

A computational framework for certified reduced basis methods: application to a multiphysic problem

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Fundings

- ANR/HAMM
- FRAE/RB4FASTSIM

- 1 Context
- 2 Feel++ : a framework for reduced basis methods
- 3 Application : design of resistive magnet
- 4 Conclusion

Usage context for reduced basis methods

Parametrized PDE

- Input-parameter examples : geometric configuration, physical properties, boundary conditions, sources.
- Output examples : mean temperature over a subdomain, flux on a boundary, etc.

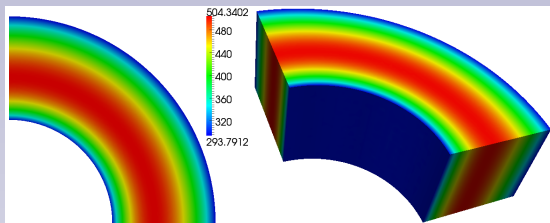
Motivation : rapid and reliable evaluation of input-output relationships

- Real-time : parameter-estimation, control...
- Many-query : sensitivity analysis, optimization....

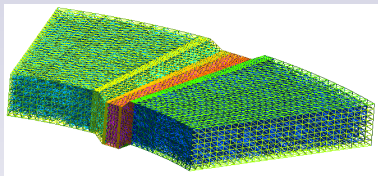
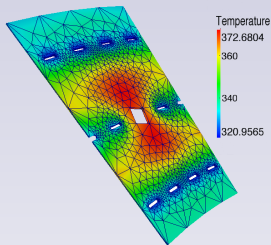
Objectives

- Generic framework ;
- Can we apply RB to a non-linear 2D/3D multiphysic problem ?

Usage context for reduced basis methods



Electro-heat problem for resistive magnets



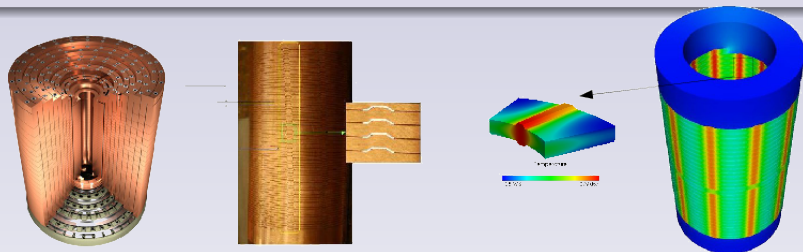
Feel++¹ reduced basis framework

Wide range of problems

- Linear or non-linear models ;
- Coercive or non-coercive models ;
- Elliptic or parabolic models ;
- Coupled multiphysics problems.

Scalability

- Models have more and more complex geometries and physics.



¹<http://www.feelpp.org/>

Feel++ reduced basis framework

Ingredients

- Set of parameters : $\mathcal{D}^\mu \subset \mathbb{R}^P$.
- Galerkin method
 - ▶ $X^\mathcal{N}$: finite element approximation of dimension $\mathcal{N} \gg 1$;
 - ▶ $u^\mathcal{N}(\boldsymbol{\mu}) \in X^\mathcal{N}$ is solution of $g(u^\mathcal{N}(\boldsymbol{\mu}), v; \boldsymbol{\mu}) = 0 \quad \forall v \in X^\mathcal{N}$.
- Sample : $S_N = \{\boldsymbol{\mu}_1 \in \mathcal{D}^\mu, \dots, \boldsymbol{\mu}_N \in \mathcal{D}^\mu\}$ with $N \ll \mathcal{N}$.
 - ▶ Several strategies for the sampling S_N selection :
 - ★ Greedy ;
 - ★ POD/Greedy (POD in time, Greedy in space) ;
 - ★ Random, Equidistributed...
- A posteriori error estimation
 - ▶ Primal and dual residuals ;
 - ▶ Successive Constraints Method.
- Recover affine decomposition even for non-linear problems
 - ▶ Using Empirical Interpolation Method.

