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A class of special subordinators with nested ranges

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Abstract

We construct, on a single probability space, a class of special subordinators $S^{(\alpha)}$, indexed by all measurable functions $\alpha : [0, 1] \rightarrow [0, 1]$. Constant functions correspond to stable subordinators. If $\alpha \leq \beta$, then the range of $S^{(\alpha)}$ is contained in the range of $S^{(\beta)}$. Other examples of special subordinators are given in the lattice case.

1 Introduction

Recall that a (possibly killed) subordinator $(S_t)_{t \geq 0}$ is a Lévy process on \mathbb{R}_+ with Laplace exponent given, for $\lambda \geq 0$, by

$$\phi(\lambda) = -\log \mathbb{E}[\exp(-\lambda S_1)] = a + b\lambda + \int_0^\infty \Pi(dx)(1 - e^{-\lambda x}) \quad (1)$$

The coefficient $b \geq 0$ is the drift, Π is the Lévy measure and $a \geq 0$ is a killing parameter. If $a > 0$, S is submarkovian. A function of the form (1) is called a *Bernstein function*.

The subordinator S is *special* if it admits a *dual* subordinator $(\widehat{S}_t)_{t \geq 0}$ with Laplace exponent $\widehat{\phi}$, such that for every $\lambda > 0$,

$$\phi(\lambda)\widehat{\phi}(\lambda) = \lambda \quad (2)$$

The canonical example is the case when S (resp. \widehat{S}) is the subordinator of the ascending (resp. descending) ladder times of a real-valued Lévy process X . In particular, if X drifts to $-\infty$, then S is a killed subordinator (that is, the parameter a in (1) is positive). If X is stable, then S and \widehat{S} are stable, with respective indices $\alpha = \mathbb{P}(X_1 > 0)$ and $1 - \alpha$. If X is symmetric, then S and \widehat{S}

are stable with index $1/2$. See, among others, Bertoin [1], Doney, [3], Schilling et al. [8] for numerous references on subordinators, Bernstein functions and the connections with fluctuation theory for Lévy processes.

The main goal of this paper is to introduce a family of special subordinators indexed by all measurable functions $\alpha : [0, 1] \rightarrow [0, 1]$. A property of this family is that we can construct the ranges of all these subordinators on a single probability space, with the property that if $\alpha \leq \beta$, then the range of $S^{(\alpha)}$ is contained in the range of $S^{(\beta)}$. Here are the statements:

Theorem 1 *For every measurable function $\alpha : [0, 1] \rightarrow [0, 1]$, there exists a special subordinator $(S_t^{(\alpha)})_{t \geq 0}$ with Laplace exponent*

$$\phi^{(\alpha)}(\lambda) = -\log \mathbb{E}[\exp(-\lambda S_1^{(\alpha)})] = \exp \int_0^1 \frac{(\lambda - 1)\alpha(x)}{1 + (\lambda - 1)x} dx$$

for $\lambda \geq 0$. Its dual is the subordinator $(S_t^{(1-\alpha)})_{t \geq 0}$.

When α is constant, $S^{(\alpha)}$ is stable with index α . Note that $\phi^{(\alpha)}(\lambda)$ can be viewed as a power series in $(\lambda - 1)$ and that the coefficients can be computed from the moments of the measure $\alpha(x)dx$ on $[0, 1]$. Therefore, if $\alpha \neq \beta$, $\phi^{(\alpha)} \neq \phi^{(\beta)}$.

Theorem 2 *Let $\mathcal{R}^{(\alpha)} \subset \mathbb{R}_+$ be the range of $S^{(\alpha)}$: $\mathcal{R}^{(\alpha)} = \{S_t^{(\alpha)}, t \geq 0\}$. One can construct, on a single probability space, the regenerative sets $\mathcal{R}^{(\alpha)}$ for all measurable functions $\alpha : [0, 1] \rightarrow [0, 1]$, with the property that if α, β are two measurable functions such that for every $x \in [0, 1]$, $\alpha(x) \leq \beta(x)$, then*

$$\mathcal{R}^{(\alpha)} \subset \mathcal{R}^{(\beta)}$$

The properties of a subordinator can be read from its Laplace exponent. In turn, the properties of this exponent can be deduced from the function α , see Proposition 1 in Section 3.

