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ABSTRACT. We study integral representations of the Gevrey series solutions of irregular hypergeometric systems. In this paper we consider the case of the systems associated with a one row matrix, for which the integration domains are one dimensional. We prove that any Gevrey series solution along the singular support of the system is the asymptotic expansion of a holomorphic solution given by a carefully chosen integral representation.

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1. INTRODUCTION

Hypergeometric systems also known as GKZ systems were introduced in [GGZ87] and [GZK89] as a far reaching generalisation of the Gauss hypergeometric differential equation. They appear as a special family of D-modules and they have been first studied in the regular case. For example in [GKZ90] the authors consider integral representations of the solutions of hypergeometric systems, at generic points of the space, which they call Euler integrals. In the irregular or confluent case, A. Adolphson considers in [A94] other integral representations of solutions which involve exponentials of polynomial functions and appropriate integration cycles. In this paper we develop new aspects in the irregular case namely the link between Gevrey series solutions and holomorphic solutions in sectors following Adolphson's approach. We aim at proving that we can materialise any Gevrey series solution as an asymptotic expansion in a sector of such an integral solution.

Let us fix some notations : D stands for the complex Weyl algebra of order n, where $n \geq 0$ is an integer. Elements in D are linear partial differential operators with polynomial coefficients. The polynomial ring $\mathbb{C}[\partial] := \mathbb{C}[\partial_1, \ldots, \partial_n]$ is a subring of the Weyl algebra D, where the ∂_i's represent the partial derivatives with respect to the variables in the space $\mathbb{C}^n$. The input data for a GKZ system is a pair (A, β) where β is a vector in $\mathbb{C}^d$ and $A = (a_{ij}) = (a^{(1)}, \cdots, a^{(n)}) \in \mathbb{Z}^{d\times n}$ is a matrix of rank d whose j$^{\text{th}}$ column is $a^{(j)}$. The toric ideal $I_A \subset \mathbb{C}[\partial]$ is the ideal generated by the family of binomials $\partial^u - \partial^v$ where $u, v \in \mathbb{N}^n$ and $Au = Av$. The ideal I_A is a prime ideal and the Krull dimension of the quotient ring $\mathbb{C}[\partial]/I_A$ equals d. Following [GGZ87, GZK89], the hypergeometric ideal associated with the pair (A, β) is :

$$H_A(\beta) = DI_A + D(E_1 - \beta_1, \ldots, E_d - \beta_d)$$

where $E_i = \sum_{j=1}^n a_{ij}x_j\partial_j$ is the i^{th} Euler operator associated with the i-th row of A. The corresponding hypergeometric D–module (or hypergeometric system) is nothing but the quotient D–module $M_A(\beta) := \frac{D}{H_A(\beta)}$.

In [GZK89] and [A94] it is proven that any hypergeometric D-module is holonomic. Moreover, a characterization of the regularity of $M_A(\beta)$ is provided in the series of papers [Ho98], [SST] and [SW08] : The holonomic D-module $M_A(\beta)$ is regular if and only if the toric ideal I_A is homogeneous for the standard grading in the polynomial ring $\mathbb{C}[\partial]$. In particular the condition to be regular for $M_A(\beta)$ is independent of β. The concept of regularity has been studied first in the case of an ordinary differential equation given by an operator $P \in A_1$. Regularity is characterised according to Fuchs theorem by the nullity of

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the irregularity number, an invariant combinatorially defined from the Newton polygon of P. In [M74] B. Malgrange proved that the irregularity is the dimension of the space of solutions at the origin of $\mathbb{C}$ with values in the space $\mathbb{C}[[x]]/\mathbb{C}\{x\}$ of formal series modulo convergent ones. Later J.P. Ramis gave a refined version of this result in [R84] calculating the space of solutions of a given Gevrey index again by using the Newton polygon of the operator.

The concept of irregularity in higher dimension is considerably more involved but generalizes the above results. Let us consider the structure sheaf $\mathcal{O}_X$ of a complex manifold X and denote $\mathcal{O}_{\widehat{X|Y}}$ its completion along a smooth hypersurface Y. In [Me90] Z. Mebkhout introduces for a holonomic $\mathcal{D}$–module $\mathcal{M}$ its irregularity complex along Y, $\mathrm{Irr}_Y(\mathcal{M}) = \mathbb{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_{\widehat{X|Y}}/\mathcal{O}_{X|Y})$, and in [LM99] the Gevrey filtration of this complex is introduced and related to other invariants of the system, the algebraic slopes in the sense of Y. Laurent.

In the case of hypergeometric D-modules the irregularity sheaves along coordinate subspaces, and Gevrey series solutions are studied and described in [F10] (see also [FC11a, FC11b]). Beforehand A. Adolphson [A94] gave a formula for the dimension of the space of holomorphic solutions at a generic point of the space and for a generic value of the parameter β, and he also described integral representations of solutions of these confluent hypergeometric systems. In the non-confluent case a analogous dimension formula was previously given in [GZK89]. In [ET12] A. Esterov and K. Takeuchi prove that these generic solution spaces are in fact completely described by integral representations along rapid decay cycles as introduced by Hien in [Hi07] and [Hi09].

We want to explore the more hidden link between these integral representations and the Gevrey series solutions described in [F10] and [FC11b, FC11a]. In this paper we first treat the case of a matrix with one row $A = (a_1, \ldots, a_n)$ with $0 < a_1 < \cdots < a_n$ a list of co-prime integers. This is the case where the integration cycles are paths. We prove that any Gevrey series solution of the system can be obtained as the asymptotic expansion of an integral representation along a well chosen path. A specificity of the one row case is that the rank of the space of Gevrey solutions is independent of β and we can treat all the values of β. For a generic β we only use the space of rapid decay cycles and for special values of β, namely for $\beta \in \mathbb{Z} \setminus (\mathbb{N}a_1 + \cdots + \mathbb{N}a_n)$ we must add an exceptional path, without the rapid decay property. The method of the proof is to reduce the statement to the case of the matrix $A = (a, b)$, using the fact that the restriction is compatible with being a Gevrey solution as well as with taking integrals on a fixed path. In parallel we know by an argument using a Gevrey version of Cauchy-Kovalevskaya theorem that the dimension of the Gevrey solution space is a_{n-1} [FC11b]. In the case $A = (a, b)$ the Gevrey expansion is with respect with the second variable x_2 and the main issue is to choose the paths of integration in a specific way to ensure a common sector of convergence for the coefficients which depend on the variable x_1.

Here is a summary of the contents of this paper. In Section 2 general facts are given about Gevrey series solutions following [F10]. In the case of a one line matrix we know that the characteristic variety of the hypergeometric system is the union $T^*_X X \cup T^*_Y X$ of the zero section and of the conormal to the hypersurface $Y \ : (x_n = 0)$. It is therefore sufficient to consider the irregularity sheaf and the germs of Gevrey series solutions at a generic point $(0, \ldots, 0, \epsilon, 0)$ of Y. We recall from [FC11b] the description given in that case of a basis of the space of Gevrey series solutions in terms of Γ-series and the proof that the dimension of this space is always equal to a_{n-1}. We also describe its behaviour by the restriction operation which consists in omitting variables among $x_1, \ldots, x_{n-2}$. In Section 3 we recall the definition of hypergeometric integrals of exponential type and the fact that in the irregular case they are solutions of the system. It is a quite general fact that these solutions have an asymptotic expansion as a Gevrey series with respect to the variable x_n provided that we choose an integration path of rapid decay both for the function $t^{-\beta-1} \exp(x_1 t^{a_1} + \cdots + x_n t^{a_n})$ and for $t^{-\beta-1} \exp(x_1 t^{a_1} + \cdots + x_{n-1} t^{a_{n-1}})$. At the beginning of Section 4 we prove that these expansions are Gevrey series of index at most $\frac{a_n}{a_{n-1}}$.

In the remainder of this Section we prove the main result of this paper about realisation of these Gevrey series as asymptotic expansion of integral solutions. We treat first in detail the case of dimension 2 and the last Subsection consists in using various restriction operations on the integral that are compared to the analogue described in Section 2 for the Gevrey series.

2. GEVREY SOLUTIONS OF HYPERGEOMETRIC SYSTEMS

In this Section we review some results on the construction of Gevrey series solutions of hypergeometric systems with respect to a coordinate hyperplane. Let us write $X = \mathbb{C}^n$, $\mathcal{O}_X$ the sheaf of holomorphic functions on X and $\mathcal{D}_X$ the sheaf of linear differential operators with holomorphic coefficients. With the hypergeometric system $M_A(\beta)$ we associate the left coherent $\mathcal{D}_X$–module $\mathcal{M}_A(\beta) := \frac{\mathcal{D}_X}{\mathcal{D}_X H_A(\beta)}$ which is called the analytic hypergeometric system associated with the pair (A, β).

Let us denote Y the hyperplane defined by $x_n = 0$ and $\mathcal{O}_{\widehat{X|Y}}$ the formal completion of $\mathcal{O}_X$ along Y. A germ f of the sheaf $\mathcal{O}_{\widehat{X|Y}}$ at a point $(p, 0) \in Y$ has the form $f = \sum_{m\geq 0} f_m(x_1, \ldots, x_{n-1})x_n^m$ where all the f_m are holomorphic functions in a common neighbourhood of p; in particular the restriction of $\mathcal{O}_X$ to Y, denoted by $\mathcal{O}_{X|Y}$, is a subsheaf of $\mathcal{O}_{\widehat{X|Y}}$.

For any real number s, we consider the sheaf $\mathcal{O}_{\widehat{X|Y}}(s)$ of *Gevrey series along* Y of order less than or equal to s defined as the subsheaf of $\mathcal{O}_{\widehat{X|Y}}$ whose germs f at any $(p, 0) \in Y$ satisfy the following convergence condition:

$$\sum_{m\geq 0} \frac{f_m(x_1, \ldots, x_{n-1})}{m!^{s-1}} x_n^m \in \mathcal{O}_{X|Y,(p,0)}.$$

If $s' < s$ then $\mathcal{O}_{\widehat{X|Y}}(s') \subset \mathcal{O}_{\widehat{X|Y}}(s)$. If a germ f belongs to $\mathcal{O}_{\widehat{X|Y}}(s)$ for some s but $f \notin \mathcal{O}_{\widehat{X|Y}}(s')$ for all $s' < s$, we say that the *index* of the Gevrey series f is s.

Let $A = (a^{(1)} \ \cdots \ a^{(n)})$ be a full rank $d \times n$ matrix with $a^{(i)} \in \mathbb{Z}^d$ for all $i = 1, \ldots, n$. In [GZK89] and [SST], the authors associate with any vector $v \in \mathbb{C}^n$ satisfying $Av = \beta$ a series expression of the form

$$\varphi_{A,\beta,v}(x) := x^v \sum_{u\in N_v} \Gamma[v; u]x^u \tag{2.1}$$

where $N_v = \{u \in \ker_{\mathbb{Z}}(A) \mid \operatorname{nsupp}(v + u) = \operatorname{nsupp}(v)\}$, $\ker_{\mathbb{Z}}(A) = \{u \in \mathbb{Z}^n : \ Au = 0\}$ and $\operatorname{nsupp}(w) := \{i \in \{1, \ldots, n\} \,|\, w_i \in \mathbb{Z}_{<0}\}$ is the negative support of $w \in \mathbb{C}^n$. The coefficient $\Gamma[v; u]$ equals $\frac{[v]_{u_-}}{[v+u]_{u_+}}$ where $[v]_u = \prod_i [v_i]_{u_i}$ and $[v_i]_{u_i} = \prod_{j=1}^{u_i}(v_i - j + 1)$ is the Pochhammer symbol for $v_i \in \mathbb{C}$, $u_i \in \mathbb{N}$.

We write $\varphi_v = \varphi_{A,\beta,v}$ is no confusion is possible. It is proved in [SST, Proposition 3.4.13] that the formal expression φ_v is annihilated by the hypergeometric ideal $H_A(\beta)$ if and only if the negative support of v is minimal, which means that $\nexists u \in \ker_{\mathbb{Z}}(A)$ with $\operatorname{nsupp}(v + u) \subsetneq \operatorname{nsupp}(v)$.

When $\beta \in \mathbb{C}^d$ is very generic, that is when β is not in a countable union of Zariski closed sets, there is a basis of the Gevrey solution space of $M_A(\beta)$ along Y at a generic point of Y, given by series φ_v for suitable vectors $v \in \mathbb{C}^n$, [F10, Th. 6.7].

In this article we restrict ourself to the case where $A = (a_1, \ldots, a_n)$ is a row matrix with $0 < a_1 < \cdots < a_n$ with $(a_1, \cdots, a_n)$ coprime and $n \geq 2$. We recall here some results about the Gevrey series solutions of $\mathcal{M}_A(\beta)$ as presented in [FC11b, Sections 4, 5]. Let us fix some notations. As before $X = \mathbb{C}^n$ and $Y \subset X$ denotes the hyperplane defined by $x_n = 0$. Let us write $Z \subset X$ the hyperplane defined by $x_{n-1} = 0$.

