - k^2 w \cdot \bar{\psi}) dx &= \int_{\Omega} (\text{curl } w - k^2 \text{curl}(\varphi(x) \mathbb{G}_k(x, z)p)) \cdot \text{curl } \bar{\psi} dx \\ &= \int_{\Omega} g \cdot \text{curl } \bar{\psi} dx, \end{aligned}$$

for all $\psi \in H_\alpha(\text{curl}, \Omega)$ with compact support, and $g := \text{curl } w - k^2 \text{curl}(\varphi(x) \mathbb{G}_k(x, z)p)$. Since g is smooth and vanishes for $|z - x| \geq r$, thus $\text{supp}(g) \subset D$. Set $f = \sqrt{|q|}^{-1} g \in L^2(D)^3$. Then we have

$$\int_{\Omega} (\text{curl } w \cdot \text{curl } \bar{\psi} - k^2 w \cdot \bar{\psi}) dx = \int_D \sqrt{|q|} f \cdot \text{curl } \bar{\psi} dx,$$

which implies that $(\hat{\Psi}_{z,j}^\pm)_{j \in \mathbb{Z}^2} \in \text{Rg}(Q)$.

Now let $z \notin D$, and on the contrary, assume that $\hat{\Psi}_{z,j}^\pm \in \text{Rg}(Q)$. That means there exists $u \in H_{\alpha, \text{loc}}(\text{curl}, \Omega)$ and $f \in L^2(D)^3$ such that u is the variational radiating solution to $\text{curl}^2 u - k^2 u = \text{curl}(\sqrt{|q|} f)$ and $\hat{u}_j = \hat{\Psi}_{z,j}^\pm$ for all $j \in \mathbb{Z}^2$. Since the Rayleigh sequences of u and Ψ_z are equal, both functions coincide in $(0, 2\pi)^2 \times \{|x_3| > h\}$ where $h > \text{supp}\{|x_3| : (x_1, x_2, x_3)^\top \in D\}$. Due to the analyticity of u and Ψ_z in $\Omega \setminus D$ and $\Omega \setminus \{z\}$, respectively, and the analytic continuation we conclude that $u = \Psi_z$ in $\Omega \setminus (D \cup \{z\})$. This is a contradiction since $u \in H(\text{curl}, B)$ for any ball B containing z but $\text{curl}(k^2 \mathbb{G}_k(\cdot, z)p) \notin H(\text{curl}, B)$ due to a strongly singularity at z . \square

Theorem 6.3. *Suppose that the contrast q satisfies the Assumption 2.1 and that the direct scattering problem (9) is uniquely solvable. For $j \in \mathbb{Z}^2$, denote by $(\lambda_n, \psi_{n,j})_{n \in \mathbb{N}}$ the orthonormal eigensystem of $(WN)_\# = |\operatorname{Re}(WN)| + \operatorname{Im}(WN)$ and by $(\hat{\Psi}_{z,j}^\pm)_{j \in \mathbb{Z}^2}$ the test sequence in Lemma 6.1. A point z belongs to the support of q if and only if*

$$\sum_{n=1}^{\infty} \frac{|\langle \hat{\Psi}_{z,j}^\pm, \psi_{j,n} \rangle_{\ell^2(\mathbb{Z}^2)}|^2}{\lambda_n} < \infty. \quad (33)$$

Proof. As we assumed in the theorem, $(\lambda_n, \psi_{n,j})_{n \in \mathbb{N}}$ is an orthonormal eigensystem of $(WN)_\#$. The assumptions of Theorem 4.1 on H , H^* and T in the factorization $WN = H^*TH$ have been checked in Lemmas 3.1 and 5.1. Therefore, an application of Theorem 4.1 yields that $\operatorname{Rg}((WN)_\#^{1/2}) = \operatorname{Rg}(H^*)$. Combining this range identity with the characterization given in Lemma 6.1 we obtain that $(\hat{\Psi}_{z,j}^\pm)_{j \in \mathbb{Z}^2} \in \operatorname{Rg}((WN)_\#^{1/2})$ if and only if $z \in D$. Then the criterion (33) follows from the Picard's range criterion. \square

7 Numerical Experiments

As mentioned in the introduction, these are to the best of our knowledge the first three-dimensional examples of the method in a biperiodic setting. These numerical examples focus on the dependence of the reconstructions on the number of the incident fields (or, equivalently, the evanescent modes), and the performance of the method when the data is perturbed by artificial noise. Further, we also indicate the number of the evanescent and propagating modes which are used for each reconstruction. These experiments use three biperiodic structures presented in one period $\Omega = (-\pi, \pi)^2 \times \mathbb{R}$ in terms of the support \overline{D} of the contrast q as follows:

