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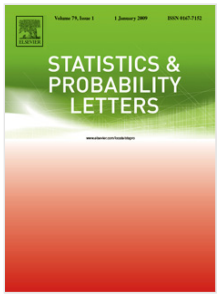

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Lévy area for Gaussian processes: A double Wiener–Itô integral approach

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Abstract. Let $\{X_1(t)\}_{0\le t\le 1}$ and $\{X_2(t)\}_{0\le t\le 1}$ be two independent continuous centered Gaussian processes with covariance functions R_1 and R_2. We show that if the covariance functions are of finite p-variation and q-variation respectively and such that $p^{-1}+q^{-1}>1$, then the Lévy area can be defined as a double Wiener–Itô integral with respect to an isonormal Gaussian process induced by X_1 and X_2. Moreover, some properties of the characteristic function of that generalised Lévy area are studied.

Keywords: Lévy area, p-variation, fractional Brownian motion, multiple Wiener-Itô integral, Young's inequality.

2000 Mathematics Subject Classification 60G15, 60G22, 60H05

1 Introduction

Let $\{W_1(t)\mid 0\le t\le 1\}$ and $\{W_2(t)\mid 0\le t\le 1\}$ be two independent standard Wiener processes defined on a probability space $(\Omega,\mathcal{F},P)$, and let A be the area included by the curve

$$x=W_1(t)\,,\quad y=W_2(t)\qquad 0\le t\le 1$$

and its chord. This random variable was first introduced by Lévy (1951), where it is described (up to the multiplicative constant 1/2, that we skip to simplify the computations) by means of stochastic integrals as

$$A=\int_0^1 W_1(t)dW_2(t)-\int_0^1 W_2(t)dW_1(t)\,. \tag{1}$$

Lévy (1951) computed its characteristic function, which is

$$\varphi(t)=\mathbb{E}[e^{itA}]=\frac{1}{\cosh(t)}\,,\qquad t\in\mathbb{R}\,. \tag{2}$$

It is easy to show that A has the law of an element of the second Wiener chaos generated by a Brownian motion. This can be proved directly due to the fact that the elements of the second Wiener chaos have a very particular characteristic function (see Janson (1997), Chapter 6), and one realizes that (2) has this form thanks to the factorisation (see Abramowitz and Stegun (1970), page 85)

$$\cosh(z)=\prod_{n\ge 0}\left(1+\frac{4z^2}{\pi^2(2n+1)^2}\right)=\prod_{n\in\mathbb{Z}}(1-2iz\alpha_n)\,e^{2iz\alpha_n}\,, \tag{3}$$

where $\alpha_n=(\pi(2n+1))^{-1}$. Alternatively, the law of A can be given as the law of a double Wiener–Itô integral

$$I_2^B(f):=\iint_{[0,1]^2} f(s,t)dB(s)dB(t)$$

for an arbitrary Brownian motion $B=\{B_t\mid t\ge 0\}$, where the kernel f is obtained in the following way: consider an orthonormal basis of $L^2([0,1])$, which for convenience - since each eigenvalue has multiplicity two (see Section 4) - we write as $\{\phi_n,\psi_n\mid n\in\mathbb{Z}\}$, and define

$$f(s,t)=\sum_{n\in\mathbb{Z}}\alpha_n\phi_n(s)\phi_n(t)+\sum_{n\in\mathbb{Z}}\alpha_n\psi_n(s)\psi_n(t)\,.$$

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Then

$$I_2^B(f) = \sum_{n\in\mathbb{Z}} \alpha_n H_2(I_1^B(\phi_n)) + \sum_{n\in\mathbb{Z}} \alpha_n H_2(I_1^B(\psi_n)) \,,$$

where $I_1^B(\phi) = \int_0^1 \phi(s)dB(s)$ and $H_2(x) = x^2 - 1$ is the Hermite polynomial of order two. Observe that $H_2(I_1^B(\phi_n))$ and $H_2(I_1^B(\psi_n))$ are all independent centered $\chi^2(1)$ random variables. From (3), $A \overset{\text{law}}{=} I_2^B(f)$.

However, it is more difficult to get a strong representation of A as a double Wiener–Itô integral with respect to the original Brownian motions W_1 and W_2. To this end, it is necessary to rely on the construction of multiple Wiener–Itô integrals for a general white noise, see for instance Nualart (2006), pages 8 and 14. There, both Brownian motions W_1 and W_2 are embedded in a Gaussian noise W on $[0,1]\times\{1,2\}$. For $h \in L^2([0,1]\times\{1,2\}, dt\otimes \text{Card}) \cong L^2([0,1],\mathbb{R}^2)$, where Card is the counting measure, we have

$$\int_{[0,1]\times\{1,2\}} h dW = \int_0^1 h(s,1)dW_1(s) + \int_0^1 h(s,2)dW_2(s) \,.$$

Moreover, for $f \in L^2(([0,1]\times\{1,2\})^2)$ symmetric in the variables $(s,i) \in [0,1]\times\{1,2\}$, we have

$$I_2(f) = 2\sum_{i,j=1}^{2} \int_0^1 \int_0^t f((s,i),(t,j))dW_i(s)dW_j(t) \,, \tag{4}$$

see Nualart (2006), page 23. For the sake of simplicity, we will indistinctly use $f_{ij}(s,t)$ for $f((s,i),(t,j))$ from now on. Define

$$f_{ij}^{\mathcal{L}}(s,t) = \begin{cases} 0, & \text{if } i=j \\ \frac{1}{2}(\mathbf{1}_{T_1}(s,t) - \mathbf{1}_{T_2}(s,t)), & \text{if } i=1, j=2 \\ \frac{1}{2}(\mathbf{1}_{T_2}(s,t) - \mathbf{1}_{T_1}(s,t)), & \text{if } i=2, j=1 \end{cases} , \tag{5}$$

where $\mathbf{1}_C$ is the indicator function of the set C and

$$T_1 := \{(s,t)\in[0,1]^2 \mid s<t\} \,, \qquad T_2 := \{(s,t)\in[0,1]^2 \mid s>t\} \,. \tag{6}$$

Note that $f^{\mathcal{L}}$ is symmetric. From (4) applied to $f^{\mathcal{L}}$, and (1) it follows that $A \overset{\text{a.s.}}{=} I_2(f^{\mathcal{L}})$. We will refer to (5) as the Lévy kernel.

The aim of this paper is to extend the above strong construction in order to define the Lévy area for general Gaussian processes under minimal conditions of their covariance functions, and to study its characteristic function. We will consider two independent continuous centered Gaussian processes $\{X_1(t) \mid 0\le t\le 1\}$ and $\{X_2(t) \mid 0\le t\le 1\}$ with (continuous) covariance functions R_1 and R_2 and we prove that if the covariance functions are of finite p-variation and q-variation respectively and such that $p^{-1} + q^{-1} > 1$, then the Lévy area can be defined as an element of the second Wiener chaos generated by X_1 and X_2. Such a kind of results have been obtained (for the non-antisymmetrized Lévy area $\int_0^1 X_1(t)\, dX_2(t)$) in the context of rough path analysis by Friz and Victoir (2010a, 2010b), but, as far as we know, in such generality they are new for classical Gaussian processes. Our results applied to two fractional Brownian motions of Hurst parameter H and H' states that the Lévy area can be defined if $H + H' > 1/2$. In particular, if $H = H'$, then the condition is $H \in (1/4, 1)$ which is a known result (see Neuenkirch *et al.* (2010) and the references therein) but we present an alternative point of view based in the Huang and Cambanis (1978) approach to stochastic integration for Gaussian processes. Our results also extends the ones given by Bardina and Tudor (2007) where the integral $\int_0^1 X_t^H\, dX_t^{H'}$ is defined using Malliavin calculus techniques for $H \in (0,1)$ and $H' > 1/2$.

We should point out that there are other ways to construct a Lévy area for Gaussian processes. In the light of Friz and Victoir (2010a, 2010b) results, under the hypothesis of the present paper, both the mollifier and Karhunen–Loeve approximations (see Friz and Victoir (2010b), page 413–416) to Lévy area are well defined. The main advantage of our approach is that the probabilistic properties of a double integral Wiener–Itô integral are well known; in particular, the structure of its characteristic function is described in terms of a Carleman–Fredholm determinant. It turns out that such properties can be transferred to the more general context of a double Wiener–Itô integral with respect to an isonormal process, where our construction takes place.