When $s < \frac{a_n}{a_{n-1}}$ one sees from the results in [FC11b] that the set of Gevrey solutions is zero if $\beta \notin \mathbb{N}A$ and is a one dimensional space generated by a polynomial if $\beta \in \mathbb{N}A$. This result follows also from what we show in this paper and we focus now on the case $s \geq \frac{a_n}{a_{n-1}}$.

2.1. **Case $a_1 = 1$.** Assume first $n \geq 3$. A basis of the free $\mathbb{Z}$–module $\ker_{\mathbb{Z}}(A)$ is formed of the vectors $\{u^{(2)}, \ldots, u^{(n)}\}$ where $u^{(n-1)} = (a_{n-1}, 0, \ldots, 0, -1, 0)$ and for $i = 2, \ldots, n$, $i \neq n-1$, we define

$$u^{(i)} = (-a_i, 0, \ldots, 0, 1, 0, \ldots, 0)$$

where 1 is in the i-th component. For each $\mathbf{m} = (m_2, \ldots, m_n) \in \mathbb{Z}^{n-1}$ we write $u(\mathbf{m}) = \sum_{i=2}^n m_i u^{(i)}$ the corresponding element in $\ker_{\mathbb{Z}}(A)$.

For $j = 0, \ldots, a_{n-1} - 1$ define $v^j = (j, 0, \ldots, 0, \frac{\beta - j}{a_{n-1}}, 0) \in \mathbb{C}^n$ and consider the associated Γ–series, see (2.1)

$$\varphi_{A,\beta,v^j} = x^{v^j} \sum_{\substack{m_2,\ldots,m_{n-1},m_n \geq 0 \\ j + a_{n-1}m_{n-1} \geq \sum_{i \neq n-1} a_i m_i}} \Gamma[v^j; u(\mathbf{m})] x^{u(\mathbf{m})}. \tag{2.2}$$

We write $\varphi^{(j)}_{A,\beta} = \varphi_{A,\beta,v^j}$ and also $\varphi^{(j)} = \varphi_{A,\beta,v^j}$ if no confusion arises. By the choice of a basis of $\ker_{\mathbb{Z}}(A)$ that we make $\varphi^{(j)}$ is a series in $x^{v^j}\mathbb{C}[[x_1, \ldots, x_{n-1}^{-1}, x_n]][x_1^{-1}]$.

Notice here that the summation in $\varphi^{(j)}$, is according to (2.1), taken over the set $N_{v^{(j)}} = \{u(\mathbf{m}) \,|\, \mathrm{nsupp}(v^{(j)} + u(\mathbf{m})) = \mathrm{nsupp}(v^{(j)}\}$. We have

$$\begin{cases} N_{v^{(j)}} = \{\mathbf{m} \in \mathbb{N}^{n-1} \,|\, j + a_{n-1}m_{n-1} \geq \sum_{i \neq n-1} a_i m_i\} & \text{if } \frac{\beta-j}{a_{n-1}} \notin \mathbb{N} \\ N_{v^{(j)}} = \{\mathbf{m} \in \mathbb{N}^{n-1} \,|\, j + a_{n-1}m_{n-1} \geq \sum_{i \neq n-1} a_i m_i,\ \frac{\beta-j}{a_{n-1}} \geq m_{n-1}\} & \text{if } \frac{\beta-j}{a_{n-1}} \in \mathbb{N}. \end{cases}$$

In this last case we can write $\beta = j + ha_{n-1}$ for unique $0 \leq j < a_{n-1}$ and $h \in \mathbb{N}$ and then the series $\varphi^{(j)}$ is a polynomial since $[h]_{m_{n-1}} = 0 = \Gamma[v^j; u(\mathbf{m})]$ if $m_{n-1} \geq h + 1$. In fact $\varphi^{(j)}$ is a polynomial if and only if we are in that case.

The negative support of each v^j is $\emptyset$ if $h \in \mathbb{N}$ and $\{n-1\}$ if $\beta - j$ is a negative multiple of a_{n-1} and it is minimal in both cases. The series $\varphi^{(j)}$ is a polynomial in the first case and a solution of the ideal $H_A(\beta)$ in both. More precisely,

Theorem 2.1. [FC11b, Th. 4.21, i)] *Let $A = (1\ a_2\ \cdots\ a_n) \in \mathbb{Z}^n$ with $1 < a_2 < \cdots < a_n$, $Y = (x_n = 0) \subset X$ and $Z = (x_{n-1} = 0) \subset X$. Then the set (of germs of) Gevrey series $\{\varphi^{(j)} \,|\, j = 0, \ldots, a_{n-1} - 1\}$ is a basis of $\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}_A(\beta), \mathcal{O}_{\widehat{X|Y}}(s))_p$ for all $\beta \in \mathbb{C}$, $p \in Y \setminus Z$ and $s \geq a_n/a_{n-1}$.*

It is also useful to consider the immersion $i : \mathbb{C}^3 \hookrightarrow X$ defined by the equations $x_2 = \cdots = x_{n-2} = 0$ and the restriction ρ' with respect to this immersion (coordinates in $\mathbb{C}^3$ are (x_1, x_{n-1}, x_n)).

The series $\varphi^{(j)}_{A,\beta}(x_1, 0, \ldots, 0, x_{n-1}, x_n)$ equals precisely $\varphi^{(j)}_{(1,a_{n-1},a_n),\beta}(x_1, x_{n-1}, x_n)$ so the restriction defines an isomorphism

$$\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}_A(\beta), \mathcal{O}_{\widehat{X|Y}}(s)) \xrightarrow{\rho'} \mathcal{H}om_{\mathcal{D}_{\mathbb{C}^3}}(\mathcal{M}_{(1,a_{n-1},a_n)}(\beta), \mathcal{O}_{\widehat{\mathbb{C}^3|Y_1}}(s)) \tag{2.3}$$

of the corresponding stalks at any point in $Y_1 \setminus Z_1$ for all $\beta \in \mathbb{C}$ and $s \in \mathbb{R}$ where we denote Y_1, Z_1 the subspaces in $\mathbb{C}^3$ with equations $x_n = 0$ and $x_{n-1} = 0$.

Remark 2.2. So far we have assumed $n \geq 3$. The case $n = 2$ is special and will be treated now, following [FC11a]. We can drop in this case the assumption on a_1 and simply write $a = a_1, b = a_2$, with $1 \leq a < b$ and $\gcd(a, b) = 1$.

The corresponding Γ–series have a slightly different shape (see [FC11a]): For $p = 0, \dots, a-1$, consider $w^p = (\frac{\beta - pb}{a}, p) \in \mathbb{C}^2$ and

$$\varphi^{(p)}_{A,\beta} = x^{w^p} \sum_{m \geq 0} \frac{[\frac{\beta - pb}{a}]_{bm}}{[am + p]_{am}} x_1^{-bm} x_2^{am}. \tag{2.4}$$

We have the following

Proposition 2.3. [FC11a, Prop. 5.3 and 5.4] *Write $X = \mathbb{C}^2$, $Y = (x_2 = 0) \subset X$. The set (of germs) of Gevrey series $\{\varphi^{(p)}_{A,\beta} \,|\, p = 0, \dots, a-1\}$ is a basis of the stalk of $\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}_A(\beta), \mathcal{O}_{\widehat{X|Y}}(s))$ at any point in $Y \setminus \{(0,0)\}$, for any real number $s \geq \frac{b}{a}$ and any $\beta \in \mathbb{C}$.*

2.2. **Case $a_1 > 1$.** Recall that $X = \mathbb{C}^n$, $Y \subset X$ (resp. $Z \subset X$) is defined by $x_n = 0$ (resp. $x_{n-1} = 0$). First of all, we follow [FC11b, Rk. 5.4] to prove the following equality

Proposition 2.4.

$$\dim_{\mathbb{C}} \left(\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}_A(\beta), \mathcal{O}_{\widehat{X|Y}}(s))_p \right) = a_{n-1} \tag{2.5}$$

if $p \in Y \setminus Z$ and $s \geq \frac{a_n}{a_{n-1}}$ for any $\beta \in \mathbb{C}$.

Proof. We apply, among other results, Cauchy-Kovalevskaya's Theorem for Gevrey series. We consider $A' = (1, a_1, \dots, a_n)$ and the hypergeometric system $\mathcal{M}_{A'}(\beta)$ on $X' := \mathbb{C}^{n+1}$ with coordinates $(x_0, x_1, \dots, x_n)$. We denote by $Y' \subset X'$ (resp. $Z' \subset X'$) the hyperplane $x_n = 0$ (resp. $x_{n-1} = 0$) and we identify $X \subset X'$ with the hyperplane $x_0 = 0$. Notice that $Y = Y' \cap X$ and $Z = Z' \cap X$.

By [CT03, Proposition 4.2] we can apply Cauchy-Kovalevskaya's Theorem for Gevrey series solutions (see [LM02, Corollary 2.2.4]) to deduce that there exists a Cauchy-Kovalevskaya's isomorphism $CK^s_{X',X}$

$$\mathcal{H}om_{\mathcal{D}_{X'}}(\mathcal{M}_{A'}(\beta), \mathcal{O}_{\widehat{X'|Y'}}(s))_{|X} \xrightarrow{CK^s_{X',X}} \mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}_{A'}(\beta)_{|X}, \mathcal{O}_{\widehat{X|Y}}(s))$$

where $\mathcal{M}_{A'}(\beta)_{|X}$ stands for the restriction in the category of $\mathcal{D}$–modules. This is true for any $s \in \mathbb{R}$ and for any β and even if $\gcd(a_1, \dots, a_n) \neq 1$.

We write $CK^s = CK^s_{X',X}$ is no confusion is possible. The isomorphism CK^s is induced by the inclusion $X \subset X'$, by the action of restriction on the modules involved. Theorem [FC11b, Th. 5.1] states that for any $\beta \in \mathbb{C}$ there exists $\beta' \in \mathbb{C}$ such that the restriction $\mathcal{M}_{A'}(\beta)_{|X}$ is isomorphic to the hypergeometric $\mathcal{D}_X$–module $\mathcal{M}_A(\beta')$. Moreover, by the same Theorem [FC11b, Th. 5.1] one can take $\beta' = \beta$ for all but finitely many β. We denote by $\mathrm{sp}(A)$ the finite set of $\beta \in \mathbb{C}$ such that $\beta' \neq \beta$. In particular, the isomorphism CK^s and Theorem 2.1 prove equality (2.5) for $p \in Y \setminus Z$, $s \geq \frac{a_n}{a_{n-1}}$ and $\beta \notin \mathrm{sp}(A)$.

Assume now $\beta^* \in \mathrm{sp}(A)$. We can take $\beta = \beta^* + A'\gamma' = \beta^* + A\gamma$ for a suitable $\gamma' = (0, \gamma) \in \mathbb{N}^{n+1} \cup (-\mathbb{N})^{n+1}$ in such a way that $\beta \notin \mathrm{sp}(A)$ and so the corresponding morphism CK^s is an isomorphism. By using [S01, Th. 2.1], [B11, Th. 6.5] one has that the morphism $\cdot \partial^\gamma : \mathcal{M}_A(\beta^*) \to \mathcal{M}_A(\beta)$ if $\gamma \in \mathbb{N}^n$ (resp. $\cdot \partial^{-\gamma} : \mathcal{M}_A(\beta) \to \mathcal{M}_A(\beta^*)$ if $\gamma \in (-\mathbb{N})^n$) is an isomorphism. This proves equality (2.5) for $\beta^* \in \mathrm{sp}(A)$. □

Once the equality (2.5) is established we give a description of a basis of the solution space of $\mathcal{M}_A(\beta)$ in $\mathcal{O}_{\widehat{X|Y}}(s))$. We use the restriction morphism

$$\mathcal{H}om_{\mathcal{D}_{X'}}(\mathcal{M}_{A'}(\beta), \mathcal{O}_{\widehat{X'|Y'}}(s))_{|X} \xrightarrow{\rho^s_{X',X}} \mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}_A(\beta), \mathcal{O}_{\widehat{X|Y}}(s)) \tag{2.6}$$

for $\beta \in \mathbb{C}$ and $s \in \mathbb{R}$. This morphism is well defined and it is induced by the restriction to $x_0 = 0$. It is useful to write $x' = (x_0, x)$ and $x = (x_1, \ldots, x_n)$. Then $\rho^s_{X',X}(\varphi(x')) = \varphi(0, x)$, since if $\varphi(x')$ is a solution in the first space then $\varphi(0, x)$ is a solution in the second one. Notice that it is an approach of restriction that could be different from the one by the CK's.