(i) Biperiodic structures of ellipsoids,

$$\begin{aligned} \overline{D} &= \{(x_1, x_2, x_3)^\top \in \Omega : \frac{x_1^2}{2.5^2} + \frac{x_2^2}{2.5^2} + \frac{x_3^2}{0.4^2} \leq 1\}, \\ q &= 0.5 \quad \text{in } D. \end{aligned}$$

(ii) Biperiodic structures of cubes,

$$\begin{aligned} \overline{D} &= \{(x_1, x_2, x_3)^\top \in \Omega : |x_1| \leq 2.5, |x_2| \leq 2.5, |x_3| \leq 0.45\}, \\ q &= (x_3 + 1)(\sin(x_1)^2 \sin(x_2)^2 + 0.3)/4 - 0.4i \quad \text{in } D. \end{aligned}$$

(iii) Biperiodic structures of plus signs,

$$\begin{aligned} \overline{D} &= \Omega \cap [(\{|x_1| \leq 1.75\} \cup \{|x_2| \leq 1.75\}) \cap \{|x_3| \leq 0.45\}], \\ q &= \begin{cases} 0.5 - 0.6i & \text{in } D_1 = \{(x_1, x_2)^\top \in D : -1 < x_1 < 1\}, \\ 0.3 & \text{in } D \setminus D_1. \end{cases} \end{aligned}$$

The data of the direct scattering problem has been obtained by the extension of the volume integral equation method studied in [24] for the scalar case to the Maxwell's equations. Of course it is not possible to numerically compute data for all incident fields $(\varphi_j^{(l)\pm})_{j \in \mathbb{Z}^2}$ in (11). Denote

$$\mathbb{Z}_{M_1, M_2}^2 = \{j = (j_1, j_2) \in \mathbb{Z}^2 : -M_1 \leq j_1, j_2 \leq M_2\}, \quad M_1, M_2 \in \mathbb{N}.$$

For the numerical experiments here we solve the direct problem for a number $j = (j_1, j_2)$ of incident fields $\varphi_j^{(l)\pm}$ where $j \in \mathbb{Z}_{M_1, M_2}^2$. Denote by \mathcal{N}_{M_1, M_2} the block matrix corresponding to the discretization of the near field operator N . Then \mathcal{N}_{M_1, M_2} is given by

$$\mathcal{N}_{M_1, M_2} = \begin{pmatrix} (\hat{u}_{1,n}^+)_j^{(1)+} & (\hat{u}_{1,n}^+)_j^{(1)-} & (\hat{u}_{1,n}^+)_j^{(2)+} & (\hat{u}_{1,n}^+)_j^{(2)-} \\ (\hat{u}_{2,n}^+)_j^{(1)+} & (\hat{u}_{2,n}^+)_j^{(1)-} & (\hat{u}_{2,n}^+)_j^{(2)+} & (\hat{u}_{2,n}^+)_j^{(2)-} \\ (\hat{u}_{1,n}^-)_j^{(1)+} & (\hat{u}_{1,n}^-)_j^{(1)-} & (\hat{u}_{1,n}^-)_j^{(2)+} & (\hat{u}_{1,n}^-)_j^{(2)-} \\ (\hat{u}_{2,n}^-)_j^{(1)+} & (\hat{u}_{2,n}^-)_j^{(1)-} & (\hat{u}_{2,n}^-)_j^{(2)+} & (\hat{u}_{2,n}^-)_j^{(2)-} \end{pmatrix}, \quad j, n \in \mathbb{Z}_{M_1, M_2}^2. \quad (34)$$

Here $\hat{u}_{(1,2),n}^\pm$ are the Rayleigh sequences defined in (10) while for $l = 1, 2$, $(\cdot)_j^{(l)\pm}$ indicate the correspondence to the incident fields $\varphi_j^{(l)\pm}$. Note that each component of \mathcal{N}_{M_1, M_2} is a matrix of size $(M_1 + M_2 + 1)^2$, thus \mathcal{N}_{M_1, M_2} is a $4(M_1 + M_2 + 1)^2 \times 4(M_1 + M_2 + 1)^2$ matrix. The matrix \mathcal{WN}_{M_1, M_2} which corresponds to the discretization of WN can be computed using (16), the symmetric matrix $\text{Re}(\mathcal{WN}_{M_1, M_2})$ can be decomposed as

$$\text{Re}(\mathcal{WN}_{M_1, M_2}) = VDV^{-1},$$

where D, V are the matrices of eigenvalues and corresponding eigenvectors of $\text{Re}(\mathcal{WN}_{M_1, M_2})$, respectively. Denote by $|D|$ the absolute value of D which is taken componentwise. Then we have

$$(\mathcal{WN}_{M_1, M_2})_\# := V|D|V^{-1} + \text{Im}(\mathcal{WN}_{M_1, M_2}).$$