The paper is organized as follows. We first introduce the general framework of the isonormal Gaussian processes, and following the scheme of Huang and Cambanis (1978), we associate an isonormal Gaussian process to a pair of independent Gaussian processes. We also give here a definition of a generalised Lévy area. In the next section we derive the conditions on the covariance functions so that $f^{\mathcal{L}}$ generates a Lévy area. As an example we explore what happens with two fractional Brownian motion (fBm) with the same covariance function, and the case with two different covariance

function. This later case allows us to let one of the processes be as irregular as desired, that is no low bounds for its Hurst parameter is assumed, provided that the other one is regular enough. Finally, we discuss about the representation of the characteristic function of a double Wiener-Itô integral in terms of a Carleman–Fredholm determinant, that we apply to compute the characteristic function of A. Under a further condition of symmetry over the stochastic processes we will show that the characteristic function of a generalised Lévy area has a Carleman-Fredholm determinant with symmetric poles and even multiplicity.

2 Isonormal Gaussian processes and multiple integrals

Gaussian multiple integrals are defined in the framework of isonormal Gaussian processes. Main references are Nualart (2006) and Peccati and Taqqu (2010). The more general abstract context of Gaussian Hilbert spaces developed by Janson (1997) is also very useful and interesting.

Let $\mathcal{H}$ be a separable Hilbert space with inner product $\langle \cdot, \cdot \rangle_{\mathcal{H}}$. An isonormal Gaussian process $X = \{X(f) \mid f \in \mathcal{H}\}$ is a centered Gaussian family of random variables such that $\mathbb{E}[X(f)X(g)] = \langle f, g \rangle_{\mathcal{H}}$. It is well known that the construction of the multiple Wiener–Itô integrals with respect to a Brownian motion can be transferred to isonormal Gaussian processes: see Nualart (2006), pages 7 and 8, or Peccati and Taqqu (2010), Section 8.4. For the convenience of the reader we recall some important facts on that construction. Let $\mathcal{H}^{\otimes n}$ (resp. $\mathcal{H}^{\odot n}$) be the nth (Hilber) tensor power of $\mathcal{H}$ (resp. the nth symmetric tensor power); for detailed constructions of these Hilbert spaces see Janson (1997), Appendix E. Given $f_1, \ldots, f_k \in \mathcal{H}$ orthonormal, and $n_1, \ldots, n_k$ nonnegative integers such that $n_1 + \cdots + n_k = n$, define

$$I_n\Big(\text{Symm}(f_1^{\otimes n_1} \otimes \cdots \otimes f_k^{\otimes n_k})\Big) = \prod_{j=1}^{k} H_{n_j}(X(f_j)),$$

where $H_n(x)$ is the Hermite polynomial of order n with leading coefficient equal to 1 (in Nualart (2006) the leading coefficient of $H_n(x)$ is $1/n!$), and Symm is a canonical symmetrization. Then I_n can be extended to a linear isometry between $\mathcal{H}^{\odot n}$ (with the norm $\sqrt{n!}\|\cdot\|_{\mathcal{H}^{\otimes n}}$) and $L^2(\Omega)$, and this extension is called the multiple Wiener–Itô integral of order n with respect to X. The image space $\{I_n(f) \mid f \in \mathcal{H}^{\odot n}\}$ is called the Wiener chaos of order n associated with X.

When $\mathcal{H} = L^2(M, \mu)$ where μ is σ-finite, the tensor power $\mathcal{H}^{\otimes n}$ may be identified with $L^2(M^n, \mu^{\otimes n})$, and the symmetric tensor power $\mathcal{H}^{\odot n}$ with the space of symmetric functions in $L^2(M^n, \mu^{\otimes n})$ (see Janson (1997), page 319).

2.1 The isonormal Gaussian process associated with two Gaussian processes

In this section we describe how two ordinary Gaussian processes can be imbedded into an isonormal Gaussian process. We extend Huang and Cambanis (1978) approach, where that construction was done for one Gaussian process. Let $X_1 = \{X_1(t) \mid t \in [0,1]\}$ and $X_2 = \{X_2(t) \mid t \in [0,1]\}$ be two independent continuous centered Gaussian processes, both starting at zero, with (continuous) covariance function $R_1(s,t)$ and $R_2(s,t)$ respectively. We start reproducing Huang and Cambanis (1978) construction adapted to our context. Let $\mathcal{E}$ denote the set of step functions on $[0,1]$

$$\phi(t) = \sum_{j=1}^{n} a_j \mathbf{1}_{(t_j, t_{j+1}]}(t), \quad a_j \in \mathbb{R}\,.$$

For ϕ of the above form, define, for $i = 1, 2$,

$$X_i(\phi) = \sum_{j=1}^{n} a_j \big(X_i(t_{j+1}) - X_i(t_j)\big) \in L^2(\Omega),$$

which is a centered Gaussian random variable, and for $\phi_1, \phi_2 \in \mathcal{E}$,

$$\mathbb{E}[X_i(\phi_1)X_i(\phi_2)] = \int_{[0,1]^2} \phi_1(s)\phi_2(t) dR_i(s,t),$$

where the above integral is defined by

$$\int_{[0,1]^2} \mathbf{1}_{(u,u']}(s)\mathbf{1}_{(v,v']}(t)\, dR_i(s,t) = R_i(u',v') - R_i(u',v) - R_i(u,v') + R_i(u,v)\,. \tag{7}$$

With the convenient identifications,

$$\langle \phi_1, \phi_2 \rangle_{\mathcal{H}_i} := \int_{[0,1]^2} \phi_1(s)\phi_2(t) dR_i(s,t), \quad \phi_1, \phi_2 \in \mathcal{E}$$

is an inner product and we can construct the Hilbert space $\mathcal{H}_i$ which is the completion of $\mathcal{E}$ with respect to $\langle \cdot, \cdot \rangle_{\mathcal{H}_i}$. In the notation of Huang and Cambanis (1978), we have $\mathcal{H}_i = \Lambda_2(R_i)$. Then, the map $X_i : \mathcal{E} \to L^2(\Omega)$ can be extended to an isometry between $\mathcal{H}_i$ and $L^2(\Omega)$. The image space $\{X_i(f) \,|\, f \in \mathcal{H}_i\}$ is an isonormal Gaussian process. It is worth to remark that when X_i is a Brownian motion, then $\mathcal{H}_i$ reduces to $L^2([0,1], dt)$. However, in general $\mathcal{H}_i$ is not a space of functions; for example, when X_i is a fractional Brownian motion of Hurst parameter $H \in (1/2, 1)$, then $\mathcal{H}_i$ contains distributions, see Pipiras and Taqqu (2001) and Jolis (2007).

In order to define an isonormal Gaussian process associated to both X_1 and X_2, the set of appropriate elementary functions, denoted by $\mathcal{E}_2$, is the set of the functions that can be written as $f(t,i) = \delta_{1i}\phi_1(t) + \delta_{2i}\phi_2(t)$, for $(t,i) \in [0,1] \times \{1,2\}$, where $\phi_1,\ \phi_2 \in \mathcal{E}$, and δ_{ij} is the Kronecker's delta. On $\mathcal{E}_2$ we can consider the inner product (with the convenient identifications):

$$\langle f, g \rangle_{\mathcal{H}} = \langle f(\cdot,1), g(\cdot,1) \rangle_{\mathcal{H}_1} + \langle f(\cdot,2), g(\cdot,2) \rangle_{\mathcal{H}_2} = \int_{[0,1]^2} f(s,1)g(t,1) dR_1(s,t) + \int_{[0,1]^2} f(s,2)g(t,2) dR_2(s,t)\,.$$

Let us call $\mathcal{H}$ the Hilbert space which is the completion of $\mathcal{E}_2$ with the above inner product. Note that $\mathcal{H} \cong \mathcal{H}_1 \oplus \mathcal{H}_2$, where $\mathcal{H}_1 \oplus \mathcal{H}_2$ is the Hilbertian direct sum of $\mathcal{H}_1$ and $\mathcal{H}_2$, that is, the Hilbert space which consists in all ordered pairs $(x_1, x_2) \in \mathcal{H}_1 \times \mathcal{H}_2$ equipped with the inner product $\langle (x_1, x_2), (y_1, y_2) \rangle_{\mathcal{H}_1 \oplus \mathcal{H}_2} := \langle x_1, y_1 \rangle_{\mathcal{H}_1} + \langle x_2, y_2 \rangle_{\mathcal{H}_2}$.