We now consider the basis $\{\varphi^{(j)}_{A',\beta}(x_0, x)\}_{j=0}^{a_{n-1}-1}$ of germs (at a point in $Y \setminus Z \subset Y' \setminus Z'$) of Gevrey series solutions (of order $\leq s$) of $\mathcal{M}_{A'}(\beta)$ described in Theorem 2.1.We simply write $\varphi^{(j)} = \varphi^{(j)}_{A',\beta}$. The terms in $x_{n-1}^{\frac{-\beta+j}{a_{n-1}}}\varphi^{(j)}$ have the form

$$\Gamma[v^{(j)}; u(\mathbf{m})]x_0^j(x')^{u(\mathbf{m})} = \Gamma[v^{(j)}; u(\mathbf{m})]x_0^{j+a_{n-1}m_{n-1}-\sum_{i\neq n-1}a_im_i}x_1^{m_1}\cdots x_{n-2}^{m_{n-2}}x_{n-1}^{-m_{n-1}}x_n^{m_n}$$

where $u(\mathbf{m})$ is a general element of $\ker_{\mathbb{Z}}(A') \subset \mathbb{Z}^{n+1}$. The summation in $\varphi^{(j)}$ (see (2.2)) is taken over the set $\{\mathbf{m} \in \mathbb{N}^n \,|\, j + a_{n-1}m_{n-1} \geq \sum_{i\neq n-1} a_im_i\}$.

It is useful to write the formal expansion of $\varphi^{(j)}(0, x)$. According to what is said before we have

$$\varphi^{(j)}(0,x) = x_{n-1}^{\frac{\beta-j}{a_{n-1}}} \sum_{\substack{m_1,\ldots,m_{n-1},m_n\geq 0\\ j+a_{n-1}m_{n-1}=\sum_{i\neq n-1}a_im_i}} \frac{[\frac{\beta-j}{a_{n-1}}]_{m_{n-1}}j!}{m_1!\cdots m_{n-2}!m_n!}x_1^{m_1}\cdots x_{n-2}^{m_{n-2}}x_{n-1}^{-m_{n-1}}x_n^{m_n}. \tag{2.7}$$

Notice that the family of non zero $\varphi^{(j)}(0, x)$ is $\mathbb{C}$–linearly independent because their supports are pairwise disjoint. This, and equality (2.5), proves the following (see [FC11b, Remark 5.4])

Theorem 2.5. *Assume that $\varphi^{(j)}(0, x)$ is non zero for $j = 0, \ldots, a_{n-1} - 1$. Then the set (of germs of) Gevrey series $\{\varphi^{(j)}(0, x) \,|\, j = 0, \ldots, a_{n-1} - 1\}$ is a basis of the stalk of $\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}_A(\beta), \mathcal{O}_{\widehat{X|Y}}(s))$ at any point in $Y \setminus Z$ for $s \geq \frac{a_n}{a_{n-1}}$.*

Remark 2.6. Assume now that for some $j = 0, \ldots, a_{n-1} - 1$ one has $\varphi^{(j)}(0, x) = 0$. By [FC11b, Rk. 5.4] this condition happens if and only if $\beta \in \mathbb{N} \setminus \mathbb{N}A$. Furthermore this j is then unique for a fixed β and the image of the morphism $\rho^s_{X',X}$ has codimension 1 in the stalk of $\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}_A(\beta), \mathcal{O}_{\widehat{X|Y}}(s))$ at any point in $Y \setminus Z$ and for all $s \geq \frac{a_n}{a_{n-1}}$. We will show in Theorem 4.8 how to describe a basis of this last solution space for all $\beta \in \mathbb{C}$.

2.3. **The restriction to $x_0 = 0$ for $A' = (1,\, ka,\, kb)$.** Let us consider the morphism $\rho^s_{X',X}$ when $X = (x_0 = 0) \subset X' = \mathbb{C}^3$ and the matrix $A' = (1,\, ka,\, kb)$ with $1 \leq a < b$, $1 < ka$ and $\gcd(a, b) = 1$. The morphism $\rho^s_{X',X}$, as defined in (2.6), sends solutions of $\mathcal{M}_{A'}(\beta)$ to solutions of $\mathcal{M}_{(ka,kb)}(\beta)$ and this last $\mathcal{D}_X$–module is isomorphic to $\mathcal{M}_{(a,b)}(\frac{\beta}{k})$. In particular if $k > 1$, $\rho^s_{X',X}$ is not an isomorphism, since the corresponding solutions spaces have dimensions ka and a respectively. Below we describe in detail a different but related morphism involving the restrictions of the derivatives up to order $k - 1$ with respect to the variable x_0.

So, instead of considering $A = (ka, kb)$ as before it is better to write $A = (a, b)$. Coordinates in X' are (x_0, x_1, x_2).

For $j = 0, \ldots, ka - 1$ let us write $v^{(j)} = (j, \frac{\beta-j}{ka}, 0)$ and

$$\varphi^{(j)}_{A',\beta} = x_1^{\frac{\beta-j}{ka}} \sum_{\substack{m_1,m_2\geq 0\\ j+kam_1\geq kbm_2}} \frac{[\frac{\beta-j}{ka}]_{m_1}j!}{m_2!(j+kam_1-kbm_2)!}x_0^{j+kam_1-kbm_2}x_1^{-m_1}x_2^{m_2}.$$

Notice that if a series φ (as for example $\varphi^{(j)}_{A',\beta}$) is a solution of $\mathcal{M}_{A'}(\beta)$ then $\frac{\partial^\ell \varphi}{\partial x_0^\ell}\Big|_{x_0=0}$ is a solution of the system $\mathcal{M}_A(\frac{\beta-\ell}{k})$ for all $\ell \geq 0$. We are going to compare this solution, for $\varphi = \varphi^{(j)}_{A',\beta}$, to the usual Gamma series solutions $\varphi^{(p)}_{A,\beta'}(x_1,x_2)$, see (2.4), of this last system $\mathcal{M}_A(\beta')$ for $\beta' = \frac{\beta-\ell}{k}$. We consider the following $\mathbb{C}$–linear map

$$\mathcal{H}om_{\mathcal{D}_{X'}}(\mathcal{M}_{A'}(\beta), \mathcal{O}_{\widehat{X'|Y'}}(s))_{|X} \xrightarrow{\varpi^s_\beta} \bigoplus_{\ell=0}^{k-1} \mathcal{H}om_{\mathcal{D}_X}\left(\mathcal{M}_A\left(\frac{\beta-\ell}{k}\right), \mathcal{O}_{\widehat{X|Y}}(s)\right)$$

which maps φ to the vector $\left(\frac{\partial^\ell \varphi}{\partial x_0^\ell}(0,x_1,x_2)\right)_{\ell=0}^{k-1}$. Here $Y' = (x_2 = 0) \subset X'$ and $Y = X \cap Y'$.

Notice that the morphism ϖ^s_β is well defined and that it coincides with the morphism (2.6) when $k = 1$. The following Proposition generalizes to an arbitrary k what is already proved for $k = 1$ in Theorem 2.5 and in Remark 2.6 :

Proposition 2.7. *1) Assume that $\beta \notin \mathbb{N}$ or $\beta \in \bigcup_{r=0}^{k-1}(r + k\mathbb{N}A)$. Then the $\mathbb{C}$–linear map ϖ^s_β is an isomorphism of vector spaces.*

2) Assume $\beta \in \bigcup_{r=0}^{k-1}(r+k(\mathbb{N}\setminus\mathbb{N}A))$ and consider the integers j_0, r_0, q_0 uniquely determined by $0 \leq j_0 < ka$, $0 \leq r_0 < k$ and $\beta = j_0 + kah = r_0 + kq_0 + kah$. Then the image of the map ϖ^s_β is the codimension 1 subspace, generated by all the $(0,\dots,0,\varphi^{(p)}_{A,\frac{\beta-r}{k}},0,\dots,0)$, with a non zero term in position r, and $(p,r) \neq (p_0,r_0)$ for $p_0 \in \{0,\dots,a-1\}$ uniquely determined by $bp_0 \equiv q_0 \pmod a$.

Proof. We consider the morphism ϖ^s_β induced on the stalks at points in $Y \setminus Z$, $Z = (x_1 = 0) \subset X$ and $s \geq \frac{b}{a}$. Let us fix $j = 0,\dots,ka-1$ and write $j = kq + r$ for unique $0 \leq q \leq a-1$ and $0 \leq r \leq k-1$. So we have

$$\frac{\beta - j}{ka} = \frac{\beta - kq - r}{ka} = \frac{\beta - r}{ka} - \frac{q}{a} = \frac{\frac{\beta-r}{k} - q}{a}.$$

The general exponent of x_0 in $\varphi^{(j)}_{A',\beta}$ is $j + k(am_1 - bm_2) = r + k(q + am_1 - bm_2)$. An exponent of x_1 in $\frac{\partial^\ell \varphi^{(j)}_{A',\beta}}{\partial x_0^\ell}\Big|_{x_0=0}$ for $0 \leq \ell \leq k-1$, can only come from a term in $\varphi^{(j)}_{A',\beta}$ for which the underlying exponent of x_0 is ℓ. Therefore the only pairs $(m_1,m_2) \in \mathbb{N}^2$ that may appear are those satisfying the relation :

$$j + k(am_1 - bm_2) - \ell = k(q + am_1 - bm_2) + r - \ell = 0. \tag{2.8}$$

This cannot happen if $\ell \neq r$ which means that $\frac{\partial^\ell \varphi^{(j)}_{A',\beta}}{\partial x_0^\ell}\Big|_{x_0=0} = 0$ if $\ell \neq r$. The equality (2.8) is equivalent to $q = bm_2 - am_1$ and $\ell = r$. For any $(m_1,m_2) \in \mathbb{N}^2$ satisfying (2.8), we write $m_2 = am + p$ with $0 \leq p \leq a-1$. We notice that the integer p depends only on q, hence on j because $q = b(am+p) - am_1$ so that a divides $bp - q$. We denote it $p = p(j)$. By a straightforward calculation the general exponent of x_1 in $\frac{\partial^r \varphi^{(j)}_{A',\beta}}{\partial x_0^r}\Big|_{x_0=0}$ is $\frac{\beta-j}{ka} - m_1 = \frac{\beta-r}{ka} - \frac{bp(j)}{a} - bm$ which equals the general exponent of x_1 in $\varphi^{(p(j))}_{A,\beta'}$ by (2.4) applied to $\beta' = \frac{\beta-r}{k}$, and the respective indices for these series (m_1,m_2) and m are in a 1-1 correspondence through the relations

$$m_1 = bm + \frac{bp(j) - q}{a} \text{ and } m_2 = am + p(j).$$

Now we compare the corresponding coefficients in $\left.\frac{\partial^r \varphi^{(j)}_{A',\beta}}{\partial x_0^r}\right|_{x_0=0}$ and in $\varphi^{(p)}_{A,\beta'}$. The quotient of these two coefficients is well defined, for all $m \geq 0$, when $\beta \notin \mathbb{N}$ and it is

$$\left(\frac{[\frac{\beta-kq-r}{ka}]_{m_1}(kq+r)!}{m_2!}\right)\left(\frac{[\frac{\beta'-pb}{a}]_{bm}p!}{(am+p)!}\right)^{-1} = \frac{\Gamma(z-bm+m_1)(kq+r)!}{\Gamma(z)p!} \tag{2.9}$$

where $z-1 = \frac{\beta'-pb}{a} = \frac{\beta-r}{ka} - \frac{pb}{a}$.

We notice here that once j is fixed, $p = p(j)$ is also fixed and the integer $bm - m_1 = \frac{q-bp}{a}$ does not depend on m, m_1. So, if $\beta \notin \mathbb{N}$ we have found a constant $\lambda_j \in \mathbb{C}^*$ such that

$$\left.\frac{\partial^r \varphi^{(j)}_{A',\beta}}{\partial x_0^r}\right|_{x_0=0} = \lambda_j \varphi^{(p(j))}_{A,\frac{\beta-r}{k}} \quad \text{and} \quad \left.\frac{\partial^\ell \varphi^{(j)}_{A',\beta}}{\partial x_0^\ell}\right|_{x_0=0} = 0 \quad \text{if } \ell \neq r.$$

When $\beta \in \mathbb{N}$, we write $\beta = j_0 + kah = r_0 + kq_0 + kah$, with unique $0 \leq r_0 \leq k-1$ and $0 \leq q_0 \leq a-1$. If $j \neq j_0 = r_0 + kq_0$ the quotient in (2.9) is still well defined for all m and the relation $\lambda_j \neq 0$ still valid.

