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Completions and simplicial complexes *

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Abstract

In this paper, we first introduce the notion of a completion. Completions are inductive properties which may be expressed in a declarative way and which may be combined. In the sequel of the paper, we show that completions may be used for describing structures or transformations which appear in combinatorial topology. We present two completions, $\langle \textsc{Cup} \rangle$ and $\langle \textsc{Cap} \rangle$, in order to define, in an axiomatic way, a remarkable collection of acyclic complexes. We give some basic properties of this collection. Then, we present a theorem which shows the equivalence between this collection and the collection made of all complexes that are acyclic in the sense of homology theory.

Keywords: Completions, simplicial complexes, collapse, computer imagery, homology, combinatorial topology.

1 Introduction

Several approaches have been proposed for the study of topological properties of digital objects in the context of computer imagery:

- The digital topology approach introduced by A. Rosenfeld [5]. Elements of $\mathbf{Z}^d$ are linked by some adjacency relations. It is not obvious, in this framework, to define certain topological notions (*e.g.*, a homotopy).

- The connected ordered topological space (COTS) approach introduced by E. Khalimsky [3]. The smallest neighborhood of each point of $\mathbf{Z}^d$ differs from one point to another. This allows to recover the structure of a topology.

- The cellular complex approach. An object is seen as a structure consisting of elements of different dimensions called cells. As noticed by V. Kovalevsky [4], it is also possible, with this approach, to recover the structure of a topology.

The underlying topology in the last two approaches corresponds to an *Alexandroff space* [2]. An Alexandroff space is a topological space in which the intersection of any arbitrary family (not necessarily finite) of open sets is open. There is a deep link between Alexandroff spaces and preorders, *i.e.*, binary relations that are reflexive and transitive. To any Alexandroff space, we may associate

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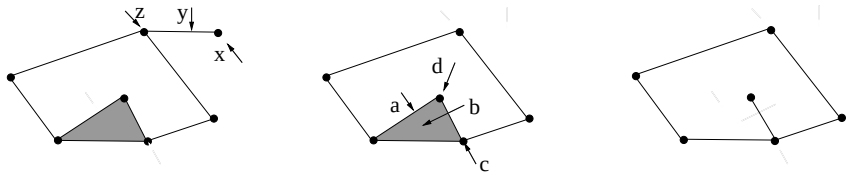


(a) (b) (c)

Figure 1: (a): A simplicial complex X, (b): A complex $Y \subseteq X$, (c): A complex $Z \subseteq Y$.

a preorder $\leq$ such that $x \leq y$ if and only if y is contained in all open sets that contain x. Conversely, a preorder determines an Alexandroff space: a set O is open for this space if and only if $x \in O$ and $x \leq y$ implies $y \in O$.
A map f between two preordered sets X and Y is monotone if $x \leq y$ in X implies $f(x) \leq f(y)$ in Y. We have the following result.
A map between two preordered sets is monotone if and only if it is a continuous map between the corresponding Alexandroff spaces. Conversely, a map between two Alexandroff spaces is continuous if and only if it is a monotone map between the corresponding preordered sets.
Thus, there is a structural equivalence between Alexandroff spaces and preorders.

Let us consider the (simplicial) objects X, Y, Z depicted Fig. 1. The object X is made of 7 vertices, 8 segments, and 1 triangle. A natural preorder between all these elements is the partial order corresponding to the relation of inclusion between sets. Thus we have $x \leq y$ and $z \leq y$. Let the map f between X and Y be such that f is the identity on all elements of Y and $f(x) = z$, $f(y) = z$. We note that f is monotone, for example we have $x \leq y$ and $f(x) \leq f(y)$. Thus, Y may be seen as a continuous retraction of X for the corresponding topology. Now, let us try to build a monotone map g between Y and Z such that g is the identity on all elements of Z. We see that this is not possible. For example, if we take $g(a) = c$, $g(b) = c$, we have $d \leq a$, but we have not $g(d) \leq g(a)$. We also observe that it is possible to build such a map between Y and $Y' = Y \setminus \{a\}$, but not between Y' and Z. Thus, in the context of this construction, the classical axioms of topology fail to interpret Z as a continuous retraction of Y.

In this paper, we propose a structural approach for describing certain transformations or certain collections of objects which appear in combinatorial topology. This approach is based on the notion of a completion, completions are inductive properties which may be expressed in a declarative way and which may be combined.

The paper is organized as follows. First, we introduce the notion of a completion. After some basic definitions for simplicial complexes, we give two examples of completions. We recall some definitions relative to the collapse operator which allows to make the two transforms illustrated Fig. 1. We present two completions, $\langle \text{Cup} \rangle$ and $\langle \text{Cap} \rangle$, in order to define a remarkable collection of acyclic complexes. We give few basic properties of this collection. Then, we present a theorem which shows the equivalence between this collection and the collection made of all complexes that are acyclic in the sense of homology theory.
Note that this paper corrects a flaw which was in a preliminary version ([6], the condition for a dendrite to be simply connected was omitted in Th. 2).

2 Completions

We introduce the notions of a constructor and a completion. See also Appendix A for more details and for the link with finitary closure operators.

In the sequel, the symbol $\mathbf{S}$ will denote an arbitrary collection. The symbol $\mathcal{K}$ will denote an arbitrary subcollection of $\mathbf{S}$, thus we have $\mathcal{K} \subseteq \mathbf{S}$.

Definition 1 *Let $\mathbf{K}$ be a binary relation on $2^{\mathbf{S}}$, thus $\mathbf{K} \subseteq 2^{\mathbf{S}} \times 2^{\mathbf{S}}$. We say that $\mathbf{K}$ is a* constructor (on $\mathbf{S}$) *if $\mathbf{K}$ is* finitary*, which means that $\mathbf{F}$ is finite whenever $(\mathbf{F}, \mathbf{G}) \in \mathbf{K}$. If $\mathbf{K}$ is a constructor on $\mathbf{S}$, we denote by $\langle \mathbf{K} \rangle$ the following property which is* the completion induced by $\mathbf{K}$*:*

-> *If $\mathbf{F} \subseteq \mathcal{K}$, then $\mathbf{G} \subseteq \mathcal{K}$ whenever $(\mathbf{F}, \mathbf{G}) \in \mathbf{K}$.* $\langle \mathbf{K} \rangle$

Let $\mathbf{K}$ be a constructor on $\mathbf{S}$ and let $\mathbf{X} \subseteq \mathbf{S}$. We define:

$$\mathbf{K}(\mathbf{X}) = \cup\{\mathbf{G} \mid \text{ there exists } (\mathbf{F}, \mathbf{G}) \in \mathbf{K} \text{ with } \mathbf{F} \subseteq \mathbf{X}\} \cup \mathbf{X}.$$

We set $\mathbf{K}^1(\mathbf{X}) = \mathbf{K}(\mathbf{X})$ and $\mathbf{K}^k(\mathbf{X}) = \mathbf{K}(\mathbf{K}^{k-1}(\mathbf{X}))$, $k \geq 2$.
We also set $\langle \mathbf{X}, \mathbf{K} \rangle = \cup\{\mathbf{K}^k(\mathbf{X}) \mid k \geq 1\}$.