Our construction generalizes a former construction for stable processes. This was used to construct Ruelle cascades, using nested stable regenerative sets obtained by subordination [6]. Other constructions of regenerative sets can be found in [4, 5, 7, 9].

We first explain, in Section 2, a similar construction in the lattice case, that is, in the framework of integer-valued regenerative sets. We use it to prove Theorems 1 and 2 in Section 3. In the lattice case, an extension is given in Section 4. In particular, this extension includes a lattice version of the special subordinators described in [10]. It should be possible to give a continuous version of the construction described in Section 4, however, we shall not handle this question here.

2 The lattice case

The lattice equivalent of a subordinator is a random walk on $\mathbb{N} \cup \{\infty\}$ (we include here the possibility of killing the random walk by sending it to ∞).

Such a random walk S has a generating function $\psi(t) = \mathbb{E}(t^{S_1})$, defined for $t < 1$. The dual of S , if it exists, is the random walk \widehat{S} with generating function $\widehat{\psi}$ such that

$$(1 - \psi(t))(1 - \widehat{\psi}(t)) = 1 - t \quad (3)$$

which is a discrete version of (2). A lattice regenerative set is the range of a random walk on \mathbb{N} started at 0.

For instance, the set of ladder times of a discrete time real-valued random walk X is a lattice regenerative set. This regenerative set has a dual, namely the set of ladder times of $-X$.

It is a classical fact that a random subset R of $\mathbb{N} \cup \{\infty\}$ is a lattice regenerative set if and only if it contains 0 and satisfies the regenerativity property: for every $n \in \mathbb{N}$, conditionally on the event that $n \in R$, the set $R \cap [n, \infty]$ is independent of $R \cap [0, n]$ and has the same law as $R + n$.

We construct a family of random walks on \mathbb{N} , indexed by measurable functions α as in Theorem 1.

Construction 1.

Fix a measurable function $\alpha : [0, 1] \rightarrow [0, 1]$. Let $(X_n, n \geq 1)$ be iid random variables, uniformly distributed on $[0, 1]^2$. We denote $X_n = (h_n, U_n)$. One should view h as a height and U as a parameter. Say that X_n is α -green if $U_n \leq \alpha(h_n)$, and α -red otherwise. Say that an integer $k \in [1, n]$ is n -visible if $h_k \geq h_m$ for all integers $m \in [k, n]$. Finally, say that n percolates for α if, for every $k \leq n$ such that k is n -visible, X_k is α -green. Let $R^{(\alpha)}$ be the set of integers that percolate for α (by convention, 0 percolates for α).

See Figure 1. Green points are represented by black circles, red points by white circles and the black squares stand for the integers that percolate. The horizontal lines express the fact that the red point at 4 prevents 5, 6 and 7 from percolating.

Remark that if α is a constant, then the X_n are green or red with probability α (resp. $1 - \alpha$), independently of the height. This is a discrete version of the construction given in [6].