If $\beta \in r_0 + k(\mathbb{N} \setminus \mathbb{N}(a,b))$, the quotient is still well defined but we have $\lambda_{j_0} = 0$ by Remark (2.6) since $\varphi^{(j_0)}_{A',\beta}$ is a polynomial solution of $\mathcal{M}_{A'}(\beta)$ and $\mathcal{M}_A\left(\frac{\beta-r_0}{k}\right)$ has no non zero polynomial solution for $\frac{\beta-r_0}{k} \notin \mathbb{N}(a,b)$.

Assume now $\frac{\beta-r_0}{k} = ha + q_0 \in \mathbb{N}(a,b)$ so that $ha + q_0 = ua + vb$ for some $u, v \in \mathbb{N}$. We can write in a unique way $q_0 = -n_1 a + n_2 b$ with $0 \leq n_2 \leq a-1$. One has $(h-n_1)a + n_2 b = ua + vb$ which forces $v \geq n_2$ and $h - n_1 - u \geq 0$. In particular, $h \geq n_1$ and $[h]_{n_1} \neq 0$ and so $\lambda_{j_0} \neq 0$. This proves 1) and 2) taking $p_0 := p(j_0)$. □

3. Hypergeometric integral of exponential type

With the pair (A, β) one associates the following integral (called hypergeometric integral of exponential type):

$$I_\gamma(A,\beta;x) = I(\beta;x) := \int_\gamma t^{-\beta-1} \exp\left(\sum_{j=1}^n x_j t^{a_j}\right) dt$$

where $A = (a_1, \ldots, a_n)$ and γ is a cycle in the rapid decay homology with closed support of M. Hien. We also write $I_\gamma(\beta;x) = I(\beta;x)$ if there is no possible confusion on the matrix A.

The expression $I(\beta;x)$ formally satisfies the equality $P(I(\beta;x)) = 0$ for any P in the hypergeometric ideal $H_A(\beta)$. So we consider $I(\beta;x)$ as a solution of $\mathcal{M}_A(\beta)$. Our goal is to give an asymptotic expansion of $I(\beta;x)$ as Gevrey series along the coordinate hyperplane appearing in the singular support of $\mathcal{M}_A(\beta)$.

We are going to prove in this case that all the Gevrey series solutions can be obtained as an asymptotic expansion of such an integral.

In the one row matrix case the rapid decay cycles are easy to describe and we prove first an asymptotic expansion statement. In the following Proposition we consider a path $\gamma : \mathbb{R} \to \mathbb{C}$ such that when $t \to \pm\infty$ the arguments of $x_n t^{a_n}$ and of $x_{n-1} t^{a_{n-1}}$ have both a limit in $]\frac{\pi}{2}, \frac{3\pi}{2}[$. These paths are exactly the rapid decay paths for both the function $t \to t^{-\beta-1}\exp\left(\sum_{j=1}^n x_j t^{a_j}\right)$ and for its restriction to the hyperplane $x_n = 0$.

Proposition 3.1. *The integral depending on $x = (x_1, \dots, x_n)$:*

$$I(\beta; x) = \int_\gamma t^{-\beta-1} \exp\left(\sum_{j=1}^{n} x_j t^{a_j}\right) dt$$

admits an asymptotic expansion $\sum_{k=0}^{\infty} c_k(x_1, \dots, x_{n-1}) x_n^k$ for x_n tending to zero whose coefficients are

$$c_k(x_1, \dots, x_{n-1}) = \frac{1}{k!} \int_\gamma t^{-\beta-1+a_n k} \exp\left(x_1 t^{a_1} + \cdots + x_{n-1} t^{a_{n-1}}\right) dt. \tag{3.1}$$

This expansion is valid for a fixed cycle γ in some product of open sectors in $\mathbb{C}^*$.

Proof. By developing the exponential $e^{x_n t^{a_n}}$ we may write

$$I(\beta; x) = \int_\gamma t^{-\beta-1} \exp\left(x_1 t^{a_1} + \cdots + x_{n-1} t^{a_{n-1}}\right) \left(\sum_{k=0}^{\infty} \frac{(x_n t^{a_n})^k}{k!}\right) dt$$

and we get

$$I(\beta; x_1, \dots, x_n) = \sum_{k=0}^{N} \frac{x_n^k}{k!} \int_\gamma t^{a_n k-\beta-1} \exp\left(x_1 t^{a_1} + \cdots + x_{n-1} t^{a_{n-1}}\right) dt + R_N(\beta; x_1, \dots, x_n) \tag{3.2}$$

where

$$R_N(\beta; x_1, \dots, x_n) = R_N(\beta; x) = \int_\gamma t^{-\beta-1} \exp\left(x_1 t^{a_1} + \cdots + x_{n-1} t^{a_{n-1}}\right) \left(\sum_{k=N+1}^{\infty} \frac{(x_n t^{a_n})^k}{k!}\right) dt.$$

These integrals are all convergent due to the choice of the path and we shall use the following elementary lemma involving an auxiliary complex variable $z \in \mathbb{C}$:

Lemma 3.2. *Let $r_N(z) = \sum_{k=N+1}^{\infty} \frac{z^k}{k!}$ be the remainder of order N of the exponential power series. There exists a positive real number C'_N, depending only on N, such that for all z with $\Re z < 0$ on has*

$$|r_N(z)| \leq C'_N |z|^{N+1}.$$

Since there is a compact set K such that for $t \in \gamma \setminus K$ we have $\Re(x_n t^{a_n}) < 0$ there is a possibly larger constant C_N depending also on γ such that

$$\forall t \in \gamma,\ |r_N(x_n t^{a_n})| \leq C_N |x_n t^{a_n}|^{N+1}$$

So we have

$$\begin{aligned} |R_N(\beta; x)| &= \left| x_n^{N+1} \int_\gamma t^{a_n(N+1)-\beta-1} \exp\left(x_1 t^{a_1} + \cdots + x_{n-1} t^{a_{n-1}}\right) \frac{r_N(x_n t^{a_n})}{(x_n t^{a_n})^{N+1}} dt \right| \\ &= |Q_N(\beta; x_1, \dots, x_n)| \cdot |x_n|^{N+1} \end{aligned}$$

this proves the existence of an asymptotic expansion which is locally uniform with respect to $(x_1, \dots, x_{n-1})$. Indeed as indicated in the statement, the domain of convergence of the last integral Q_N contains the product of $\mathbb{C}^{n-2}$ by a product of sectors in the variables x_{n-1}, x_n . It is convergent since the integrand is bounded by the integrable function :

$$C_N \left| t^{a_n(N+1)-\beta-1} \exp\left(x_1 t^{a_1} + \cdots + x_{n-1} t^{a_{n-1}}\right) \right|.$$

□

4. GEVREY EXPANSIONS OF HYPERGEOMETRIC INTEGRALS FOR $A = (a_1, \ldots, a_n)$

First of all we prove the following

Proposition 4.1. *The asymptotic expansion of the integral depending on* $(x_1, \ldots, x_n)$*:*

$$I(\beta; x) = \int_\gamma t^{-\beta-1} \exp\left(\sum_{j=1}^n x_j t^{a_j}\right) dt$$

given in Proposition 3.1 is a Gevrey series of order less than or equal to $s = \frac{a_n}{a_{n-1}}$ *with respect to* $x_n = 0$*.*

Proof. We set $(a_{n-1}, a_n) = (da, db)$ with $\gcd(a, b) = 1$ and for each $k \in \mathbb{N}$, $k = aq + j$ with $0 \leq j \leq a - 1$. The coefficients c_k in (3.1) satisfy by derivation under the sign $\int_\gamma$ the relation

$$(qa + j)! c_{qa+j}(x_1, \ldots, x_{n-1}) = j! \frac{\partial^{qb} c_j}{\partial {x_{n-1}}^{qb}}(x_1, \ldots, x_{n-1})$$

Since each of the functions $c_0, \ldots, c_{a-1}$ is holomorphic, we have in a small enough neighbourhood of a point $x_1, \ldots, x_{n-1}$ with $\Re x_{n-1} < 0$ a uniform upper bound involving a constant K that we can choose common to all the indices $j = 0, \ldots, a - 1$:

$$(qa + j)! \, |c_{qa+j}(x_1, \ldots, x_{n-1})| \leq j!(qb)! K^{qb}$$

Then a local upper bound of $c_k(x_1, \ldots, x_{n-1})$ is the quotient

$$\frac{(a-1)!(\lfloor \frac{kb}{a} \rfloor)! K^{\frac{kb}{a}}}{k!}.$$

This proves that the asymptotic expansion that we consider is a Gevrey series of order $\leq \frac{b}{a}$. □

Remark 4.2. We will see, as a consequence of Theorem 4.8, that in fact these Gevrey series have index equal to $\frac{a_n}{a_{n-1}}$ when β is generic enough. See Subsection 4.3.

4.1. **Case** $A = (a, b)$**.** The aim of this Subsection is to compare, in the case $A = (a, b)$, the Gevrey asymptotic expansions of the hypergeometric integrals to the Gevrey solutions described in Proposition 2.3. This comparison is proved in Theorem 4.7. We consider here $a, b \in \mathbb{Z}$, $1 \leq a < b$ and a, b are relative primes. We consider the integrals

$$I_\gamma(A, \beta; x) = I_\gamma(\beta; x) = \int_\gamma t^{-\beta-1} \exp\left(x_1 t^a + x_2 t^b\right) dt.$$

with respect to various specific cycles of rapid decay $\gamma = C_p$ as in Figure 1.

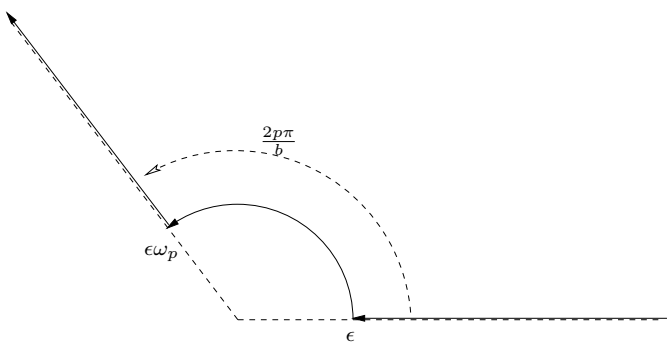

Fig. 1. The cycle C_p.

This path consists of the real half line $[\epsilon, +\infty)$ negatively oriented, and the half line $\omega_p \cdot [\epsilon, +\infty)$ where $\omega_p = e^{\frac{2ip\pi}{b}}$ is a b–th root of unity joined by the arc of a circle $\{\epsilon e^{i\theta} \mid 0 \leq \theta \leq \frac{2p\pi}{b}\}$ with $\epsilon > 0$.

The cycles $C_p \otimes \sigma$, for $1 \leq p \leq b$ are a basis of the rapid decay homology with closed support as defined in [Hi07] where σ is a section of the local system $\mathbb{C} \cdot t^{-\beta-1} \exp(x_1 t^a + x_2 t^b)$ in $\mathbb{C}^*$. We choose the determination of $t^{-\beta-1}$ as being real on $[\epsilon, +\infty)$.

When β is generic, in practise here when $\beta \notin \mathbb{Z}$, they are all non compact and when β is an integer the last cycle C_b is compact being equivalent to the circle of radius ϵ because the determinations $t^{-\beta-1}$ are the same for the argument 0 and 2π.

The integral along C_p does not depend of the choice of ϵ and we have an asymptotic expansion which is just a particular case of Proposition 3.1:

Proposition 4.3. *The integral depending on* (x_1, x_2)*:*

$$I_{C_p}(\beta; x_1, x_2) = \int_{C_p} t^{-\beta-1} e^{x_1 t^a + x_2 t^b} dt$$

is defined when $\Re x_2 < 0$ *and admit an asymptotic expansion* $\sum_{k=0}^{\infty} c_k(x_1) x_2^k$ *whose coefficients are*

$$c_k(x_1) = \frac{1}{k!} \int_{C_p} t^{-\beta-1+bk} e^{x_1 t^a} dt.$$

This expansion is valid in the open sector defined by $\Re x_1 < 0, \Re(\omega_p^a x_1) < 0$.

□

Remark 4.4. Notice that the open sector of the variable x_1 described in Proposition 4.3 is empty when $\omega_p^a = -1$ which happens if $(2k+1)b = 2pa$. In a forthcoming Proposition we will show that if we restrict the domain of the variable x_2 to some sector $\pi - \epsilon < \arg x_2 < \pi + \epsilon$ for a small $\epsilon > 0$ we may enlarge the domain of validity in the variable x_1.