Let $\langle \mathrm{K} \rangle$ be a property which depends on $\mathcal{K}$. We say that a given collection $\mathbf{X} \subseteq \mathbf{S}$ satisfies $\langle \mathrm{K} \rangle$ if the property $\langle \mathrm{K} \rangle$ is true for $\mathcal{K} = \mathbf{X}$.

Theorem 1 *Let $\mathbf{K}$ be a constructor on $\mathbf{S}$ and let $\mathbf{X} \subseteq \mathbf{S}$. There exists, under the subset ordering, a unique minimal collection which contains $\mathbf{X}$ and which satisfies $\langle \mathbf{K} \rangle$, this collection is precisely $\langle \mathbf{X}, \mathbf{K} \rangle$.*
Furthermore, we have $\langle \mathbf{X}, \mathbf{K} \rangle = \cap\{\mathbf{Y} \subseteq \mathbf{S} \mid \mathbf{X} \subseteq \mathbf{Y}$ and $\mathbf{Y}$ satisfies $\langle \mathbf{K} \rangle\}$.

We say that a property $\langle \mathrm{K} \rangle$ is a *completion (property)* if there exists a constructor $\mathbf{K}$ such that $\langle \mathrm{K} \rangle = \langle \mathbf{K} \rangle$ which means that, for each $\mathcal{K} \subseteq \mathbf{S}$, $\langle \mathrm{K} \rangle$ is true if and only if $\langle \mathbf{K} \rangle$ is true. If $\langle \mathrm{K} \rangle$ is a completion and if $\mathbf{X} \subseteq \mathbf{S}$, we write $\langle \mathbf{X}, \mathrm{K} \rangle$ for the unique minimal collection which contains $\mathbf{X}$ and which satisfies $\langle \mathrm{K} \rangle$.

Let $\langle \mathrm{K} \rangle$ and $\langle \mathrm{Q} \rangle$ be two completions induced, respectively, by the constructors $\mathbf{K}$ and $\mathbf{Q}$. We see that $\mathbf{K} \cup \mathbf{Q}$ is a constructor and that $\langle \mathrm{K} \rangle \wedge \langle \mathrm{Q} \rangle$ is the property induced by $\mathbf{K} \cup \mathbf{Q}$, the symbol $\wedge$ standing for the logical "and". Thus, if $\langle \mathrm{K} \rangle$ and $\langle \mathrm{Q} \rangle$ are completions, then $\langle \mathrm{K} \rangle \wedge \langle \mathrm{Q} \rangle$ is a completion.

In the sequel of the paper, we write $\langle \mathrm{K}, \mathrm{Q} \rangle$ for the completion $\langle \mathrm{K} \rangle \wedge \langle \mathrm{Q} \rangle$. Thus, if $\mathbf{X} \subseteq \mathbf{S}$, the notation $\langle \mathbf{X}, \mathrm{K}, \mathrm{Q} \rangle$ stands for the smallest collection which contains $\mathbf{X}$ and which satisfies $\langle \mathrm{K} \rangle \wedge \langle \mathrm{Q} \rangle$.

Remark 1 *We may build a counter-example which shows that, if $\langle \mathrm{K} \rangle$ and $\langle \mathrm{Q} \rangle$ are completions, then $\langle \mathrm{K} \rangle \vee \langle \mathrm{Q} \rangle$ is not necessarily a completion, the symbol $\vee$ standing for the logical "or".*

A constructor $\mathbf{K}$ is *one-to-one* if $Card(\mathbf{F}) = Card(\mathbf{G}) = 1$ whenever $(\mathbf{F}, \mathbf{G}) \in \mathbf{K}$. If $\mathbf{K}$ is a one-to-one constructor, we set $-\mathbf{K} = \{(\mathbf{G}, \mathbf{F}) \mid (\mathbf{F}, \mathbf{G}) \in \mathbf{K}\}$, the constructor $-\mathbf{K}$ is the *inverse of* $\mathbf{K}$.
It may be seen that, if $\mathbf{K}$ and $\mathbf{Q}$ are two one-to-one constructors, we have $\langle \mathbf{K} \rangle = \langle \mathbf{Q} \rangle$ if and only if $\langle -\mathbf{K} \rangle = \langle -\mathbf{Q} \rangle$.
A *one-to-one completion* $\langle \mathrm{K} \rangle$ is a completion induced by some one-to-one constructor $\mathbf{K}$, we write $\langle -\mathrm{K} \rangle$ for the completion induced by $-\mathbf{K}$. By the preceding property, this notation is consistent since it does not depend on the choice of $\mathbf{K}$.

3 Basic definitions for simplicial complexes

A *complex* or an *hypergraph* is a finite family composed of finite sets. We denote by $\mathbb{H}$ the collection of all complexes.

Let $X \in \mathbb{H}$. The *simplicial closure of* X is the complex X^- such that $X^- = \{y \subseteq x \mid x \in X\}$. The complex X is a *simplicial complex* if $X = X^-$. We denote by $\mathbb{S}$ the collection of all simplicial complexes. Observe that $\emptyset \in \mathbb{S}$ and $\{\emptyset\} \in \mathbb{S}$, $\emptyset$ is the *void complex*, and $\{\emptyset\}$ is the *empty complex*.

Let $X \in \mathbb{S}$. An element of X is *a simplex of* X or *a face of* X. A *facet of* X is a simplex of X which is maximal for inclusion.

A *simplicial subcomplex* of $X \in \mathbb{S}$ is any subset Y of X which is a simplicial complex. If Y is a subcomplex of X, we write $Y \preceq X$.

A complex $A \in \mathbb{S}$ is *a cell* if $A = \emptyset$ or if A has precisely one non-empty facet x, we set $A^\circ = A \setminus \{x\}$ and $\emptyset^\circ = \emptyset$. We write $\mathbb{C}$ for the collection of all cells.