Computing singular value decomposition of $(\mathcal{WN}_{M_1, M_2})_\#$ implies that

$$(\mathcal{WN}_{M_1, M_2})_\#^{1/2} = U|S|^{1/2}V^{-1},$$

where S is the diagonal matrix of singular values λ_m of $(\mathcal{WN}_{M_1, M_2})_\#$. Also $U = [\psi_{n,m}]$ is a $4(M_1 + M_2 + 1)^2 \times 4(M_1 + M_2 + 1)^2$ matrix of ‘‘left’’ singular vectors. We now reshape $[\psi_{n,m}]$ into 4 arrays $[\psi_{j+M_1+1,m}^{(l)}]$, $l = 1, \dots, 4$, $j \in \mathbb{Z}_{M_1, M_2}^2$, where each of them consists of $4(M_1 + M_2 + 1)^2$ square matrices of size $M_1 + M_2 + 1$. Note that the elements of $[\psi_{j+M_1+1,m}^{(l)}]$ are taken columnwise from $[\psi_{n,m}]$.

Now recall that we have $\hat{\Psi}_{z,j}^\pm = (\hat{\Psi}_{1,z,j}^+, \hat{\Psi}_{2,z,j}^+, \hat{\Psi}_{1,z,j}^-, \hat{\Psi}_{2,z,j}^-)^\top$ which can be rewritten as

$$\hat{\Psi}_{z,j} = (\hat{\Psi}_{z,j}^{(1)}, \hat{\Psi}_{z,j}^{(2)}, \hat{\Psi}_{z,j}^{(3)}, \hat{\Psi}_{z,j}^{(4)})^\top.$$

Then the criterion (33) for computing the image can be approximated as follows

$$P(z) = \left[\sum_{n=1}^{4(M_1+M_2+1)^2} \frac{A_n(z)}{\lambda_n} \right]^{-1}, \quad (35)$$

where

$$A_n(z) = \left| \sum_{l=1}^4 \sum_{j \in \mathbb{Z}_{M_1, M_2}^2} \hat{\Psi}_{z,j}^{(l)} \overline{\psi_{j+M_1+1,n}^{(l)}} \right|^2.$$

Note that P should be small outside of D and big inside of D .

To show the performance of the method with noisy data, we perturb our synthetic data by artificial noise. More particularly, we add the noise matrix \mathcal{X} of uniformly distributed random entries to the data matrix $(\mathcal{WN}_{M_1, M_2})_\#^{1/2}$. Denote by δ the noise level, then the noise data matrix $(\mathcal{WN}_{M_1, M_2})_\#^{1/2, \delta}$ is given by

$$(\mathcal{WN}_{M_1, M_2})_\#^{1/2, \delta} := (\mathcal{WN}_{M_1, M_2})_\#^{1/2} + \delta \frac{\mathcal{X}}{\|\mathcal{X}\|_2} \|(\mathcal{WN}_{M_1, M_2})_\#^{1/2}\|_2,$$

where $\|\cdot\|_2$ is the matrix 2-norm. Note that from the latter equation we also have

$$\frac{\|(\mathcal{WN}_{M_1, M_2})_\#^{1/2, \delta} - (\mathcal{WN}_{M_1, M_2})_\#^{1/2}\|_2}{\|(\mathcal{WN}_{M_1, M_2})_\#^{1/2}\|_2} = \delta.$$

Since we apply Tikhonov regularization [12], instead of implementing (35) we consider

$$P(z) = \left[\sum_{n=1}^{4(M_1+M_2+1)^2} \left(\frac{\lambda_n^{1/2}}{\lambda_n + \gamma} \right)^2 A_n(z) \right]^{-1}, \quad (36)$$

where

$$A_n(z) = \left| \sum_{l=1}^4 \sum_{j \in \mathbb{Z}_{M_1, M_2}^2} \hat{\Psi}_{z,j}^{(l)} \overline{\psi}_{j+M_1+1,n}^{(l)} \right|^2.$$

Here $\lambda_n, \psi_{j,n}$ are the singular values and vectors of $(\mathcal{WN}_{M_1, M_2})_{\#,\delta}$, respectively. The parameter γ is chosen by Morozov's generalized discrepancy principle which can be obtained by solving the equation

$$\sum_{n=1}^{4(M_1+M_2+1)^2} \frac{\gamma^2 - \delta^2 \lambda_n}{(\lambda_n + \gamma)^2} A_n(z) = 0,$$

for each sampling point z . For the following experiments, we choose the wave number $k = 2\pi/3$. The number of the incident fields used is $4(M_1 + M_2 + 1)^2$. Further, the reconstructions have been smoothed using the command `smooth3` in Matlab, and we plot the pictures in 3×3 periods.