We give now a more precise description of the asymptotic expansion of the integral $I_{C_p}(\beta; x_1, x_2)$. The coefficient $c_k(x_1)$ can be decomposed as a sum $(-I_{k,1}) + J_k + I_{k,2}$ where

$$I_{k,1}(\epsilon, x_1) = \frac{1}{k!} \int_{\epsilon}^{+\infty} s^{bk-\beta-1} e^{x_1 s^a} ds$$

$$J_k(\epsilon, x_1) = \frac{1}{k!} \int_0^{\frac{2p\pi}{b}} \epsilon^{bk-\beta-1} e^{-i\theta(\beta+1-bk)} e^{x_1 \epsilon^a e^{ia\theta}} i\epsilon\, e^{i\theta} d\theta$$

$$I_{k,2}(\epsilon, x_1) = \frac{1}{k!} \int_{\omega_p \cdot [\epsilon, +\infty)} t^{-\beta-1+bk} e^{x_1 t^a} dt = \frac{1}{k!} \int_{\epsilon}^{\infty} e^{\frac{2ip\pi}{b}(bk-\beta)} s^{bk-\beta-1} e^{x_1 \omega_p^a s^a} ds$$

For k large enough $\Re(bk - \beta) > 0$ and the limit of $J_k(\epsilon, x_1)$ is 0 when ϵ tends to 0. Under the same condition the limit of the sum $I_{k,1}$ and of $I_{k,2}$ exist and then

$$c_k(x_1) = \frac{1}{k!} \int_0^{+\infty} e^{\frac{2ip\pi}{b}(bk-\beta)} s^{bk-\beta-1} e^{x_1 \omega_p^a s^a} ds - \frac{1}{k!} \int_0^{+\infty} s^{bk-\beta-1} e^{x_1 s^a} ds \tag{4.1}$$

Les us denote $c_k(x_1) = I_{k,2}(x_1) - I_{k,1}(x_1)$ this decomposition of the coefficient $c_k(x_1)$ as a difference. We transform $I_{k,1}(x_1)$ for x_1 real negative by the change of variable $u = |x_1| s^a$, $ds = \frac{sdu}{au}$.

$$I_{k,1}(x_1) = \frac{1}{k!} \int_0^{\infty} s^{bk-\beta-1} e^{x_1 s^a} ds = \frac{1}{k!} \int_0^{+\infty} \left(\frac{u}{|x_1|}\right)^{\frac{bk-\beta}{a}} \frac{e^{-u} du}{au} \tag{4.2}$$

The final result is

$$I_{k,1}(x_1) = \frac{1}{a\,k!} e^{-i\pi(\frac{\beta-kb}{a})} x_1^{\frac{\beta-kb}{a}} \int_0^{+\infty} u^{\frac{bk-\beta}{a}-1} e^{-u} du$$

when we choose π as a determination of the argument of x_1. The equality is valid on the half plane $\Re x_1 < 0$ because both sides are holomorphic and coincide on the real negative half line by the equation (4.2). The last integral equals $\Gamma(\frac{bk-\beta}{a})$.

Similarly the first part of the equation (4.1) for $c_k(x_1)$ can be evaluated for the values of $x_1 \in \mathbb{C}$ such that $x_1\omega_p^a$ is real negative and setting $u = |x_1|s^a$ for such a fixed x_1

$$I_{k,2}(x_1) = \frac{1}{k!} \int_0^{+\infty} e^{\frac{2ip\pi}{b}(bk-\beta)} s^{bk-\beta-1} e^{x_1\omega_p^a s^a} ds = \frac{1}{k!} \int_0^{+\infty} e^{\frac{2ip\pi}{b}(bk-\beta)} \left(\frac{u}{|x_1|}\right)^{\frac{bk-\beta}{a}} \frac{e^{-u}du}{au}$$

The appropriate argument for is ω_p^a is $\frac{2ap\pi}{b}$, therefore in order to get obtain the argument π for $x_1\omega_p^a$ it is natural to set $\arg x_1 = \pi - \frac{2ap\pi}{b}$. With this choice of $\arg x_1$ and by a calculation identical to the one used for $I_{k,1}(x_1)$ we get :

$$|x_1|^{\frac{\beta-kb}{a}} = x_1^{\frac{\beta-kb}{a}} e^{-i\pi(\frac{\beta-kb}{a})} e^{\frac{2iap\pi}{b}(\frac{\beta-kb}{a})} = x_1^{\frac{\beta-kb}{a}} e^{-i\pi(\frac{\beta-kb}{a})} e^{\frac{2ip\pi}{b}\beta}$$

$$I_{k,2}(x_1) = \frac{1}{a\,k!} x_1^{\frac{\beta-kb}{a}} e^{-i\pi(\frac{\beta-kb}{a})} \int_0^{+\infty} u^{\frac{bk-\beta}{a}-1} e^{-u} du.$$

We get an expression of $I_{k,2}(x_1)$ formally identical to the one for $I_{k,1}(x_1)$. However the determination of the argument of x_1 is in general different in the intersection S_p of the two half planes $\Re x_1 < 0$ and $\Re x_1\omega_p^a < 0$. Therefore in this common sector there is a constant $c(k,p)$ such that $I_{k,1}(x_1) = c(k,p)I_{k,2}(x_1)$.

Assume that $\omega_p^a \neq -1$. We have $\arg(\omega_p^{-a}) = -\frac{2\pi ap}{b}$ and we define the integer $\ell \in \{0,\dots,a\}$ by the property that $\alpha = 2\pi(\ell - \frac{ap}{b})$ is the determination of the argument of $\arg(\omega^{-a})$ in $]-\pi,\pi[$

$$-\pi < 2\pi\ell - \frac{2\pi ap}{b} < \pi \tag{4.3}$$

in other words $\ell = \lfloor \frac{ap}{b} - \frac{1}{2} \rfloor$, and since $\omega_p^a \neq -1$, ℓ is the unique integer such that $|\frac{ap}{b} - \ell| < \frac{1}{2}$. The argument used for x_1 in the sector of validity of $I_{k,2}(x_1)$ is centered on the value $\pi - \frac{2ap\pi}{b} \in]-2\pi\ell, -2\pi(\ell-1)[$. Therefore to obtain in the common sector the determination of $\arg x_1$ used for $I_{k,1}(x_1)$ we must add $2\pi\ell$ and therefore we obtain at any point the sector S_p by treating the effect of this difference on the monomial $x_1^{\frac{\beta-kb}{a}}$:

$$I_{k,2}(x_1) = I_{k,1}(x_1) \times e^{-2i\pi\ell(\frac{\beta-kb}{a})} \tag{4.4}$$

If we set $k = am + j$ it results in

$$I_{am+j,2}(x_1) = I_{am+j,1}(x_1) \times \left(e^{2i\pi\frac{jb-\beta}{a}}\right)^{\ell}.$$

The asymptotic expansion of $I_{C_p}(x_1,x_2)$ is valid in the same sector by the proof of Proposition 3.1 and we obtain still limiting ourselves to k hence to m large enough:

$$\begin{aligned} c_{am+j}(x_1) &= \left(\left(e^{2i\pi\frac{jb-\beta}{a}}\right)^{\ell} - 1\right) I_{am+j,1}(x_1) \\ &= \frac{\left(e^{2i\pi\frac{jb-\beta}{a}}\right)^{\ell} - 1}{a(am+j)!} e^{i\pi(\frac{jb-\beta}{a})} e^{i\pi bm} x_1^{\frac{\beta-jb}{a}-bm} \Gamma\left(bm + \frac{jb-\beta}{a}\right) \end{aligned} \tag{4.5}$$

Assume that $\beta \notin \mathbb{Z}$. Using the well known identity

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z} \text{ and } \frac{\Gamma(z+1)}{\Gamma(z-m+1)} = [z]_m = z(z-1)\dots(z-m+1)$$

$$\Gamma\left(bm + \frac{bj-\beta}{a}\right) = \frac{(-1)^{bm}\pi}{\sin(\pi \cdot \frac{bj-\beta}{a})\Gamma(\frac{\beta-bj}{a} - bm + 1)} = \frac{(-1)^{bm}\pi}{\sin(\pi \cdot \frac{bj-\beta}{a})\Gamma(\frac{\beta-bj}{a} + 1)} \cdot \left[\frac{\beta-bj}{a}\right]_{bm},$$

the expression of the function c_{am+j} is finally :

$$c_{am+j}(x_1) = \left(\left(e^{2i\pi\frac{jb-\beta}{a}}\right)^{\ell} - 1\right) \cdot \frac{\pi e^{-i\pi(\frac{\beta-jb}{a})}}{a\, j! \sin(\pi \cdot \frac{bj-\beta}{a})\Gamma(\frac{\beta-bj}{a}+1)} x_1^{\frac{\beta-bj}{a}} \frac{\left[\frac{\beta-bj}{a}\right]_{bm}}{[am+j]_{am}} x_1^{-mb} \tag{4.6}$$

Now we are ready to prove a more precise statement about asymptotic expansion of the integral along the cycle C_p. Recall that we denote $\varphi^{(j)}$, see (2.4), the elements of a basis of the Gevrey solution space of order less than or equal to $s \geq \frac{b}{a}$ at generic points in $Y = (x_2 = 0)$, see Proposition 2.3. We recognize in the expression (4.6) of $c_{am+j}(x_1)$ a multiple by a factor depending only on j of the coefficient of x_2^{am+j} in the expansion of $\varphi^{(j)}$.

Proposition 4.5. (1) *The integral $I_{C_p}(x_1, x_2)$ has, provided that $e^{\frac{2i\pi pa}{b}} \neq -1$, an asymptotic expansion which is a linear combination of the Gevrey series $\varphi^{(j)}$. When $\beta \notin \mathbb{Z}$ the coefficient of $\varphi^{(j)}$ in this linear combination is equal to the product of $q_j^\ell - 1 := \left(e^{2i\pi\frac{jb-\beta}{a}}\right)^{\ell} - 1$ by a non zero constant λ_j which does not depend on p while $|\ell - \frac{ap}{b}| < \frac{1}{2}$. When $\beta \in \mathbb{Z}$ the same is true if $\frac{\beta-bj}{a} \notin \mathbb{Z}$. Finally when $\beta \in \mathbb{Z} \setminus \mathbb{N}(a,b)$ the coefficient of $\varphi^{(j_0)}$ is zero for the unique j_0 such that $\frac{\beta-bj_0}{a} \in \mathbb{Z}_{<0}$.*

(2) *This expansion is valid for $\Re x_2 < 0$ and x_1 in the sector of angular width $\pi - |\alpha|$, with $\alpha = 2\pi(\ell - \frac{ap}{b})$ defined by the condition:*

$$\arg x_1 \in \begin{cases} \left]\frac{\pi}{2} + \alpha, \frac{3\pi}{2}\right[\text{ if } \alpha \geq 0 \\ \left]\frac{\pi}{2}, \frac{3\pi}{2} + \alpha\right[\text{ if } \alpha \leq 0. \end{cases}$$

(3) *If we restrict the domain for the variable x_2 to a sector $|\arg(x_2) - \pi| < \epsilon$ with ϵ sufficiently small we can extend the domain of validity with respect to the variable x_1 to a larger sector, in such a way that for each $\ell \in \{1, \dots, a\}$ there exist a p such that $|\ell - \frac{ap}{b}| < \frac{1}{2}$ and the open extended sector contains the real negative half line.*

Proof. By Proposition 2.3 and since the integral $I_{C_p}(x_1, x_2)$ is a solution of the hypergeometric system $\mathcal{M}_A(\beta)$, the asymptotic expansion that we found in Proposition 4.3 is a linear combination of the Gevrey series $\varphi^{(j)}$, described in equation (2.4). Let us call μ_j the coefficient of this linear combination. Since the set of exponents of these series are mutually disjoint the sum $\sum_{m\geq 0} c_{am+j}(x_1)x_2^{am+j}$ must be the multiple $\mu_j\varphi^{(j)}$ of the series $\varphi^{(j)}$. Assuming first that $\beta \notin \mathbb{Z}$ the calculation for m large enough in equation (4.5) is sufficient to determine μ_j and comparing formula (4.6) with the expression of the series $\varphi^{(j)}$ gives the result with the value $\lambda_j = \pi e^{-i\pi(\frac{\beta-jb}{a})}\left(a\, j! \sin(\pi \cdot \frac{bj-\beta}{a})\Gamma(\frac{\beta-bj}{a}+1)\right)^{-1}$, which yields $\mu_j \neq 0$.

When $\beta \in \mathbb{Z}$ here is a unique $j_0 \in \{0, \dots, a-1\}$ such that $\frac{\beta-bj_0}{a}$ is an integer. For $j \neq j_0$, the same argument applies for the determination of μ_j and again $\mu_j \neq 0$. Let us write $\beta = bj_0 + aq$, with $q \in \mathbb{Z}$. In that case $c_{am+j_0}(x_1) = 0$ for m big enough. When $q < 0$ or equivalently $\beta \in \mathbb{Z} \setminus \mathbb{N}(a,b)$ the series $\varphi^{(j_0)}$ has an infinite number of terms and again the argument based on equation (4.5) gives the announced result but now with $\mu_{j_0} = 0$ by equation (4.5). Finally when $\beta \in \mathbb{N}(a,b)$ the series $\varphi^{(j_0)}$ reduces to a polynomial and equation (4.5) gives no information about μ_{j_0}.