Let $X \in \mathbb{S}$. The *dimension* of $x \in X$, written $dim(x)$, is the number of its elements minus one. The *dimension of* X, written $dim(X)$, is the largest dimension of its simplices, the *dimension of* $\emptyset$ is defined to be -1.
If $\mathbb{X} \subseteq \mathbb{S}$, we set:

$$\mathbb{X}[d] = \{X \in \mathbb{X} \mid dim(X) = d\} \text{ and } \mathbb{X}\langle d\rangle = \{X \in \mathbb{X} \mid dim(X) \leq d\}.$$

The *ground set* of $X \in \mathbb{H}$ is the set $\underline{X} = \cup\{x \in X\}$. Let $X, Y \in \mathbb{H}$ such that $\underline{X} \cap \underline{Y} = \emptyset$. The *join of* X *and* Y is the complex XY such that $XY = \{x \cup y \mid x \in X, y \in Y\}$. Thus, $XY = \emptyset$ if $Y = \emptyset$ and $XY = X$ if $Y = \{\emptyset\}$.

In this paper, if $X, Y \in \mathbb{H}$, we implicitly assume that X and Y have disjoint ground sets whenever we write XY.
If $X \in \mathbb{S}$, we say that AX is a *simple cone* if $A \in \mathbb{C}[0]$, and a *cone* if $A \in \mathbb{C}$. Note that $AB \in \mathbb{C}$ whenever $A \in \mathbb{C}$ and $B \in \mathbb{C}$.

Let $X \in \mathbb{S}$, and let $A, B \in \mathbb{C}[0]$, $A \neq B$. The *suspension of* X *(by* A *and* B*)* is the complex $AX \cup BX$.

The *(reduced) Euler characteristic of* $X \in \mathbb{S}$ is the number $\chi(X)$ such that $\chi(X) = \sum\{(-1)^{dim(x)} \mid x \in X\}$ if $X \neq \emptyset$, and $\chi(\emptyset) = 0$. Note that $\chi(X)$ is equal to the ordinary Euler characteristic minus one.

Let $A \in \mathbb{C}$ and $X \preceq A$. The *dual of* X *for* A is the simplicial complex, written $(X; A)^*$, such that $(X; A)^* = \{\underline{A} \setminus x \mid x \in A \setminus X\}$. Thus, we have $(X; A)^* = \{x \in A \mid (\underline{A} \setminus x) \notin X\}$. For any $A \in \mathbb{C}$, we have the following:
- If $X \preceq A$, then $((X; A)^*; A)^* = X$.
- If $X \preceq A$ and $Y \preceq A$, then $(X \cup Y; A)^* = (X; A)^* \cap (Y; A)^*$.
- If $X \preceq A$ and $Y \preceq A$, then $(X \cap Y; A)^* = (X; A)^* \cup (Y; A)^*$.
- We have $(\emptyset; A)^* = A$ and $(\{\emptyset\}; A)^* = A^\circ$.

In the rest of this paper, we set $\mathbf{S} = \mathbb{S}$. Thus, we will have $\mathcal{K} \subseteq \mathbb{S}$.

In the next two sections, we give some basic examples of completions on $\mathbb{S}$.

4 Connectedness

The family composed of all connected simplicial complexes may be defined by means of completions. We define the one-to-one completion $\langle$**PATH**$\rangle$ as follows.

-> If $S \in \mathcal{K}$, then $S \cup C \in \mathcal{K}$ whenever $C \in \mathbb{C}$, and $S \cap C \neq \{\emptyset\}$. $\langle$**PATH**$\rangle$

We may easily verify that $\langle$**PATH**$\rangle$ is indeed a (one-to-one) completion. The

property $\langle \textbf{PATH} \rangle$ is the completion induced by the (one-to-one) constructor:

$$\textbf{PATH} = \{(\{S\}, \{S \cup C\}) \mid S \in \mathbb{S},\ C \in \mathbb{C} \text{ and } S \cap C \neq \{\emptyset\}\}.$$

We set $\Pi = \langle \mathbb{C}, \textbf{PATH} \rangle$. We say that a complex $X \in \mathbb{S}$ is *connected* if $X \in \Pi$.

It may be checked that this definition of a connected complex is equivalent to the classical definition based on paths. Now, let us define the completion $\langle \Upsilon \rangle$ as follows.

-> If $S, T \in \mathcal{K}$, then $S \cup T \in \mathcal{K}$ whenever $S \cap T \neq \{\emptyset\}$. $\langle \Upsilon \rangle$

Again, we may easily verify that $\langle \Upsilon \rangle$ is indeed a completion. We have the following result which shows that Υ provides another way to generate Π.

Proposition 1 *We have* $\Pi = \langle \mathbb{C}, \Upsilon \rangle$

A property similar to $\langle \Upsilon \rangle$ has been introduced by J. Serra and G. Matheron who proposed, through the notion of a connection, a new set of axioms for connectedness [17]. The main difference between a connection and $\langle \mathbb{C}, \Upsilon \rangle$ is that a connection may be seen as a "static structure" for modeling connectedness, on the contrary Υ is used in $\langle \mathbb{C}, \Upsilon \rangle$ in a "dynamic way" for generating all elements of Π.

5 Trees

A tree is classically defined as a graph which is path-connected and which does not contain any cycle. We give here a definition based on the following one-to-one completion $\langle \textbf{TREE} \rangle$.

We set $\mathbb{V} = \mathbb{C}\langle 0 \rangle$ and $\mathbb{E} = \mathbb{C}[1]$. A complex X, $X \neq \emptyset$, is a *vertex* if $X \in \mathbb{V}$. A complex X is *an edge* if $X \in \mathbb{E}$.

-> If $S \in \mathcal{K}$, then $S \cup A \in \mathcal{K}$ whenever $A \in \mathbb{E}$ and $S \cap A \in \mathbb{V}$. $\langle \textbf{TREE} \rangle$

Let $\mathbb{T}\mathrm{ree} = \langle \mathbb{V}, \textbf{TREE} \rangle$. We say that a complex $X \in \mathbb{S}$ is a *tree* if $X \in \mathbb{T}\mathrm{ree}$.

It may be checked that this definition of a tree is equivalent to the classical one. Through this example, we observe that completions allow to define a collection in a constructive way rather by the means of properties of this collection.

We can express, in a concise manner, a fundamental property of trees.