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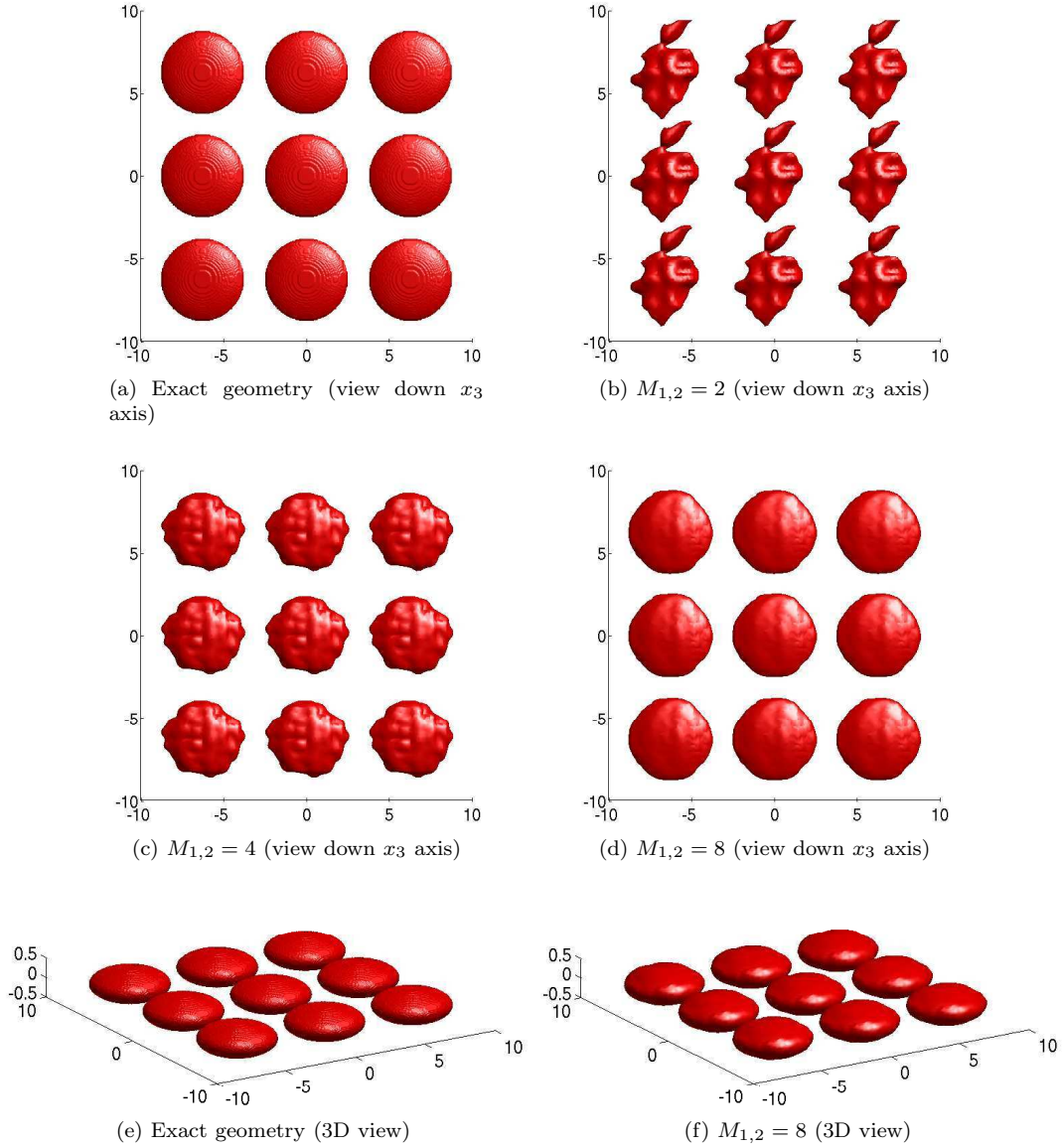


Figure 1: Reconstructions of biperiodic shapes of ellipsoids for different number of incident fields without noise. The number of Rayleigh coefficients measured in each reconstruction is $4(M_1 + M_2 + 1)^2$. The contrast $q = 0.5$ in D . (b) 48 propagating modes, 52 evanescent modes, isovalue 7 (c) 52 propagating modes, 312 evanescent modes, isovalue 0.1 (d) 52 propagating modes, 1104 evanescent modes, isovalue 0.01.