In the exceptional case $\beta \in \mathbb{N}(a,b)$ we find directly that $\varphi^{(j_0)}$ comes from an integral solution. Indeed by inspection of the Γ-series in (2.4) we find that $\varphi^{(j_0)}$ is then the polynomial

$$\sum_{m=0}^{\lfloor \frac{q}{b} \rfloor} \frac{[q]_{bm}}{[am+j_0]_{am}} x_1^{q-bm} x_2^{am+j_0}$$

and this is exactly the integral

$$\frac{1}{2\pi i\, q!\, j_0!} I_\gamma(A,\beta;x) = \frac{1}{2\pi i\, q!\, j_0!} \int_\gamma t^{-\beta-1} \exp\left(x_1 t^a + x_2 t^b\right) dt$$

along the compact cycle $\gamma = (|t| = \epsilon)$ for $\epsilon > 0$ small enough.

Recall that $\alpha = 2\pi\ell - \frac{2\pi a p}{b} \in]-\pi, \pi[$ and consider the two half planes where the function $I_{k,1}$ and $I_{k,2}$ are defined. They intersect along the common sector of complex numbers with arguments in $]\frac{\pi}{2}, \frac{3\pi}{2}[\cap]\frac{\pi}{2} + \alpha, \frac{3\pi}{2} + \alpha[$. This gives the second statement.

Furthermore for each $\ell \in \{1, \ldots, a\}$ we choose p such that $|\alpha|$ the smallest possible which yields :

$$|\alpha| = |2\pi\ell - \frac{2\pi a p}{b}| \leq \frac{\pi a}{b}$$

This remark is useful for the proof of the statement (3) and shows that the restriction $\omega_p^a \neq -1$ is harmless since we can avoid it and reach any given ℓ.

In view of the last statement let us consider the family of paths $e^{i\theta} C_p$, for $|\theta| < \frac{\pi}{2b} - \frac{\epsilon}{b}$. When x_2 remains in the open sector $|\arg x_2 - \pi| < \epsilon$ centered on real negative axis they are all rapid decay cycles for the function $t^{-\beta-1}\exp(x_1 t^a + x_2 t^b)$ and are equivalent in the rapid decay homology. Therefore we have :

$$I_{C_p}(x_1,x_2) = \int_{C_p} t^{-\beta-1}\exp(x_1 t^a + x_2 t^b) dt = \int_{e^{i\theta}C_p} t^{-\beta-1}\exp(x_1 t^a + x_2 t^b) dt.$$

By Proposition 3.1 the Gevrey asymptotic expansion of $I_{C_p}(x_1,x_2)$ that we have obtained in statement (1) of the Proposition can also be written $\sum_{k=0}^\infty c_{\theta,k}(x_1) x_n^k$ with :

$$c_{\theta,k}(x_1) = \frac{1}{k!}\int_{e^{i\theta}C_p} t^{bk-\beta-1}\exp(x_1 t^a) dt = \frac{1}{k!}\int_{C_p} t^{bk-\beta-1}\exp(x_1 e^{ia\theta} t^a) dt$$

The domain of validity with respect to the variable x_1 of this asymptotic expansion therefore contains a sector S_θ which is the image by a rotation with an angle $-a\theta$ of the initial sector obtained from the part 1 of the Proposition. Since we can continuously deform the paths of integration from C_p to $e^{i\theta}C_p$ the domain of validity of the asymptotic expansion of I_{C_p} is the union of all the sectors S_θ. The original sector S_0 is centered at $\frac{\alpha}{2} + \pi$ and its width is $\pi - |\alpha|$. When $|\arg(x_2) - \pi| < \epsilon$ we have enlarged this width by an angle $\frac{a}{b}(\pi - 2\epsilon)$. The width of the enlarged sector is :

$$\pi - |\alpha| + \frac{a}{b}(\pi - 2\epsilon) > \pi - 2\epsilon.$$

Since this sector is centered at $\frac{\alpha}{2} + \pi \in]\frac{\pi}{2}, \frac{3\pi}{2}[$ this proves that it contains the real negative half line for $\epsilon > 0$ small enough. □

Proposition 4.6. *Assume that $\beta \notin \mathbb{Z}$. Then for every Gevrey series φ of order less than or equal to $s \geq \frac{b}{a}$, which is a solution of the hypergeometric system $\mathcal{M}_A(\beta)$, there is an holomorphic solution defined in a product of sectors, which is a neighbourhood of the product of the real negative axes, and which admits φ as an asymptotic expansion. This solution can be described as an integral of the function $t^{-\beta-1}\exp(x_1 t^a + x_2 t^b)$ along a rapid decay cycle.*

Proof. According to Proposition 2.3 it is sufficient to prove that each of the series $\varphi^{(j)}$ or what amount to the same each series $\lambda_j\varphi^{(j)}$ is the asymptotic expansion of such an integral. In Proposition 4.5, we have described such integrals and asymptotic expansions as linear combinations of the $\lambda_j\varphi^{(j)}$. The number a of these integrals is equal to the dimension of the space of Gevrey solutions. Therefore in order to prove the statement we just have to show that the square matrix of the coefficients of these linear combinations is invertible. In the notations of Proposition 4.5 this matrix is :

$$M = \begin{pmatrix} q_0 - 1 & q_0^2 - 1 & \dots & q_0^a - 1 \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ q_{a-1} - 1 & q_{a-1}^2 - 1 & \dots & q_{a-1}^a - 1 \end{pmatrix}$$

and one show by elementary calculations

$$\det M = \begin{vmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & q_0 & q_0^2 & \dots & q_0^a \\ \vdots & \dots & \dots & \dots & \dots \\ \vdots & \dots & \dots & \dots & \dots \\ 1 & q_{a-1} & q_{a-1}^2 & \dots & q_{a-1}^a \end{vmatrix} = \prod_{i=0}^{a-1}(q_i - 1)\prod_{i<j}(q_j - q_i) \neq 0$$

where the last inequality follows from $q_j = e^{2i\pi\frac{jb-\beta}{a}}$, with $\beta \notin \mathbb{Z}$ and a, b co-prime. □

Theorem 4.7. *For any $\beta \in \mathbb{C}$ and for every Gevrey series $\varphi \in \mathcal{O}_{\widehat{X|Y}}(s)$, $s \geq \frac{b}{a}$, which is a solution of the hypergeometric system $\mathcal{M}_A(\beta)$, there is an holomorphic solution defined in a product of sectors, which is a neighbourhood of the product of the real negative axes, and which admits φ as an asymptotic expansion. All these solutions can be described as an integral of the function $t^{-\beta-1}\exp(x_1t^a + x_2t^b)$ along a rapid decay cycle when $\beta \notin \mathbb{Z}\setminus\mathbb{N}(a,b)$. When $\beta \in \mathbb{Z}\setminus\mathbb{N}(a,b)$, the above integral solutions span a codimension one subspace and there is a one dimensional supplementary space obtained by expanding an integral along $[0, +\infty)$.*

Proof. When $\beta \notin \mathbb{Z}$, Proposition 4.6 give a complete proof of the statement. When $\beta \in \mathbb{Z}$ we shall write $\beta = p_0b + aq$, $0 \leq p_0 < a$ and the proof of the same Proposition shows that all the Gevrey series $\varphi^{(p)}$ for $p \neq p_0$ are obtained as integrals on rapid decay cycles. The non obtained Gevrey series is :

$$\varphi^{(p_0)} = \sum_{m\geq 0} \frac{[q]_{bm}}{[am + p_0]_{am}} x_1^{q-bm} x_2^{am+p_0}$$

It is a polynomial if and only if $q \geq 0$, that is when $\beta \in \mathbb{N}(a,b)$. We notice that it is exactly the case where the integral along a circle of radius $\epsilon > 0$ centered at the origin which is equal to

$$\int_{C_b} t^{-\beta-1}e^{x_1t^a+x_2t^b}dt = 2\pi i \sum_{\ell_1a+\ell_2b=\beta} \frac{x_1^{\ell_1}x_2^{\ell_2}}{\ell_1!\ell_2!}$$

is non zero. Since this is a polynomial solution we are done in this case.

Finally when $\beta \in \mathbb{Z}\setminus\mathbb{N}(a,b)$, that is when $q > 0$, we notice that there is an integral holomorphic solution given by the formula :

$$J_\beta(x_1, x_2) = \int_0^\infty t^{-\beta-1}\left(e^{x_1t^a+x_2t^b} - P_\beta(x_1, x_2, t)\right)dt \tag{4.7}$$

where $P_\beta(x_1,x_2,t)=0$ if $\beta<0$, and if $\beta\in\mathbb{N}\setminus\mathbb{N}(a,b)$, we take the Taylor polynomial of degree $\leq\beta$ in t :

$$P_\beta(x_1,x_2,t)=\sum_{\ell_1a+\ell_2b\leq\beta}\frac{x_1^{\ell_1}x_2^{\ell_2}t^{\ell_1a+\ell_2b}}{\ell_1!\ell_2!}.$$

It is in fact of degree $<\beta$, by the condition $\beta\notin\mathbb{N}(a,b)$ which guarantees the convergence of J_β.

The fact that J_β is a solution of the system $H_A(\beta)$ is completely similar to the proof in [A94, Section 2] for the case of rapid decay cycle. The reason is first that since $\partial_1^b-\partial_2^a$ annihilates $e^{x_1t^a+x_2t^b}$, it annihilates as well the coefficients of all the monomials t^j in its power expansion of hence also the polynomial $P_\beta(x_1,x_2,t)$. Concerning the Euler operator $\chi=ax_1\partial_1+bx_2\partial_2$ we just have to notice that the relation for the integrand which leads to the proof is still valid and it is :

$$\chi\left(e^{x_1t^a+x_2t^b}-P_\beta(x_1,x_2,t)\right)=t\partial_t\left(e^{x_1t^a+x_2t^b}-P_\beta(x_1,x_2,t)\right).$$

In order to conclude the proof we just have to check that J_β admits under the same conditions as in Section 4.1 an asymptotic expansion (of Gevrey order less than or equal to $\frac{b}{a}$) which must then be a linear combination $\sum\mu_p\varphi^{(p)}$ of all the $\varphi^{(p)}$. We rewrite it as follows

$$J_\beta(x_1,x_2)=\int_0^\infty\left(\sum_{k=0}^\infty t^{-\beta-1+bk}\left(e^{x_1t^a}-\sum_{\ell a\leq\beta-kb}\frac{x_1^\ell t^{\ell a}}{\ell!}\right)\frac{x_2^k}{k!}\right)dt$$

and this lead to the existence of an asymptotic expansion completely similar to the one given in the general case, but with a coefficient of x_2^k given by :

$$c_k(x_1)=\frac{1}{k!}\int_0^\infty t^{-\beta-1+bk}\left(e^{x_1t^a}-\sum_{\ell a\leq\beta-kb}\frac{x_1^\ell t^{\ell a}}{\ell!}\right)dt \tag{4.8}$$

which is again a convergent integral. The change of variable $s=-x_1t^a$ for $x_1\in\mathbb{R}_{<0}$ works exactly in the same way as in Remark 4.4 and Proposition 4.5. In fact as soon as k is large enough the correcting term $\sum_{\ell a\leq\beta-kb}\frac{x_1^\ell t^{\ell a}}{\ell!}$ is zero and we find explicitly :

$$c_k(x_1)=\frac{1}{ak!}e^{-i\pi\frac{\beta-bk}{a}}\Gamma\left(\frac{bk-\beta}{a}\right)x_1^{\frac{\beta-kb}{a}}.$$

Considering the coefficients $c_{am+p_0}(x_1)$ for m large enough, we see that the coefficient μ_{p_0} of $\varphi_{A,\beta}^{(p_0)}$ is non zero in the asymptotic expansion of J_β as expected. □

4.2. **A basis of Gevrey asymptotic expansions.** Let $A=(a_1,\dots,a_n)$ with integers $0<a_1<\cdots<a_n$ and $\gcd(a_1,\dots,a_n)=1$. For any $\beta\in\mathbb{C}$ and any rapid decay 1-cycle γ we write $\Phi_\gamma(A,\beta;x)$ the asymptotic expansion of the integral $I_\gamma(A,\beta,x)$, that is

$$\Phi_\gamma(A,\beta;x)=\text{a.e.}(I_\gamma(A,\beta,x))=\text{a.e.}\left(\int_\gamma t^{-\beta-1}\exp(\sum_j x_jt^{a_j})dt\right).$$

In Propositions 3.1 and 4.1 we have proved that for some well defined cycles $\gamma:\mathbb{R}\to\mathbb{C}^*$ there exists an asymptotic expansion of the integral $I_\gamma(\beta;x)=I_\gamma(A,\beta,x)$ and that this asymptotic expansion $\Phi_\gamma(A,\beta;x)$ is a germ of Gevrey series in $\mathcal{O}_{\widehat{X|Y}}(s)$ at any point in $Y\setminus Z$ for all $s\geq\frac{a_n}{a_{n-1}}$. When $\beta\in\mathbb{Z}\setminus\mathbb{N}A$, and restricting to a product of sector in the variable x_{n-1},x_n containing the real negative axes we may, as in the previous Section, define an asymptotic expansion for the integral

$$J_\beta(x_1,\dots,x_n)=\int_0^\infty\left(t^{-\beta-1}\exp\left(x_1t^{a_1}+\cdots+x_nt^{a_n}\right)-P_\beta(x_1,\dots,x_n,t)\right)dt$$

where $P_\beta(x_1,\dots,x_n,t)$ is the Taylor polynomial of degree $\leq \beta$ in t of the function $\exp\left(\sum_{i=1}^n x_i t^{a_i}\right)$. When $\beta \in \mathbb{Z}\setminus\mathbb{N}A$ and $\gamma = [0,+\infty)$ we still denote $\Phi_\gamma(A,\beta;x)$ this asymptotic expansion, and by extension we will still call cycle an element of the formal direct sum of $\mathbb{C}\cdot[0,+\infty)$ and of the space of a rapid decay cycle.

