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On the Cauchy problem for gravity water waves

T. Alazard, N. Burq, C. Zuily

ABSTRACT. We are interested in the system of gravity water waves equations without surface tension. Our purpose is to study the optimal regularity thresholds for the initial conditions. In terms of Sobolev embeddings, the initial surfaces we consider turn out to be only of $C^{3/2}$ class and consequently have unbounded curvature, while the initial velocities are only Lipschitz. We also take benefit from our low regularity result and an elementary (though seemingly yet unnoticed) observation to solve a question raised by Boussinesq on the water waves problem in a canal. We reduce the system using a paradifferential approach.

CONTENTS

1. Introduction	1
2. Paradifferential calculus	8
3. The Dirichlet-Neumann operator	23
4. A priori estimates in Sobolev spaces	45
5. Contraction	62
6. Well-posedness of the Cauchy problem	73
7. Three-dimensional waves in a canal	77
References	86

1. Introduction

We are interested in this work in the study of the Cauchy problem for the water waves system in arbitrary dimension, without surface tension.

An important question in the theory is the possible emergence of singularities (see [16, 17, 20, 25, 58]) and as emphasized by Craig and Wayne [30], it is important to decide whether some physical or geometric quantities control the equation. In terms of the velocity field, a natural criterium (in view of Cauchy-Lipschitz theorem) is given by the Lipschitz regularity threshold. Indeed, this is necessary for the “fluid particles” motion (i.e. the integral curves of the velocity field) to be well-defined.

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In terms of the free boundary, there is no such natural criterium. In fact, the systematic use of the Lagrangian formulation in most previous works [9, 54, 55], and the intensive use of Riemannian geometry tools (parallel transport, vector fields,...) by Shatah-Zeng [49, 50, 51], Christodoulou–Lindblad [21] or Lindblad [41] seem to at least require *bounded curvature assumptions* (see also [24] where a logarithmic divergence is allowed). In this direction, the beautiful work by Christodoulou–Lindblad [21], gives *a priori* bounds as long as the second fundamental form of the free surface is bounded, and the first-order derivatives of the velocity are bounded. This could lead to the natural conjecture that the regularity threshold for the water waves system is indeed given by Christodoulou–Lindblad’s result and that the domain has to be assumed to be essentially C^2 . Our first contribution in this work is that this is *not* the case and that the relevant threshold is actually only the Lipschitz regularity of the velocity field. Indeed (see Theorem 1.2), our local existence result involves assumptions which, in view of Sobolev embeddings, require only (in terms of Hölder regularity) the initial free domain to be $C^{3/2}$.

Our second contribution is an illustration of the relevance of the analysis of low regularity solutions in a domain with a rough boundary. We give an application of our analysis to the local Cauchy theory of three-dimensional gravity water waves in a canal. This question was historically at the heart of the analysis of the water waves system and goes back to the work by Boussinesq at the beginning of the 20th century (see [15]). Our result seems to be the first one in this setting.

Our analysis require the introduction of new techniques and new tools. In [1, 2] we started a para-differential study of the water waves system in the presence of surface tension and were able to prove that the equations can be reduced to a simple form

$$(1.1) \quad \partial_t u + T_V \cdot \nabla u + iT_\gamma u = f,$$

where T_V is a para-product and T_γ is a para-differential operator of order $3/2$. Here the main step in the proof is to perform the same task without surface tension, with T_γ of order $1/2$. It has to be noticed however that performing our reduction is considerably more difficult here than in our previous papers ([1, 2]). Indeed, in the case with non vanishing surface tension, the natural regularity threshold forces the velocity field to be Lipschitz while the domain is actually much smoother ($C^{5/2}$). In the present work, the velocity field is also Lipschitz, but the domain is merely $C^{3/2}$. To overcome these difficulties, we had to give a micro-local description (and contraction estimates) of the Dirichlet-Neumann operator which is non trivial in the whole range of C^s domains, $s > 1$ (see the work by Dahlberg-Kenig [31] and Craig-Schwarz-Sulem [28] for results on the Dirichlet-Neumann operator in Lipschitz domains). We think that this analysis is of independent interest.

Finally, let us mention that, as we proceed by energy estimates, our results are proved in L^2 -based Sobolev spaces and our initial data (η_0, v_0) which describe respectively the initial domain as the graph of the function η_0 and the trace of the initial velocity on the free surface, are assumed to be in $H^{s+\frac{1}{2}}(\mathbf{R}^d) \times H^s(\mathbf{R}^d)$, $s > 1 + \frac{d}{2}$. The gravity water waves system enjoys a scaling invariance for which the critical threshold is $s_c = \frac{1}{2} + \frac{d}{2}$. In other terms our well-posedness result is $1/2$ above the scaling critical index. We intend to lower the threshold $1 + \frac{d}{2}$ by using dispersive estimates in a forthcoming paper [3] (see also [2]), following a strategy successfully developed in the framework of quasi-linear wave equations by Bahouri-Chemin and Tataru [10, 53] (notice that in [3], the Lipschitz threshold is still relevant as even though the initial velocity field is only $C^{1-\epsilon}$, the solution itself is still $L^2((-T, T); C^{1+\epsilon})$).

1.1. Assumptions on the domain. Hereafter, $d \geq 1$, t denotes the time variable and $x \in \mathbf{R}^d$ and $y \in \mathbf{R}$ denote the horizontal and vertical spatial variables. We work in a time-dependent fluid domain Ω located underneath a free surface Σ and moving in a fixed container denoted by \mathcal{O} . This fluid domain

$$\Omega = \{ (t, x, y) \in [0, T] \times \mathbf{R}^d \times \mathbf{R} : (x, y) \in \Omega(t) \},$$

is such that, for each time t , one has

$$\Omega(t) = \{ (x, y) \in \mathcal{O} : y < \eta(t, x) \},$$

where η is an unknown function and \mathcal{O} is a given open domain which contains a fixed strip around the free surface

$$\Sigma = \{ (t, x, y) \in [0, T] \times \mathbf{R}^d \times \mathbf{R} : y = \eta(t, x) \}.$$

This implies that there exists $h > 0$ such that, for all $t \in [0, T]$,

$$(1.2) \quad \Omega_h(t) := \left\{ (x, y) \in \mathbf{R}^d \times \mathbf{R} : \eta(t, x) - h < y < \eta(t, x) \right\} \subset \Omega(t).$$

We also assume that the domain \mathcal{O} (and hence the domain $\Omega(t)$) is connected.

- REMARK 1.1. (i) Two classical examples are given by $\mathcal{O} = \mathbf{R}^d \times \mathbf{R}$ (infinite depth case) or $\mathcal{O} = \mathbf{R}^d \times [-1, +\infty)$ (flat bottom). Notice that, in the following, no regularity assumption is made on the bottom $\Gamma := \partial\mathcal{O}$.
- (ii) Notice that Γ does not depend on time. However, our method applies in the case where the bottom is time dependent (with the additional assumption in this case that the bottom is Lipschitz).

1.2. The equations. Below we use the following notations

$$\nabla = (\partial_{x_i})_{1 \leq i \leq d}, \quad \nabla_{x,y} = (\nabla, \partial_y), \quad \Delta = \sum_{1 \leq i \leq d} \partial_{x_i}^2, \quad \Delta_{x,y} = \Delta + \partial_y^2.$$

We consider an incompressible inviscid liquid, having unit density. The equations by which the motion is to be determined are well known. Firstly, the eulerian velocity field $v: \Omega \rightarrow \mathbf{R}^{d+1}$ solves the incompressible Euler equation

$$(1.3) \quad \partial_t v + v \cdot \nabla_{x,y} v + \nabla_{x,y} P = -g e_y, \quad \operatorname{div}_{x,y} v = 0 \quad \text{in } \Omega,$$

where $-g e_y$ is the acceleration of gravity ($g > 0$) and where the pressure term P can be recovered from the velocity by solving an elliptic equation. The problem is then given by three boundary conditions. They are

$$(1.4) \quad \begin{cases} v \cdot n = 0 & \text{on } \Gamma, \\ \partial_t \eta = \sqrt{1 + |\nabla \eta|^2} v \cdot \nu & \text{on } \Sigma, \\ P = 0 & \text{on } \Sigma, \end{cases}$$

where n and ν are the exterior unit normals to the bottom Γ and the free surface $\Sigma(t)$. The first condition in (1.4) expresses the fact that the particles in contact with the rigid bottom remain in contact with it. Notice that to fully make sense, this condition requires some smoothness on Γ , but in general, it has a weak variational meaning (see Section 3). The second condition in (1.4) states that the free surface moves with the fluid and the last condition is a balance of forces across the free surface. Notice that the pressure at the upper surface of the fluid may be indeed supposed to be zero, provided we afterwards add the atmospheric pressure to the pressure so determined.

The fluid motion is supposed to be irrotational. The velocity field is therefore given by $v = \nabla_{x,y} \phi$ for some potential $\phi: \Omega \rightarrow \mathbf{R}$ satisfying

$$\Delta_{x,y} \phi = 0 \quad \text{in } \Omega, \quad \partial_n \phi = 0 \quad \text{on } \Gamma.$$

Using the Bernoulli integral of the dynamical equations to express the pressure, the condition $P = 0$ on the free surface implies that (recalling that $\nabla = \nabla_x$)

$$(1.5) \quad \begin{cases} \partial_t \eta = \partial_y \phi - \nabla \eta \cdot \nabla \phi & \text{on } \Sigma, \\ \partial_t \phi + \frac{1}{2} |\nabla_{x,y} \phi|^2 + gy = 0 & \text{on } \Sigma, \\ \partial_n \phi = 0 & \text{on } \Gamma. \end{cases}$$

Many results have been obtained on the Cauchy theory for System (1.5), starting from the pioneering works of Nalimov [45], Shinbrot [52], Yoshihara [59], Craig [26]. In the framework of Sobolev spaces and without smallness assumptions on the data, the well-posedness of the Cauchy problem was first proved by Wu for the case without surface tension (see [54, 55]) and by Beyer-Günther in [13] in the case with surface tension. Several extensions of their results have been obtained by different methods (see [23, 32, 33, 34, 37, 42, 56, 57, 61] for recent results and the surveys [12, 30, 38] for more references). Here we shall use the Eulerian formulation. Following Zakharov [60] and Craig-Sulem [29], we reduce the analysis to a system on the free surface $\Sigma(t) = \{y = \eta(t, x)\}$. If ψ is defined by

$$\psi(t, x) = \phi(t, x, \eta(t, x)),$$

then ϕ is the unique variational solution of

$$\Delta_{x,y} \phi = 0 \text{ in } \Omega, \quad \phi|_{\Sigma} = \psi, \quad \partial_n \phi = 0 \text{ on } \Gamma.$$

Define the Dirichlet-Neumann operator by

$$\begin{aligned} (G(\eta)\psi)(t, x) &= \sqrt{1 + |\nabla \eta|^2} \partial_n \phi|_{y=\eta(t,x)} \\ &= (\partial_y \phi)(t, x, \eta(t, x)) - \nabla \eta(t, x) \cdot (\nabla \phi)(t, x, \eta(t, x)). \end{aligned}$$

For the case with a rough bottom, we recall the precise construction later on (see §3.1). Now (η, ψ) solves (see [29] or [38, chapter 1] for instance)

$$(1.6) \quad \begin{cases} \partial_t \eta - G(\eta)\psi = 0, \\ \partial_t \psi + g\eta + \frac{1}{2} |\nabla \psi|^2 - \frac{1}{2} \frac{(\nabla \eta \cdot \nabla \psi + G(\eta)\psi)^2}{1 + |\nabla \eta|^2} = 0. \end{cases}$$

1.3. The Taylor condition. Introduce the so-called Taylor coefficient

$$(1.7) \quad a(t, x) = -(\partial_y P)(t, x, \eta(t, x)).$$

The stability of the waves is dictated by the Taylor sign condition, which is the assumption that there exists a positive constant c such that

$$(1.8) \quad a(t, x) \geq c > 0.$$

This assumption is now classical and we refer to [12, 21, 22, 39, 54, 55] for various comments. Here we only recall some basic facts. First of all, as proved by Wu ([54, 55]), this assumption is automatically satisfied in the infinite depth case (that is when $\Gamma = \emptyset$) or for flat bottoms (when $\Gamma = \{y = -k\}$). Notice that the proof remains valid for any $C^{1,\alpha}$ -domain, $0 < \alpha < 1$ (by using the fact that the Hopf Lemma is true for such domains, see [47] and the references therein). There are two other cases where this assumption is known to be satisfied. For instance under a smallness assumption. Indeed, if $\partial_t \phi = O(\varepsilon^2)$ and $\nabla_{x,y} \phi = O(\varepsilon)$ then directly from the definition of the pressure we have $P - gy = O(\varepsilon^2)$. Secondly, it was proved by Lannes ([39]) that the Taylor's assumption is satisfied under a smallness assumption on the curvature of the bottom (provided that the bottom is at least C^2). However, for general bottom we will assume that (1.8) is satisfied at time $t = 0$.