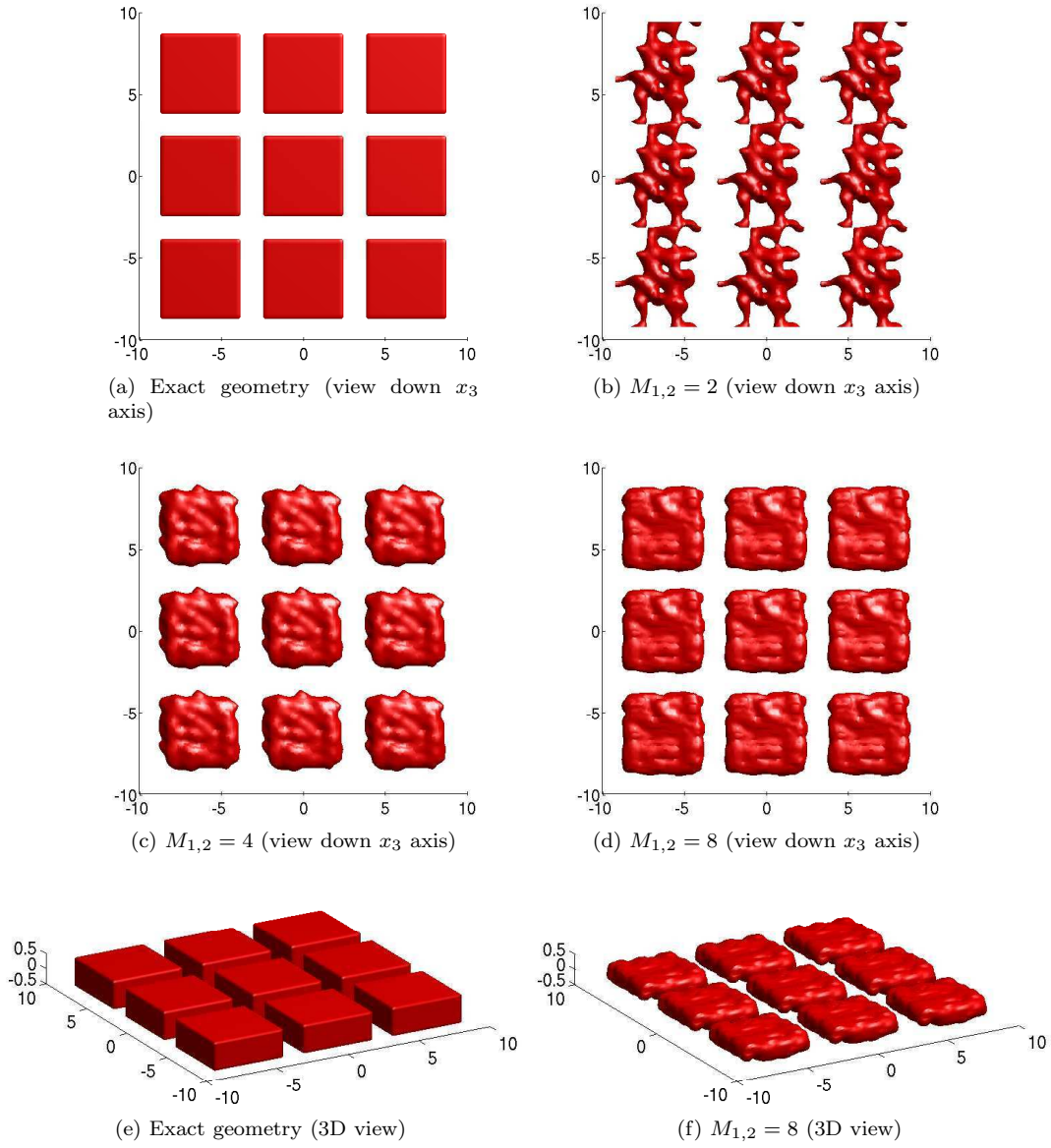


Figure 2: Reconstructions of biperiodic shapes of cubes for different number of incident fields without noise. The number of Rayleigh coefficients measured in each reconstruction is $4(M_1 + M_2 + 1)^2$. The contrast $q = (x_3 + 1)(\sin(x_1)^2 \sin(x_2)^2 + 0.3)/4 - 0.4i$ in D . (b) 48 propagating modes, 52 evanescent modes, isovalue 40 (c) 52 propagating modes, 312 evanescent modes, isovalue 1.8 (d) 52 propagating modes, 1104 evanescent modes, isovalue 0.008.