Define $X : \mathcal{E}_2 \to L^2(\Omega)$ by

$$X(f) = X_1(f(\cdot,1)) + X_2(f(\cdot,2)).$$

From the independence between X_1 and X_2, it follows that

$$\mathbb{E}[X(f)X(g)] = \langle f, g \rangle_{\mathcal{H}},\ f, g \in \mathcal{E}_2.$$

So the map X can be extended to an isometry on $\mathcal{H}$, and $X = \{X(f) \,|\, f \in \mathcal{H}\}$ is an isonormal Gaussian process.

2.2 Generalised Lévy area

Our plan is to use a double Wiener–Itô integral to define the Lévy area, so we need to consider the Hilbert space $\mathcal{H}^{\otimes 2}$. A construction of that space can be done in the following way: Let $\mathcal{E}_2^{\otimes 2}$ be the linear span of elements of type $f_1 \otimes f_2$ for $f_1, f_2 \in \mathcal{E}_2$, where $f_1 \otimes f_2\big((s,i),(t,j)\big) = f_1(s,i)f_2(t,j)$. On this set consider the inner product induced by

$$\langle f_1 \otimes f_2, g_1 \otimes g_2 \rangle_{\mathcal{H}^{\otimes 2}} := \langle f_1, g_1 \rangle_{\mathcal{H}} \langle f_2, g_2 \rangle_{\mathcal{H}} \tag{8}$$

Then $\mathcal{H}^{\otimes 2}$ is the completion of $\mathcal{E}_2^{\otimes 2}$ by the inner product (8). Furthermore, we have

$$\mathcal{H}^{\otimes 2} \cong \mathcal{H}_1^{\otimes 2} \oplus (\mathcal{H}_1 \otimes \mathcal{H}_2) \oplus (\mathcal{H}_2 \otimes \mathcal{H}_1) \oplus \mathcal{H}_2^{\otimes 2}\,, \tag{9}$$

which gives a very appropriate interpretation of the elements $f \in \mathcal{H}^{\otimes 2}$ as 2 by 2 matrices with entries $f_{ij} \in \mathcal{H}_i \otimes \mathcal{H}_j$ for $i, j = 1, 2$. The above isomorphism also induces the decomposition

$$\langle f, g \rangle_{\mathcal{H}^{\otimes 2}} = \sum_{i,j=1}^{2} \langle f_{ij}, g_{ij} \rangle_{\mathcal{H}_i \otimes \mathcal{H}_j}, \quad \text{where } f = (f_{ij})_{i,j=1,2},\ g = (g_{ij})_{i,j=1,2},\ \text{with } f_{ij},\ g_{ij} \in \mathcal{H}_i \otimes \mathcal{H}_j. \tag{10}$$

In particular, when $f, g \in \mathcal{E}_2^{\otimes 2}$, the product can be expressed in terms of the covariance functions. In next Lemma, the double integral is defined in an obvious way since we are integrating rectangles.

Lemma 2.1. *Consider $f, g \in \mathcal{E}_2^{\otimes 2}$, with $f = (f_{ij})_{i,j=1,2}$ and $g = (g_{ij})_{i,j=1,2}$. Then*

$$\langle f, g \rangle_{\mathcal{H}^{\otimes 2}} = \sum_{i,j=1}^{2} \int_{[0,1]^2} \int_{[0,1]^2} f_{ij}(s,u) g_{ij}(t,v) dR_i(s,t) dR_j(u,v) \tag{11}$$

Proof. By (10) it suffices to consider a term $\langle f_{ij}, g_{ij}\rangle_{\mathcal{H}_i\otimes\mathcal{H}_j}$. Let $f_{ij} = f_1\otimes f_2$ and $g_{ij} = g_1\otimes g_2$, where $f_1, f_2, g_1, g_2 \in \mathcal{E}_2$. Then

$$\langle f_{ij}, g_{ij}\rangle_{\mathcal{H}_i\otimes\mathcal{H}_j} = \langle f_1, g_1\rangle_{\mathcal{H}_i}\langle f_2, g_2\rangle_{\mathcal{H}_j} = \int_{[0,1]^2} f_1(s)g_1(t)dR_i(s,t)\int_{[0,1]^2} f_2(u)g_2(v)dR_j(u,v)$$
$$= \int_{[0,1]^2}\int_{[0,1]^2} f_1(s)g_1(t)f_2(u)g_2(v)dR_i(s,t)dR_j(u,v) = \int_{[0,1]^2}\int_{[0,1]^2} f_{ij}(s,u)g_{ij}(t,u)(v)dR_i(s,t)dR_j(u,v). \blacksquare$$

Following again Huang and Cambanis (1978), the symmetric tensor Hilbert space $\mathcal{H}^{\odot 2}$ can be constructed using the fact that $\mathcal{H}^{\otimes 2}$ is generated by functions, and transferring to this space the notion of symmetric function. Specifically, given a function $f\big((s,i),(t,j)\big)$ the symmetrized function of f is defined as

$$\widetilde{f}\big((s,i),(t,j)\big) = \frac{1}{2}\Big(f\big((s,i),(t,j)\big) + f\big((t,j),(s,i)\big)\Big).$$

A function f is symmetric if and only if $f = \widetilde{f}$. With the same steps as Huang and Cambanis (1978), page 593, the symmetryzation procedure can be extended from $\mathcal{E}_2^{\otimes 2}$ to the whole $\mathcal{H}^{\otimes 2}$, and an element of $f \in \mathcal{H}^{\otimes 2}$ is called symmetric if $f = \widetilde{f}$. Then, the space $\mathcal{H}^{\odot 2}$ can be identified with the subspace of the symmetric elements in $\mathcal{H}^{\otimes 2}$. On $\mathcal{H}^{\odot 2}$ we can define the double integral I_2 in agreement with the comments in the first part of Section 2.

Now we are ready to define the Lévy area. The desirable definition would be $I_2(f^{\mathcal{L}})$ whenever $f^{\mathcal{L}} \in \mathcal{H}^{\odot 2}$, where $f^{\mathcal{L}}$ was defined in (5). Unfortunately this is very difficult, if possible at all, to prove. We will circumvent this problem by finding an element in $\mathcal{H}^{\odot 2}$ which is indistinguishable (in a particular sense) from $f^{\mathcal{L}}$ and to which we will apply the isometry $I_2(\cdot)$. In other words, we will say that a symmetric function $f\big((s,i),(t,j)\big)$ can be identified with an element $\hat{f} \in \mathcal{H}^{\odot 2}$ if

$$\langle \hat{f}, g\rangle_{\mathcal{H}^{\otimes 2}} = \sum_{i,j=1}^{2}\int_{[0,1]^2}\int_{[0,1]^2} f_{ij}(s,u)g_{ij}(t,v)dR_i(s,t)dR_j(u,v), \text{ for all } g = (g_{ij})_{i,j=1,2} \in \mathcal{E}_2^{\otimes 2},$$

where the above integrals are iterated two-dimensional Young–Stieltjes integrals (see Definition (3.6)),

$$\int_{[0,1]^2}\Big(\int_{[0,1]^2} f_{ij}(s,u)g_{ij}(t,v)dR_i(s,t)\Big)dR_j(u,v).$$

Note that we are not enlarging the space $\mathcal{H}^{\odot 2}$ but renaming the element $\hat{f}$ by f, since under the inner product in $\mathcal{H}^{\odot 2}$ they are indistinguishable. Thus the map I_2 is well defined for $\hat{f}$ and we put $I_2(f) := I_2(\hat{f})$. This is a common procedure to ease the identification of the elements of Hilbert spaces which have been constructed by completion, see Huang and Cambanis (1978). Therefore we will define the generalised Lévy area in the following way:

Definition 2.2 (Generalised Lévy area)**.** *We say that $I_2(f^{\mathcal{L}})$ is a generalised Lévy area if there exists $\hat{f}^{\mathcal{L}} \in \mathcal{H}^{\odot 2}$ such that*

$$\langle \hat{f}^{\mathcal{L}}, g\rangle_{\mathcal{H}^{\otimes 2}} = \sum_{i,j=1}^{2}\int_{[0,1]^2}\int_{[0,1]^2} f_{ij}^{\mathcal{L}}(s,u)g_{ij}(t,v)dR_i(s,t)dR_j(u,v)\,, \tag{12}$$

for all $g = (g_{ij})_{i,j=1,2} \in \mathcal{E}_2^{\otimes 2}$. Then $I_2(f^{\mathcal{L}}) := I_2(\hat{f}^{\mathcal{L}})$.