1.4. Main result. We work below with the vertical and horizontal traces of the velocity on the free boundary, namely

$$B := (\partial_y \phi)|_{y=\eta}, \quad V := (\nabla_x \phi)|_{y=\eta}.$$

These can be defined only in terms of η and ψ by means of the formulas

$$(1.9) \quad B = \frac{\nabla \eta \cdot \nabla \psi + G(\eta)\psi}{1 + |\nabla \eta|^2}, \quad V = \nabla \psi - B \nabla \eta.$$

Also, recall that the Taylor coefficient a defined in (1.7) can be defined in terms of η, V, B, ψ only (see Section 1.6 below).

THEOREM 1.2. *Let $d \geq 1$, $s > 1 + d/2$ and consider (η_0, ψ_0) such that*

- (1) $\eta_0 \in H^{s+\frac{1}{2}}(\mathbf{R}^d)$, $\psi_0 \in H^{s+\frac{1}{2}}(\mathbf{R}^d)$, $V_0 \in H^s(\mathbf{R}^d)$, $B_0 \in H^s(\mathbf{R}^d)$,
- (2) *there exists $h > 0$ such that condition (1.2) holds initially for $t = 0$,*
- (3) *there exists a positive constant c such that, for all $x \in \mathbf{R}^d$, $a_0(x) \geq c$.*

Then there exists $T > 0$ such that the Cauchy problem for (1.6) with initial data (η_0, ψ_0) has a unique solution $(\eta, \psi) \in C^0([0, T]; H^{s+\frac{1}{2}}(\mathbf{R}^d) \times H^{s+\frac{1}{2}}(\mathbf{R}^d))$, such that

- (1) *we have $(V, B) \in C^0([0, T]; H^s(\mathbf{R}^d) \times H^s(\mathbf{R}^d))$,*
- (2) *the condition (1.2) holds for $0 \leq t \leq T$, with h replaced by $h/2$,*
- (3) *for all $0 \leq t \leq T$ and for all $x \in \mathbf{R}^d$, $a(t, x) \geq c/2$.*

REMARK 1.3. The main novelty is that, in view of Sobolev embeddings, the initial surfaces we consider turn out to be only of $C^{3/2}$ class and consequently have unbounded curvature.

REMARK 1.4. Assumption 1 in the above theorem is automatically satisfied if

$$\eta_0 \in H^{s+\frac{1}{2}}(\mathbf{R}^d), \quad \psi_0 \in H^{\frac{1}{2}}(\mathbf{R}^d), \quad V_0 \in H^s(\mathbf{R}^d), \quad B_0 \in L^2(\mathbf{R}^d).$$

The only point where the estimates depend on ψ (and not only on η, V, B) come from the fact that we consider a general domain without assumption on the bottom. Otherwise, we shall prove *a priori* estimates for the fluid velocity and not for the fluid potential (notice that the fluid potential is defined up to a constant).

1.5. Water waves in a canal. We give here an illustration of the relevance of the analysis of low regularity solutions in a domain with a rough boundary. We show that the above result (or rather a slight adaptation of this result) allows to prove the existence of three-dimensional gravity water waves in a canal having vertical walls near the free surface. The propagation of waves whose crests are orthogonal to the walls is one of the main motivation for the analysis of 2D waves. It was historically at the heart of the analysis of water waves. The study of the propagation of three-dimensional water waves for the linearized equations goes back to Boussinesq (see [15]). However, there are no existence results for the nonlinear equations in the general case where the waves can be reflected on the walls of the canals (except the analysis of 3D-periodic travelling waves which correspond to the reflexion of a 2D-wave off a vertical wall, see Reeder-Shinbrot [46], Craig and Nicholls [27] and Iooss-Plotnikov [36]). More precisely, we consider a fluid domain which at time t is of the form

$$\Omega(t) = \{(x_1, x_2, y) \in (0, 1) \times \mathbf{R} \times \mathbf{R} : b(x) < y < \eta(t, x)\},$$

for some given function b . We do not make any regularity assumption on b ; again, our only assumptions are that: (i) there exists a positive constant h such that $\eta(t, x) \geq b(x) + h$ and (ii) the Taylor sign condition holds initially.

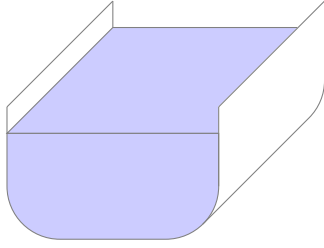


FIGURE 1. A non rectangular canal with vertical walls near the free surface.

In this context, we are able to prove that the Cauchy problem for the water waves system in this domain is well posed (see Section 7 for a precise statement). The analysis is based on an elementary (though seemingly yet unnoticed) observation: for any solution of the free boundary Euler equation having non vanishing Taylor coefficient, the free surface necessarily makes a right-angle with the rigid walls (see Figure 2 and Proposition 7.1). This suggests, following Boussinesq (see [15, page 37]) to perform a symmetrization process, as illustrated on Figure 3.

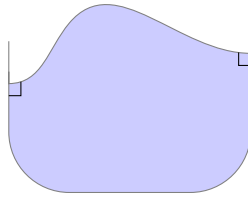


FIGURE 2. Two-dimensional section of the fluid domain.

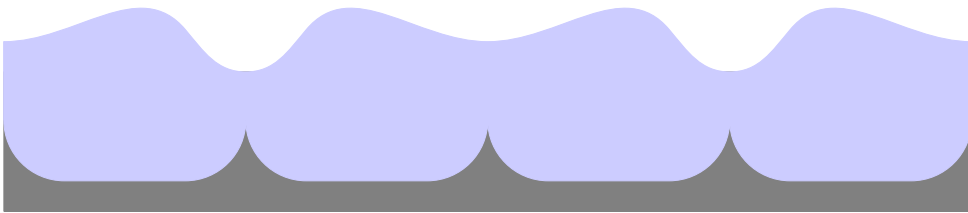


FIGURE 3. Two-dimensional section of the extended fluid domain.

Notice on Figure 3 that, to handle non rectangular canals, we have to work with rough bottoms (even if initially the bottom is smooth, after symmetry/periodization this is no more the case). Of course, the symmetry should yield in general a Lipschitz singularity for the symmetrized free surface. However, here the possible singularities are weaker since the above physical observation about the right angles at the interface implies that $\partial_{x_1}\eta(t, 0, x_2) = 0$ and $\partial_{x_1}\eta(t, 1, x_2) = 0$. Therefore, we are in position to apply our low regularity Cauchy theory. Namely, we apply Theorem 1.2 for some $s < 3$. Notice however that we need a small adaptation of our main result as we consider domains in $\mathbf{T} \times \mathbf{R}^2$ instead of \mathbf{R}^3 . We refer to Section 7 for details on how to adapt the result to the periodic framework.

1.6. The pressure. The purpose of this paragraph is to clarify, for low regularity solutions of the water waves system in rough domains, the definition of the pressure which is required if one wants to come back from solutions to the Zakharov system to solutions to the free boundary Euler equation. This definition will also provide the basic *a priori* estimates which will be later the starting point when establishing higher order elliptic regularity estimates required when studying the Taylor coefficient $a = -\partial_y P|_{\Sigma}$. On a physics point of view, the pressure is the Lagrange multiplier which is required by the incompressibility of the fluid (preservation of the null divergence condition). As a consequence, taking the divergence in (1.3), it is natural to define the pressure as a solution of

$$(1.10) \quad \Delta_{x,y}P = -\operatorname{div}_{x,y}(v \cdot \nabla_{x,y}v), \quad P|_{y=\eta} = 0.$$

Notice however that the solution of such problem may not be unique as can be seen in the simple case when $\Omega = (-\infty, 0) \times \mathbf{R}^d$. Indeed, if P is a solution, then $P + cy$ is another. Notice also that if P satisfies (1.10), then

$$\Delta_{x,y}\left(P + gy + \frac{1}{2}|v|^2\right) = 0.$$

DEFINITION 1.5. Let $(\eta, \psi) \in (W^{1,\infty} \cap H^{1/2}(\mathbf{R}^d)) \times H^{1/2}(\mathbf{R}^d)$. Assume that the variational solution (as defined in §3.1) of the equation

$$(1.11) \quad \Delta_{x,y}\phi = 0, \quad \phi|_{y=\eta} = \psi,$$

satisfies

$$|\nabla_{x,y}\phi|^2(x, \eta(x)) \in H^{\frac{1}{2}}(\mathbf{R}^d).$$

Let R be the variational solution of

$$\Delta_{x,y}R = 0 \text{ in } \Omega, \quad R|_{y=\eta} = -\left(g\eta + \frac{1}{2}|\nabla_{x,y}\phi|^2\right)|_{y=\eta}.$$

We define the pressure P in the domain Ω by

$$P(x, y) := R(x, y) - gy - \frac{1}{2}|\nabla_{x,y}\phi(x, y)|^2.$$

REMARK 1.6. The main advantage of defining the pressure as the solution of a variational problem is that it will satisfy automatically an *a priori* estimate (the estimate given by the variational theory).

It remains to link the solutions to the Zakharov/Craig-Sulem system (1.6) to solutions of the free boundary Euler system (1.3) with boundary conditions (1.4). To do so, we proved in [4] that if (η, ψ) is a solution of System (1.6), if we consider the variational solution to (1.11), then the velocity field $v = \nabla_{x,y}\phi$ satisfies (1.3), which is of course equivalent to

$$(1.12) \quad P = -\partial_t\phi - gy - \frac{1}{2}|\nabla_{x,y}\phi|^2.$$

THEOREM 1.7 (from [4]). Assume that $(\eta, \psi) \in C([0, T]; H^{s+\frac{1}{2}}(\mathbf{R}^d) \times H^{s+\frac{1}{2}}(\mathbf{R}^d))$, with $s > 1 + d/2$, is a solution of the Zakharov/Craig-Sulem system (1.6). Then the assumptions required to define the pressure are satisfied, and (1.12) is satisfied, and the distribution $\partial_t\phi$ is well defined for fixed t and belongs to the space $H^{1,0}(\Omega(t))$ (see Definition 3.3).

1.7. Plan of the paper. At first glance, Theorem 1.2 looks very similar to our previous result in presence of surface tension [1, Theorem 1.1]. Indeed, the regularity threshold exhibited by the velocity field (namely $V, B \in H^s(\mathbf{R}^d)$, $s > 1 + d/2$) is the same in both results and (as explained above) appears to be the natural one. However, an important difference between both cases is that the algebraic nature of (1.6) (and its counter-part in presence of surface tension) requires that the free domain is $3/2$ smoother than the velocity field in presence of surface tension and only $1/2$ smoother without surface tension. This algebraic rigidity of the system implies that in order to lower the regularity threshold to the natural one (Lipschitz velocities), we are forced to work with $C^{3/2}$ domains (compared to the much smoother $C^{5/2}$ regularity in [1]). This in turn poses new challenging questions in the study of the Dirichlet–Neumann operator. Indeed, at this level of regularity the regularity of the remainder term in the paradifferential description of the Dirichlet–Neumann operator $G(\eta)\psi$ is not given by the regularity of the function ψ itself, but rather by the regularity of the domain. This is this phenomenon which forces us to work with the new unknowns V, B rather than with ψ .