Proposition 2 *We have* $\mathbb{T}\mathrm{ree} = \langle \mathbb{T}\mathrm{ree}, -\textbf{TREE} \rangle$.

6 Collapse

We now present some completions related to the collapse operator introduced by J.H.C. Whitehead [14]. Let us recall a classical definition of collapse.

Let $X \in \mathbb{S}$. We say that a face $x \in X$ is *free for* X if x is a proper face of exactly one face y of X, such a pair (x, y) is said to be a *free pair for* X. If (x, y) is a free pair for X, we say that the complex $Y = X \setminus \{x, y\}$ is *an elementary collapse of* X or that X is *an elementary expansion* of Y.

We define the one-to-one completion $\langle \textbf{EXP} \rangle$:

-> If $S \in \mathcal{K}$, then $S \cup AB \in \mathcal{K}$ whenever $A \in \mathbb{C}[0]$, $AB \in \mathbb{C}$, $S \cap AB = AB^\circ$. $\langle \textbf{EXP} \rangle$

Observe that, if $AB \in \mathbb{C}$, and if $A \in \mathbb{C}[0]$, then necessarily $B \in \mathbb{C}$ or $B = \{\emptyset\}$. If $B = \{\emptyset\}$, B° is defined to be $\emptyset$.

It may be seen that, if S and $S \cup AB$, with $B \neq \emptyset$, fulfill the above conditions, then S is an elementary collapse of $S \cup AB$. Conversely, we may formulate any elementary collapse by such an expression. Thus $\langle \textbf{EXP} \rangle$ is an alternative definition of collapse.

Let $X \preceq Y \preceq A$, $A \in \mathbb{C}$, with $X = Y \setminus \{x, y\}$ where x and y are two distinct faces of Y. If y is the only face of Y which contains x, then $\underline{A} \setminus x$ is the only face of $(X; A)^*$ which contains $\underline{A} \setminus y$. Thus, we have the following result [8].

Proposition 3 *Let $A \in \mathbb{C}$, $X \preceq A$, $Y \preceq A$. The complex X is an elementary collapse of Y if and only if $(Y; A)^*$ is an elementary collapse of $(X; A)^*$.*

If $X, Y \in \mathbb{S}$, we say that Y *collapses onto* X if $Y \in \langle \{X\}, \textbf{EXP} \rangle$, and Y is *simple homotopic to* X if $Y \in \langle \{X\}, \textbf{EXP}, -\textbf{EXP} \rangle$. The complex $X \in \mathbb{S}$ is *collapsible* if X collapses onto $\emptyset$, and X is *simply contractible* if X is simple homotopic to $\emptyset$.

Let us consider the one-to-one completion:

-> If $S \in \mathcal{K}$, then $S \cup A \in \mathcal{K}$ whenever $A \in \mathbb{C}$, and A collapses onto $S \cap A$. $\langle \textbf{SIM} \rangle$

When A satisfies $\langle \textbf{SIM} \rangle$, we say that the cell A is *simple for* S. We have $\langle \textbf{SIM} \rangle = \langle \textbf{EXP} \rangle$. This completion leads to a notion of simplicity introduced in the context of computer imagery [18] (see also [19, 20]) where an object is often seen as a set of cells (*e.g.*, a set of *voxels* in 3D) rather than a set of faces.

Nonevasive complexes [21, 22, 9] constitute another example of a collection which may be defined by means of completions.

Let us consider the completion $\langle \textbf{NEV} \rangle$:

-> If $S, T \in \mathcal{K}$, then $S \cup AT \in \mathcal{K}$ whenever $A \in \mathbb{C}[0]$ and $S \cap AT = T$. $\langle \textbf{NEV} \rangle$

A complex $X \in \mathbb{S}$ is *nonevasive* if $X \in \langle \mathbb{V}, \textbf{NEV} \rangle$. Nonevasive complexes are collapsible [22], the converse being, in general, not true.

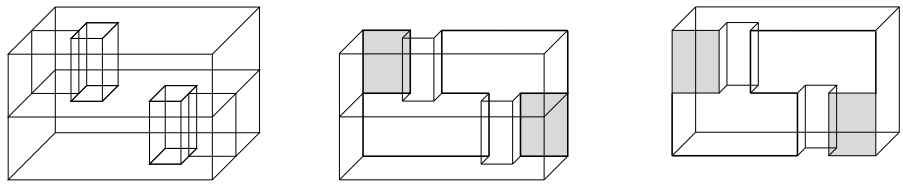
7 The Cup/Cap completions

We introduce the notion of a dendrite for defining a remarkable collection made of acyclic complexes.

Definition 2 *We define the two completions $\langle \text{Cup} \rangle$ and $\langle \text{Cap} \rangle$:*
-> *If S, $T \in \mathcal{K}$, then $S \cup T \in \mathcal{K}$ whenever $S \cap T \in \mathcal{K}$.* $\langle \text{Cup} \rangle$
-> *If S, $T \in \mathcal{K}$, then $S \cap T \in \mathcal{K}$ whenever $S \cup T \in \mathcal{K}$.* $\langle \text{Cap} \rangle$
We set $\mathbb{R} = \langle \mathbb{C}, \text{Cup} \rangle$ and $\mathbb{D} = \langle \mathbb{C}, \text{Cup}, \text{Cap} \rangle$.
Each element of $\mathbb{R}$ is a ramification *and each element of $\mathbb{D}$ is a* dendrite.

Note that we have $\mathbb{R} \subseteq \mathbb{D} \subseteq \mathbb{S}$. The Bing's house with two rooms [15] is a classical example of an object which is contractible but not collapsible, this object is depicted Fig. 2 (a). Let us consider the two complexes Y and Z of Fig. 2 (b) and (c). They are such that $X = Y \cup Z$. If X is correctly triangulated, then Y, Z, and $Y \cap Z$ are ramifications. Thus, the Bing's house X is a ramification.

It may be easily seen that we have $\langle \{\emptyset\}, \textbf{EXP} \rangle \subseteq \langle \mathbb{C}, \text{Cup} \rangle$, *i.e.*, that any collapsible complex is a ramification. Since the Bing's house is not collapsible, this inclusion is strict.



(a) (b) (c)

Figure 2: (a): A Bing's house X with two rooms, (b): An object $Y \subseteq X$, (c): An object $Z \subseteq X$. We have $X = Y \cup Z$, the object $Y \cap Z$ is outlined in (b) and (c).