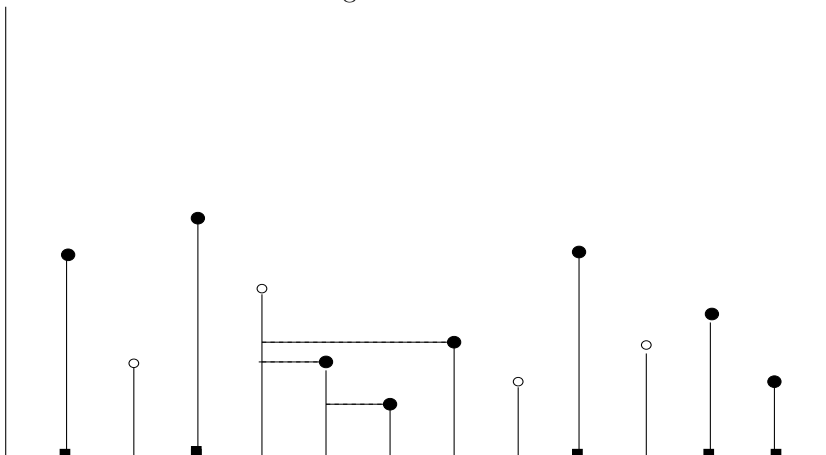
Theorem 3 *The set $R^{(\alpha)}$ defined by Construction 1 is a lattice regenerative set. It can be viewed as the image of a random walk $(S_n^{(\alpha)}, n \geq 0)$, where $S_n^{(\alpha)} = Y_1^{(\alpha)} + \dots + Y_n^{(\alpha)}$, the $Y_i^{(\alpha)}$ being iid random variables taking values in $\mathbb{N} \cup \{\infty\}$, with generating function*

$$\psi^{(\alpha)}(t) = \mathbb{E}(t^{Y_1^{(\alpha)}}) = 1 - \exp\left(-\int_0^1 \frac{t\alpha(x)}{1-tx} dx\right)$$

Moreover, $R^{(\alpha)}$ has a dual, namely $R^{(1-\alpha)}$.

From the very definition, the nested property of the sets $R^{(\alpha)}$ is obvious: if $\alpha \leq \beta$ and if X_n is α -green, then it is also β -green. Therefore $R^{(\alpha)} \subset R^{(\beta)}$. So we have immediately:

Figure 1: Construction 1



Theorem 4 *By construction, the sets $R^{(\alpha)}$ are defined on a single probability space, for all measurable functions $\alpha : [0, 1] \rightarrow [0, 1]$. Moreover, if α, β are two measurable functions with $\alpha \leq \beta$, then*

$$R^{(\alpha)} \subset R^{(\beta)}$$

Proof of Theorem 3.

Let $n \in \mathbb{N}$ and let E_n be the event that n is α -green. Conditionally on E_n , all the n -visible points are green. Moreover, for every $N \geq n$ and every $k \leq n$, if k is N -visible, then k is also n -visible. Therefore, for every $N \geq n$, conditionally on E_n , N is α -green if and only if all N -visible points in $[n+1, N]$ are α -green. This is independent of $(X_i, i \in [1, n])$ and has the same probability as the probability that $N - n$ is α -green. Hence $R^{(\alpha)}$ satisfies the regenerativity property.

Let us compute the probability that $n \in R^{(\alpha)}$. If n is α -green, then there is a left-most n -visible point, say n_1 , with height $x_1 = h_{n_1}$. Then n_1 has to be green, which occurs with probability $h(x_1)$, and for all $i \in [1, n_1 - 1]$, $h_i \leq x_1$, which occurs with probability $x_1^{n_1-1}$. If $n_1 \neq n$, then there is second left-most n -visible point, say $n_1 + n_2$, and so on. So we have

$$\begin{aligned} \mathbb{P}(n \in R^{(\alpha)}) &= \sum_k \sum_{n_1 + \dots + n_k = n} \\ &\int_0^1 dx_1 \int_0^{x_1} dx_2 \dots \int_0^{x_{k-1}} dx_k \alpha(x_1) x_1^{n_1-1} \dots \alpha(x_k) x_k^{n_k-1} \end{aligned}$$

By symmetrization,

$$\mathbb{P}(n \in R^{(\alpha)}) = \sum_k \frac{1}{k!} \sum_{n_1 + \dots + n_k = n} \int_0^1 dx_1 \dots \int_0^1 dx_k \alpha(x_1) x_1^{n_1-1} \dots \alpha(x_k) x_k^{n_k-1}$$

Summing over n ,

$$\sum_n \mathbb{P}(n \in R^{(\alpha)})t^n = \exp\left(\int_0^1 \frac{t\alpha(x)}{1-tx} dx\right)$$