In Section 2, we wrote a review of paradifferential calculus and proved various technical results useful in the article. In Section 3 we study the Dirichlet–Neumann operator and prove the main estimate (see Theorem 3.17) that we believe is of independent interest. In Section 4, we symmetrize the system and prove *a priori* estimates. In Section 5 we prove the contraction estimates required to show uniqueness and stability of solutions. In particular we prove a contraction estimate for the difference of two Dirichlet–Neumann operators, involving only the $C^{\frac{1}{2}}$ norm of the difference of the functions defining the domains (see Theorem 5.3). In Section 6 we prove the existence of solutions by a regularization process. Finally, in Section 7 we study the Cauchy problem in the canal.

2. Paradifferential calculus

Let us review notations and results about Bony’s paradifferential calculus. We refer to [14, 35, 43, 44] for the general theory. Here we follow the presentation by Métivier in [43].

2.1. Paradifferential operators. For $k \in \mathbf{N}$, we denote by $W^{k,\infty}(\mathbf{R}^d)$ the usual Sobolev spaces. For $\rho = k + \sigma$, $k \in \mathbf{N}$, $\sigma \in (0, 1)$ denote by $W^{\rho,\infty}(\mathbf{R}^d)$ the space of functions whose derivatives up to order k are bounded and uniformly Hölder continuous with exponent σ .

DEFINITION 2.1. *Given $\rho \in [0, 1]$ and $m \in \mathbf{R}$, $\Gamma_\rho^m(\mathbf{R}^d)$ denotes the space of locally bounded functions $a(x, \xi)$ on $\mathbf{R}^d \times (\mathbf{R}^d \setminus \{0\})$, which are C^∞ with respect to ξ for $\xi \neq 0$ and such that, for all $\alpha \in \mathbf{N}^d$ and all $\xi \neq 0$, the function $x \mapsto \partial_\xi^\alpha a(x, \xi)$ belongs to $W^{\rho,\infty}(\mathbf{R}^d)$ and there exists a constant C_α such that,*

$$\forall |\xi| \geq \frac{1}{2}, \quad \|\partial_\xi^\alpha a(\cdot, \xi)\|_{W^{\rho,\infty}(\mathbf{R}^d)} \leq C_\alpha (1 + |\xi|)^{m-|\alpha|}.$$

Then $\dot{\Gamma}_\rho^m(\mathbf{R}^d)$ denotes the subspace of $\Gamma_\rho^m(\mathbf{R}^d)$ which consists of symbols $a(x, \xi)$ which are homogeneous of degree m with respect to ξ .

Given a symbol a , we define the paradifferential operator T_a by

$$(2.1) \quad \widehat{T_a u}(\xi) = (2\pi)^{-d} \int \chi(\xi - \eta, \eta) \widehat{a}(\xi - \eta, \eta) \widehat{\psi}(\eta) \widehat{u}(\eta) d\eta,$$

where $\widehat{a}(\theta, \xi) = \int e^{-ix \cdot \theta} a(x, \xi) dx$ is the Fourier transform of a with respect to the first variable; χ and ψ are two fixed C^∞ functions such that:

$$(2.2) \quad \psi(\eta) = 0 \quad \text{for } |\eta| \leq 1, \quad \psi(\eta) = 1 \quad \text{for } |\eta| \geq 2,$$

and $\chi(\theta, \eta)$ satisfies, for $0 < \varepsilon_1 < \varepsilon_2$ small enough,

$$\chi(\theta, \eta) = 1 \quad \text{if } |\theta| \leq \varepsilon_1 |\eta|, \quad \chi(\theta, \eta) = 0 \quad \text{if } |\theta| \geq \varepsilon_2 |\eta|,$$

and such that

$$\forall(\theta, \eta) : \quad \left| \partial_\theta^\alpha \partial_\eta^\beta \chi(\theta, \eta) \right| \leq C_{\alpha, \beta} (1 + |\eta|)^{-|\alpha| - |\beta|}.$$

REMARK 2.2 (Choice of the cut-off function). It is known that one can change the cut-off function χ above, up to adding some smoothing remainders (see [43, Proposition 5.1.8]). However, this requires to work with smooth enough symbols. Below, we shall need to work with rough symbols, so that the choice of the cut-off function can, sometimes matter. It will therefore be convenient to fix the cut-off function so that all the computations below can be done with the same choice. Since we need to work with paraproducts, we chose a cut-off function χ such that when $a = a(x)$, T_a is given by the usual expression in terms of the Littlewood-Paley operators. Namely, we introduce $\kappa \in C_0^\infty(\mathbf{R}^d)$ such that

$$\kappa(\theta) = 1 \quad \text{for } |\theta| \leq 1.1, \quad \kappa(\theta) = 0 \quad \text{for } |\theta| \geq 1.9.$$

Then we define

$$\chi(\theta, \eta) = \sum_{k=0}^{+\infty} \kappa_{k-3}(\theta) \varphi_k(\eta)$$

where

$$\kappa_k(\theta) = \kappa(2^{-k}\theta) \quad \text{for } k \in \mathbf{Z}, \quad \varphi_0 = \kappa_0, \quad \text{and} \quad \varphi_k = \kappa_k - \kappa_{k-1} \quad \text{for } k \geq 1.$$

2.2. Symbolic calculus. We shall use quantitative results from [43] about operator norms estimates in symbolic calculus. To do so, introduce the following semi-norms.

DEFINITION 2.3. For $m \in \mathbf{R}$, $\rho \in [0, 1]$ and $a \in \Gamma_\rho^m(\mathbf{R}^d)$, we set

$$(2.3) \quad M_\rho^m(a) = \sup_{|\alpha| \leq \frac{3d}{2} + 1 + \rho} \sup_{|\xi| \geq 1/2} \left\| (1 + |\xi|)^{|\alpha| - m} \partial_\xi^\alpha a(\cdot, \xi) \right\|_{W^{\rho, \infty}(\mathbf{R}^d)}.$$

DEFINITION 2.4 (Zygmund spaces). Consider a dyadic decomposition of the identity: $I = \Delta_{-1} + \sum_{q=0}^{\infty} \Delta_q$. If s is any real number, we define the Zygmund class $C_*^s(\mathbf{R}^d)$ as the space of tempered distributions u such that

$$\|u\|_{C_*^s} := \sup_q 2^{qs} \|\Delta_q u\|_{L^\infty} < +\infty.$$

REMARK 2.5. It is known that $C_*^s(\mathbf{R}^d)$ is the usual Hölder space $W^{s, \infty}(\mathbf{R}^d)$ if $s > 0$ is not an integer.

DEFINITION 2.6. Let $m \in \mathbf{R}$. An operator T is said to be of order m if, for all $\mu \in \mathbf{R}$, it is bounded from H^μ to $H^{\mu-m}$ and from C_*^μ to $C_*^{\mu-m}$.

The main features of symbolic calculus for paradifferential operators are given by the following theorem.

THEOREM 2.7. *Let $m \in \mathbf{R}$ and $\rho \in [0, 1]$.*

(i) *If $a \in \Gamma_0^m(\mathbf{R}^d)$, then T_a is of order m . Moreover, for all $\mu \in \mathbf{R}$ there exists a constant K such that*

$$(2.4) \quad \|T_a\|_{H^\mu \rightarrow H^{\mu-m}} \leq KM_0^m(a), \quad \|T_a\|_{C_*^\mu \rightarrow C_*^{\mu-m}} \leq KM_0^m(a).$$

(ii) *If $a \in \Gamma_\rho^m(\mathbf{R}^d)$, $b \in \Gamma_\rho^{m'}(\mathbf{R}^d)$ then $T_a T_b - T_{ab}$ is of order $m + m' - \rho$. Moreover, for all $\mu \in \mathbf{R}$ there exists a constant K such that*

$$(2.5) \quad \begin{aligned} \|T_a T_b - T_{ab}\|_{H^\mu \rightarrow H^{\mu-m-m'+\rho}} &\leq KM_\rho^m(a)M_0^{m'}(b) + KM_0^m(a)M_\rho^{m'}(b), \\ \|T_a T_b - T_{ab}\|_{C_*^\mu \rightarrow C_*^{\mu-m-m'+\rho}} &\leq KM_\rho^m(a)M_0^{m'}(b) + KM_0^m(a)M_\rho^{m'}(b). \end{aligned}$$

(iii) *Let $a \in \Gamma_\rho^m(\mathbf{R}^d)$. Denote by $(T_a)^*$ the adjoint operator of T_a and by \bar{a} the complex conjugate of a . Then $(T_a)^* - T_{\bar{a}}$ is of order $m - \rho$. Moreover, for all μ there exists a constant K such that*

$$(2.6) \quad \|(T_a)^* - T_{\bar{a}}\|_{H^\mu \rightarrow H^{\mu-m+\rho}} \leq KM_\rho^m(a), \quad \|(T_a)^* - T_{\bar{a}}\|_{C_*^\mu \rightarrow C_*^{\mu-m+\rho}} \leq KM_\rho^m(a).$$

We shall need in this article to consider paradifferential operators with negative regularity. As a consequence, we need to extend our previous definition.

DEFINITION 2.8. *For $m \in \mathbf{R}$ and $\rho \in (-\infty, 0)$, $\Gamma_\rho^m(\mathbf{R}^d)$ denotes the space of distributions $a(x, \xi)$ on $\mathbf{R}^d \times (\mathbf{R}^d \setminus 0)$, which are C^∞ with respect to ξ and such that, for all $\alpha \in \mathbf{N}^d$ and all $\xi \neq 0$, the function $x \mapsto \partial_\xi^\alpha a(x, \xi)$ belongs to $C_*^\rho(\mathbf{R}^d)$ and there exists a constant C_α such that,*

$$(2.7) \quad \forall |\xi| \geq \frac{1}{2}, \quad \|\partial_\xi^\alpha a(\cdot, \xi)\|_{C_*^\rho} \leq C_\alpha (1 + |\xi|)^{m-|\alpha|}.$$

Then $\dot{\Gamma}_\rho^m(\mathbf{R}^d)$ denotes the subspace of $\Gamma_\rho^m(\mathbf{R}^d)$ which consists of symbols $a(x, \xi)$ which are homogeneous of degree m with respect to ξ . For $a \in \Gamma_\rho^m$, we define

$$(2.8) \quad M_\rho^m(a) = \sup_{|\alpha| \leq \frac{3d}{2} + \rho + 1} \sup_{|\xi| \geq 1/2} \left\| (1 + |\xi|)^{|\alpha| - m} \partial_\xi^\alpha a(\cdot, \xi) \right\|_{C_*^\rho(\mathbf{R}^d)}.$$

2.3. Paraproducts and product rules. If $a = a(x)$ is a function of x only, the paradifferential operator T_a is called a paraproduct. A key feature of paraproducts is that one can replace nonlinear expressions by paradifferential expressions, to the price of error terms which are smoother than the main terms. Also, one can define paraproducts T_a for rough functions a which do not belong to $L^\infty(\mathbf{R}^d)$ but merely $C_*^{-m}(\mathbf{R}^d)$ with $m > 0$.

DEFINITION 2.9. *Given two functions a, b defined on \mathbf{R}^d we define the remainder*

$$R(a, u) = au - T_a u - T_u a.$$

We record here various estimates about paraproducts (see chapter 2 in [11] or [19]).