The goal of the remaining part of this Section is to prove the following

Theorem 4.8. *There exists a family of cycles $\gamma_1,\dots,\gamma_{a_{n-1}}$ in $\mathbb{C}^*$ such that the set of (germs of) Gevrey asymptotic expansions*

$$\left\{\Phi_{\gamma_j}(A,\beta;x)\,|\,j=1,\dots,a_{n-1}\right\}$$

is a basis of the stalk of $\mathcal{H}om_{\mathcal{D}_X}\left(\mathcal{M}_A(\beta),\mathcal{O}_{\widehat{X|Y}}(s)\right)$ at any point in $Y\setminus Z$, for $\beta\in\mathbb{C}$ and for $s\geq\frac{a_n}{a_{n-1}}$.

Remark 4.9. 1) By [FC11b, Remark 4.12] it is enough to prove the Theorem for points of the form $(0,\dots,0,\varepsilon,0)\in X$ with $\varepsilon\neq 0$.

2) In the proof of this Theorem we will see that for $\beta\notin\mathbb{Z}\setminus\mathbb{N}A$ the cycles γ_j are all rapid decay cycles. When $\beta\in\mathbb{Z}\setminus\mathbb{N}A$ there exists a unique $\Phi_{\gamma_{j_0}}(A,\beta;x)$ which is obtained by expanding an integral along $[0,+\infty)$ the other cycles being again cycles with the rapid decay property.

3) In the proof of Theorem 4.8 we obtain integral solutions in the neighbourhood of a product of sectors for the variables x_{n-1},x_n centered along the real negative axes, while the other variables $x_1,\dots,x_{n-2}$ are arbitrary in $\mathbb{C}$. We can deduce from it the same statement for products of sectors centered along arbitrary rays $]0,+\infty)\cdot e^{i\theta_1}\times]0,+\infty)\cdot e^{i\theta_2}$. For the proof we use cycles of the form $e^{i\theta}\cdot\gamma_j$ and $e^{i\theta}\cdot[0,+\infty)$ when the argument of x_2 is near $\theta_2 = b\theta \pmod 2\pi$. The result can be obtain from this data by using carefully the width of the sector of validity with respect to x_1 as described in Proposition 4.5 and playing with the different determinations of $\theta=\theta_2/b$.

Corollary 4.10. [of Theorem 4.7] *Theorem 4.8 holds for $n=2$.*

Proof. In the proof of Theorem 4.7 one describes, by using Propositions 4.5 and 4.6, how to build the cycles $\gamma_1,\dots,\gamma_a$ for $A=(a,b)$, $1\leq a<b$, $\gcd(a,b)=1$. If $\beta\notin\mathbb{Z}\setminus\mathbb{N}(a,b)$ all the γ_p are rapid decay cycles. When $\beta\in\mathbb{Z}\setminus\mathbb{N}(a,b)$ among them a unique cycle is $[0,\infty)$. □

In order to simply notations we will denote $\mathrm{Sol}(\mathcal{M}) = \mathcal{H}om_{\mathcal{D}}(\mathcal{M},\mathcal{O}_{\widehat{X|Y}}(\frac{a_n}{a_{n-1}}))$ the sheaf of Gevrey solutions of order less than or equal to $\frac{a_n}{a_{n-1}}$ of a holonomic $\mathcal{D}_X$–module $\mathcal{M}$ on $X=\mathbb{C}^n$ for $Y=(x_n=0)\subset X$.

Remark 4.11. By Corollary 4.10, for all $\beta\in\mathbb{C}$ and for each $r=0,\dots,k-1$ there exists a family $\gamma_{1,r},\dots,\gamma_{a,r}$ of cycles such that the family

$$\left\{\Phi_{\gamma_{1,r}}\left((a,b),\frac{\beta-r}{k};x_1,x_2\right),\dots,\Phi_{\gamma_{a,r}}\left((a,b),\frac{\beta-r}{k};x_1,x_2\right)\right\}$$

is a basis of the stalk of the solution space $\mathrm{Sol}(\mathcal{M}_{(a,b)}(\frac{\beta-r}{k}))$ at any point in $Y_1\setminus Z_1$. Here $Y_1\subset\mathbb{C}^2$ (resp. $Z_1\subset\mathbb{C}^2$) is the line $x_2=0$ (resp. $x_1=0$). If there exists $0\leq r_0<k$ (necessarily unique) such that $\frac{\beta-r_0}{k}\in\mathbb{Z}\setminus\mathbb{N}(a,b)$ then among them one of these cycles is $[0,\infty)$.

In the statement of the following Proposition we are going to use the morphism $\varpi_\beta := \varpi_\beta^s$, for $s=\frac{a_n}{a_{n-1}}$, defined in Subsection 2.3.

Proposition 4.12. *Theorem 4.7 holds for the matrix $(1,ka,kb)$ with $\gcd(a,b)=1$ and $k\geq 1$. More precisely, there exists a set of cycles $\{\widetilde{\gamma}_{p,r}\,|\,0\leq p<a, 0\leq r<k\}$ such that the image (of the germ) of $\Phi_{\widetilde{\gamma}_{p,r}}((1,ka,kb),\beta)$ by the morphism ϖ_β equals $(0,\cdots,0,\frac{1}{k}\Phi_{\gamma_{p,r}}((a,b),\frac{\beta-r}{k}),0,\cdots,0)$ in $\bigoplus_{r=0}^{k-1}\mathrm{Sol}(\mathcal{M}_{(a,b)}(\frac{\beta-r}{k}))$.*

Proof. We consider the germs at a point $(0,\varepsilon,0)$ with $\varepsilon\neq 0$ and the stalk of the direct sum at $(\varepsilon,0)$. Write $B'=(1,ka,kb)$ and $B=(a,b)$. We write (x_0,x_1,x_2) for coordinates in $\mathbb{C}^3$. The proof here depend heavily on the Proposition 2.7 in Section 2. When $\beta\notin\mathbb{Z}$ or when $\beta\in\bigcup_{r=0}^{k-1}(r+k\mathbb{N}(a,b))$ the morphism ϖ_β^s studied in this Proposition is an isomorphism and all the cycles that we obtain are rapid decay cycles. When $\beta\in\mathbb{Z}\setminus\bigcup_{r=0}^{k-1}(r+k\mathbb{N}(a,b))$ one of these cycles is special. This last cose containing the one case $\beta\in\mathbb{N}\setminus\bigcup_{r=0}^{k-1}(r+k\mathbb{N}(a,b))$ where ϖ_β^s is not an isomorphism.

Let us assume first $\beta\notin\mathbb{Z}$. We fix $\gamma_{p,r}$ one of the cycles described in Corollary 4.10 with respect to the matrix B and $\frac{\beta-r}{k}$. Consider any cycle γ in $\mathbb{C}^*$ such that $\gamma^k=\gamma_{p,r}$. We develop the integral

$$I_\gamma=I_\gamma(x_0,x_1,x_2)=\int_\gamma t^{-\beta-1}\exp(x_0t+x_1t^{ka}+x_2t^{kb})dt$$

as

$$I_\gamma(x_0,x_1,x_2)=\sum_{\ell=0}^{k-1}x_0^\ell J_\gamma^\ell(x_0^k,x_1,x_2).$$

Notice that $\frac{\partial^\ell I_\gamma}{\partial x_0^\ell}(0,x_1,x_2)=\ell!J_\gamma^\ell(0,x_1,x_2)$. Notice also that

$$\frac{\partial^r I_\gamma}{\partial x_0^r}(0,x_1,x_2)=\frac{1}{k}\int_{\gamma_{p,r}}s^{-\frac{\beta-r}{k}-1}\exp(x_1s^a+x_2s^b)ds=\frac{1}{k}I_{\gamma_{p,r}}\left(B,\frac{\beta-r}{k};x_1,x_2\right).$$

Write $\omega=e^{\frac{2i\pi}{k}}$ and for $\nu=0,\dots,k-1$, $\gamma^{(\nu)}:=\omega^{-\nu}\gamma$. We can write

$$I_\gamma(x_0,x_1,x_2)=\omega^{-\nu\beta}\int_{\gamma^{(\nu)}}s^{-\beta-1}\exp(x_0\omega^\nu s+x_1s^{ka}+x_2s^{kb})ds=\omega^{-\nu\beta}I_{\gamma^{(\nu)}}(x_0\omega^\nu,x_1,x_2)$$

and then

$$I_{\gamma^{(\nu)}}(x_0,x_1,x_2)=\omega^{\nu\beta}I_\gamma(x_0\omega^{-\nu},x_1,x_2)=\omega^{\nu\beta}\sum_{\ell=0}^{k-1}\omega^{-\nu\ell}x_0^\ell J_\gamma^\ell(x_0^k,x_1,x_2).$$

The matrix $(\omega^{-\nu\ell})_{0\leq\nu,\ell\leq k-1}$ being invertible we can write

$$x_0^\ell J_\gamma^\ell(x_0^k,x_1,x_2)=\sum_{\nu=0}^{k-1}\lambda_{\ell,\nu}\omega^{-\nu\beta}I_{\gamma^{(\nu)}}(x_0,x_1,x_2) \tag{4.9}$$

for some $\lambda_{\ell,\nu}\in\mathbb{C}$.

Consider now the cycle $\widetilde{\gamma}_{p,r}:=\sum_{\nu=0}^{k-1}\lambda_{r,\nu}\omega^{-\nu\beta}\gamma^{(\nu)}$ and the integral $I_{\widetilde{\gamma}_{p,r}}(x_0,x_1,x_2)$. Then we have

$$\frac{\partial^\ell I_{\widetilde{\gamma}_{p,r}}}{\partial x_0^\ell}(0,x_1,x_2)=0$$

if $\ell\neq r$ and

$$\frac{\partial^r I_{\widetilde{\gamma}_{p,r}}}{\partial x_0^r}(0,x_1,x_2)=r!J_{\widetilde{\gamma}_{p,r}}^r(0,x_1,x_2)=\frac{1}{k}I_{\gamma_{p,r}}\left(B,\frac{\beta-r}{k};x_1,x_2\right).$$

Assume now $\beta\in\mathbb{Z}$. Write $\beta=\beta'k+r_0$ for unique integers β' and $0\leq r_0<k$. Write by Euclidean division $\beta'=ha+p_0$, $0\leq p_0<a$. We can apply the argument for $\beta\notin\mathbb{Z}$ to any $(p,r)\neq(p_0,r_0)$ to prove that there exists a cycle $\widetilde{\gamma}_{p,r}$ such that the image of $\Phi_{\widetilde{\gamma}_{p,r}}(B',\beta)$ by the morphism ϖ_β is $(0,\dots,0,\frac{1}{k}\Phi_{\gamma_{p,r}}(B,\frac{\beta-r}{k};x_1,x_2),0,\dots,0)$.

Let us treat now the case of (p_0,r_0). If $\beta\in\mathbb{Z}\setminus\bigcup_{r=0}^{k-1}(r+k\mathbb{N}(a,b))$ then we consider the expression

$$J_\beta=\int_0^\infty t^{-\beta-1}(\exp(x_0t+x_1t^{ka}+x_2t^{kb})-P_\beta(x_0,x_1,x_2,t))dt$$

with analogous notation to the one used in the proof of Theorem 4.7. It can be proved as in Theorem 4.7 that J_β is a holomorphic solution of the system $\mathcal{M}_{B'}(\beta)$ and that it admits an asymptotic expansion which is of Gevrey order less than or equal to $\frac{a_n}{a_{n-1}}$. In fact J_β is holomorphic in a product of sectors which is a neighbourhood of the product of the real negative axes. Then we conclude that this asymptotic expansion is a linear combination $\sum_j \lambda_j \varphi_{B'}^{(j)}$ with $\lambda_{p_0} \neq 0$.