On the other hand,

$$\sum_n \mathbb{P}(n \in R^{(\alpha)})t^n = \sum_i \mathbb{P}(S_i^{(\alpha)} = n)t^n = \sum_i \mathbb{E}(t^{S_i^{(\alpha)}}) = \frac{1}{1 - \mathbb{E}(t^{Y_1^{(\alpha)}})}$$

Finally, the duality property follows from a straightforward computation:

$$(1 - \psi^{(\alpha)}(t))(1 - \psi^{(1-\alpha)}(t)) = \exp\left(-\int_0^1 \frac{t}{1-tx} dx\right) = 1 - t$$

3 From the lattice case to the continuous case

3.1 Proof of Theorem 1

From Construction 1 we derive a continuous process. Consider the random walk $(S_n^{(\alpha)}, n \geq 0)$, let $(N_t, t \geq 0)$ be an independent Poisson process with parameter $\theta > 0$ and let e_1, e_2, \dots be iid exponential random variables with parameter $q > 0$, independent of $(N_t, t \geq 0)$ and $(S_n^{(\alpha)}, n \geq 0)$. Let $X_t = e_1 + \dots + e_{S_{N_t}^{(\alpha)}}$ for each $t \geq 0$. Then $(X_t, t \geq 0)$ is a subordinator and

$$\mathbb{E}(\exp(-\lambda X_1)) = \sum_{n \geq 0} \frac{\theta^n}{n!} e^{-\theta} (\psi^{(\alpha)}(q/(q-\lambda)))^n = \exp[-\theta(1 - \psi^{(\alpha)}(q/(q-\lambda)))]$$

Thus the Laplace exponent of X is

$$\phi(\lambda) = -\log \mathbb{E}(\exp(-\lambda X_1)) = \theta[1 - \psi^{(\alpha)}(q/(q-\lambda))]$$

That is,

$$\phi(\lambda) = \theta \exp\left(-\int_0^1 \frac{q\alpha(x)}{\lambda + q - qx} dx\right)$$

Remark that if $F : [0, 1] \rightarrow [0, 1]$ is a bijective, increasing function, then by a change of variable,

$$\phi(\lambda) = \theta \exp\left(-\int_0^1 \frac{q\alpha(F(x))}{\lambda + q - qx} F'(x) dx\right)$$

Now take an integer $m > 0$ and choose

$$F'(x) = \frac{\mathbf{1}_{\{x \in [1/m, 1]\}}}{(m-1)x^2}$$

$$q = m - 1$$

and

$$\theta = \int_{1/m}^1 \frac{\alpha(F(x))}{x}$$

Then

$$\phi_m^{(\alpha)}(\lambda) = \exp \left(- \int_{1/m}^1 \frac{\alpha(F(x))}{x[1 + (\lambda - 1)x]} dx + \int_{1/m}^1 \frac{\alpha(F(x))}{x} dx \right)$$

That is,

$$\phi_m^{(\alpha)}(\lambda) = \exp \left(\int_{1/m}^1 \frac{(\lambda - 1)\alpha(F(x))}{1 + (\lambda - 1)x} dx \right)$$

and any function $\phi_m^{(\alpha)}$ of this form is a Bernstein function. It is known [8] that every limit of Bernstein functions is a Bernstein function. Therefore, denoting $\beta(x) = \alpha(F(x))$, the function

$$\phi^{(\beta)}(\lambda) = \exp \left(\int_0^1 \frac{(\lambda - 1)\beta(x)}{1 + (\lambda - 1)x} dx \right)$$

is the Laplace exponent of a subordinator. Likewise, the function $\phi^{(1-\beta)}$ is a Bernstein function and the duality relation follows from an elementary computation. This establishes Theorem 1.