Intrusive / Non-intrusive method

$W_N = \text{span}\{\xi_1, \dots, \xi_N\}$: reduced approximation space.

Intrusive method

- Apply EIM to $g(u^{\mathcal{N}}(\boldsymbol{\mu}), v; \boldsymbol{\mu})$:
 - ▶ $g(u^{\mathcal{N}}(\boldsymbol{\mu}), v; \boldsymbol{\mu}) \approx \sum_{q=1}^{Q^g} \sum_{m=1}^{M_q^g} \beta_{qm}^g(u^{\mathcal{N}}(\boldsymbol{\mu}); \boldsymbol{\mu}) \hat{g}_{qm}(u^{\mathcal{N}}(\boldsymbol{\mu}), v)$.
- Galerkin projection on W_N and solve the reduced non-linear problem.
- Solution is then given by $u_N(\boldsymbol{\mu}) = \sum_{i=1}^N u_{N,i}(\boldsymbol{\mu}) \xi_i$.

Non-intrusive method

- Reduced basis elements $(\xi_i)_{1 \leq i \leq N}$ are on a fine grid.
- Computation of the coarse solution $u_N^H(\boldsymbol{\mu})$ via black box FEM code.
- Solution is given by $u_N^h(\boldsymbol{\mu}) = \sum_{i=1}^N u_{N,i}^H(\boldsymbol{\mu}) \xi_i$ where :
 - ▶ $u_{N,i}^H(\boldsymbol{\mu}) = \int_{\Omega} u_N^H(\boldsymbol{\mu}) \xi_i$.

A language embedded in C++ : EIM expansion

Let T be the solution of $g(T, v; \boldsymbol{\mu}) = 0 \forall v \in X^{\mathcal{N}}$ and $\sigma(T)$ a non linear expression defined by $\sigma(T) = \frac{\sigma_0}{1+\alpha(T-T_0)}$.

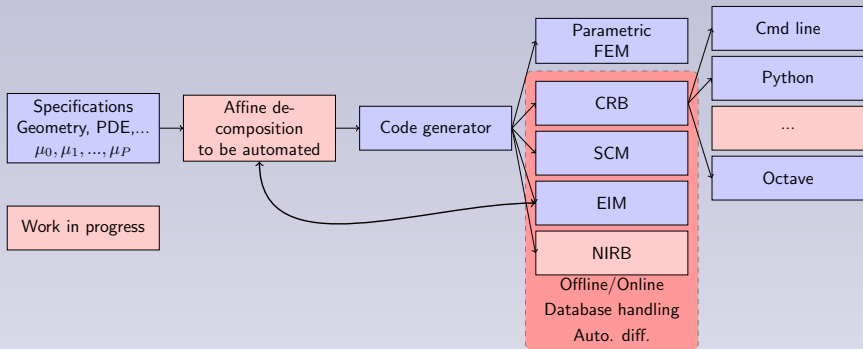
Let $\boldsymbol{\mu} \in \mathcal{D}^{\boldsymbol{\mu}}$ with $\boldsymbol{\mu}_0 = \sigma_0$, $\boldsymbol{\mu}_2 = \alpha$.

```
parameterspace_ptrtype Dmu; //Dμ
parameter_type mu;
auto Pset = Dmu->sampling();
int eim_sampling_size = 1000;
Pset->randomize(eim_sampling_size);
auto sigma = ref(mu(0))/(1+ref(mu(2))*(idv(T)-T0));

auto eim_sigma = eim( _model=solve( g(T,v;μ;x) = 0 ),
                     _element=T,
                     _parameter=mu, // μ
                     _expr=sigma,  // σ(T)
                     _space=XN,
                     _name="eim_sigma",
                     _sampling=Pset );

//then we can have access to β coefficients
std::vector<double> beta_sigma = eim_sigma->beta( mu );
```

Reduced basis framework



Sensitivity analysis

OpenTURNS^a can be used and is interfaced with python scripts.