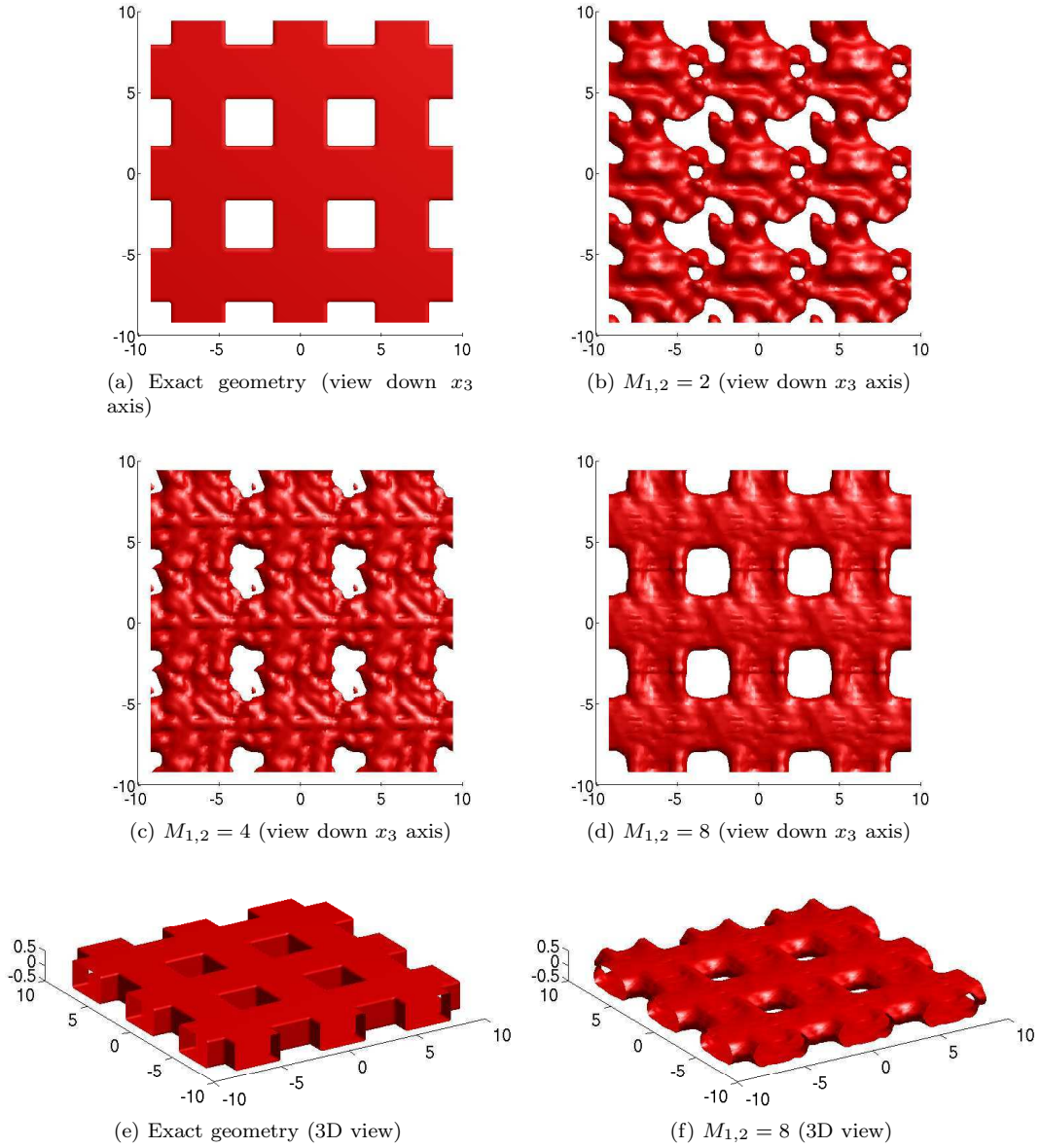
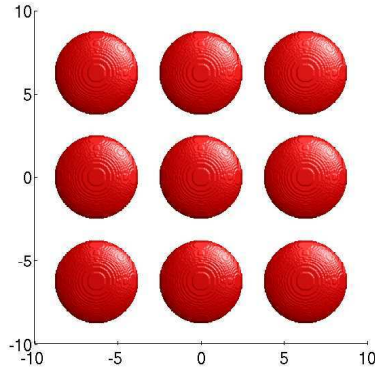
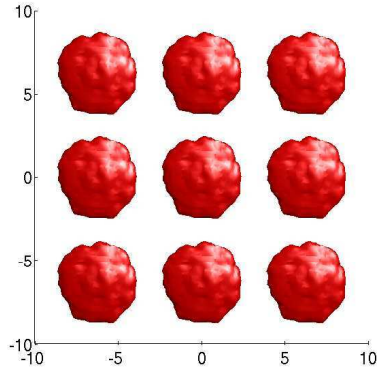