3.2 Some properties

The basic properties of a subordinator can be read easily from the asymptotic behaviour of its Laplace exponent. It turns out that the small-time properties of $S^{(\alpha)}$ depend on the behaviour of α near 0, while the large-time properties depend on the behaviour of α near 1. More precisely,

Proposition 1 *Let the subordinator $S^{(\alpha)}$ be as in Theorem 1.*

(i) *The drift of $S^{(\alpha)}$ is given by*

$$\exp \int_0^1 \frac{-\alpha(x)}{x} dx$$

(ii) *The killing factor of $S^{(\alpha)}$ is positive if and only if*

$$\int_0^1 \frac{\alpha(x)}{1-x} dx < \infty$$

(iii) *If $\alpha(x) \rightarrow \beta$ as $x \rightarrow 0$, then the Hausdorff dimension of $\mathcal{R}^{(\alpha)}$ is β almost surely.*

Proof

Recall the following elementary results, which can be found for instance in [1], Chapter 1. The drift of a subordinator is given by $\lim_{\lambda \rightarrow \infty} \phi(\lambda)/\lambda$. The

killing rate is $\phi(0)$. Together with the expression of $\phi^{(\alpha)}$, this easily gives (i) and (ii).

Next, recall that index of the exponent $\phi^{(\alpha)}$ is given by

$$I = \lim_{\lambda \rightarrow \infty} \frac{\log \phi^{(\alpha)}(\lambda)}{\log \lambda}$$

if this limit exists. If so, the Hausdorff dimension of the range is equal to the index. See [2], or [1], Chapter 5. It is easy to check that that if $\alpha(x) \rightarrow \beta$ as $x \rightarrow 0$, then the index of the exponent is β .

3.3 Proof of Theorem 2

Consider a Poisson Point process \mathcal{N} on $\mathbb{R}_+ \times [0, 1] \times [0, 1]$ with intensity $dx \otimes y^{-2}dy \otimes dz$.

For $m > 0$ an integer, consider the restriction $\mathcal{N}^{(m)}$ of \mathcal{N} to the subset $\mathbb{R}_+ \times [1/m, 1] \times [0, 1]$. Let $(X_n^{(m)}, n \geq 1)$ be the set of points of $\mathcal{N}^{(m)}$, ranked by increasing x -coordinate. Denote $X_n^{(m)} = (t_n^{(m)}, h_n^{(m)}, U_n^{(m)})$.

Given a measurable function α and using the coordinates h, U , we can define the set $R_m^{(\alpha)}$ of integers that percolate as in Construction 1. We then say that a positive real t percolates if $t = t_n^{(m)}$ for some integer $n \in R_m^{(\alpha)}$.

Let $\mathcal{R}_m^{(\alpha)}$ be the set of times that percolate. Then it is easy to check that $\mathcal{R}_m^{(\alpha)}$ is a regenerative set with characteristic exponent

$$\phi_m^{(\alpha)}(\lambda) = \exp \left(\int_{1/m}^1 \frac{(\lambda - 1)\alpha(x)}{1 + (\lambda - 1)x} dx \right)$$

Moreover, if $m < m'$,

$$\mathcal{R}_m^{(\alpha)} \subset \mathcal{R}_{m'}^{(\alpha)} \tag{4}$$

and if $\alpha \leq \beta$, $\mathcal{R}_m^{(\alpha)} \subset \mathcal{R}_m^{(\beta)}$. Define now the set

$$\mathcal{R}^{(\alpha)} = \overline{\cup_{m>0} \mathcal{R}_m^{(\alpha)}}$$

Then the nesting property of the sets $\mathcal{R}^{(\alpha)}$ as stated in Theorem 2 is straightforward.