Another problem we have to face in order to make this definition tractable is that we do not know how to compute the inner product $\langle \hat{f}^{\mathcal{L}}, g\rangle_{\mathcal{H}^{\otimes 2}}$. Indeed, Lemma 2.1 only shows how to calculate the inner product of elements of $\mathcal{E}_2^{\otimes 2}$. Hence we will need to approximate $\hat{f}^{\mathcal{L}}$ by elements of $\mathcal{E}_2^{\otimes 2}$ and check equality (12) using a limit procedure.

3 Existence of a generalised Lévy area

This section gives the sufficient conditions on the processes $\{X_1(t) \mid 0 \le t \le 1\}$ and $\{X_2(t) \mid 0 \le t \le 1\}$ so a generalised Lévy area exists (see Definition 2.2). In fact the conditions on the processes will be constrains on their covariance functions, and indeed this is what the previous sections suggest as the Hilbert space of the domain of I_2 is characterised by the covariance functions of X_1 and X_2. Friz and Victoir (2010a) claim the p–variation of the covariance function of a Gaussian process to be a fundamental quantity related to the process. Therefore, we first recall some definitions on the p–variation of a function.

3.1 Functions of finite p–variation and Young–Stieltjes integral

For the sake of completeness and to introduce notation, we here give some definitions on the p-variation of a function and the Young–Stieltjes integral.

For a given interval of the real line $[s,t]$ such that $s \le t$, we will denote the set of all partitions of $[s,t]$ by

$$\mathcal{P}([s,t]) := \{\{t_0,\dots,t_n\} \mid s = t_0 < t_1 < \ldots < t_n = t,\ n \in \mathbb{N}\}\,.$$

For $D \in \mathcal{P}([s,t])$ we write $|D| := \max_{t_i \in D}\{|t_i - t_{i-1}|\}$.

Definition 3.1. *Let $f : [s,t] \to \mathbb{R}$ be a function and $p \ge 1$. We say that f has finite p–variation if $V_p^1(f,[s,t]) < \infty$, where*

$$V_p^1(f,[s,t]) := \sup_{D\in\mathcal{P}([s,t])} \left(\sum_{t_i \in D} |f(t_{i+1}) - f(t_i)|^p \right)^{1/p}.$$

The superscript on V_p^1 is to stress that f is 1–dimensional in contrast to the p–variation of a 2–dimensional function, which it is defined below.

Definition 3.2. *Let $f : [s,t]\times[u,v] \to \mathbb{R}$ be a function and $p \ge 1$. We say that f has finite p–variation if $V_p^2(f,[s,t]\times[u,v]) < \infty$, where*

$$V_p^2(f,[s,t]\times[u,v]) := \sup_{\substack{D\in\mathcal{P}([s,t])\\ D'\in\mathcal{P}([u,v])}} \left(\sum_{(t_i,t'_j)\in D\times D'} \left| f\begin{pmatrix} t_i & t'_j \\ t_{i+1} & t'_{j+1}\end{pmatrix}\right|^p \right)^{1/p}$$

and

$$f\begin{pmatrix} t_i & t'_j \\ t_{i+1} & t'_{j+1}\end{pmatrix} := f(t_{i+1},t'_{j+1}) - f(t_{i+1},t'_j) - f(t_i,t'_{j+1}) + f(t_i,t'_j)\,.$$

Another important concept related to the p-variation is the *control map* (see Friz and Victoir (2010b, 2011a). Write $\Delta = \{(s,t) :\ 0 \le s \le t \le 1\}$. In the following definition we will deal with a function ω defined on $\Delta\times\Delta$, and we will interpret a *point* $\big((s,t),(u,v)\big)$ as a rectangle $[s,t]\times[u,v]$.

Definition 3.3. *A $2D$ control is a map $\omega :\ \Delta\times\Delta \longrightarrow [0,\infty)$ continuous, zero if the rectangle has area zero, and super–additive in the sense that if $R \subset [0,1]^2$ is a rectangle, and $\{R_1,\dots,R_n\}$ is a finite partition of R in rectangles, then*

$$\sum_{i=1}^n \omega(R_i) \le \omega(R).$$

The relationship between the control and the p–variation is given by the following lemma, which is a direct consequence of Theorem 1 of Friz and Victoir (2011a) (see also Friz and Victoir (2011b) for a correction of Friz and Victoir (2010b)).

Lemma 3.4. *Let $f : [0,1]^2 \to \mathbb{R}$ be a continuous function of finite p–variation. Then for every $\varepsilon > 0$ there is a $2D$ control map ω such that for every rectangle $R \subset [0,1]^2$*

$$V_{p+\varepsilon}^2(f,R) \le \omega^{1/(p+\varepsilon)}(R)\,.$$

We need the following technical result about the product of control maps; see Friz and Victoir (2010b).

Lemma 3.5. *Let ω_1 and ω_2 be $2D$ control maps and $p,q > 0$ such that $p^{-1}+q^{-1} \ge 1$, then $\omega_1^{1/p}\omega_2^{1/q}$ is also a $2D$ control map.*

Finally, we recall the definition of the two-dimensional Young–Stieltjes integral and the statement of Young's inequality for a 2-dimensional function as it is given in Towghi (2002) (see also Friz and Victoir (2010a)):

Definition 3.6. *Let $f,g : [0,1]^2 \to \mathbb{R}$ be two functions, and let $D = \{t_i\}$ and $D' = \{t'_j\}$ be two partitions of $[0,1]$. Consider the Riemman–Stieltjes type sum*

$$L(f,g,D,D') = \sum_{i,j} f(s_i,s'_j)\, g\begin{pmatrix} t_i & t'_j \\ t_{i+1} & t'_{j+1}\end{pmatrix}$$

where $s_i, s'_j \in [t_i,t_{i+1}]\times[t'_j,t'_{j+1}]$. If $L(f,g,D,D')$ converges when $\lim\max\{|D|,|D'|\} = 0$, then the limit, denoted by $\int_{[0,1]^2} f\,dg$, is called the Young–Stieltjes integral of f with respect to g.

Remark 3.7. Note that $\mathbf{1}_{(0,u]\times(0,v]}$ and $\mathbf{1}_{[0,u]\times[0,v]}$ are Young–Stieltjes integrable with respect to R_i, and both integrals are equal to $R_i(u,v)$, so notation (7) is coherent with this definition.

Theorem 3.8. (Towghi (2002)). *Assume $p^{-1}+q^{-1}>1$. Let $f,g:[0,1]^2\to\mathbb{R}$ be functions and g continuous such that*

$$||f||_{W_p^2([0,1]^2)} := V_p^2(f,[0,1]^2) + V_p^1(f(0,\cdot),[0,1]) + V_p^1(f(\cdot,0),[0,1]) + |f(0,0)| < \infty$$

and $V_q^2(g,[0,1]^2)<\infty$. Then the Young–Stieltjes integral of f with respect to g exists, and

$$\left|\int_{[0,1]^2} f\,dg\right| \le c(p,q)||f||_{W_p^2([0,1]^2)} V_q^2(g,[0,1]^2)\ ,$$

where $c(p,q)$ is a constant independent of f and g.