If $\beta \in \bigcup_{r=0}^{k-1}(r + k\mathbb{N}(a,b))$, the integral along the cycle C_b (see notations in Subsection 4.1)

$$\int_{C_b} t^{-\beta-1} \exp(x_0 t + x_1 t^{ka} + x_2 t^{kb}) dt$$

is a polynomial and its image by ϖ_β is a non-zero scalar multiple of $(0, \dots, 0, \varphi_B^{(p_0)}(\frac{\beta - r_0}{k}), 0, \dots, 0)$.

This finishes the proof. □

Proof. [of Theorem 4.8, general case]. We may assume $n \geq 3$. We simply write $ka = a_{n-1}, kb = a_n$ for some integer $k \geq 1$ and $\gcd(a,b) = 1$.

We first assume $a_1 > 1$ and denote as usual $A' = (1, a_1, \cdots, a_n)$. We also denote $B' = (1, a_{n-1}, a_n) = (1, ka, kb)$ and $B = (a,b)$. Let us consider the following diagram

$$\begin{array}{ccc} \mathrm{Sol}(\mathcal{M}_{A'}(\beta))_p & \xrightarrow{\rho'} & \mathrm{Sol}(\mathcal{M}_{B'}(\beta))_p \\ \big\downarrow \rho & & \\ \mathrm{Sol}(\mathcal{M}_A(\beta))_p. & & \end{array}$$

Let us explain the restriction morphisms in the above diagram: ρ' is the restriction defined as

$$\rho'(f(x_0, x_1, \dots, x_n)) = f(x_0, 0, \dots, 0, x_{n-1}, x_n).$$

Similarly $\rho(f(x_0, x_1, \dots, x_n)) = f(0, x_1, \dots, x_n)$. We consider a point $p = (\varepsilon, 0) \in \mathbb{C}^2$ with $\varepsilon \neq 0$ and we also denote p the image of this point in the different considered spaces.

The morphism ρ' is an isomorphism for any $\beta \in \mathbb{C}$, see Subsection 2.1. The morphism ρ is an isomorphism if $\beta \in \mathbb{N} \setminus \mathbb{N}A$, see Theorem 2.5 and Remark 2.6.

Let us consider the set of asymptotic expansions

$$\{\Phi_{\widetilde{\gamma}_{p,r}}(A', \beta) \,|\, 0 \leq p < a, 0 \leq r < k\}$$

where $\widetilde{\gamma}_{p,r}$ is the cycle built in the proof of Proposition 4.12. The image by ρ' of $\Phi_{\widetilde{\gamma}_{p,r}}(A', \beta)$ is just $\Phi_{\widetilde{\gamma}_{p,r}}(B', \beta)$. This prove the theorem for A' and then for A if $\beta \notin \mathbb{N} \setminus \mathbb{N}A$ (because in this case ρ is an isomorphism and the image of $\Phi_{\widetilde{\gamma}_{p,r}}(A', \beta)$ is precisely $\Phi_{\widetilde{\gamma}_{p,r}}(A, \beta)$).

If $\beta \in \mathbb{N} \setminus \mathbb{N}A$ then we have again $\rho(\Phi_{\widetilde{\gamma}_{p,r}}(A', \beta)) = \Phi_{\widetilde{\gamma}_{p,r}}(A, \beta)$ for all $(p,r) \neq (p_0, r_0)$. The integral

$$J_\beta := \int_0^\infty t^{-\beta-1} \left(\exp\left(\sum_{i=1}^n x_i t^{a_i} \right) - P_\beta(x_1, \dots, x_n, t) \right) dt \tag{4.10}$$

defines a holomorphic function in a domain $\mathbb{C}^{n-2} \times S_{n-1} \times S_n$ where S_i is a sector in $\mathbb{C}$ which is a neighbourhood of the real negative axis for $i = n-1, n$. Here P_β is the Taylor polynomial, for the exponential, of degree $\leq \beta$ in t.

As usual we consider, for k big enough,

$$c_k(x_1,\dots,x_{n-1}) = \frac{1}{k!}\int_0^\infty t^{-\beta-1+a_nk}\exp\left(\sum_{i=1}^{n-1}x_it^{a_i}\right)dt \tag{4.11}$$

the coefficient of x_n^k in the expansion of J_β. By developing we get

$$c_k(x_1,\dots,x_{n-1}) = \sum_{m_1,\dots,m_{n-2}\geq 0} c_{m_1,\dots,m_{n-2},k}(x_{n-1})x_1^{m_1}\cdots x_{n-2}^{m_{n-2}}$$

where, writing $m_n = k$ big enough,

$$c_{m_1,\dots,m_{n-2},m_n}(x_{n-1}) = \frac{1}{m_1!\cdots m_{n-2}!m_n!}\int_0^\infty t^{-\beta-1+\sum_{i\neq n-1}a_im_i}e^{x_{n-1}t^{a_{n-1}}}dt.$$

Up to a scalar multiple this last integral equals

$$x_{n-1}^{\frac{\beta-\sum_{i\neq n-1}a_im_i}{a_{n-1}}}\Gamma\left(\frac{-\beta+\sum_{i\neq n-1}a_im_i}{a_{n-1}}\right).$$

The condition $\beta\in\mathbb{N}\setminus\mathbb{N}A$ implies that the argument of the Gamma factor is never a non-positive integer. Writing $\beta = qa_{n-1}+j_0$ with $0\leq j_0<a_{n-1}$ and choosing $m_1,\dots,m_{n-2},m_{n-1},m_n\geq 0$ such that $j_0+a_{n-1}m_{n-1} = \sum_{i\neq n-1}a_im_i$ we see that the corresponding exponent of x_{n-1} in the expansion of J_β is $\frac{\beta-\sum_{i\neq n-1}a_im_i}{a_{n-1}} = q-m_{n-1}$ which is a negative integer if m_{n-1} is large enough. Moreover, the asymptotic expansion of J_β is a Gevrey series solution of $\mathcal{M}_A(\beta)$ of order less than or equal to $\frac{a_n}{a_{n-1}}$ which is linearly independent of the set $\{\varphi^j_{A',\beta}(0,x)\,|\,j\neq j_0\}$. This finishes the proof of the theorem for A if $a_1>1$.

If finally $a_1=1$ then we apply previous discussion by using the restriction to the case $(1,a_{n-1},a_n)$. □

4.3. **Gevrey solutions modulo convergent solutions.** We can also give a description of the stalk of $\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}_A(\beta),\mathcal{Q}_Y(s))$ at any point of $Y\setminus Z$ where, for $s\geq 1$, $\mathcal{Q}_Y(s)$ is the quotient of $\mathcal{O}_{\widehat{X|Y}}(s)$ by $\mathcal{O}_{X|Y}$. By [FC11b, Th. 5.3] this space is just (0) if $1\leq s<\frac{a_n}{a_{n-1}}$ and has dimension a_{n-1} if $s\geq\frac{a_n}{a_{n-1}}$. We will assume in this Subsection that $s\geq\frac{a_n}{a_{n-1}}$.

Let $\phi(t)$ a C^∞ function with compact support locally constant with value 1 near the origin. We consider the following integral, see (4.10):

$$J_{\phi,\beta}(x) = J_{\phi,\beta}(x_1,\dots,x_n) := \int_0^\infty t^{-\beta-1}\left(e^{x_1t^{a_1}+\cdots+x_nt^{a_n}}-P_\beta(x,t\phi(t))\right)dt. \tag{4.12}$$

We write $J_{\phi,\beta}(A;x)$ if we want to emphasize the dependence of this integral on the matrix A. This integral defines a holomorphic function in $\mathbb{C}^{n-1}\times S_{n-1}\times S_n$ for some open sectors $S_i\subset\mathbb{C}$, $i=n-1,n$, each of them containing the real negative axis. In general this integral is not a solution of the hypergeometric system $\mathcal{M}_A(\beta)$ but it is a solution modulo convergent power series.

Now we come back to the Gevrey solutions modulo convergent ones. We will treat the case $A=(a,b)$ first. According to Corollary 4.10, for $\beta\notin\mathbb{Z}$ the family of asymptotic expansions $\{\Phi_\ell(A,\beta) := \Phi_{\gamma_\ell}(A,\beta),\ell=1,\dots,a\}$ is a basis of Gevrey solutions of $\mathcal{M}_A(\beta)$ in $\mathcal{O}_{\widehat{X|Y}}(s)$. For $\beta\notin\mathbb{Z}$, the family $\{\varphi^{(p)}_{A,\beta}\,|\,p=0,\dots,a-1\}$ is also a basis of Gevrey solutions and their classes modulo convergent series form a basis of $\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}_A(\beta),\mathcal{Q}_Y(s))$ [FC11a, Th. 5.9 (i)]. Then, the classes of $\{\Phi_\ell(A,\beta),\ell=1,\dots,a\}$ modulo convergent series forms a basis of $\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}_A(\beta),\mathcal{Q}_Y(s))$.

Previous situation is still valid when $\beta\in\mathbb{Z}\setminus\mathbb{N}(a,b)$, taking into account that we have denoted $\Phi_a(A,\beta)$ the asymptotic expansion of an integral $J_\beta(x_1,x_2)$ over $[a,\infty)$, see (4.7).

Assume now $\beta \in \mathbb{N}(a,b)$ and write $\beta = p_0 b + qa$ with $0 \leq p_0 < a$ and $q \geq 0$. We know that the Gamma series $\phi_{A,\beta}^{(p_0)}$ is a polynomial and then its class modulo convergent series is zero. We denote by $\Phi_a(A,\beta)$ the asymptotic expansion of $J_{\phi,\beta}(x_1,x_2)$. The coefficient $c_k(x_1)$ of this expansion is, for k big enough, exactly the same as in (4.8). In particular, the exponent of x_1 in $c_{am+p_0}(x_1)$ for m big enough is the negative integer $q-bm$. This proves that the family $\{\Phi_\ell(A,\beta),\, \ell = 1,\ldots,a\}$ is still linearly independent modulo convergent power series. Hence, it defines a basis of $\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}_A(\beta),\mathcal{Q}_Y(s))$.

We treat now the case $A = (a_1,\ldots,a_n)$ and $n \geq 3$. We write as usual $a_{n-1} = ka, a_n = kb$ for $k \geq 1$ and $\gcd(a,b) = 1$. According to Theorem 4.8, for $\beta \notin \mathbb{N}$ the family of asymptotic expansions $\{\Phi_{\widetilde{\gamma}_{p,r}}(A,\beta),\, 0 \leq p < a,\, 0 \leq r < k\}$ is a basis of Gevrey solutions of $\mathcal{M}_A(\beta)$ in $\mathcal{O}_{\widehat{X|Y}}(s)$. For $\beta \notin \mathbb{N}$, the family $\{\widetilde{\varphi}_{A,\beta}^{(j)} := \varphi_{A',\beta}^{(j)}(0,x) \,|\, j = 0,\ldots,a_{n-1}-1\}$ is also a basis of Gevrey solutions and their classes modulo convergent series form a basis of $\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}_A(\beta),\mathcal{Q}_Y(s))$ [FC11b, Th. 5.5, (i)]. Then, the classes of $\{\Phi_{\widetilde{\gamma}_{p,r}}(A,\beta),\, 0 \leq p < a,\, 0 \leq r < k\}$ modulo convergent series form a basis of $\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}_A(\beta),\mathcal{Q}_Y(s))$.

Previous situation is still valid when $\beta \in \mathbb{N} \setminus \mathbb{N}A$, taking into account that we have denoted $\Phi_{\widetilde{\gamma}_{p_0,r_0}}(A,\beta)$ the asymptotic expansion of an integral $J_\beta(x)$ over $[a,\infty)$, see (4.10) and we have considered this asymptotic expansion as the generator of a complement space of $\{\widetilde{\varphi}_{A,\beta}^{(j)},\, j = 0,\ldots,a_{n-1}-1\}$.

Assume now $\beta \in \mathbb{N}A$. We denote by $\Phi_{\widetilde{\gamma}_{p_0,r_0}}(A,\beta)$ the asymptotic expansion of $J_{\phi,\beta}(x)$, see (4.12), with respect to x_n. The coefficient $c_k(x_1,\ldots,x_{n-1})$ of this expansion is, for k big enough, exactly the same as in (4.11). We can proceed as in the proof of the general case of Theorem 4.8 to see that the classes modulo convergent power series of the asymptotic expansions $\Phi_{\widetilde{\gamma}_{p,r}}(A,\beta)$ form a basis of the solution space $\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}_A(\beta),\mathcal{Q}_Y(s))$.

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