It remains to show that for every measurable function α , $\mathcal{R}^{(\alpha)}$ is a regenerative set with the Laplace exponent given in Theorem 1. To this end, consider a Poisson Point process \mathcal{N}' on $\mathbb{R}_+ \times \mathbb{R}_+$ with intensity $dx \otimes y^{-2}dy$, independent of \mathcal{N} . For every integer $m > 0$, let $N_s^{(m)}$ be the number of points of \mathcal{N}' in the strip $[0, s] \times [c_m, \infty)$, where

$$c_m = \frac{1}{\int_{1/m}^1 dx \frac{\alpha(x)}{x}}$$

Then $(N_s^{(m)}, s \geq 0)$ is a Poisson process with intensity $1/c_m$. Define now the process $(X_s^{(m)}, s \geq 0)$ as follows:

(i) The jump times of $(X_s^{(m)}, s \geq 0)$ are the same as the jump times of $(N_s^{(m)}, s \geq 0)$.

(ii) For every integer $k \geq 1$, the length of the k -th jump of $(X_s^{(m)}, s \geq 0)$ is the length of the k -th interval of the complement of $\mathcal{R}_m^{(\alpha)}$.

Then it is clear that $(X_s^{(m)}, s \geq 0)$ is a subordinator with Laplace exponent $\phi_m^{(\alpha)}$. The convergence of the Laplace exponents entails that the processes $(X_s^{(m)}, s \geq 0)$ converge in law to the subordinator $X^{(\alpha)}$ with Laplace exponent $\phi^{(\alpha)}$. Besides, the ranges of these processes $(X_s^{(m)}, s \geq 0)$ converge almost surely to $\mathcal{R}^{(\alpha)}$ in every compact set for the Hausdorff distance, because of the inclusion property (4). Therefore, $\mathcal{R}^{(\alpha)}$ has the law of the range of $X^{(\alpha)}$.

4 A generalization in the lattice case

Construction 2.

Let $(S_n, n \geq 0)$ be a real-valued random walk started at 0, with increments $(X_n, n \geq 1)$. Let $(H_n, n \geq 1)$ be iid real-valued random variables with law ν and $(\hat{H}_n, n \geq 0)$ be iid real-valued random variables with law $\hat{\nu}$. Assume that the X_n, H_n and \hat{H}_n are independent.

For $n \geq 1$, say that an integer $k \in [0, n-1]$ is an n -obstacle if, for every $m \in [k+1, n]$,

$$S_m + H_m < S_k + \hat{H}_k \quad (5)$$

Say that $n \geq 1$ percolates if, for every $k \in [0, n-1]$, k is not an n -obstacle. By convention, say that 0 percolates. Let \mathcal{R} be the set of integers that percolate.

Theorem 5 *The random set \mathcal{R} defined by Construction 2 is a lattice regenerative set. Its dual is obtained by replacing the random walk $(S_n, n \geq 0)$ with $(-S_n, n \geq 0)$ and exchanging the role of the random variables (H_n) and (\hat{H}_n) .*

Proof

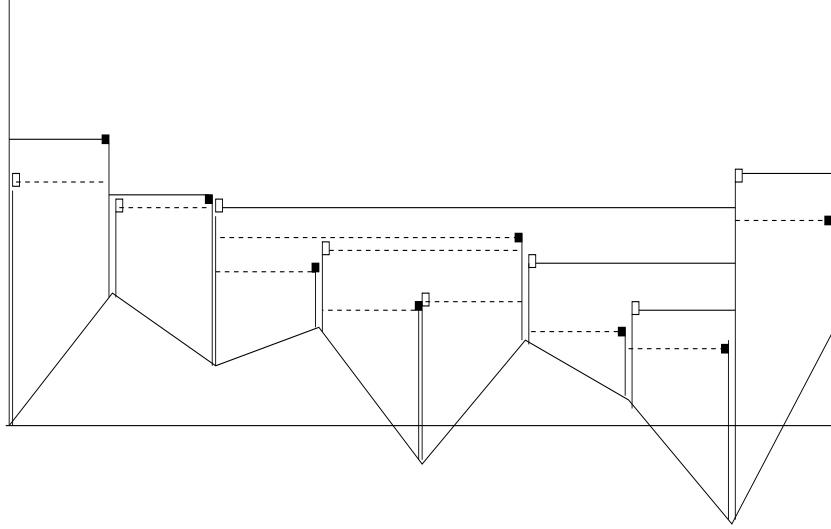
The regenerativity property is established by the same argument as for Theorem 3.