Remark 3.9. The definition of finite p–variation could be stated for $p>0$ both in the 1–dimensional and in the 2–dimensional case, but we restrict ourselves to $p\ge 1$. This is because a 1–dimensional continuous function of finite p–variation for $p<1$ is constant (see Friz and Victoir (2010a), page 78). This is not true for the 2–dimensional case: For example the function $f(x,y)=x+y$ has finite p–variation for all $p>0$. However, for continuous covariance functions coming from processes that start at 0 it is true (see next result). We will see in the next section that the hypotheses related with the finite variation are always with respect to continuous covariance functions. Therefore, without lost of generality, we consider p–variations for $p\ge 1$.

Lemma 3.10. *A continuous function f on $[0,1]^2$ such that $f(0,0)=f(s,0)=f(0,t)$ for all $t,s\in[0,1]$ and of finite p–variation with $p<1$ is constant.*

Proof. For a fixed $a\in[0,1]$, the function $y\to f(a,y)-f(0,y)$ is a 1–dimensional continuous function of finite p–variation and hence constant. Indeed, it is zero since $f(a,0)=f(0,0)$, and the result follows. ■

3.2 Main result

The main result of the paper is proved in this section. We construct a sequence of elements of $\mathcal{E}_2^{\otimes 2}$ which converges almost everywhere to the Lévy kernel and show that it is a Cauchy sequence in $\mathcal{H}^{\otimes 2}$, and finally, we also show that its limit satisfies Definition 2.2.

We start by a technical lemma which will ease the proof of the main result, but, before that, let us introduce some notation. Let $\{t_i^n = i2^{-n} \mid i=0,\dots,2^n-1\}$ be the dyadic partition of the interval $[0,1]$ for a given n, and consider the dyadic partition of the triangles T_1 and T_2 (see (6))

$$T_1^n := \bigcup_{i<j} I_i^n\times I_j^n \qquad T_2^n := \bigcup_{i>j} I_i^n\times I_j^n\ ,$$

where $I_i^n := (t_i^n, t_{i+1}^n]$. Then a natural approximation of the Lévy kernel by symmetric functions of $\mathcal{E}_2^{\otimes 2}$ is

$$f_n((s,i),(t,j)) := \begin{cases} 0, & \text{if } i=j \\ \frac{1}{2}(\mathbf{1}_{T_1^n}(s,t)-\mathbf{1}_{T_2^n}(s,t)), & \text{if } i=1,j=2 \\ \frac{1}{2}(\mathbf{1}_{T_2^n}(s,t)-\mathbf{1}_{T_1^n}(s,t)), & \text{if } i=2,j=1 \end{cases}\ .$$

Lemma 3.11. *Let R_1 and R_2 be two continuous covariance functions on $[0,1]^2$. Let R_1 be of finite p–variation and R_2 of finite q–variation and assume that $p^{-1}+q^{-1}>1$, then*

$$\lim_{\substack{n\to\infty\\ m\to\infty}} \underbrace{\sum_{i,j=1}^{2}\int_{[0,1]^2}\int_{[0,1]^2} (f^{\mathcal{L}}-f_n)_{ij}(s,u)\cdot(f^{\mathcal{L}}-f_m)_{ij}(t,v)dR_i(s,t)dR_j(u,v)}_{(\star)} = 0\ .$$

Proof. Write $J_{k,l}^{n,m} := I_k^n\times I_l^m$ and note that

$$(f^{\mathcal{L}}-f_n)_{ij}(s,u)\cdot(f^{\mathcal{L}}-f_m)_{ij}(t,v) = \frac{(1-\delta_{ij})}{4}\sum_{k=0}^{2^n-1}\sum_{l=0}^{2^m-1}\mathbf{1}_{(J_{k,l}^{n,m})^2}(s,t,u,v)(\mathbf{1}_{\{(v-t)(u-s)>0\}}-\mathbf{1}_{\{(v-t)(u-s)<0\}})\ .$$

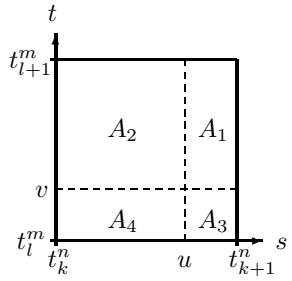

Figure 1. Decomposition of rectangle $J^{n,m}_{k,l}$ for a fixed $(u,v) \in J^{n,m}_{k,l}$ used in the integration (**).

Therefore the double integral of the $(\star)$-term is split into a sum of double integrals over $(J^{n,m}_{k,l})^2$. These integrals are iterated integrals and they can be further reduced, according to Figure 1, to

$$(\star) = \frac{1}{4} \sum_{\substack{i,j=1 \\ i\neq j}}^{2} \underbrace{\sum_{k=0}^{2^n-1} \sum_{l=0}^{2^m-1} \int_{J^{n,m}_{k,l}} F^{n,m,i}_{k,l}(u,v) dR_j(u,v)}_{(\star\star)} ,$$

where

$$\begin{aligned} F^{n,m,i}_{k,l}(u,v) &:= \int_{A_1} dR_i(s,t) - \int_{A_2} dR_i(s,t) - \int_{A_3} dR_i(s,t) + \int_{A_4} dR_i(s,t) \\ &= R_i \begin{pmatrix} u & , & v \\ t^n_{k+1} & , & t^m_{l+1} \end{pmatrix} - R_i \begin{pmatrix} t^n_k & , & v \\ u & , & t^m_{l+1} \end{pmatrix} - R_i \begin{pmatrix} u & , & t^m_l \\ t^n_{k+1} & , & v \end{pmatrix} + R_i \begin{pmatrix} t^n_k & , & t^m_l \\ u & , & v \end{pmatrix} . \end{aligned}$$

Note that in the above definition we have used Remark 3.7.

It is enough to prove that the $(\star\star)$-term goes to zero as $n, m \to \infty$ for $i = 1$ and $j = 2$. The key point is to apply Young's inequality to each sumand of the $(\star\star)$-term, and in that way we will prove the existence of the integrals and get a bound for them. In order to do so note the following identities which relate the function $F^{n,m,1}_{k,l}$ with the function R_1

$$F^{n,m,1}_{k,l} \begin{pmatrix} u & , & v \\ u' & , & v' \end{pmatrix} = 4R_1 \begin{pmatrix} u & , & v \\ u' & , & v' \end{pmatrix} , \quad F^{n,m,1}_{k,l}(t^n_k, v) - F^{n,m,1}_{k,l}(t^n_k, v') = 2R_1 \begin{pmatrix} t^n_k & , & v \\ t^n_{k+1} & , & v' \end{pmatrix} ,$$

$$F^{n,m,1}_{k,l}(t^n_k, t^m_l) = R_1 \begin{pmatrix} t^n_k & , & t^m_l \\ t^n_{k+1} & , & t^m_{l+1} \end{pmatrix} \text{ and } F^{n,m,1}_{k,l}(u, t^m_l) - F^{n,m,1}_{k,l}(u', t^m_l) = 2R_1 \begin{pmatrix} u & , & t^m_l \\ u' & , & t^m_{l+1} \end{pmatrix} .$$