^a<http://www.openturns.org>

Parallelization and Genericity

Parallelization

- Strategy :
 - ▶ Data partitionning using seamless parallel features of Feel++.
- Some difficulties :
 - ▶ Robust scalable solvers (linear, non linear and eigen value solvers) for the range of parameters under study ;
 - ▶ Parallel interpolation tools for eim ;
 - ▶ Parameter sampling that needs to be exactly the same on all processors.
- Challenges ahead
 - ▶ Realtime visualization ;
 - ▶ Enrichment of an existing database .

Genericity

- Variational language and data structures need to provide quite advanced features :
 - ▶ Meta-expression for expressions requiring evaluation using finite element or reduced basis functions (e.g. eim) ;
 - ▶ Parallelism **must** be totally seamless for the user.

Laboratoire National des Champs Magnétiques Intenses

Large scale user facility in France

- High magnetic field : from 24 T .
- Highest magnetic field at Grenoble : 35 T .

Application domains

- Magnetoscience ;
- Solide state physic (NMR) ;
- Applied supraconductivity .

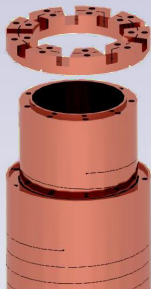


Figure: Helix

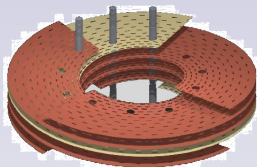


Figure: Bitter

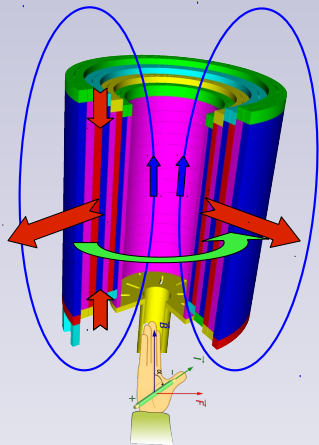
Magnetic Field

- Earth's field : $5.8 \cdot 10^{-4} T$.
- Supraconductors : $24 T$.
- **Continuous field : $35 T$.**
- Pulsed field : $90 T$.

Magnet study requires multi-physics modeling

Equations of interest

- Highest magnetic field or specific profile
 - ▶ Depend on current density $j = -\sigma \nabla V$
 - ▶ → **Electrical potential**
 - ▶ → **Magnetostatic / Biot & Savart's law**
- Joule effect
 - ▶ Thermic energy dissipation
 - ▶ → **Temperature**
- Magnet cooling
 - ▶ Thermic Exchange conductor/water
 - ▶ → **Hydraulic - Transfert coefficient**
- Mechanical stresses
 - ▶ Lorentz forces (induced by interaction between magnetic field and current density)
 - ▶ Thermic dilatation
 - ▶ → **Mechanic - Elasticity**



Electro-heat model

Magnet conception (materials, design) requires accurate prediction of temperature
 \Rightarrow **Electro-heat coupled model** .

Input parameters

Design

- Material properties
 - ▶ Conductivities : $\sigma(T), k(T)$
 - ▶ Lorentz number : L
- Ratio $\frac{\text{resistivity}}{\text{temperature}}$: α

In-the-field

- Applied potential : V_D
- Water temperature : T_w
- Heat transfert coefficient : h

$$\mu = (\sigma_0, k_0, \alpha, L, V_D, h, T_w) \subset \mathbb{R}^7$$

Output of interest

- Mean temperature.

Sensitivity analysis

Determine :

- Influence of parameters on the output ;
- A threshold for the output (quantile).