THEOREM 2.10. *i) Let $\alpha, \beta \in \mathbf{R}$. If $\alpha + \beta > 0$ then*

$$(2.9) \quad \|R(a, u)\|_{H^{\alpha+\beta-\frac{d}{2}}(\mathbf{R}^d)} \leq K \|a\|_{H^\alpha(\mathbf{R}^d)} \|u\|_{H^\beta(\mathbf{R}^d)},$$

$$(2.10) \quad \|R(a, u)\|_{C_*^{\alpha+\beta}(\mathbf{R}^d)} \leq K \|a\|_{C_*^\alpha(\mathbf{R}^d)} \|u\|_{C_*^\beta(\mathbf{R}^d)},$$

$$(2.11) \quad \|R(a, u)\|_{H^{\alpha+\beta}(\mathbf{R}^d)} \leq K \|a\|_{C_*^\alpha(\mathbf{R}^d)} \|u\|_{H^\beta(\mathbf{R}^d)}.$$

ii) Let $m > 0$ and $s \in \mathbf{R}$. Then

$$(2.12) \quad \|T_a u\|_{H^{s-m}} \leq K \|a\|_{C_*^{-m}} \|u\|_{H^s},$$

$$(2.13) \quad \|T_a u\|_{C_*^{s-m}} \leq K \|a\|_{C_*^{-m}} \|u\|_{C_*^s}.$$

$$(2.14) \quad \|T_a u\|_{C_*^s} \leq K \|a\|_{L^\infty} \|u\|_{C_*^s}.$$

iii) Let s_0, s_1, s_2 be such that $s_0 \leq s_2$ and $s_0 < s_1 + s_2 - \frac{d}{2}$, then

$$(2.15) \quad \|T_a u\|_{H^{s_0}} \leq K \|a\|_{H^{s_1}} \|u\|_{H^{s_2}}.$$

By combining the two previous points with the embedding $H^\mu(\mathbf{R}^d) \subset C_*^{\mu-d/2}(\mathbf{R}^d)$ (for any $\mu \in \mathbf{R}$) we immediately obtain the following results that we shall need in the sequel.

PROPOSITION 2.11. Let $r, \mu \in \mathbf{R}$ be such that $r + \mu > 0$. If $\gamma \in \mathbf{R}$ satisfies

$$\gamma \leq r \quad \text{and} \quad \gamma < r + \mu - \frac{d}{2},$$

then there exists a constant K such that, for all $a \in H^r(\mathbf{R}^d)$ and all $u \in H^\mu(\mathbf{R}^d)$, we have

$$\|au - T_a u\|_{H^\gamma} \leq K \|a\|_{H^r} \|u\|_{H^\mu}.$$

COROLLARY 2.12. i) If $u_j \in H^{s_j}(\mathbf{R}^d)$ ($j = 1, 2$) with $s_1 + s_2 > 0$ then $u_1 u_2 \in H^{s_0}(\mathbf{R}^d)$ and

$$(2.16) \quad \|u_1 u_2\|_{H^{s_0}} \leq K \|u_1\|_{H^{s_1}} \|u_2\|_{H^{s_2}},$$

if

$$s_0 \leq s_j, \quad j = 1, 2, \quad \text{and} \quad s_0 < s_1 + s_2 - d/2.$$

ii) (Tame estimate in Sobolev spaces) If $s \geq 0$ then

$$(2.17) \quad \|u_1 u_2\|_{H^s} \leq K (\|u_1\|_{H^s} \|u_2\|_{L^\infty} + \|u_1\|_{L^\infty} \|u_2\|_{H^s}).$$

iii) (Tame estimate in Zygmund spaces) If $s \geq 0$ then

$$(2.18) \quad \|u_1 u_2\|_{C_*^s} \leq K (\|u_1\|_{C_*^s} \|u_2\|_{L^\infty} + \|u_1\|_{L^\infty} \|u_2\|_{C_*^s}).$$

iv) Let $\mu, m \in \mathbf{R}$ be such that $\mu, m > 0$ and $m \notin \mathbf{N}$. Then

$$(2.19) \quad \|u_1 u_2\|_{H^\mu} \leq K (\|u_1\|_{L^\infty} \|u_2\|_{H^\mu} + \|u_2\|_{C_*^{-m}} \|u_1\|_{H^{\mu+m}}).$$

v) Let $\beta > \alpha > 0$. Then

$$(2.20) \quad \|u_1 u_2\|_{C_*^{-\alpha}} \leq K \|u_1\|_{C_*^\beta} \|u_2\|_{C_*^{-\alpha}}.$$

vi) Let $s > d/2$ and consider $F \in C^\infty(\mathbf{C}^N)$ such that $F(0) = 0$. Then there exists a non-decreasing function $\mathcal{F}: \mathbf{R}^+ \rightarrow \mathbf{R}^+$ such that

$$(2.21) \quad \|F(U)\|_{H^s} \leq \mathcal{F}(\|U\|_{L^\infty}) \|U\|_{H^s},$$

for any $U \in H^s(\mathbf{R}^d)^N$.

vii) Let $s \geq 0$ and consider $F \in C^\infty(\mathbf{C}^N)$ such that $F(0) = 0$. Then there exists a non-decreasing function $\mathcal{F}: \mathbf{R}^+ \rightarrow \mathbf{R}^+$ such that

$$(2.22) \quad \|F(U)\|_{C_*^s} \leq \mathcal{F}(\|U\|_{L^\infty}) \|U\|_{C_*^s},$$

for any $U \in C_*^s(\mathbf{R}^d)^N$.

PROOF. The first three estimates are well-known, see Hörmander [35] or Chemin [19]. To prove *iv*) and *v*) we write

$$u_1 u_2 = T_{u_1} u_2 + T_{u_2} u_1 + R(u_1, u_2).$$

Then (2.19) follows from

$$\begin{aligned} \|T_{u_1} u_2\|_{H^\mu} &\lesssim \|u_1\|_{L^\infty} \|u_2\|_{H^\mu} && \text{(see (2.4)),} \\ \|T_{u_2} u_1\|_{H^\mu} &\lesssim \|u_2\|_{C_*^{-m}} \|u_1\|_{H^{\mu+m}} && \text{(see (2.12)),} \\ \|R(u_1, u_2)\|_{H^\mu} &\lesssim \|u_2\|_{C_*^{-m}} \|u_1\|_{H^{\mu+m}} && \text{(see (2.11)),} \end{aligned}$$

while similarly (2.20) follows from

$$\begin{aligned} \|T_{u_1} u_2\|_{C_*^{-\alpha}} &\lesssim \|u_1\|_{L^\infty} \|u_2\|_{C_*^{-\alpha}} \lesssim \|u_1\|_{C_*^\beta} \|u_2\|_{C_*^{-\alpha}}, \\ \|T_{u_2} u_1\|_{C_*^{-\alpha}} &\lesssim \|u_2\|_{C_*^{-\alpha}} \|u_1\|_{C_*^0} \leq \|u_2\|_{C_*^{-\alpha}} \|u_1\|_{C_*^\beta}, \\ \|R(u_1, u_2)\|_{C_*^{-\alpha}} &\leq \|R(u_1, u_2)\|_{C_*^{\beta-\alpha}} \lesssim \|u_2\|_{C_*^{-\alpha}} \|u_1\|_{C_*^\beta}. \end{aligned}$$

(With regards to the last inequality, to apply (2.10) we do need $\beta > \alpha > 0$.) Finally, *vi*) and *vii*) are due to Meyer [44, Théorème 2.5 and remarque], in the line of the work by Bony [14]. \square

Finally, let us finish this section with a generalization of (2.12)

PROPOSITION 2.13. *Let $\rho < 0$, $m \in \mathbf{R}$ and $a \in \dot{\Gamma}_\rho^m$. Then the operator T_a is of order $m - \rho$:*

$$(2.23) \quad \begin{aligned} \|T_a\|_{H^s \rightarrow H^{s-(m-\rho)}} &\leq CM_\rho^m(a), \\ \|T_a\|_{C_*^s \rightarrow C_*^{s-(m-\rho)}} &\leq CM_\rho^m(a). \end{aligned}$$

PROOF. Let us prove the first estimate. The proof of the second is similar. Notice that if $m = 0$ and $a(x, \xi) = a(x)$, then (2.23) is simply (2.12). Furthermore, if $a(x, \xi) = b(x)p(\xi)$, then

$$T_a = T_b(\chi p)(|D_x|),$$

where χ is a cutoff function vanishing near 0 and equal to 1 for $|\xi| \geq 1$. As a consequence, in this particular case, we get

$$\|T_a\|_{H^s \rightarrow H^{s-(m-\rho)}} \leq C \|b\|_{C_*^\rho} \|p\|_{\mathbf{S}^{d-1}} \|L^\infty.$$

In the general case, we can expand, for fixed x , $a(x, \xi)$ in terms of spherical harmonics. Let $(\tilde{h}_\nu)_{\nu \in \mathbf{N}^*}$ be an orthonormal basis of $L^2(\mathbf{S}^{d-1})$ consisting of eigenfunctions of the (self-adjoint) Laplace–Beltrami operator, $\Delta_\omega = \Delta_{\mathbf{S}^{d-1}}$ on $L^2(\mathbf{S}^{d-1})$, *i.e.*

$$\Delta_\omega \tilde{h}_\nu = \lambda_\nu^2 \tilde{h}_\nu.$$

By the Weyl formula, we know that $\lambda_\nu \sim c\nu^{\frac{1}{d}}$. Setting

$$h_\nu(\xi) = |\xi|^m \tilde{h}_\nu(\omega), \quad \omega = \frac{\xi}{|\xi|}, \quad \xi \neq 0,$$

we can write

$$a(x, \xi) = \sum_{\nu \in \mathbf{N}^*} a_\nu(x) h_\nu(\xi) \quad \text{where} \quad a_\nu(x) = \int_{\mathbf{S}^{d-1}} a(x, \omega) \overline{\tilde{h}_\nu(\omega)} d\omega.$$

Since

$$\lambda_\nu^{2k} a_\nu(t, x) = \int_{\mathbf{S}^{d-1}} \Delta_\omega^k a(x, \omega) \overline{\tilde{h}_\nu(\omega)} d\omega,$$

we have, for all $\nu \geq 1$,

$$(2.24) \quad \|a_\nu(\cdot)\|_{C_*^\rho} \leq C \lambda_\nu^{-\frac{3d}{2}+1} \leq \nu^{-\frac{3}{2}-\frac{1}{d}} M_\rho^m(p).$$

Moreover,

$$(2.25) \quad \|\tilde{h}_\nu\|_{L^\infty} \leq C\lambda_\nu^{\frac{(d-1)}{2}} \leq C\nu^{\frac{1}{2}-\frac{1}{2d}}.$$

and the result follows because

$$\|T_a\|_{H^s \rightarrow H^{s-(m-\rho)}} \leq C \sum_\nu \nu^{-1-\frac{1}{2d}} M_\rho^m(p).$$

This completes the proof. \square

We shall also need the following technical result.