Now, since $p^{-1} + q^{-1} > 1$, there exist $p'' > p' > p$ and $q' > q$ such that $(p'')^{-1} + (q')^{-1} > 1$. Thus, R_1 is also of finite $p'$and p''–variation, and R_2 is of finite q'-variation. Now use the above equalities to obtain

$$\begin{aligned} V^2_{p''}(F^{n,m,1}_{k,l}, \overline{J^{n,m}_{k,l}}) &= 4V^2_{p''}(R_1, \overline{J^{n,m}_{k,l}}) \quad , \quad V^1_{p''}(F^{n,m,1}_{k,l}(t^n_k, \cdot), [t^m_l, t^m_{l+1}]) \leq 2V^2_{p''}(R_1, \overline{J^{n,m}_{k,l}}) , \\ \left| F^{n,m,1}_{k,l}(t^n_k, t^m_l) \right| &\leq V^2_{p''}(R_1, \overline{J^{n,m}_{k,l}}) \quad \text{and} \quad V^1_{p''}(F^{n,m,1}_{k,l}(\cdot, t^m_l), [t^n_k, t^n_{k+1}]) \leq 2V^2_{p''}(R_1, \overline{J^{n,m}_{k,l}}) , \end{aligned}$$

where $\overline{J^{n,m}_{k,l}}$ is the closure of $J^{n,m}_{k,l}$. This consideration is a technicality required by the definition of the finite variation. According to the notation of Theorem 3.8 the above inequalities imply that $||F^{n,m,1}_{k,l}||_{W^2_{p''}(\overline{J^{n,m}_{k,l}})} \leq 9V^2_{p''}(R_1, \overline{J^{n,m}_{k,l}})$, and thus we can apply Young's inequality to every integral of the $(\star\star)$-term to get

$$|(\star\star)| \leq c(p'', q') \sum_{k=0}^{2^n-1} \sum_{l=0}^{2^m-1} ||F^{n,m,1}_{k,l}||_{W^2_{p''}(\overline{J^{n,m}_{k,l}})} V^2_{q'}(R_2, \overline{J^{n,m}_{k,l}}) \leq 9c(p'', q') \sum_{k=0}^{2^n-1} \sum_{l=0}^{2^m-1} V^2_{p''}(R_1, \overline{J^{n,m}_{k,l}}) V^2_{q'}(R_2, \overline{J^{n,m}_{k,l}}) .$$

Recall Lemma 3.4 to associate the finite p'–variation of R_1 (respectively the q'–variation of R_2) to a control map ω_1 (resp.

ω_2). Thus

$$|(\star\star)| \leq C \sum_{k=0}^{2^n-1}\sum_{l=0}^{2^m-1} \left(\sup_{\substack{D\in\mathcal{P}(\overline{I_k^n})\\ D'\in\mathcal{P}(\overline{I_l^m})}} \left(\sum_{(t_i,t_j')\in D\times D'} \left| R_1 \begin{pmatrix} t_i & , & t_j' \\ t_{i+1} & , & t_{j+1}' \end{pmatrix} \right|^{p''} \right)^{1/p''} \right) V_{q'}^2(R_2, \overline{J_{k,l}^{n,m}})$$

$$\leq C \sup_{\substack{|u-u'|\leq 2^{-n}\\ |v-v'|\leq 2^{-m}}} \left| R_1 \begin{pmatrix} u & , & v \\ u' & , & v' \end{pmatrix} \right|^{\frac{p''-p'}{p''}} \sum_{k=0}^{2^n-1}\sum_{l=0}^{2^m-1} (V_{p'}^2(R_1, \overline{J_{k,l}^{n,m}}))^{p'/p''} V_{q'}^2(R_2, \overline{J_{k,l}^{n,m}})$$

$$\leq C \sup_{\substack{|u-u'|\leq 2^{-n}\\ |v-v'|\leq 2^{-m}}} \left| R_1 \begin{pmatrix} u & , & v \\ u' & , & v' \end{pmatrix} \right|^{\frac{p''-p'}{p''}} \sum_{k=0}^{2^n-1}\sum_{l=0}^{2^m-1} \omega_1^{1/p''}(\overline{J_{k,l}^{n,m}}) \omega_2^{1/q'}(\overline{J_{k,l}^{n,m}})$$

$$\leq C \sup_{\substack{|u-u'|\leq 2^{-n}\\ |v-v'|\leq 2^{-m}}} \left| R_1 \begin{pmatrix} u & , & v \\ u' & , & v' \end{pmatrix} \right|^{\frac{p''-p'}{p''}} \sum_{k=0}^{2^n-1}\sum_{l=0}^{2^m-1} \hat{\omega}(\overline{J_{k,l}^{n,m}}) \leq C\hat{\omega}([0,1]^2) \sup_{\substack{|u-u'|\leq 2^{-n}\\ |v-v'|\leq 2^{-m}}} \left| R_1 \begin{pmatrix} u & , & v \\ u' & , & v' \end{pmatrix} \right|^{\frac{p''-p'}{p''}},$$

where C is a constant which is renamed when necessary, and we applied Lemma 3.5 to deduce that $\hat{\omega} := \omega_1^{1/p''}\omega_1^{1/q'}$ is a control map. Finally note that the last supremum goes to zero as $n, m \to \infty$ by the uniform continuity of R_1. ■

Now to finalize the construction of the generalized Lévy area we need to prove that the sequence $\{f_n\}$ in Lemma 3.11 is a Cauchy sequence and its limit, $\hat{f}^{\mathcal{L}}$, satisfies Definition 2.2.

Theorem 3.12. *Let $\{X_1(t) \mid 0 \leq t \leq 1\}$ and $\{X_2(t) \mid 0 \leq t \leq 1\}$ be two continuous centered Gaussian processes such that $X_1(0) = X_2(0) = 0$, independent, and with covariance functions R_1 and R_2 respectively. Let R_1 be of finite p-variation and R_2 be of finite q-variation and assume that $p^{-1} + q^{-1} > 1$, then the sequence $\{f_n\}_{n\geq 1}$ is a Cauchy sequence in $\mathcal{H}^{\otimes 2}$. Moreover, if we denote its limit by $\hat{f}^{\mathcal{L}}$, we have that*

$$\langle \hat{f}^{\mathcal{L}}, g\rangle_{\mathcal{H}^{\otimes 2}} = \sum_{i,j=1}^{2} \int_{[0,1]^2}\int_{[0,1]^2} f_{ij}^{\mathcal{L}}(s,u) g_{ij}(t,v) dR_i(s,t) dR_j(u,v) \tag{13}$$

for all $g = (g_{ij})_{i,j=1,2} \in \mathcal{E}_2^{\otimes 2}$.

Proof. Note that $f_n - f_m \in \mathcal{E}_2^{\otimes 2}$. Therefore by Lemma 2.1 we have that

$$||f_n - f_m||_{\mathcal{H}^{\otimes 2}} = \sum_{\substack{i,j=1\\ i\neq j}}^{2} \int_{[0,1]^2}\int_{[0,1]^2} (f_n - f_m)_{ij}(s,u)\cdot(f_n - f_m)_{ij}(t,v) dR_i(s,t) dR_j(u,v)$$

$$= \sum_{\substack{i,j=1\\ i\neq j}}^{2} \int_{[0,1]^2}\int_{[0,1]^2} (f_n - f^{\mathcal{L}} + f^{\mathcal{L}} - f_m)_{ij}(s,u)\cdot(f_n - f^{\mathcal{L}} + f^{\mathcal{L}} - f_m)_{ij}(t,v) dR_i(s,t) dR_j(u,v)\,.$$

Each term of the above product was denoted as a $(\star)$-term in Lemma 3.11 and thus goes to zero as $n, m \to \infty$.

For the second part of the proof it suffices to prove the equality for a function g such that $g_{12}(s,t) = \mathbf{1}_{[a,b]\times[c,d]}(s,t)$ where $[a,b]\times[c,d] \subseteq [0,1]^2$ and $g_{ij}(s,t) \equiv 0$ for $i = j$. Since $\hat{f}^{\mathcal{L}}$ is the limit of $\{f_n\}_{n\geq 1}$ in $\mathcal{H}^{\otimes 2}$ then

$$\lim_{n\to\infty} \langle f_n, g\rangle_{\mathcal{H}^{\otimes 2}} = \langle \hat{f}^{\mathcal{L}}, g\rangle_{\mathcal{H}^{\otimes 2}}\,.$$

Our objective is to prove that $\lim_{n\to\infty}\langle f_n, g\rangle_{\mathcal{H}^{\otimes 2}}$ equals the right hand side of equation (13). From the definition of the

Lévy kernel we have that

$$\begin{aligned}
&\sum_{i,j=1}^{2}\int_{[0,1]^2}\int_{[0,1]^2} f_{ij}^{\mathcal{L}}(s,u)g_{ij}(t,v)dR_i(s,t)dR_j(u,v) \\
&\qquad = \frac{1}{2}\int_{[0,1]^2}\int_{[0,1]^2}(\mathbf{1}_{s>u}(s,u)-\mathbf{1}_{s<u}(s,u))\mathbf{1}_{[a,b]\times[c,d]}(t,v)dR_1(s,t)dR_2(u,v) \\
&\qquad = \frac{1}{2}\int_{[0,1]\times[c,d]} dR_2(u,v)\int_{[0,1]\times[a,b]}(\mathbf{1}_{s>u}(s,u)-\mathbf{1}_{s<u}(s,u))dR_1(s,t) \\
&\qquad = \frac{1}{2}\int_{[0,1]\times[c,d]}\left[R_1\begin{pmatrix} u & , & a \\ 1 & , & b\end{pmatrix} - R_1\begin{pmatrix} 0 & , & a \\ u & , & b\end{pmatrix}\right] dR_2(u,v)\,. \qquad (14)
\end{aligned}$$