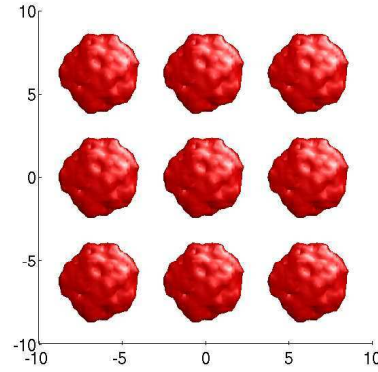
Figure 3: Reconstructions of biperiodic shapes of plus signs for different number of incident fields without noise. The number of Rayleigh coefficients measured in each reconstruction is $4(M_1 + M_2 + 1)^2$. The contrast $q = 0.5 - 0.6i$ in $D_1 = \{(x_1, x_2)^T \in D : -1 < x_1 < 1\}$ and $q = 0.3$ in $D \setminus D_1$. (b) 48 propagating modes, 52 evanescent modes, isovalue 15 (c) 52 propagating modes, 312 evanescent modes, isovalue 2 (d) 52 propagating modes, 1104 evanescent modes, isovalue 0.05.



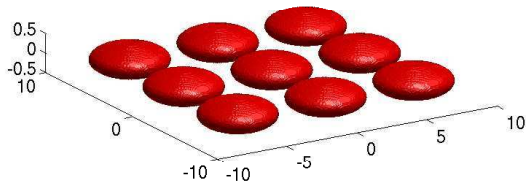
(a) Exact geometry (view down x_3 axis)



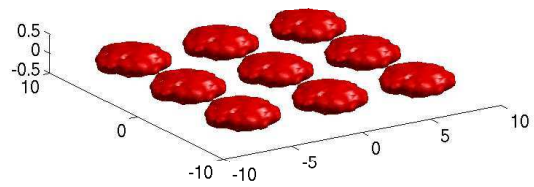
(b) 2% artificial noise, $M_{1,2} = 8$ (view down x_3 axis)



(c) 5% artificial noise, $M_{1,2} = 8$ (view down x_3 axis)



(d) Exact geometry (3D view)



(e) 5% artificial noise, $M_{1,2} = 8$ (3D view)

Figure 4: Reconstructions of biperiodic shapes of ellipsoids for artificial noise. The number of Rayleigh coefficients measured in each reconstruction is $4(M_1 + M_2 + 1)^2$. The contrast $q = 0.5$ in D . (b) 52 propagating modes, 1104 evanescent modes, isovalue 0.0012 (c) 52 propagating modes, 1104 evanescent modes, isovalue 0.0023.

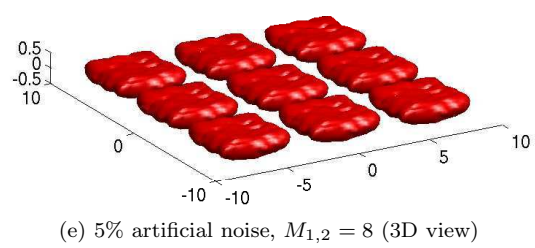
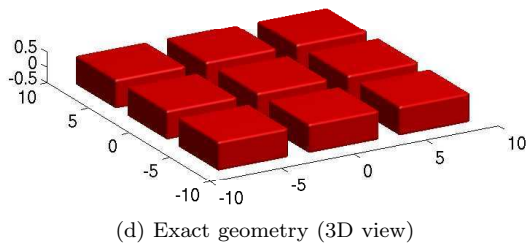
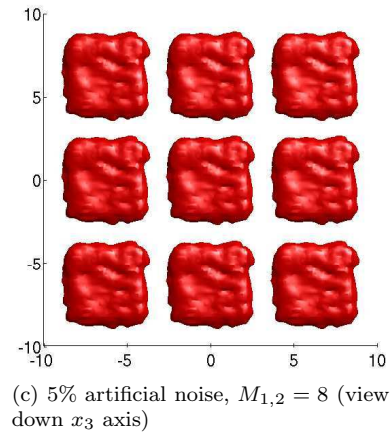
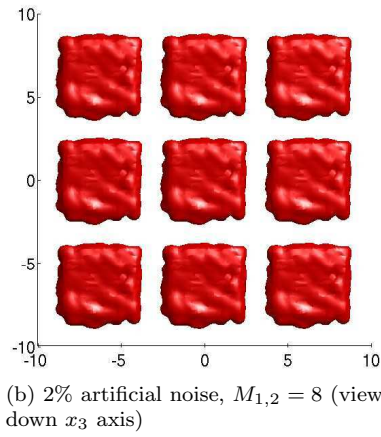
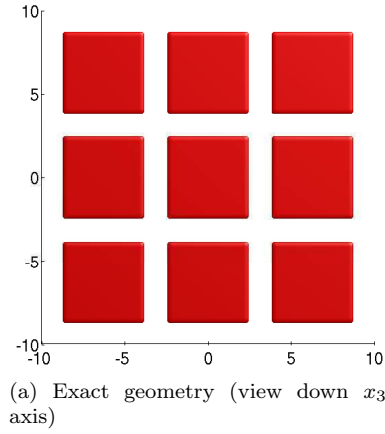