PROPOSITION 2.14. Set $\langle D_x \rangle = (I - \Delta)^{1/2}$.

i) Let $s > \frac{1}{2} + \frac{d}{2}$ and $\sigma \in \mathbf{R}$ be such that $\sigma \leq s$. Then there exists $K > 0$ such that for all $V \in W^{1,\infty}(\mathbf{R}^d) \cap H^s(\mathbf{R}^d)$ and $u \in H^{\sigma-\frac{1}{2}}(\mathbf{R}^d)$ one has

$$\|[\langle D_x \rangle^\sigma, V]u\|_{L^2(\mathbf{R}^d)} \leq K \{ \|V\|_{W^{1,\infty}(\mathbf{R}^d)} + \|V\|_{H^s(\mathbf{R}^d)} \} \|u\|_{H^{\sigma-\frac{1}{2}}(\mathbf{R}^d)}.$$

ii) Let $s > 1 + \frac{d}{2}$ and $\sigma \in \mathbf{R}$ be such that $\sigma \leq s$. Then there exists $K > 0$ such that for all $V \in H^s(\mathbf{R}^d)$ and $u \in H^{\sigma-1}(\mathbf{R}^d)$ one has

$$\|[\langle D_x \rangle^\sigma, V]u\|_{L^2(\mathbf{R}^d)} \leq K \|V\|_{H^s(\mathbf{R}^d)} \|u\|_{H^{\sigma-1}(\mathbf{R}^d)}.$$

iii) Let $s > \frac{1}{2} + \frac{d}{2}$ and $V \in H^s(\mathbf{R}^d)$. Then

$$\|[\langle D_x \rangle^{\frac{1}{2}}, V]u\|_{L^\infty(\mathbf{R}^d)} \leq K \|V\|_{H^s(\mathbf{R}^d)} \|u\|_{L^\infty(\mathbf{R}^d)}.$$

PROOF. To prove *i)* we write

$$\begin{cases} \|[\langle D_x \rangle^\sigma, V]u\|_{L^2} \leq A + B, \\ A = \|[\langle D_x \rangle^\sigma, T_V]u\|_{L^2}, \quad B = \|[\langle D_x \rangle^\sigma, V - T_V]u\|_{L^2}. \end{cases}$$

By (2.5) we have

$$A \leq K \|V\|_{W^{1,\infty}} \|u\|_{H^{\sigma-1}}.$$

On the other hand one can write

$$B \leq \|\langle D_x \rangle^\sigma (V - T_V)u\|_{L^2} + \|(V - T_V)\langle D_x \rangle^\sigma u\|_{L^2} = B_1 + B_2.$$

We use Proposition 2.11 two times. To estimate B_1 we take $\gamma = \sigma, r = s, \mu = \sigma - \frac{1}{2}$. To estimate B_2 we take $\gamma = 0, r = s, \mu = -\frac{1}{2}$ and we obtain,

$$B \leq K \|V\|_{H^s} \|u\|_{H^{\sigma-\frac{1}{2}}}.$$

To prove *ii)*, to estimate B_1 (resp. B_2) we use again Proposition 2.11 with $\gamma = \sigma, r = s, \mu = \sigma - 1$ (resp. $\gamma = 0, r = s, \mu = -1$). Finally, let us prove *iii)*. Using (2.5) with $m = \frac{1}{2}, m' = 0, \rho = \frac{1}{2} + \epsilon$, we obtain

$$\|[\langle D_x \rangle^{\frac{1}{2}}, T_V]u\|_{C_*^\epsilon} \leq C \|V\|_{H^s} \|u\|_{C_*^0} \leq C \|V\|_{H^s} \|u\|_{L^\infty}.$$

On the other hand,

$$[\langle D_x \rangle^{\frac{1}{2}}, V - T_V]u = \langle D_x \rangle^{\frac{1}{2}} (V - T_V)u - (V - T_V)\langle D_x \rangle^{\frac{1}{2}} u.$$

Let $\frac{1}{2} < r < s - \frac{d}{2}$ so that

$$\|V\|_{C_*^r} \leq C \|V\|_{H^s}.$$

According to (2.14) and (2.10), $V - T_V$ is bounded from L^∞ to C_*^r by $K \|V\|_{C_*^r}$ and according to (2.13) and (2.10), from $C_*^{-\frac{1}{2}}$ to $C_*^{r-\frac{1}{2}}$ by $K \|V\|_{C_*^r}$, which implies

$$\|[\langle D_x \rangle^{\frac{1}{2}}, V - T_V]u\|_{C_*^{r-\frac{1}{2}}} \leq K \|V\|_{H^s} \|u\|_{L^\infty}.$$

This completes the proof. \square

We shall need elementary estimates on the solutions of transport equations that we recall now.

PROPOSITION 2.15. *Let $I = [0, T]$ and consider the Cauchy problem*

$$(2.26) \quad \begin{cases} \partial_t u + V \cdot \nabla u = f, & t \in I, \\ u|_{t=0} = u_0. \end{cases}$$

We have the following estimates

$$(2.27) \quad \|u(t)\|_{L^\infty(\mathbf{R}^d)} \leq \|u_0\|_{L^\infty(\mathbf{R}^d)} + \int_0^t \|f(\sigma, \cdot)\|_{L^\infty(\mathbf{R}^d)} d\sigma.$$

There exists a non decreasing function $\mathcal{F} : \mathbf{R}^+ \rightarrow \mathbf{R}^+$ such that

$$(2.28) \quad \|u(t)\|_{L^2(\mathbf{R}^d)} \leq \mathcal{F}(\|V\|_{L^1(I; W^{1,\infty}(\mathbf{R}^d))}) (\|u_0\|_{L^2(\mathbf{R}^d)} + \int_0^t \|f(t', \cdot)\|_{L^2(\mathbf{R}^d)} dt').$$

If $s > 1 + \frac{d}{2}$ and $\sigma \leq s$ there exists a non decreasing function $\mathcal{F} : \mathbf{R}^+ \rightarrow \mathbf{R}^+$ such that

$$(2.29) \quad \|u(t)\|_{H^\sigma(\mathbf{R}^d)} \leq \mathcal{F}(\|V\|_{L^1(I; H^s(\mathbf{R}^d))}) (\|u_0\|_{H^\sigma(\mathbf{R}^d)} + \int_0^t \|f(t', \cdot)\|_{H^\sigma(\mathbf{R}^d)} dt').$$

PROOF. The estimate (2.27) follows from the exact formula for the solution, in terms of the solution to the differential equation

$$\dot{X} = -V(t, X(t)),$$

while (2.28) is obtained by computing $\frac{d}{dt} \|u(t, \cdot)\|_{L^2(\mathbf{R}^d)}^2$ and (2.29) follows from (2.28) applied to $\langle D_x \rangle^\sigma u$, Proposition 2.14 and Gronwall inequality. \square

2.4. Commutation with a vector field. We prove in this paragraph a commutator estimate between a paradifferential operator T_p and the convective derivative $\partial_t + V \cdot \nabla$. Inspired by Chemin [18] and Alinhac [7], we prove an estimate which depends on estimates on $\partial_t p + V \cdot \nabla p$ and not on $\nabla_{t,x} p$.

When a and u are symbols and functions depending on $t \in I$, we still denote by $T_a u$ the spatial paradifferential operator (or paraproduct) such that for all $t \in I$, $(T_a u)(t) = T_{a(t)} u(t)$. Given a symbol $a = a(t; x, \xi)$ depending on time, we use the notation

$$\mathcal{M}_0^m(a) := \sup_{t \in [0, T]} \sup_{|\alpha| \leq \frac{3d}{2} + 1 + \rho} \sup_{|\xi| \geq 1/2} \left\| (1 + |\xi|)^{|\alpha| - m} \partial_\xi^\alpha a(t; \cdot, \xi) \right\|_{L^\infty(\mathbf{R}^d)}.$$

Given a scalar symbol $p = p(t, x, \xi)$ of order m , it follows directly from the symbolic calculus rules for paradifferential operators (see (2.4) and (2.5)) that,

$$\| [T_p, \partial_t + T_V \cdot \nabla] u \|_{H^\mu} \leq K \{ \mathcal{M}_0^m(\partial_t p) + \mathcal{M}_0^m(\nabla p) \} \|V\|_{W^{1,\infty}} \|u\|_{H^{\mu+m}}.$$

A technical key point in our analysis is that one can replace this estimate by a tame estimate which does not involve the first order derivatives of p , but instead $\partial_t p + V \cdot \nabla p$.

LEMMA 2.16. Let $V \in C^0([0, T]; C_*^{1+\varepsilon}(\mathbf{R}^d))$ for some $\varepsilon > 0$ and consider a symbol $p = p(t, x, \xi)$ which is homogeneous in ξ of order m . Then there exists $K > 0$ (independent of p, V) such that for any $t \in [0, T]$ and any $u \in C^0([0, T]; H^m(\mathbf{R}^d))$.

$$(2.30) \quad \begin{aligned} & \left\| [T_p, \partial_t + T_V \cdot \nabla] u(t) \right\|_{L^2(\mathbf{R}^d)} \\ & \leq K \left\{ \mathcal{M}_0^m(p) \|V(t)\|_{C_*^{1+\varepsilon}} + \mathcal{M}_0^m(\partial_t p + V \cdot \nabla p) \|V(t)\|_{L^\infty} \right\} \|u(t)\|_{H^m(\mathbf{R}^d)}. \end{aligned}$$

PROOF. Set $I = [0, T]$ and denote by \mathcal{R} the set of continuous operators $R(t)$ from $H^m(\mathbf{R}^d)$ to $L^2(\mathbf{R}^d)$ with norm satisfying

$$\|R(t)\|_{\mathcal{L}(H^m(\mathbf{R}^d), L^2(\mathbf{R}^d))} \leq K \left\{ \mathcal{M}_0^m(p) \|V(t)\|_{C_*^{1+\varepsilon}} + \mathcal{M}_0^m(\partial_t p + V \cdot \nabla p) \|V(t)\|_{L^\infty} \right\}.$$

We begin by noticing that it is sufficient to prove that

$$(2.31) \quad (\partial_t + V \cdot \nabla) T_p = T_p (\partial_t + T_V \cdot \nabla) + R, \quad R \in \mathcal{R}.$$

Indeed, by Theorem 5.2.9 in [43], we have (for fixed t)

$$\|(V - T_V) \cdot \nabla T_p u\|_{L^2} \lesssim \|V\|_{W^{1,\infty}} \|T_p u\|_{L^2},$$

and hence, by using the operator norm estimate (2.4), we find that

$$\|(V - T_V) \cdot \nabla T_p u\|_{L^2} \lesssim \|V\|_{W^{1,\infty}} \mathcal{M}_0^m(p) \|u\|_{H^m},$$

which implies that $(V - T_V) \cdot \nabla T_p \in \mathcal{R}$.

We split the proof of (2.31) into three steps. By decomposing p into a sum of spherical harmonic, we shall reduce the analysis to establishing (2.31) for the special case when T_p is a paraproduct. In the first step we prove (2.31) for $m = 0$ and $p = p(t, x)$. In the second step we prove (2.31) for $p = a(t, x)h(\xi)$ where h is homogeneous in ξ of order m . Then we consider the general case.

Step 1: Paraproduct, $m = 0, p = p(t, x)$. Notice that in this case

$$\mathcal{M}_0^0(p) = \|p\|_{L^\infty}.$$

We have

$$(2.32) \quad \begin{cases} \partial_t T_p u = T_{\partial_t p} u + T_p \partial_t u, \\ V \cdot \nabla T_p u = V \cdot T_{\nabla p} u + V T_p \cdot \nabla u =: A + B. \end{cases}$$

Decompose $V = S_{j-3}(V) + S^{j-3}(V)$, with

$$S_{j-3}(V) = \sum_{k \leq j-2} \Delta_k V, \quad S^{j-3}(V) = \sum_{k \geq j-3} \Delta_k V,$$

to obtain

$$(2.33) \quad \begin{cases} A = A_1 + A_2, \\ A_1 := \sum_j S_{j-3}(V) S_{j-3}(\nabla p) \Delta_j u, \\ A_2 := \sum_j S^{j-3}(V) S_{j-3}(\nabla p) \Delta_j u. \end{cases}$$

Let us consider the term A_2 . Since

$$\|S^{j-3}(V)\|_{L^\infty} \leq \sum_{k \geq j-3} \|\Delta_k V\|_{L^\infty} \lesssim \sum_{k \geq j-3} 2^{-k(1+\varepsilon)} \|V\|_{C_*^{1+\varepsilon}} \lesssim 2^{-j(1+\varepsilon)} \|V\|_{C_*^{1+\varepsilon}}$$

and $\|S_{j-3}(\nabla p)\|_{L^\infty} \lesssim 2^j \|p\|_{L^\infty}$, we obtain

$$(2.34) \quad \begin{aligned} \|A_2\|_{L^2} &\lesssim \sum_j 2^{-j\varepsilon} \|V\|_{C_*^{1+\varepsilon}} \|p\|_{L^\infty} \|u\|_{L^2} \\ &\lesssim \mathcal{M}_0^0(p) \|V\|_{C_*^{1+\varepsilon}} \|u\|_{L^2}. \end{aligned}$$