The above integral is a well defined Young–Stieltjes integral. Then, for $D\in\mathcal{P}([0,1])$ and $D'\in\mathcal{P}([c,d])$,

$$\begin{aligned}
&\sum_{i,j=1}^{2}\int_{[0,1]^2}\int_{[0,1]^2} f_{ij}^{\mathcal{L}}(s,u)g_{ij}(t,v)dR_i(s,t)dR_j(u,v) \\
&\qquad = \frac{1}{2}\lim_{\substack{|D|\to 0\\ |D'|\to 0}}\sum_{\substack{\xi_i\in D\\ \zeta_j\in D'}}\left[R_1\begin{pmatrix} \nu_i & , & a \\ 1 & , & b\end{pmatrix} - R_1\begin{pmatrix} 0 & , & a \\ \nu_i & , & b\end{pmatrix}\right] R_2\begin{pmatrix} \xi_i & , & \zeta_j \\ \xi_{i+1} & , & \zeta_{j+1}\end{pmatrix} \\
&\qquad = \frac{1}{2}\lim_{|D|\to 0}\sum_{\xi_i\in D}\left[R_1\begin{pmatrix} \nu_i & , & a \\ 1 & , & b\end{pmatrix} - R_1\begin{pmatrix} 0 & , & a \\ \nu_i & , & b\end{pmatrix}\right] R_2\begin{pmatrix} \xi_i & , & c \\ \xi_{i+1} & , & d\end{pmatrix}\,, \qquad (15)
\end{aligned}$$

where $\nu_i\in[\xi_i,\xi_{i+1}]$. On the other hand, from Lemma 2.1 we have that

$$\begin{aligned}
\langle f_n, g\rangle_{\mathcal{H}^{\otimes 2}} &= \frac{1}{2}\sum_{\substack{k,l=0\\ k>l}}^{2^n-1}\int_{I_k^n\times[a,b]}\int_{I_l^n\times[c,d]} dR_1(s,t)dR_2(u,v) - \frac{1}{2}\sum_{\substack{k,l=0\\ k<l}}^{2^n-1}\int_{I_k^n\times[a,b]}\int_{I_l^n\times[c,d]} dR_1(s,t)dR_2(u,v) \\
&= \frac{1}{2}\sum_{\substack{k,l=0\\ k>l}}^{2^n-1} R_1\begin{pmatrix} t_k^n & , & a \\ t_{k+1}^n & , & b\end{pmatrix} R_2\begin{pmatrix} t_l^n & , & c \\ t_{l+1}^n & , & d\end{pmatrix} - \frac{1}{2}\sum_{\substack{k,l=0\\ k<l}}^{2^n-1} R_1\begin{pmatrix} t_k^n & , & a \\ t_{k+1}^n & , & b\end{pmatrix} R_2\begin{pmatrix} t_l^n & , & c \\ t_{l+1}^n & , & d\end{pmatrix} \\
&= \frac{1}{2}\sum_{l=0}^{2^n-1}\left[R_1\begin{pmatrix} t_{l+1}^n & , & a \\ 1 & , & b\end{pmatrix} - R_1\begin{pmatrix} 0 & , & a \\ t_l^n & , & b\end{pmatrix}\right] R_2\begin{pmatrix} t_l^n & , & c \\ t_{l+1}^n & , & d\end{pmatrix}\,. \qquad (16)
\end{aligned}$$

Note that in equation (15) we could replace the first ν_i by ν_i^1 and the second ν_i by ν_i^2, where $\nu_i^1,\nu_i^2\in[\xi_i,\xi_{i+1}]$. This is because the integral (14) could be split into two. Indeed we can choose $\nu_i^1=\xi_i$ and $\nu_i^2=\xi_{i+1}$, and then equation (16) becomes a particular election of the partition D in equation (15) and thus

$$\lim_{n\to\infty}\langle f_n,g\rangle_{\mathcal{H}^{\otimes 2}} = \int_{[0,1]^2}\int_{[0,1]^2} f_{12}^{\mathcal{L}}(s,u)g_{12}(t,v)dR_1(s,t)dR_2(u,v)$$

from where the result follows. ■

3.3 The case of the fractional Brownian motion

One case of special interest is to explore what happens with the generalised Lévy area for a fractional Brownian motion (fBm) of Hurst parameter $H\in(0,1)$. It is known that when fBm has Hurst parameter $H\leq\frac{1}{2}$ then its covariance function is of finite $\frac{1}{2H}$-variation (see Friz and Victoir (2010b) page 405), and when $H>\frac{1}{2}$ then its covariance function is of bounded variation. Then the Lévy area can be defined for two independent fBm with parameters H and H' as long as $H+H'>\frac{1}{2}$. Hence we can let one of the processes be as irregular as desired provided that the other one is regular enough. That is, we can let one of the fBm be of Hurst parameter $H<\frac{1}{2}$ as long as the other independent fBm has Hurst parameter $H'>\frac{1}{2}-H$. When $H=H'$ then the condition is $H>\frac{1}{4}$.

4 On the characteristic function of the Lévy area

Consider an isonormal Gaussian process $X = \{X(h) \mid h \in \mathcal{H}\}$, where the separable Hilbert space $\mathcal{H}$ is of the form $L^2(M, \mu)$, where μ is σ–finite. As we commented before, in this case $\mathcal{H}^{\otimes 2}$ is isomorphic to $L^2(M^2, \mu^{\otimes 2})$. To each element $f \in L^2(M^2, \mu^{\otimes 2})$ corresponds a Hilbert–Schmidt operator $F : \mathcal{H} \to \mathcal{H}$ defined by

$$(F(h))(t) = \int_M f(s,t)h(s)\mu(ds) \; .$$

Assume that f is symmetric. Then it can be expressed as $f(s,t) = \sum_n \alpha_n \varphi(s)\varphi(t)$, where $\{\varphi_n \mid n \geq 1\}$ is an enumeration of the (orthonormal) eignefunctions of F and $\{\alpha_n \mid n \geq 1\}$ are the corresponding eigenvalues (that appear repeated according to its multiplicity) (see, for example, Dunford and Schwartz (1988), page 1087, or Neveu (1968), Proposition 6.18). Working in a similar way as in the example of the Lévy area in the Introduction, it is deduced that

$$\mathbb{E}[e^{zI_2(f)}] = \frac{1}{\sqrt{\prod_{n=1}^{\infty}(1 - 2z\alpha_n)e^{2z\alpha_n}}} \qquad \text{for } \; 2|\Re(z)|\sigma(F) < 1 \; ,$$

where $\sigma(F) = \sup_{n \geq 1}\{|\alpha_n|\}$, which is the formula given by Janson (1997) page 78. The infinite product of the above expression is called a generalised determinant or a Carleman–Fredholm determinant of F (see Dunford and Schwartz (1988), page 1036).