M. Hochster [12] (see also [10] [11]) introduced the notion of a constructible complex. This notion may be expressed using the following completion:

-> If S, $T \in \mathcal{K}[d]$, then $S \cup T \in \mathcal{K}$ whenever $S \cap T \in \mathcal{K}[d-1]$, $d \geq 0$. $\langle \textsc{Cons} \rangle$

A simplicial complex is *constructible* if it is an element of $\langle \mathbb{C} \cup \{\{\emptyset\}\}, \textsc{Cons} \rangle$.

M. Hachimori [11], [13] showed that the Bing's house with two rooms and the dunce hat [16] are not constructible. Observe that we have $\langle \mathbb{C}, \textsc{Cons} \rangle \subseteq \langle \mathbb{C}, \textsc{Cup} \rangle$. Since the Bing's house is a ramification, the previous inclusion is strict.

With the notion of a buildable complex, J. Jonsson [9] drops the condition for dimension which appears in $\langle \textsc{Cons} \rangle$. The definition of a buildable complex is a recursive definition of what we call a ramification.[1] It was shown [9] that any buildable complex is contractible, *i.e.*, is homotopy equivalent to a single point. As far as we know, the above definition for dendrites has never been proposed.

8 Some basic properties

In this section, we give some basic properties which may be derived directly from the definitions of $\mathbb{R}$ and $\mathbb{D}$ using inductive arguments. Perhaps the simplest property which may be proved in such a way is the following.

Proposition 4 *If $X \in \mathbb{D}$, then $\chi(X) = 0$.*

Proof. We have $\chi(X) = 0$ for each $X \in \mathbb{C}$. Since the Euler characteristic is such that $\chi(S \cup T) = \chi(S) + \chi(T) - \chi(S \cap T)$, the result follows by induction. $\square$

Let us consider the two completions:

-> If $S, T \in \mathcal{K}$, then $S \cup T \in \mathcal{K}$. $\langle \textsc{Union} \rangle$

-> If $S, T \in \mathcal{K}$, then $S \cap T \in \mathcal{K}$. $\langle \textsc{Inter} \rangle$

Let $\mathbb{X} \subseteq \mathbb{R}$. An element of $\langle \mathbb{X}, \textsc{Union} \rangle$ is, in general, not necessarily a ramification (nor a dendrite), but this property is true in the following case.

Proposition 5 *Let $\mathbb{X} \subseteq \mathbb{R}$ and let $\mathbb{Y} = \langle \mathbb{X}, \textsc{Union} \rangle$. If $\langle \mathbb{Y}, \textsc{Inter} \rangle = \mathbb{Y}$, i.e., if $\mathbb{Y}$ satisfies the property $\langle \textsc{Inter} \rangle$, then we have $\mathbb{Y} \subseteq \mathbb{R}$.*

Proof. Suppose $\langle \mathbb{Y}, \textsc{Inter} \rangle = \mathbb{Y}$. We set $\mathbb{Y}^k = \{X \in \mathbb{Y} \mid Card(X) \leq k\}$.
i) We have $\mathbb{Y}^0 = \emptyset$ or $\mathbb{Y}^0 = \{\emptyset\}$. In both cases $\mathbb{Y}^0 \subseteq \mathbb{R}$.

[1] We suggest the name "ramification" rather than "buildable complex" since these objects may be seen as natural extensions of trees.

ii) Suppose $\mathbb{Y}^{k-1} \subseteq \mathbb{R}$, for some $k \geq 1$. Let $X \in \mathbb{Y}^k$. If $X \in \mathbb{X}$, then $X \in \mathbb{R}$. If $X \notin \mathbb{X}$, then there exists $S, T \in \mathbb{Y}$ such that $X = S \cup T$, and $S \not\subseteq T$, $T \not\subseteq S$. Thus $Card(S) \leq k-1$, $Card(T) \leq k-1$, and $Card(S \cap T) \leq k-1$. Furthermore, we have $S \cap T \in \mathbb{Y}$. Therefore, by the induction hypothesis, S, T and $S \cap T$ are ramifications, which means that X is a ramification. It follows that $\mathbb{Y}^k \subseteq \mathbb{R}$. $\square$

Let $A \in \mathbb{C}$ and let $\mathbb{X} = \{A\} \cup \{BA \mid B \in \mathbb{C}\}$. Thus, $\mathbb{X} \subseteq \mathbb{C}$. Since $XA \cup YA = (X \cup Y)A$, it may be seen that we have $\langle \mathbb{X}, \text{Union} \rangle = \{XA \mid X \in \mathbb{S}\}$. Furthermore, since $XA \cap YA = (X \cap Y)A$, $\langle \mathbb{X}, \text{Union} \rangle$ satisfies the property $\langle \text{Inter} \rangle$. Therefore, since $\mathbb{X} \subseteq \mathbb{R}$, the following is a consequence of Prop. 5.

Proposition 6 *Let $X \in \mathbb{S}$ and $A \in \mathbb{C}$. Then XA is a ramification.*

The following fact will be used for the proof of Prop. 8.

Proposition 7 *Let $A, B \in \mathbb{C}$. Then $(AB)^\circ = AB^\circ \cup A^\circ B$.*

Proof. The property is true if $A = \emptyset$ or $B = \emptyset$. Suppose $A \neq \emptyset$ and $B \neq \emptyset$. In this case, we have $\underline{AB} = \underline{A} \cup \underline{B}$, thus $(AB)^\circ = \{x \subset \underline{A} \cup \underline{B}\}$. Then $(AB)^\circ = \{y \cup z \mid y \subseteq \underline{A}, z \subset \underline{B}\} \cup \{y \cup z \mid y \subset \underline{A}, z \subseteq \underline{B}\} = AB^\circ \cup A^\circ B$. $\square$

Proposition 8 *Let $X \in \mathbb{S}$ and $A \in \mathbb{C}$, $A \neq \emptyset$. The complex XA° is a dendrite if and only if X is a dendrite.*

Proof. If $A \in \mathbb{C}[0]$, we have $A^\circ = \{\emptyset\}$ and $XA^\circ = X$. Suppose the property is true for any $A \in \mathbb{C}\langle i-1 \rangle$, $i \geq 1$. Let $Y = XA^\circ$, with $X \in \mathbb{S}$ and $A \in \mathbb{C}[i]$. Since $i \geq 1$, there exists $B, C \in \mathbb{C}\langle i-1 \rangle$, such that $A = BC$. By Prop. 7, we have $Y = XBC^\circ \cup XB^\circ C$. By Prop. 6, XBC° and $XB^\circ C$ are dendrites. But $XBC^\circ \cap XB^\circ C = XB^\circ C^\circ$. By the induction hypothesis, $XB^\circ C^\circ$ is a dendrite iff XB° is a dendrite, and XB° is a dendrite iff X is a dendrite. Thus, by $\langle \text{Cup} \rangle$ and $\langle \text{Cap} \rangle$, Y is a dendrite iff X is a dendrite. $\square$

We now give a property for duality which will be used in the sequel through two corollaries.