We now estimate $A_1 = A_{11} + A_{12}$, with

$$(2.35) \quad \begin{cases} A_{11} := \sum_j S_{j-3} \{S_{j-3}(V) \cdot \nabla p\} \Delta_j u, \\ A_{12} := \sum_j \{[S^{j-3}(V), S_{j-3}] \nabla p\} \Delta_j u. \end{cases}$$

Write $S_{j-3}(V) = V - S^{j-3}(V)$, to obtain

$$\begin{aligned} A_{11} &= \sum_j S_{j-3}(V \cdot \nabla p) \Delta_j u - \sum_j S_{j-3} \{S^{j-3}(V) \cdot \nabla p\} \Delta_j u \\ &= T_{V \cdot \nabla p} u + I + II \end{aligned}$$

where

$$I = - \sum_j (\nabla \cdot S_{j-3} \{S^{j-3}(V)p\}) \Delta_j u, \quad II = \sum_j S_{j-3} \{S^{j-3}(\nabla \cdot V)p\} \Delta_j u.$$

Then

$$\begin{aligned} \|I\|_{L^2} &\lesssim \sum_j 2^j \|S^{j-3}(V)p\|_{L^\infty} \|\Delta_j u\|_{L^2} \\ &\lesssim \sum_j 2^j 2^{-j(1+\varepsilon)} \|V\|_{C_*^{1+\varepsilon}} \|p\|_{L^\infty} \|u\|_{L^2} \\ &\lesssim \|V\|_{C_*^{1+\varepsilon}} \|p\|_{L^\infty} \|u\|_{L^2}. \end{aligned}$$

Moreover,

$$\begin{aligned} \|II\|_{L^2} &\lesssim \sum_j \|S^{j-3}(\nabla V)\|_{L^\infty} \|p\|_{L^\infty} \|\Delta_j u\|_{L^2} \\ &\lesssim \|V\|_{C_*^{1+\varepsilon}} \|p\|_{L^\infty} \|u\|_{L^2}. \end{aligned}$$

Therefore

$$(2.36) \quad A_{11} = T_{V \cdot \nabla p} u + Ru, \quad R \in \mathcal{R}.$$

In order to estimate A_{12} , note that one can replace ∇p by $\tilde{S}_{j-3}(\nabla p)$ where $\tilde{S}_{j-3} = \tilde{\psi}(2^{-(j-3)}D)$ for some function $\tilde{\psi} \in C_0^\infty(\mathbf{R}^d)$ such that $\tilde{\psi}(\xi) = 1$ for $|\xi| \leq 2$. Next, observe that

$$A_{12} = \sum_j \{[S^{j-3}(V), S_{j-3}] \nabla \tilde{S}_{j-3}(p)\} \Delta_j u = \sum_j w_j,$$

where w_j is spectrally supported in an annulus $\{c_1 2^j \leq |\xi| \leq c_2 2^j\}$, $c_j > 0$. These annuli have only finite overlap, thus by Plancherel we have

$$\begin{aligned} \|A_{12}\|_{L^2}^2 &\lesssim \sum_j \left\| \{[S^{j-3}(V), S_{j-3}] \nabla \tilde{S}_{j-3}(p)\} \Delta_j u \right\|_{L^2}^2 \\ &\lesssim \sum_j 2^{-2j} \|V\|_{C_*^{1+\varepsilon}}^2 2^{2j} \|p\|_{L^\infty}^2 \|\Delta_j u\|_{L^2}^2 \\ &\lesssim \|V\|_{C_*^{1+\varepsilon}} \|p\|_{L^\infty} \|u\|_{L^2}, \end{aligned}$$

where we used the fact that the commutator $[S^{j-3}(V), S_{j-3}]$ is of order -1 (uniformly in j), since $V \in C^0([0, T]; W^{1, \infty})$. It follows that $A_{12} = Ru$ with $R \in \mathcal{R}$. Consequently, we deduce from (2.35) and (2.36) that

$$A_1 = T_{V \cdot \nabla p} u + Ru, \quad R \in \mathcal{R}.$$

It thus follows from (2.33) and (2.34) that

$$A = T_{V \cdot \nabla p} u + Ru, \quad R \in \mathcal{R}.$$

It remains to estimate the term B introduced in (2.32). Again, we split this term as follows:

$$\begin{aligned} B &= V \cdot (T_p \nabla u) = V \cdot \sum_j S_{j-3}(p) \nabla \Delta_j u \\ &= \sum_j S_{j-3}(V) S_{j-3}(p) \Delta_j \nabla u + \sum_j S^{j-3}(V) S_{j-3}(\nabla p) \Delta_j \nabla u =: B_1 + B_2. \end{aligned}$$

We have

$$\begin{aligned} \|B_2\|_{L^2} &\leq \sum_j \|S^{j-3}(V)\|_{L^\infty} \|S_{j-3}(p)\|_{L^\infty} \|\Delta_j \nabla u\|_{L^2} \\ &\lesssim \sum_j 2^{-j(1+\varepsilon)} \|V\|_{C_*^{1+\varepsilon}} 2^j \|p\|_{L^\infty} \|u\|_{L^2}. \end{aligned}$$

and hence $B_2 = Ru$ with $R \in \mathcal{R}$. To deal with the term B_1 , let us introduce

$$(2.37) \quad C := T_p T_V \cdot \nabla u = \sum_j S_{j-3}(p) \Delta_j \sum_k S_{k-3}(V) \cdot \nabla \Delta_k u.$$

Since the spectrum of $S_{k-3}(V) \cdot \nabla \Delta_k u$ is contained in $\{(3/8)2^k \leq |\xi| \leq (2+1/8)2^k\}$, the term $\Delta_j(S_{k-3}(V) \cdot \nabla \Delta_k u)$ vanishes unless $|k-j| \leq 3$. On the other hand, for $|k-j| \leq 3$, $S_{k-3}(V) - S_{j-3}(V) = \pm \sum_{\ell=-N_0}^{N_0} \Delta_{\ell+j} V$, and hence we can write C under the form

$$C = C_1 + C_2 = C_1 + \sum_j S_{j-3}(p) \Delta_j \left\{ S_{j-3}(V) \cdot \sum_{|k-j| \leq 3} \nabla \Delta_k u \right\}$$

where C_1 is given by

$$C_1 = \sum_j S_{j-3}(p) \Delta_j \sum_{i=1}^3 \sum_{\ell=-1}^{i-2} \left\{ \Delta_{\ell+j}(V) \nabla \Delta_{i+j}(u) - \Delta_{\ell+j-i}(V) \nabla \Delta_{j-i}(u) \right\},$$

so that

$$\|C_1\|_{L^2} \lesssim \sum_j \|p\|_{L^\infty} 2^{-j(1+\varepsilon)} \|V\|_{C_*^{1+\varepsilon}} 2^j \|u\|_{L^2},$$

which implies that $C_1 = Ru$ with $R \in \mathcal{R}$. To estimate C_2 , as before we write $C_2 = C_{21} + C_{22}$ where

$$\begin{aligned} C_{21} &:= \sum_j S_{j-3}(p) [\Delta_j, S_{j-3}(V)] \cdot \sum_{|k-j| \leq 3} \nabla \Delta_k u, \\ C_{22} &:= \sum_j S_{j-3}(p) S_{j-3}(V) \cdot \Delta_j \sum_{|k-j| \leq 3} \nabla \Delta_k u, \end{aligned}$$

where (using frequency localization in dyadic annuli and Plancherel formula)

$$\|C_{21}\|_{L^2}^2 \lesssim \sum_j \|p\|_{L^\infty}^2 2^{-2j} \|V\|_{W^{1, \infty}}^2 2^{2j} \sum_{|k-j| \leq 3} \|\Delta_k u\|_{L^2}^2 \lesssim \|p\|_{L^\infty} \|V\|_{C_*^{1+\varepsilon}} \|u\|_{L^2}.$$

On the other hand, since $\Delta_j \sum_{|k-j|\leq 3} \Delta_k = \Delta_j$, we have

$$C_{22} = \sum_j S_{j-3}(V) S_{j-3}(p) \nabla \Delta_j u = B_1.$$

We thus end up with

$$(2.38) \quad II = T_p T_V \cdot \nabla u + Ru, \quad R \in \mathcal{R}.$$

It follows from (2.32) and (2.38) that

$$(2.39) \quad (\partial_t + V \cdot \nabla) T_p u = T_p (\partial_t + T_V \cdot \nabla) u + T_{\partial_t p + V \cdot \nabla p} u + Ru, \quad R \in \mathcal{R}.$$

The symbolic calculus shows that $T_{\partial_t p + V \cdot \nabla p} \in \mathcal{R}$, which proves (2.31) and concludes the proof of the first step.

Step 2 : Higher order paraproducts. We now assume that $p(t, x, \xi) = a(t, x)h(\xi)$ where $h(\xi) = |\xi|^m \tilde{h}(\xi/|\xi|)$ with $\tilde{h} \in C^\infty(\mathbf{S}^{d-1})$. Then, directly from the definition (2.1), we have $T_p = T_a \psi(D_x) h(D_x)$ where ψ satisfies (2.2). We have

$$[T_p, \partial_t + T_V \cdot \nabla] = [T_a, \partial_t + T_V \cdot \nabla] \psi(D_x) h(D_x) + T_a [\psi(D_x) h(D_x), \partial_t + T_V \cdot \nabla].$$

The norm from H^m to L^2 of the first term in the right-hand side is estimated by means of the previous step by

$$K \|a\|_{L^\infty} \|V\|_{C_*^{1+\varepsilon}} + \|\partial_t a + V \cdot \nabla a\|_{L^\infty} \|V\|_{L^\infty},$$

while the norm of the second term simplifies to $T_a [\psi(D_x) h(D_x), T_V \cdot \nabla]$ and is easily estimated using (2.4) and (2.5) by

$$\|a\|_{L^\infty} \|V\|_{C_*^{1+\varepsilon}} (\|\tilde{h}\|_{L^\infty} + \|\nabla_\xi h|_{\mathbf{S}^{d-1}}\|_{L^\infty}).$$

Step 3 : Paradifferential operators. Consider an orthonormal basis $(\tilde{h}_\nu)_{\nu \in \mathbf{N}^*}$ of $L^2(\mathbf{S}^{d-1})$ consisting of eigenfunctions of the (self-adjoint) Laplace–Beltrami operator, $\Delta_\omega = \Delta_{\mathbf{S}^{d-1}}$ on $L^2(\mathbf{S}^{d-1})$, i.e. $\Delta_\omega \tilde{h}_\nu = \lambda_\nu^2 \tilde{h}_\nu$. By the Weyl formula, we know that $\lambda_\nu \sim c\nu^{\frac{1}{d}}$. Setting $h_\nu(\xi) = |\xi|^m \tilde{h}_\nu(\omega)$, $\omega = \xi/|\xi|$, $\xi \neq 0$, we can write

$$p(t, x, \xi) = \sum_{\nu \in \mathbf{N}^*} a_\nu(t, x) h_\nu(\xi) \quad \text{where} \quad a_\nu(t, x) = \int_{\mathbf{S}^{d-1}} p(t, x, \omega) \overline{\tilde{h}_\nu(\omega)} d\omega.$$

Since

$$\lambda_\nu^{2k} a_\nu(t, x) = \int_{\mathbf{S}^{d-1}} \Delta_\omega^k p(t, x, \omega) \overline{\tilde{h}_\nu(\omega)} d\omega,$$

we deduce

$$(2.40) \quad \sup_{t \in I} \|a_\nu(t)\|_{L^\infty} \leq C \lambda_\nu^{-\frac{3d}{2}-1} \mathcal{M}_0^m(p).$$

Moreover, there exists a positive constant K such that, for all $\nu \geq 1$,

$$(2.41) \quad \|\tilde{h}_\nu\|_{L^\infty} \leq C \lambda_\nu^{\frac{d-1}{2}}.$$

Now we can write

$$\|[\partial_t + V \cdot \nabla, T_p] u\|_{L^2} \leq \sum_{\nu \in \mathbf{N}^*} \|[\partial_t + V \cdot \nabla, T_{a_\nu h_\nu}] u\|_{L^2}.$$

So using the estimates obtained in the previous steps for every $\nu \geq 1$ and the estimates (2.40)–(2.41), we obtain (2.31), since the sum

$$\sum_\nu \lambda_\nu^{\frac{(d-1)}{2}+1} \lambda_\nu^{-(\frac{3d}{2}+1)} \sim \sum_\nu \nu^{-1-\frac{1}{2d}}$$

is finite. This completes the proof of the lemma. \square

We have also a Sobolev analogue of Lemma 2.16 which can be proved similarly.