There are many works about the characteristic function of quadratic Wiener functionals. In our case we are interested in a particular functional viewed in different Wiener spaces, therefore the aim of this section is to explicit the procedure to compute the eigenvalues as much as possible. It is worthwhile to state the relationship between elements of the second Wiener chaos and Hilbert–Schmidt operators by working out a particular example and then extending the results to the general case. To this end we study the case of the Lévy area for two standard Wiener processes, as we commented in the Introduction. In this case, $R_1(s,t) = R_2(s,t) = s \wedge t$, and the Hilbert space $\mathcal{H}$ is isomorphic to $L^2([0,1] \times \{1,2\}, dt \otimes \text{Card})$. Then the Lévy kernel $f^{\mathcal{L}}$ defines the Hilbert–Schmidt operator

$$\begin{aligned} F : L^2([0,1] \times \{1,2\}) &\to L^2([0,1] \times \{1,2\}) \\ h &\mapsto \int_{[0,1]\times\{1,2\}} f_{ij}^{\mathcal{L}}(s,t)h_i(s)ds \otimes \text{Card} \; , \end{aligned}$$

which is reduced to the form

$$F(h)_j(t) = \frac{\delta_{2j}}{2}\left(\int_0^t h_1(s)ds - \int_t^1 h_1(s)ds\right) - \frac{\delta_{1j}}{2}\left(\int_0^t h_2(s)ds - \int_t^1 h_2(s)ds\right) \; . \tag{17}$$

Let h be an eigenvector of eigenvalue α, $F(h) = \alpha h$, therefore it is continuous because it is defined by an integral, and applying again the same argument it is differentiable. Then we differentiate the above expression and obtain the matrix differential equation

$$\begin{pmatrix} h_1'(t) \\ h_2'(t) \end{pmatrix} = \frac{1}{\alpha}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} h_1(t) \\ h_2(t) \end{pmatrix} = \frac{1}{\alpha}M\begin{pmatrix} h_1(t) \\ h_2(t) \end{pmatrix} \tag{18}$$

with solution given by $h(t) = e^{M\frac{t}{\alpha}}h(0)$. From (17) it is clear that $h(1) + h(0) = 0$ and thus the eigenvalues satisfy the equation

$$e^{M\frac{1}{\alpha}} = \begin{pmatrix} \cos(\alpha^{-1}) & -\sin(\alpha^{-1}) \\ \sin(\alpha^{-1}) & \cos(\alpha^{-1}) \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -Id_2 \; .$$

Therefore the eigenvalues, α_n, are $\{\pm(\pi(2n+1))^{-1} \mid n \in \mathbb{N}\}$ with multiplicity 2 since the space of solutions of the ordinary differential equation has dimension 2. Finally we compute the Carleman-Fredholm determinant to obtain $\mathbb{E}[e^{itA}] = \cosh(t)^{-1}$.

Now we use the same sort of ideas into the abstract setting presented in Section 2. Let $f \in \mathcal{H}^{\otimes 2}$ and define the operator $F := \Psi \circ \Phi_f : \mathcal{H} \to \mathcal{H}$, where $\Psi : \mathcal{H}^* \to \mathcal{H}$ is the duality isomorphism, and $\Phi_f : \mathcal{H} \to \mathcal{H}^*$ is defined in the following way: for $g \in \mathcal{H}$ set $\Phi_f(g) : \mathcal{H} \to \mathbb{R}$ as

$$\Phi_f(g)(h) := \langle f, g \otimes h\rangle_{\mathcal{H}^{\otimes 2}}, \qquad h \in \mathcal{H} \; .$$

Then F is a Hilbert–Schmidt operator (see Neveu (1968) Proposition 6.16). Note that $g \in \mathcal{H}$ is an eigenvector of the operator F with eigenvalue α if and only if $\langle f, g \otimes h\rangle_{\mathcal{H}^{\otimes 2}} = \alpha\langle g, h\rangle_{\mathcal{H}}$ for all $h \in \mathcal{H}$. Writing $g(t,i) = \delta_{1i}g_1(t) + \delta_{2i}g_2(t)$

where $g_i \in \mathcal{H}_i$ and similarly for h, then $g \in \mathcal{H}$ is an eigenvector of eigenvalue α if and only if

$$\sum_{i,j=1}^{2} \langle f_{ij}, g_i \otimes h_j \rangle_{\mathcal{H}_i \otimes \mathcal{H}_j} = \alpha \sum_{i=1}^{2} \langle g_i, h_i \rangle_{\mathcal{H}_i} , \quad \forall h_1 \in \mathcal{H}_1, \, \forall h_2 \in \mathcal{H}_2 .$$

We will say that two covariance functions R_1 and R_2 are equivalent if the associated Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ are the same, i.e. they are composed by the same functions with the same scalar product. Under this symmetry of the processes we recover the spectrum structure of the classical Lévy area.

Proposition 4.1. *Let $\{X_1(t) \mid 0 \leq t \leq 1\}$ and $\{X_2(t) \mid 0 \leq t \leq 1\}$ be continuous centered independent Gaussian processes with equivalent covariance functions R_1 and R_2 respectively. Assume that the generalized Lévy area exits. Then the corresponding Hilbert–Schmidt operator has eigenvalues with even multiplicity and symmetric with respect to zero. As a consequence, the characteristic function of the generalized Lévy area is of the form*

$$\varphi(t) = \prod_{n \geq 1} \frac{1}{(1 + 4\alpha_n^2 t^2)^{m_n}} ,$$

where $m_n \geq 1$.

Proof. From the factorisation (9) it is clear that the symmetry of the approximation of the Lévy kernel $\{f_n\}_{n \geq 1}$ is transferred to $\hat{f}^{\mathcal{L}}$. Then, from equation

$$\sum_{\substack{i,j=1 \\ i \neq j}}^{2} \langle \hat{f}_{ij}^{\mathcal{L}}, g_i \otimes h_j \rangle_{\mathcal{H}_i \otimes \mathcal{H}_j} = \alpha \sum_{i=1}^{2} \langle g_i, h_i \rangle_{\mathcal{H}_i} \quad \forall h_1, h_2 \in \mathcal{H}_1 = \mathcal{H}_2 , \tag{19}$$

it is deduced that if $g(t,i) = \delta_{1i} g_1(t) + \delta_{2i} g_2(t)$ is an eigenvector with eigenvalue α, then $\tilde{g}(t,i) = \delta_{1i} g_2(t) - \delta_{2i} g_1(t)$ is an eigenvector with eigenvalue α, and $\hat{g}(t,i) = \delta_{1i} g_2(t) + \delta_{2i} g_1(t)$ is an eigenvector with eigenvalue $-\alpha$. If $g = \lambda \tilde{g}$ for $\lambda \neq 0$, then $g_1 = \lambda g_2 = -\lambda^2 g_1$ and hence $g \equiv 0$, thus g and $\tilde{g}$ are linear independent. On the other hand, it can be proved that if $\{g_1(t,i), \tilde{g}_1(t,i), \ldots, g_k(t,i), \tilde{g}_k(t,i), \, h(t,i)\}$ is a family of linear independent eigenvectors of eigenvalue α, then also $\tilde{h}$ is independent of that set. Therefore α has even multiplicity. Note that this suffices to deduce the same property for the eigenvalue $-\alpha$ and by construction the multiplicity of α and $-\alpha$ is the same. Finally, we recover the same structure for the spectrum of the Hilbert–Schmidt operator that we have in the classical Lévy area. ■

Example. From the explicit calculations made for the classical Lévy area we can easily work out a bit more general case. Let $X_i(t) = \int_0^t f(s) dW_i(s)$, for two independent Brownian motions W_1 and W_2, where $f \in L^2([0,1], dx)$. Then $R_i(s,v) = \int_0^{s \wedge v} f^2(u) du$, $\mathcal{H}_i = L^2([0,1], f^2(u)du)$ and equation (18) can be written as

$$\begin{pmatrix} h_1'(t) \\ h_2'(t) \end{pmatrix} = \frac{f^2(t)}{\alpha} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} h_1(t) \\ h_2(t) \end{pmatrix} . \tag{20}$$

Therefore the general solution is

$$h(t) = \exp\left\{ \int_0^t \frac{f^2(u)}{\alpha} \, du \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right\} h(0) .$$

Finally the characteristic function of A in this setting is $\mathbb{E}[e^{i\xi A}] = \operatorname{sech}\left(\xi ||f||_{L^2}^2\right)$.

A comment on the fractional Brownian motion. To find the eigenvalues of the Lévy area for the fractional Brownian motion is an open problem. The main difficulty is that the Hilbert space $\mathcal{H}$ corresponding to such a process is complicate; for example, as we comment in Subsection 2.1 for a Hurst parameter $H > 1/2$ that space is not a space of functions. Then it is not evident how to get an equation for the eigenfunctions as equation (18) in the Brownian motion or equation (20) in the previous example.

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