To show the duality, consider a regenerative set \mathcal{R}' defined as in Construction 2, using independent random variables $(X'_n, n \geq 1)$, $(H'_n, n \geq 1)$ and $(\hat{H}'_n, n \geq 0)$, where S'_1 has the same law as $-S_1$, H'_1 has the same law as \hat{H}_1 , and \hat{H}'_1 has the same law as H_1 . The only difference is that we define an obstacle using a large inequality, in contrast to the strict inequality in (5). To avoid any ambiguity, we shall use the terms *dual obstacle*, *dual-percolate* for the construction of \mathcal{R}' .

One can construct the sets $\mathcal{R} \cap [0, N]$ and $\mathcal{R}' \cap [0, N]$ on the same probability space, using the random variables $X_n, n \in [0, N]$, $H_m, m \in [1, N]$, $\hat{H}_l, l \in [0, N-1]$, by putting

$$S'_n = S_{N-n} - S_n, \quad n \in [0, N]$$

Figure 2: Construction 2



$$H'_n = \tilde{H}_{N-n}, \quad n \in [1, N]$$

$$\tilde{H}'_n = H_{N-n}, \quad n \in [0, N-1]$$

See Figure 2. The black squares stand for the variables $S_n + H_n$, the white squares for the variables $S_n + \tilde{H}_n$. Black squares “look to the left”, white squares “look to the right”. The horizontal dashed lines express the fact that they see an obstacle when looking to the left, or a dual obstacle when looking to the right. In turn, the plain lines express the fact that they see no obstacle or dual obstacle and, therefore, percolate or dual-percolate.

Let $G_N = \max(\mathcal{R} \cap [0, N])$, $G'_N = \max(\mathcal{R}' \cap [0, N])$. We claim that

(i) $N - G_N$ dual-percolates.

(ii) For every $n \in [N - G_N + 1, N]$, n does not dual-percolate.

To show (i), suppose that $N - G_N$ does not dual-percolate. Let k be the largest integer that is a dual obstacle for $N - G_N$. Then from the definition of a dual obstacle, there exists no $(N - k)$ -obstacle in $[G_N, k - 1]$. Moreover, from the definition of G_N , there is no $(N - k)$ -obstacle either in $[0, G_N - 1]$. Therefore, k percolates, but this contradicts the definition of G_N . This proves (i). One proves (ii) by similar arguments. As a consequence, $G'_N = N - G_N$.

This being true for every $N \geq 1$, we find that

(iii) for every $N > 0$, $N - G_N$ and G'_N have the same law.

It is then standard to check that (iii) is equivalent to the analytical property (3).

Some examples

1. If both ν and $\hat{\nu}$ are the Dirac mass at 0, then \mathcal{R} is the set of strict ascending ladder times of the random walk S , that is, the set of integers n such that $S_n > \max_{k \leq n-1} S_k$. On the other hand, \mathcal{R}' is the set of weak descending

ladder times of S , ie the set of integers n such that $S_n \leq \min_{k \leq n-1} S_k$.

2. Suppose that $\hat{\nu}$ is the Dirac mass at 0 and that

$$\nu(dx) = (1-r)\delta_0 + r\delta_{-\infty}$$

for some fixed $r \in [0, 1]$. Then the event that $T_1 > n$ is the event that for every ladder time $k \leq n$, $H_k = -\infty$. Therefore,

$$\phi(t) = \sum_{n=1}^{\infty} t^n \mathbb{E}(r^{L_{n-1}} - r^{L_n})$$

where L_n is the number of ladder times between time 1 and time n . Put

$$\psi(t) = \mathbb{E}(t^\tau)$$

where τ is the first ladder time. Then by standard computations, we find

$$\phi(t) = \frac{(1-r)\psi(t)}{1-r\psi(t)}$$

3. Suppose that $\hat{\nu}$ is the Dirac mass at 0 and that

$$\nu(dx) = c \exp(-c|x|) \mathbf{1}_{\{x < 0\}} dx$$

Then the event that $T_1 > n$ is the event that for every ladder time $k \leq n$,