Proposition 9 *Let $A, B \in \mathbb{C}$, and let $X, Y \in \mathbb{S}$ such that $X \preceq A$, $Y \preceq B$. We have $(XY; AB)^* = A(Y; B)^* \cup B(X; A)^*$.*

Proof. We have $(XY; AB)^* = \{x \in AB \mid (\underline{AB} \setminus x) \notin XY\}$.
For any $x \in AB$, if we set $y = x \cap \underline{A}$ and $z = x \cap \underline{B}$, we have $x = y \cup z$ and $\underline{AB} \setminus x = (\underline{A} \setminus y) \cup (\underline{B} \setminus z)$. Thus we have:
$(XY; AB)^* = \{y \cup z \mid y \in A, z \in B, (\underline{A} \setminus y) \notin X \text{ or } (\underline{B} \setminus z) \notin Y\}$
$= \{y \cup z \mid y \in A, z \in B, (\underline{A} \setminus y) \notin X\} \cup \{y \cup z \mid y \in A, z \in B, (\underline{B} \setminus z) \notin Y\}$
$= \{y \cup z \mid y \in (X; A)^*, z \in B\} \cup \{y \cup z \mid y \in A, z \in (Y; B)^*\}$
$= B(X; A)^* \cup A(Y; B)^*$. $\square$

The following corollary may be obtained from Prop. 9 by setting $X = A$. It shows that the collection of cones is closed by duality. More precisely, if we set $\mathbb{K} = \{AX \mid A \in \mathbb{C}, X \in \mathbb{S}\}$ and $\mathbb{K}^* = \{(X; B)^* \mid X \in \mathbb{K}, B \in \mathbb{C}, X \preceq B\}$, then we have $\mathbb{K} = \mathbb{K}^*$. Note that the collection $\mathbb{C}$ is not closed by duality (By Cor. 2 below, we have $(A; AB)^* = B^\circ A$ which is not a cell).

Corollary 1 *Let $A, B \in \mathbb{C}$, and let $X \preceq B$. We have $(AX; AB)^* = A(X; B)^*$.*

The second corollary may be obtained from Prop. 9 by setting $Y = \{\emptyset\}$.

Corollary 2 *Let $A, B \in \mathbb{C}$, and $X \preceq A$. We have $(X; AB)^* = B(X; A)^* \cup B^\circ A$.*

We see that this last formula allows us to calculate the dual of an object in a given space (a cell) from the dual of this object in a smaller space.

Proposition 10 *Let $A, B \in \mathbb{C}$, and $X \preceq A$. Then $(X; AB)^*$ is a dendrite if and only if $(X; A)^*$ is a dendrite.*

Proof. Let $Y = (X; AB)^*$. By Cor. 2, we have $Y = B(X; A)^* \cup B^\circ A$. By Prop. 6, $B(X; A)^*$ and $B^\circ A$ are dendrites. Since $B(X; A)^* \cap B^\circ A = B^\circ(X; A)^*$, by Prop. 8, it follows that Y is a dendrite if and only if $(X; A)^*$ a dendrite. □

Proposition 11 *Let $X \in \mathbb{S}$, $A \in \mathbb{C}$, with $X \preceq A$. The complex X is a dendrite if and only if $(X; A)^*$ is a dendrite.*

Proof. Let $X \in \mathbb{D}$. There exists a finite collection $\mathbb{F} \subseteq \mathbb{C}$ such that $X \in \langle \mathbb{F}, \mathrm{C}_{\mathrm{UP}}, \mathrm{C}_{\mathrm{AP}} \rangle$. Let $C \in \mathbb{C}$ such that C contains all cells in $\mathbb{F}$. Let $F \in \mathbb{F}$. By Cor. 2, $(F; C)^* = D^\circ F$, with $C = DF$. Thus, $(F; C)^*$ is a cone and we have $(F; C)^* \in \mathbb{D}$. Therefore, since C_{UP} and C_{AP} are dual, by induction we have $(X; C)^* \in \mathbb{D}$. The result follows from Prop. 10. □

9 Contractible and acyclic complexes

In this section, we give some results relative to dendrites, simply contractible, and acyclic complexes.

Proposition 12 *Let $X \in \mathbb{D}$. If Y is simple homotopic to X, then Y is a dendrite.*

Proof. Let $X \in \mathbb{D}$. The result follows by induction from i) and ii).
i) Suppose Y is an elementary expansion of X. Then $Y = X \cup AB$, with $A \in \mathbb{C}[0]$, $AB \in \mathbb{C}$, $X \cap AB = AB^\circ$. By Prop. 6, AB and AB° are both dendrites. Thus, by $\langle \mathrm{C}_{\mathrm{UP}} \rangle$, Y is a dendrite.
ii) Suppose Y is an elementary collapse of X. Let $C \in \mathbb{C}$ such that $X \preceq C$. By Prop. 3, $(Y; C)^*$ is an elementary expansion of $(X; C)^*$. Furthermore, by Prop. 11, $(X; C)^*$ is a dendrite. By the preceding result, $(Y; C)^*$ is a dendrite. Using again Prop. 11, it follows that Y is a dendrite. □

Since the void complex is a dendrite, we have the following direct consequence of Prop. 12.

Corollary 3 *Let $X \in \mathbb{S}$. If X is simply contractible, then X is a dendrite.*

The following theorem makes clear the link between dendrites and complexes that are acyclic in the sense of integral homology. It is a direct consequence of several classical theorems of algebraic topology. Recall that a a complex $X \in \mathbb{S}$ is *acyclic (over $\mathbb{Z}$)* if its reduced integral homology vanishes in all dimensions [23]. Any contractible complex is acyclic, the converse is, in general, not true. The punctured Poincaré homology sphere provides an example of a complex which is acyclic but not simply contractible.