Figure 5: Reconstructions of biperiodic shapes of cubes for artificial noise. The number of Rayleigh coefficients measured in each reconstruction is $4(M_1 + M_2 + 1)^2$. The contrast $q = (x_3 + 1)(\sin(x_1)^2 \sin(x_2)^2 + 0.3)/4 - 0.4i$ in D . (b) 52 propagating modes, 1104 evanescent modes, isovalue 0.1 (c) 52 propagating modes, 1104 evanescent modes, isovalue 0.02.

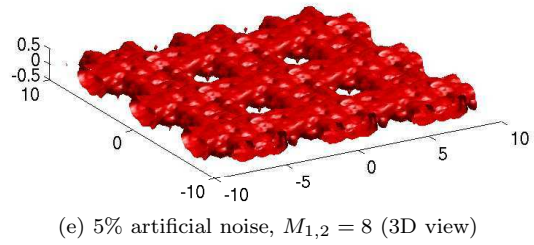
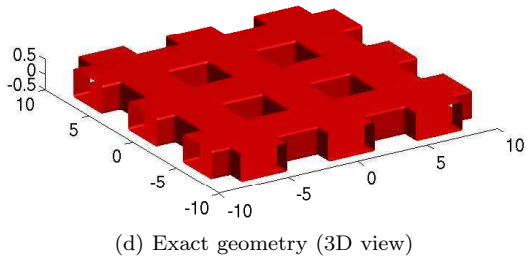
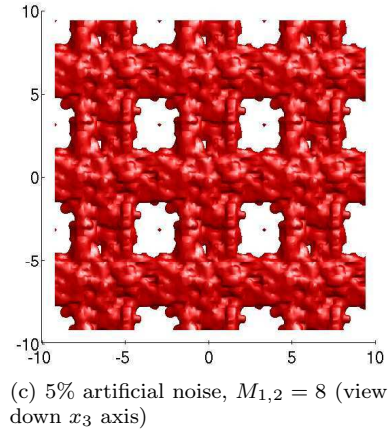
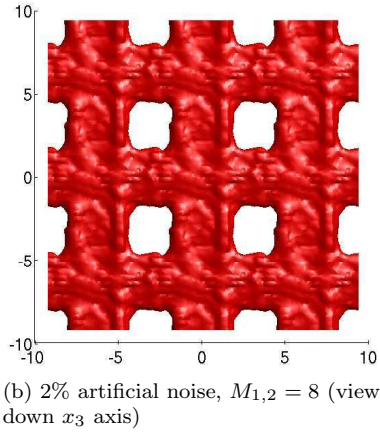
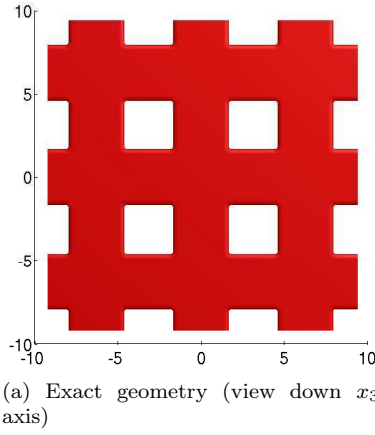


Figure 6: Reconstructions of biperiodic shapes of plus signs for artificial noise. The number of Rayleigh coefficients measured in each reconstruction is $4(M_1 + M_2 + 1)^2$. The contrast $q = 0.5 - 0.6i$ in $D_1 = \{(x_1, x_2)^\top \in D : -1 < x_1 < 1\}$ and $q = 0.3$ in $D \setminus D_1$. (b) 52 propagating modes, 1104 evanescent modes, isovalue 0.1 (c) 52 propagating modes, 1104 evanescent modes, isovalue 0.02.

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