$$|H_k| \geq S_k - \sup_{i < k} S_i$$

Conditionnally on S_k and $\sup_{i < k} S_i$, the latter event has probability $\exp[-c(S_k - \sup_{i < k} S_i)]$. Therefore,

$$\mathbb{P}(T_1 > n) = \mathbb{E} \exp[-c \sup_{i \leq n} S_i]$$

By time reversal, we have:

$$\mathbb{P}(\hat{T}_1 > n) = \int_0^\infty c e^{-cx} \mathbb{P}(\forall k \in [1, n], S_k \geq x) dx$$

Note that in the limit $c \rightarrow \infty$, we recover the first example.

4. Let S be deterministic, $S_n = -n$. Also, suppose that $\hat{\nu}$ is the Dirac mass at 0. Then the event that $T_1 > n$ is the intersection of the events $\{H_1 \leq 1\}$, $\{H_2 \leq 2\}$, \dots , $\{H_n \leq n\}$, all these events being independent. Therefore,

$$\mathbb{P}(T_1 > n) = \prod_{i=1}^n \nu([0, i])$$

In particular, the sequence of ratios

$$\frac{\mathbb{P}(T_1 > n + 1)}{\mathbb{P}(T_1 > n)} = \nu([0, n + 1])$$

can be chosen to be any nondecreasing sequence of reals $\in [0, 1]$. If we consider the dual process, we see that

$$\mathbb{P}(n \in \widehat{\mathcal{R}}) = \prod_{i=1}^n \nu([0, n])$$

This is the lattice equivalent of Corollary 2.5 in [10]. In particular, if the support of ν is bounded, say $\text{supp}(\nu) \subset [0, A]$, then $\mathbb{P}(n \in \widehat{\mathcal{R}})$ is constant for $n \geq A$. This corresponds to the examples given in Section 3 in [10].

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References

- [1] Bertoin, Jean. *Subordinators: examples and applications*. Lectures on probability theory and statistics (Saint-Flour, 1997), 191, Lecture Notes in Math., **1717**, Springer, Berlin, 1999.
- [2] Blumenthal, R. M.; Gettoor, R. K. Sample functions of stochastic processes with stationary independent increments. *J. Math. Mech.* **10** 1961 493516.
- [3] Doney, Ronald A. *Fluctuation theory for Levy processes*. Lectures from the 35th Summer School on Probability Theory held in Saint-Flour, July 623, 2005. Edited and with a foreword by Jean Picard. Lecture Notes in Mathematics, **1897**. Springer, Berlin, 2007.
- [4] Fitzsimmons, P. J., Fristedt, B., and Shepp, L. A. (1985). The set of real numbers left uncovered by random covering intervals. *Z. Wahr. Verw. Geb.* **70**, 175189.
- [5] Mandelbrot, B. B. (1972). Renewal sets and random cutouts. *Z. Wahr. Verw. Geb.* **22**, 145157.
- [6] Marchal, Philippe. Nested regenerative sets and their associated fragmentation process. *Mathematics and computer science*. III, 461470, Trends Math., Birkhäuser, Basel, 2004.
- [7] Rivero, Victor Manuel On random sets connected to the partial records of Poisson point process. *J. Theoret. Probab.* **16** (2003), no. 1, 277307
- [8] Schilling, René L.; Song, Renming; Vondracek, Zoran *Bernstein functions. Theory and applications*. de Gruyter Studies in Mathematics, **37**. Walter de Gruyter, Berlin, 2010.

- [9] Shepp, L. A. (1972). Covering the line with random intervals. *Z. Wahr. Verw. Geb.***22**, 163170.
- [10] Song, Renming; Vondracek, Zoran Some remarks on special subordinators. *Rocky Mountain J. Math.* **40** (2010), no. 1, 321337.