Theorem 2 *A simplicial complex is a dendrite if and only if it is acyclic.*

Proof.
i) A complex is acyclic whenever it is a cell ([23], Th. 8.2). Suppose S and T are acyclic complexes, with $S, T \neq \emptyset$. The reduced Mayer-Vietoris sequence yields an exact sequence ([23], Th. 25.1)

$$\tilde{H}_p(S) \oplus \tilde{H}_p(T) \to \tilde{H}_p(S \cup T) \to \tilde{H}_{p-1}(S \cap T) \to \tilde{H}_{p-1}(S) \oplus \tilde{H}_{p-1}(T).$$

Since S and T are acyclic, both end terms vanish. Therefore $\tilde{H}_p(S \cup T)$ is isomorphic to $\tilde{H}_{p-1}(S \cap T)$. Thus, $S \cup T$ is acyclic if and only if $S \cap T$ is acyclic. By induction on $\langle \text{C}_{\text{UP}} \rangle$ and $\langle \text{C}_{\text{AP}} \rangle$, it follows that any dendrite is acyclic.
ii) Suppose X is an acyclic complex, $X \neq \emptyset$. Let $Y = AX \cup BX$ be a suspension of X, thus $A, B \in \mathbb{C}[0]$, and $AX \cap BX = X$. Let $Y' = A'X' \cup B'X'$ be a geometric realization [23] of $AX \cup BX$. The complexes A' and B' are connected and simply connected, and the complex X' is connected. It follows that, by Van Kampen theorem, Y' is simply connected.
By the Hurewicz theorem ([24], Th. 4.32), since Y' is acyclic and simply connected, all homotopy groups of Y' are trivial. By Whitehead's theorem ([24], Th. 4.5), it implies that Y' is homotopy equivalent to a single point, *i.e.*, that Y' is contractible ([25], Cor. 8.3.11). Then, by the Whitehead's theorem on simple homotopy ([14], Th. 21), the complex Y is simply contractible (see also [26]). By Cor. 3, Y is a dendrite. By $\langle \text{C}_{\text{AP}} \rangle$, since AX, BX, and $AX \cup BX$ are dendrites, the complex X is a dendrite. □

A complex is acyclic if and only if its suspension is contractible. Therefore we have the following result which may be derived using the arguments of the preceding proof.

Theorem 3 *A simplicial complex is a dendrite if and only if its suspension is simply contractible.*

10 Conclusion

We have seen that (one-to-one) completions make possible to formulate recursive transformations of objects. More remarkably, completions allow us to define structures on objects (*e.g.*, a connection). We introduced two completions, $\langle \text{C}_{\text{UP}} \rangle$ and $\langle \text{C}_{\text{AP}} \rangle$, in order to define, in an axiomatic way, a collection of acyclic complexes. We gave a theorem which shows the equivalence between this collection and the collection made of complexes that are acyclic in the sense of homology. In a future work, we will further investigate completions-based structures for combinatorial topology.

Appendix A: completions and closure operators

In the sequel, the symbol $\mathbf{S}$ denotes an arbitrary collection.

Let γ be a map from $2^{\mathbf{S}}$ to $2^{\mathbf{S}}$, such a map is said to be an *operator (on* $\mathbf{S}$*)*.

If γ is an operator on $\mathbf{S}$, we set $\gamma^1 = \gamma$ and $\gamma^k = \gamma \circ \gamma^{k-1}$, $k \geq 2$. We define $\widehat{\gamma}$ to be the operator such that, for each $\mathbf{X} \subseteq \mathbf{S}$, $\widehat{\gamma}(\mathbf{X}) = \cup\{\gamma^k(\mathbf{X}) \mid k \geq 1\}$.

Let γ be an operator on $\mathbf{S}$. We say that:
- γ is *extensive* if, for all $\mathbf{X} \subseteq \mathbf{S}$, we have $\mathbf{X} \subseteq \gamma(\mathbf{X})$.
- γ is *increasing* if, whenever $\mathbf{X} \subseteq \mathbf{Y} \subseteq \mathbf{S}$, we have $\gamma(\mathbf{X}) \subseteq \gamma(\mathbf{Y})$.
- γ is *idempotent* if, for all $\mathbf{X} \subseteq \mathbf{S}$, we have $\gamma^2(\mathbf{X}) = \gamma(\mathbf{X})$.

The operator γ is a *closure operator (on* $\mathbf{S}$*)* if γ is extensive, increasing, and idempotent. If γ is a closure operator, $\mathbf{X} \subseteq \mathbf{S}$ is *closed for* γ if $\gamma(\mathbf{X}) = \mathbf{X}$.

Let γ be a closure operator on $\mathbf{S}$. We have the following result, this is a basic property of closure operators:

For any $\mathbf{X} \subseteq \mathbf{S}$, we have $\gamma(\mathbf{X}) = \cap\{\mathbf{Y} \subseteq \mathbf{S} \mid \mathbf{X} \subseteq \mathbf{Y} \text{ and } \gamma(\mathbf{Y}) = \mathbf{Y}\}$.

Let γ be an operator on $\mathbf{S}$. We say that γ is *finitary* if, for all $\mathbf{X} \subseteq \mathbf{S}$, we have $\gamma(\mathbf{X}) = \cup\{\gamma(\mathbf{F}) \mid \mathbf{F} \subseteq \mathbf{X} \text{ and } \mathbf{F} \text{ finite}\}$.

Alfred Tarski [7] introduced finitary closure operators (also called "finite consequence operators") as an abstract theory of logical deductions. In this context, the set $\mathbf{S}$ represents a set of statements in some language. Given a subset $\mathbf{X}$ of $\mathbf{S}$, the set $\gamma(\mathbf{X})$ represents the set of all statements that may be deduced from $\mathbf{X}$.

Observe that any finitary operator is increasing. Thus, an operator γ is a finitary closure operator if and only if γ is extensive, finitary, and idempotent.

We say that an operator κ is a Λ-*operator* if κ is extensive and finitary.

The following fixed point theorem is essential for our purpose. See [27, 28, 29, 30] for more general fixed point theorems.

Theorem 4 *Let* κ *be a* Λ*-operator. Then, for any* $\mathbf{X} \subseteq \mathbf{S}$, $\widehat{\kappa}(\mathbf{X})$ *is a* fixed point for κ, i.e., *we have* $\kappa(\widehat{\kappa}(\mathbf{X})) = \widehat{\kappa}(\mathbf{X})$.

Proposition 13 *If* κ *is a* Λ*-operator, then* $\widehat{\kappa}$ *is a finitary closure operator.*

Remark 2 *If* γ *is an extensive and increasing operator (not necessarily finitary), then* $\widehat{\kappa}(\mathbf{X})$ *is not necessarily a fixed point for* κ.