Modeling Electro-heat

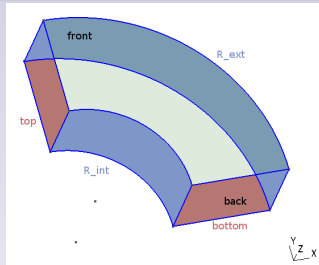
Steady non linear coupled model

$$\left\{ \begin{array}{ll} -\nabla \cdot (\sigma(T) \nabla V) = 0 & \text{in } \Omega \\ -\nabla \cdot (k(T) \nabla T) = \sigma(T) \nabla V \cdot \nabla V & \text{in } \Omega \\ V = 0 & \text{on } Bottom \\ V = V_D & \text{on } Top \\ -\sigma(T) \nabla V \cdot \mathbf{n} = 0 & \text{on cooling borders} \\ -k(T) \nabla T \cdot \mathbf{n} = 0 & \text{on } Top \cup Bottom \\ -k(T) \nabla T \cdot \mathbf{n} = h (T - T_w) & \text{on cooling borders} \end{array} \right.$$

Properties of materials

$$\sigma(T) = \frac{\sigma_0}{1 + \alpha(T - T_0)}$$

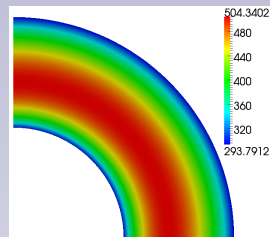
$$k(T) = L \sigma(T) T$$



2D numerical experiments

Approximations

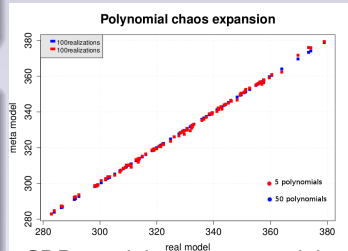
- FEM:
 - ▶ Dimension of FEM space : 2 312 (P1)
 - ▶ Picard iterations ≈ 20 (relative tol 10^{-8})
- RB :
 - ▶ Dimension of RB space : 15
 - ▶ Error : $\approx 10^{-6}$
 - ▶ Picard iterations ≈ 20 (relative tol 10^{-8})
 - ▶ EIM terms ≈ 40 (relative tol 10^{-10})



Temperature (K)

Meta model

- $Y = F(X)$
 - ▶ X and Y are stochastic variables
 - ▶ X follows uniform distribution
 - ▶ use polynomial chaos expansion (Legendre)

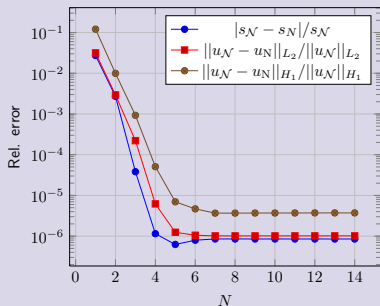


CRB model vs meta-model

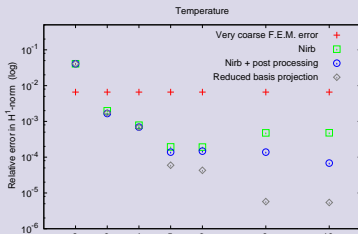
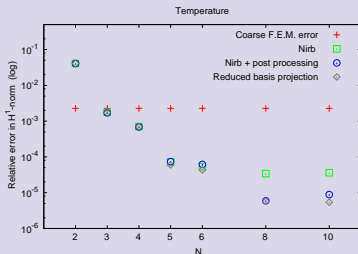
2D numerical experiments - convergence rates

Evaluation at a random μ (on 10 processors for reduced basis)

Reduced Basis



Non-Intrusive Reduced Basis



2D numerical experiments - sobol indices and quantiles

Parameters range

- $\sigma_0 \in [43.5.10^6, 58.10^6](S.m^{-1})$
- $k_0 \in [300, 400](W.m^{-1}.K^{-1})$
- $\alpha \in [3.10^{-3}, 4.10^{-3}](K^{-1})$
- $L \in [2.5.10^{-8}, 2.9.10^{-8}]$
- $V_D \in [0.05, 0.2](V)$
- $h \in [6.10^4, 9.10^4](W.m^{-2}.K^{-1})$
- $T_w \in [273, 323](K)$