LEMMA 2.17. *Let $s > 1 + d/2$ and $V \in C^0([0, T]; H^s(\mathbf{R}^d))$. There exists a positive constant K such that for any symbol $p = p(t, x, \xi)$ which is homogeneous in ξ of order $m \in \mathbf{R}$ and all $u \in C^0([0, T]; H^{s+m}(\mathbf{R}^d))$,*

$$\begin{aligned} & \left\| [T_p, \partial_t + T_V \cdot \nabla] u(t) \right\|_{H^s(\mathbf{R}^d)} \\ & \leq K \{ \mathcal{M}_0^m(p) \|V(t)\|_{H^s} + \mathcal{M}_0^m(\partial_t p + V \cdot \nabla p) \|V(t)\|_{L^\infty} \} \|u(t)\|_{H^{s+m}(\mathbf{R}^d)}. \end{aligned}$$

2.5. Parabolic evolution equation. Consider the parabolic evolution equation

$$\partial_z w + |D_x| w = 0,$$

where $z \in \mathbf{R}$ and $x \in \mathbf{R}^d$. By using the Fourier transform, one easily checks that

$$(2.42) \quad \sup_{z \in [0, 1]} \|w(z)\|_{H^r} + \left(\int_0^1 \|w(z)\|_{H^{r+\frac{1}{2}}}^2 dz \right)^{\frac{1}{2}} \leq K \|w(0)\|_{H^r},$$

and, for $r \in]0, +\infty[\setminus \mathbf{N}$,

$$(2.43) \quad \|w(z)\|_{C^0([0, 1]; C_*^r)} \leq K \|w(0)\|_{C_*^r}.$$

The purpose of this section is to prove similar results when the constant coefficient operator $|D_x|$ is replaced by an elliptic paradifferential operator T_a of order 1 with regularity C_*^ρ in x for some $\rho > 0$.

2.5.1. *Tangential paradifferential calculus.* Given $I \subset \mathbf{R}$, $z_0 \in I$ and a function $\varphi = \varphi(x, z)$ defined on $\mathbf{R}^d \times I$, we denote by $\varphi(z_0)$ the function $x \mapsto \varphi(x, z_0)$. For $I \subset \mathbf{R}$ and a normed space E , $\varphi \in C_z^0(I; E)$ means that $z \mapsto \varphi(z)$ is a continuous function from I to E . Similarly, for $1 \leq p \leq +\infty$, $\varphi \in L_z^p(I; E)$ means that $z \mapsto \|\varphi(z)\|_E$ belongs to the Lebesgue space $L^p(I)$. We endow the spaces $C_z^0(I; E)$ and $L_z^p(I; E)$ with the usual norms.

In this section, when a and u are symbols and functions depending on z , we still denote by $T_a u$ the function defined by $(T_a u)(z) = T_{a(z)} u(z)$ where $z \in I$ is seen as a parameter. We denote by $\Gamma_\rho^m(\mathbf{R}^d \times I)$ the space of symbols $a = a(z; x, \xi)$ such that $z \mapsto a(z; \cdot)$ is bounded from I into the space $\Gamma_\rho^m(\mathbf{R}^d)$ introduced in Definition 2.3. This space is equipped with the semi-norm

$$(2.44) \quad \mathcal{M}_\rho^m(a) = \sup_{z \in I} \sup_{|\alpha| \leq \frac{3d}{2} + \rho + 1} \sup_{|\xi| \geq 1/2} \left\| (1 + |\xi|)^{|\alpha| - m} \partial_\xi^\alpha a(z; \cdot, \xi) \right\|_{W^{\rho, \infty}(\mathbf{R}^d)}.$$

2.5.2. *Estimates in Sobolev spaces.* Given $\mu \in \mathbf{R}$ we define the spaces

$$(2.45) \quad \begin{aligned} X^\mu(I) &= C_z^0(I; H^\mu(\mathbf{R}^d)) \cap L_z^2(I; H^{\mu+\frac{1}{2}}(\mathbf{R}^d)), \\ Y^\mu(I) &= L_z^1(I; H^\mu(\mathbf{R}^d)) + L_z^2(I; H^{\mu-\frac{1}{2}}(\mathbf{R}^d)). \end{aligned}$$

PROPOSITION 2.18. *Let $r \in \mathbf{R}$, $\rho \in (0, 1)$, $J = [z_0, z_1] \subset \mathbf{R}$ and let $p \in \Gamma_\rho^1(\mathbf{R}^d \times J)$ satisfying*

$$\operatorname{Re} p(z; x, \xi) \geq c |\xi|,$$

for some positive constant c . Then for any $f \in Y^r(J)$ and $w_0 \in H^r(\mathbf{R}^d)$, there exists $w \in X^r(J)$ solution of the parabolic evolution equation

$$(2.46) \quad \partial_z w + T_p w = f, \quad w|_{z=z_0} = w_0,$$

satisfying

$$\|w\|_{X^r(J)} \leq K \left\{ \|w_0\|_{H^r} + \|f\|_{Y^r(J)} \right\},$$

for some positive constant K depending only on r, ρ, c and $\mathcal{M}_\rho^1(p)$. Furthermore, this solution is unique in $X^s(J)$ for any $s \in \mathbf{R}$.

PROOF. Let $r \in \mathbf{R}$. Denote by $\langle \cdot, \cdot \rangle_{H^r}$ the scalar product in $H^r(\mathbf{R}^d)$ and chose F_1 and F_2 such that $f = F_1 + F_2$ with

$$\|F_1\|_{L_z^1(J; H^r)} + \|F_2\|_{L^2(J; H^{r-\frac{1}{2}})} \leq \|f\|_{Y^r(J)} + \delta, \quad \delta > 0.$$

Let us consider for $\varepsilon > 0$ the equation

$$(2.47) \quad \partial_z w_\varepsilon + \varepsilon(-\Delta + \text{Id})w_\varepsilon + T_p w_\varepsilon = f, \quad w_\varepsilon|_{z=z_0} = w_0.$$

Then standard methods in parabolic equations show that for any $z_1 > z_0$, this equation have a unique solution in

$$C^0([z_0, z_1]; H^r(\mathbf{R}^d)) \cap L^2((z_0, z_1); H^{r+2}(\mathbf{R}^d))$$

(here we only used that T_p is a Sobolev first order operator). To pass to the limit $\varepsilon \rightarrow 0$, we need to establish uniform estimates with respect to ε . Taking the scalar product in H^r , directly from (2.47), we obtain

$$\begin{aligned} \frac{1}{2} \frac{d}{dz} \|w_\varepsilon(z)\|_{H^r}^2 + \varepsilon \langle (-\Delta + \text{Id})w_\varepsilon(z), w_\varepsilon(z) \rangle_{H^r} + \text{Re} \langle T_p(z)w_\varepsilon(z), w_\varepsilon(z) \rangle_{H^r} \\ \leq \|F_1(z)\|_{H^r} \|w_\varepsilon(z)\|_{H^r} + \|F_2(z)\|_{H^{r-\frac{1}{2}}} \|w_\varepsilon(z)\|_{H^{r+\frac{1}{2}}}. \end{aligned}$$

It follows from Gårding's inequality (see [43, Section 6.3.2]) that there exist two constants $C_1, C_2 > 0$ depending only on $\mathcal{M}_\rho^1(p)$ such that for any $u \in H^r$,

$$\text{Re} \langle T_p(z)u(z), u(z) \rangle_{H^r} \geq C_1 \|u(z)\|_{H^{r+\frac{1}{2}}}^2 - C_2 \|u(z)\|_{H^{r+\frac{1-\rho}{2}}}^2,$$

for each fixed $z \in J$. Therefore, we obtain

$$\begin{aligned} \frac{1}{2} \frac{d}{dz} \|w_\varepsilon(z)\|_{H^r}^2 + \varepsilon \langle (-\Delta + \text{Id})w_\varepsilon(z), w_\varepsilon(z) \rangle_{H^r} + C_1 \|w_\varepsilon(z)\|_{H^{r+\frac{1}{2}}}^2 \\ \leq \|F_1(z)\|_{H^r} \|w_\varepsilon(z)\|_{H^r} + \|F_2(z)\|_{H^{r-\frac{1}{2}}} \|w_\varepsilon(z)\|_{H^{r+\frac{1}{2}}} + C_2 \|w_\varepsilon(z)\|_{H^{r+\frac{1-\rho}{2}}}^2. \end{aligned}$$

Integrating in z we obtain that, for all $z \in [z_0, z_1]$,

$$\begin{aligned} A(z) := \frac{1}{2} \left\{ \|w_\varepsilon(z)\|_{H^r}^2 - \|w_\varepsilon(z_0)\|_{H^r}^2 \right\} + \varepsilon \int_{z_0}^z \|w_\varepsilon(z')\|_{H^{r+1}}^2 dz' \\ + C_1 \int_{z_0}^z \|w_\varepsilon(z')\|_{H^{r+\frac{1}{2}}}^2 dz' \end{aligned}$$

is bounded by

$$\begin{aligned} B := \|F_1\|_{L^1(J; H^r)} \|w_\varepsilon\|_{L^\infty(J; H^r)} + \|F_2\|_{L^2(J; H^{r-\frac{1}{2}})} \|w_\varepsilon\|_{L^2(J; H^{r+\frac{1}{2}})} \\ + C_2 \|w_\varepsilon\|_{L^2(J; H^{r+\frac{1-\rho}{2}})}^2. \end{aligned}$$

Since

$$(2.48) \quad \begin{aligned} B \leq 4 \|F_1\|_{L^1(J; H^r)}^2 + \frac{1}{16} \|w_\varepsilon\|_{L^\infty(J; H^r)}^2 \\ + \frac{4}{C_1} \|F_2\|_{L^2(J; H^{r-\frac{1}{2}})}^2 + \frac{C_1}{16} \|w_\varepsilon\|_{L^2(J; H^{r+\frac{1}{2}})}^2 + C_2 \|w_\varepsilon\|_{L^2(J; H^{r+\frac{1-\rho}{2}})}^2 \end{aligned}$$

and since

$$(2.49) \quad \frac{1}{2} \|w_\varepsilon\|_{L^\infty(J; H^r)}^2 - \frac{1}{2} \|w_\varepsilon(z_0)\|_{H^r}^2 + C_1 \|w_\varepsilon\|_{L^2(J; H^{r+\frac{1}{2}})}^2 \leq \sup_{z \in J} A(z) \leq B,$$

one can absorb the second and fourth terms in the right-hand side of (2.48) by the left-hand side of (2.49) to obtain (notice that the constants C, C_1 are uniform with respect to $\varepsilon > 0$),

$$(2.50) \quad \|w_\varepsilon\|_{L^\infty(J; H^r)}^2 + C_1 \|w_\varepsilon\|_{L^2(J; H^{r+\frac{1}{2}})}^2 \leq \|w_0\|_{H^r}^2 \\ + C(\|F_1\|_{L^1(J; H^r)}^2 + \|F_2\|_{L^2(J; H^{r-\frac{1}{2}})}^2 + \|w_\varepsilon\|_{L^2(J; H^{r+\frac{1-\rho}{2}})}^2).$$