Let κ be a binary relation on $2^{\mathbf{S}}$, thus $\kappa \subseteq 2^{\mathbf{S}} \times 2^{\mathbf{S}}$. We say that κ is a *constructor (on* $\mathbf{S}$*)* if κ is *finitary*, which means that $\mathbf{F}$ is finite whenever $(\mathbf{F}, \mathbf{G}) \in \kappa$.

Let κ be a constructor on $\mathbf{S}$. We also denote by κ the Λ-operator such that, for each $\mathbf{X} \subseteq \mathbf{S}$, we have:

$$\kappa(\mathbf{X}) = \cup\{\mathbf{G} \mid \text{there exists } (\mathbf{F}, \mathbf{G}) \in \kappa \text{ with } \mathbf{F} \subseteq \mathbf{X}\} \cup \mathbf{X}.$$

We say that the operator κ is the Λ-*operator induced by the constructor* κ.

Let κ be a Λ-operator on $\mathbf{S}$, the *constructor induced by* κ is the constructor $\underline{\kappa}$ such that $\underline{\kappa} = \{(\mathbf{F}, \kappa(\mathbf{F})) \mid \mathbf{F} \text{ is a finite subset of } \mathbf{S}\}$.

The following result is a direct consequence of the above definitions. It shows that specifying a Λ-operator is (in a certain sense) equivalent to specifying a constructor.

Proposition 14 *Let κ be a Λ-operator, and let $\underline{\kappa}$ be the constructor induced by κ. The two Λ-operators $\underline{\kappa}$ and κ are equal.*
Let κ be a constructor on $\mathbf{S}$, and let $\underline{\kappa}$ be the constructor induced by the Λ-operator κ. The two Λ-operators κ and $\underline{\kappa}$ are equal.

Let κ be a constructor. We say that κ is *many-to-one* if $Card(\mathbf{G}) = 1$ whenever $(\mathbf{F}, \mathbf{G}) \in \kappa$.

To each arbitrary constructor κ we may associate a many-to-one constructor $\underline{\kappa}$, we define $\underline{\kappa}$ to be $\{(\mathbf{F}, \{\mathbf{x}\}) \mid$ there exists $(\mathbf{F}, \mathbf{G}) \in \kappa$ and $x \in \mathbf{G}\}$. We see that the two Λ-operators κ and $\underline{\kappa}$ are equal. Thus, specifying a constructor is equivalent to specifying a many-to-one constructor.

Remark 3 *Let κ be a binary relation over $2^{\mathbf{S}}$ and $\mathbf{S}$, thus $\kappa \subseteq 2^{\mathbf{S}} \times \mathbf{S}$. We say that κ is* finitary *if $\mathbf{F}$ is finite whenever $(\mathbf{F}, \mathbf{x}) \in \kappa$. It may be seen that specifying a many-to-one constructor is equivalent to specifying such a relation.*

In the sequel, the symbol $\mathcal{K}$ will denote an arbitrary subcollection of $\mathbf{S}$, thus we have $\mathcal{K} \subseteq \mathbf{S}$.

If κ is a constructor on $\mathbf{S}$, we denote by $\langle\kappa\rangle$ the following property which is *the completion induced by* κ:

-> If $\mathbf{F} \subseteq \mathcal{K}$, then $\mathbf{G} \subseteq \mathcal{K}$ whenever $(\mathbf{F}, \mathbf{G}) \in \kappa$. $\langle\kappa\rangle$

Let $\langle \mathrm{K} \rangle$ be a property which depends on $\mathcal{K}$. We say that a given collection $\mathbf{X} \subseteq \mathbf{S}$ satisfies $\langle \mathrm{K} \rangle$ if the property $\langle \mathrm{K} \rangle$ is true for $\mathcal{K} = \mathbf{X}$.

Thus, if κ is a constructor, a collection $\mathbf{X} \subseteq \mathbf{S}$ satisfies $\langle\kappa\rangle$ if and only if $\kappa(\mathbf{X}) = \mathbf{X}$.

The following propositions are direct consequences of Th. 4 and Prop. 13. Prop. 15 and 16 are equivalent to Th. 1 given in section 2. The collection $\langle \mathbf{X}, \mathrm{K} \rangle$ of Th. 1 is equal to $\widehat{\kappa}(\mathbf{X})$.

Proposition 15 *Let κ be a constructor on $\mathbf{S}$ and let $\mathbf{X} \subseteq \mathbf{S}$. Then $\widehat{\kappa}(\mathbf{X})$ is, under the subset ordering, the unique minimal collection which contains $\mathbf{X}$ and which satisfies $\langle\kappa\rangle$.*

Let $\langle \mathrm{K} \rangle$ be a property which depends on $\mathcal{K}$.
If $\mathbf{X} \subseteq \mathbf{S}$, we set $\Delta(\mathbf{X}, \mathrm{K}) = \cap\{\mathbf{Y} \subseteq \mathbf{S} \mid \mathbf{X} \subseteq \mathbf{Y}$ and $\mathbf{Y}$ satisfies $\langle \mathrm{K} \rangle\}$.

Proposition 16 *If κ is a constructor on $\mathbf{S}$ and if $\mathbf{X} \subseteq \mathbf{S}$, then $\widehat{\kappa}(\mathbf{X}) = \Delta(\mathbf{X}, \mathrm{K})$.*

We say that a property $\langle \mathrm{K} \rangle$ is a *completion (property)* if there exists a constructor κ such that $\langle \mathrm{K} \rangle = \langle\kappa\rangle$ which means that, for each $\mathcal{K} \subseteq \mathbf{S}$, $\langle \mathrm{K} \rangle$ is true if and only if $\langle\kappa\rangle$ is true.

Proposition 17 *Let $\langle \mathrm{K} \rangle$ be a property which depends on $\mathcal{K}$. The property $\langle \mathrm{K} \rangle$ is a completion if and only if, for each $\mathbf{X} \subseteq \mathbf{S}$:*
i) $\Delta(\mathbf{X}, \mathrm{K})$ satisfies $\langle \mathrm{K} \rangle$, and
ii) $\Delta(\mathbf{X}, \mathrm{K}) = \cup\{\Delta(\mathbf{F}, \mathrm{K}) \mid \mathbf{F} \subseteq \mathbf{X}$ and $\mathbf{F}$ finite$\}$.

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