Mean, Standard deviation

```
--- Mean = [328.473]
--- Standard deviation = 22.0728297966
```

Sobol indices

```
--- Sobol 0 = 3.24405836584e-05
--- Sobol 1 = 5.81262321421e-06
--- Sobol 2 = 3.45831134532e-05
--- Sobol 3 = 0.00242817982199
--- Sobol 4 = 0.633300283167 ( $V_D$ )
--- Sobol 5 = 1.76545493422e-05
--- Sobol 6 = 0.362812453403 ( $T_w$ )
```

Quantiles

Determine a threshold $q(\gamma)$ such that $P(Y_i < q(\gamma)) > \gamma$

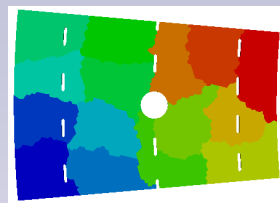
99.0 -quantile = [374.123]

80.0 -quantile = [354.55]

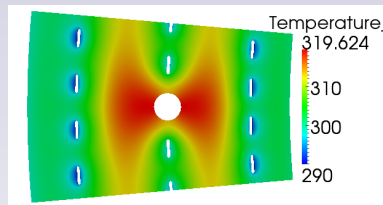
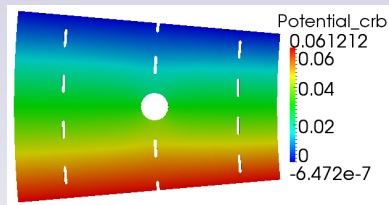
3D numerical experiments : bitter magnet

Approximations

- FEM:
 - ▶ Dimension of FEM space : 114108 (P1)
 - ▶ Picard iterations ≈ 40 (relative tol 10^{-8})
- RB :
 - ▶ Dimension of RB space : 15 basis functions
 - ▶ Error : $\approx 10^{-5}$
 - ▶ Picard iterations ≈ 40 (relative tol 10^{-8})
 - ▶ EIM terms ≈ 40 (relative tol 10^{-10})



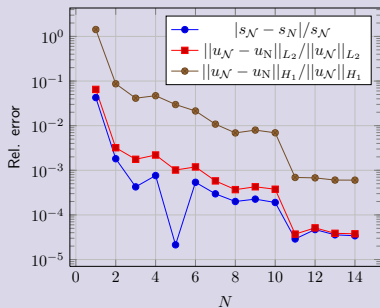
15 Partitions



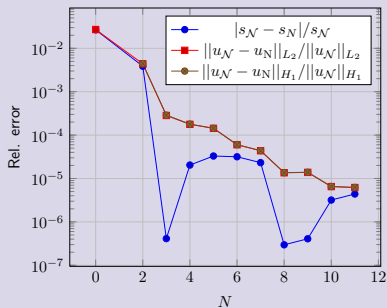
3D numerical experiments (Bitter, HRSector) - convergence rates

Evaluation at a random μ on 16 processors

HRSector



Bitter



Conclusion and perspectives

Conclusion

- RB framework deals with multiphysics complex problems.
- Sensitivity analysis :
 - ▶ Reduce parameter space dimension.

Perspectives

- Electro-heat model :
 - ▶ Improvements :
 - ★ Add geometric parameters;
 - ★ Add physics (Magneto-statics , Elasticity , Hydraulics) ;
 - ★ Transient problem .
 - ▶ Optimize magnets design ;
 - ▶ Real time monitoring of magnets (magnetic field, temperature , stress , ...) .
- Aerothermal problems (Navier-Stokes+Heat Transfer) ;
- Lego simulation (with domain decomposition) ;
- More large scale applications ;
- *hp*-RB/*hp*-EIM methods .

References I