Finally, to eliminate the term $\|w_\varepsilon\|_{L^2(J; H^{r+\frac{1-\rho}{2}})}^2$ in the right hand side of (2.50), it is enough to notice that the left hand side controls by interpolation $c \|w_\varepsilon\|_{L^p(J; H^{r+\frac{1-\rho}{2}})}^2$, for some $p > 2$, hence by Hölder in the z variable, there exists $\kappa > 0$ (depending only on p) such that if $|z_0 - z_1| \leq \kappa$, we have

$$C \|w_\varepsilon\|_{L^2(J; H^{r+\frac{1-\rho}{2}})}^2 \leq \frac{1}{2} (\|w_\varepsilon\|_{L^\infty(J; H^r)}^2 + C_1 \|w_\varepsilon\|_{L^2(J; H^{r+\frac{1}{2}})}^2).$$

we consequently obtain

$$(2.51) \quad \|w_\varepsilon\|_{L^\infty(J; H^r)}^2 + C_1 \|w_\varepsilon\|_{L^2(J; H^{r+\frac{1}{2}})}^2 \\ \leq 2 \|w_0\|_{H^r}^2 + C(\|F_1\|_{L^1(J; H^r)}^2 + \|F_2\|_{L^2(J; H^{r-\frac{1}{2}})}^2).$$

We can now iterate the estimate between $z_0 + \kappa$ and $z_0 + 2\kappa, \dots$ to get rid of the assumption $|z_1 - z_0| \leq \kappa$ (and of course the constants will depend on z_1). By using the equation, we obtain now that sequence (w_ε) is bounded in

$$L^\infty(J; H^r) \cap L^2(J; H^{r+\frac{1}{2}}) \cap C^1(J; H^{r-2}).$$

It follows from the Banach–Alaoglu theorem that, up to a subsequence, (w_ε) converges in the sense of distributions to $w \in X^r(J)$, which satisfies the equation $\partial_z w + T_p w = f$. Then $\partial_z w$ belongs to $Y^r(J)$ which implies that w belongs to $C^0([z_0, z_1]; H^r(\mathbf{R}^d))$. Moreover, by the Ascoli theorem, up to a subsequence, (w_ε) converges in $C^0([z_0, z_1]; H_{loc}^{r-\mu})$ for some $\mu > 0$. Since $w_\varepsilon|_{z=0} = w_0$ we obtain that $w|_{z=0} = w_0$, which completes the existence part in Proposition 2.18. The proof of uniqueness follows the same steps and we omit it. \square

2.5.3. Estimates in Hölder spaces. The next proposition is the analog in Hölder spaces of the previous result. Here we follow a similar strategy previously used in [48, 40, 5, 1].

PROPOSITION 2.19. *Let $\rho \in (0, 1)$, $J = [z_0, z_1] \subset \mathbf{R}$, $p \in \Gamma_\rho^1(\mathbf{R}^d \times J)$ with the assumption that*

$$\operatorname{Re} p(z; x, \xi) \geq c |\xi|,$$

for some positive constant c . Assume that w solves

$$\partial_z w + T_p w = F_1 + F_2, \quad w|_{z=z_0} = w_0.$$

Then for any $q \in [1, +\infty]$, $(r_0, r) \in \mathbf{R}^2$ with $r_0 < r$, if

$$w \in L^\infty(J; C_*^{r_0}), \quad F_1 \in L^1(J; C_*^r), \quad F_2 \in L^q(J; C_*^{r-1+\frac{1}{q}+\delta}) \quad \text{with } \delta > 0,$$

and $w_0 \in C_*^r(\mathbf{R}^d)$, we have $w \in C^0(J; C_*^r)$ and

$$\|w\|_{C^0(J; C_*^r)} \leq K \left\{ \|w_0\|_{C_*^r} + \|F_1\|_{L^1(J; C_*^r)} + \|F_2\|_{L^q(J; C_*^{r-1+\frac{1}{q}+\delta})} + \|w\|_{L^\infty(J; C_*^{r_0})} \right\},$$

for some positive constant K depending only on $r_0, r, \rho, c, \delta, q$ and $\mathcal{M}_\rho^1(p)$.

PROOF. For this proof, we denote by K various constants which depend only on r_0, r, ρ, c and $M_\rho^1(p)$. Given $y \in J$ introduce the symbol $e = e(y, z; x, \xi)$ defined by

$$e(y, z; x, \xi) = \exp\left(-\int_z^y p(s; x, \xi) ds\right) \quad (z \in [z_0, y]).$$

This symbol satisfies $\partial_z e = ep$, so that

$$\partial_z(T_e w) = (T_{ep} - T_e T_p)w + T_e F, \quad F = F_1 + F_2.$$

Integrating on $[z_0, y]$ the function $\frac{d}{dz} T_{e(y, z, x, \xi)} w(z)$, we find

$$(2.52) \quad T_1 w(y) = T_{e|z=z_0} w_0 + \int_{z_0}^y (T_e F)(z) dz + \int_{z_0}^y (T_{ep} - T_e T_p)(z) w(z) dz.$$

(Notice that the paraproduct T_1 differs from the identity only by a smoothing operator.) Introduce $G(y) = T_{e|z=z_0} w_0 + \int_{z_0}^y (T_e F)(z) dz$ and the operator R defined on functions $u: J \rightarrow C^m(\mathbf{R}^d)$ by

$$(Ru)(y) = \int_{z_0}^y (T_{ep} - T_e T_p)(z) u(z) dz$$

so that $T_1 w = G + Rw$. Now, by a bootstrap argument, to complete the proof we see that it is enough to prove that the function G belongs to $L^\infty(J; C_*^r)$ and that R is a smoothing operator of order $-a$ for some $a > 0$, which means that R maps $L^\infty(J; C_*^t)$ to $L^\infty(J; C_*^{t+a})$. Indeed, by writing

$$w = (I + R + \dots + R^N)G - R^{N+1}w,$$

and choosing N large enough, we can estimate the second term in the right-hand side in $L^\infty(J; C_*^r)$ by means of any $L^\infty(J; C_*^{r_0})$ -norm of w .

In the analysis, we need to take into account how the semi-norms $M_\rho^{-m}(e(z))$ (see Definition 2.3) depend on z . Then the key estimates are stated in the following lemma.

LEMMA 2.20. *For any $m \geq 0$ there exists a positive constant K depending only on $\sup_J M_\rho^1(p(\cdot; x, \xi))$ such that, for all $y \in (0, -z_1]$ and all $z \in [0, y)$,*

$$(2.53) \quad M_\rho^{-m}(e(z)) \leq \frac{K}{(y-z)^m}.$$

This follows easily from the assumptions $p \in \Gamma_\rho^1$, $\operatorname{Re} p(s; x, \xi) \geq c|\xi|$, and the elementary inequalities (valid for any $a \geq 0$)

$$(y-z)^a |\xi|^a \exp((z-y)|\xi|) \lesssim 1.$$

By using the bound (2.53), applied with $m = 0$, it follows from the operator norm estimate (2.5) that, for any $z \leq y$ and any function $f = f(x)$, we have

$$(2.54) \quad \|T_{e(y, z)} f\|_{C_*^r} \lesssim M_0^0(e(y, z)) \|f\|_{C_*^r} \leq K \|f\|_{C_*^r}.$$

This implies that

$$\left\| T_{e|z=z_0} w_0 + \int_{z_0}^y (T_e F_1)(z) dz \right\|_{L^\infty(J; C_*^r)} \leq K \|w_0\|_{C_*^r} + K \|F_1\|_{L^1(J; C_*^r)}.$$

On the other hand, by using the bound (2.53), applied with $m = 1 - \frac{1}{q} - \delta$, we obtain that

$$\left\| \int_{z_0}^y (T_e F_2)(z) dz \right\|_{L^\infty(J; C_*^r)} \leq K \int_{z_0}^y \frac{1}{|y-z|^m} \|F_2(z)\|_{C_*^{r-m}} dz,$$

which implies by Hölder inequality that

$$\|G\|_{L^\infty(J; C_*^r)} \leq K \|w_0\|_{C_*^r} + K \|F_1\|_{L^1(J; C_*^r)} + \|F_2\|_{L^q(J; C_*^{r-1+\frac{1}{q}+\delta})}.$$

It remains to show that R is a smoothing operator. To do that, we first use the operator norm estimate (2.5) (applied with (m, m', ρ) replaced with $(-m, 1, \rho)$) to obtain

$$\|(T_{ep} - T_e T_p)(z)\|_{C_*^t \rightarrow C_*^{t+m-1+\rho}} \lesssim M_\rho^{-m}(e(z)) M_\rho^1(p(z)).$$

Taking $m = 1 - \rho/2$, it follows from the previous bound and Lemma 2.20 that

$$\|(T_{ep} - T_e T_p)v(z)\|_{C_*^{t+\rho/2}} \leq \frac{K}{(y-z)^m} \|v(z)\|_{C_*^t}.$$

Since $0 \leq m < 1$ we have $\int_0^y (y-z)^{-m} dz < +\infty$ and hence

$$(2.55) \quad \|Ru(y)\|_{C_*^{t+\rho/2}} \leq \int_0^y \|(T_{ep} - T_e T_p)u(z)\|_{C_*^{t+\rho/2}} dz \leq K \|u\|_{L^\infty(J; C_*^t)},$$

which completes the proof. \square

3. The Dirichlet-Neumann operator

A notable technical step in the analysis of the Euler equation with free surface consists in describing the Dirichlet-Neumann operator in domains with free boundaries of limited regularity. Here we shall prove some results about elliptic regularity which complement previous works (see Propositions 3.18 and 3.19). To do this we shall use a paradifferential approach.

3.1. Definition and continuity. In this paragraph we recall from [1] the definition of the Dirichlet-Neumann operator under very general assumptions on the bottom. One of the novelty with respect to our previous work is that we first clarify the regularity assumptions: assuming only that $\eta \in W^{1,\infty}(\mathbf{R}^d)$ (which is satisfied if $\eta \in H^{s+\frac{1}{2}}(\mathbf{R}^d)$ with $s > 1/2+d/2$) and $f \in H^{\frac{1}{2}}(\mathbf{R}^d)$, we show how to define $G(\eta)\psi$ and prove that the map

$$\psi \in H^{\frac{1}{2}}(\mathbf{R}^d) \mapsto G(\eta)\psi \in H^{-\frac{1}{2}}(\mathbf{R}^d)$$

is continuous. Our second contribution is to prove that the map $\eta \mapsto G(\eta)$ is Lipschitz (in a proper topology). Finally, we prove also that in some weak sense, the Dirichlet-Neumann operator thus defined is a local operator (see Theorem 3.9). We also refer to chapter 3 in [38] for an introduction to the analysis of the Dirichlet-Neumann operator.

The goal is to study the boundary value problem

$$(3.1) \quad \Delta_{x,y}\phi = 0, \quad \phi|_\Sigma = f, \quad \partial_n \phi|_\Gamma = 0.$$

See §1.1 for the definitions of Ω, Σ, Γ . Since we make no assumption on Γ , the definition of ϕ requires some care. We recall here the definition of ϕ as given in [1].

NOTATION 3.1. Denote by \mathcal{D} the space of functions $u \in C^\infty(\Omega)$ such that $\nabla_{x,y}u \in L^2(\Omega)$. We then define \mathcal{D}_0 as the subspace of functions $u \in \mathcal{D}$ such that u is equal to 0 in a neighborhood of the top boundary Σ .

PROPOSITION 3.2 ([1, Proposition 2.2]). *There exists a positive weight $g \in L_{loc}^\infty(\Omega)$, equal to 1 near the top boundary of Ω and a constant $C > 0$ such that for all $u \in \mathcal{D}_0$,*

$$(3.2) \quad \int_\Omega g(x,y)|u(x,y)|^2 dx dy \leq C \int_\Omega |\nabla_{x,y}u(x,y)|^2 dx dy.$$

DEFINITION 3.3. Denote by $H^{1,0}(\Omega)$ the space of functions u on Ω such that there exists a sequence $(u_n) \in \mathcal{D}_0$ such that,

$$\nabla_{x,y} u_n \rightarrow \nabla_{x,y} u \text{ in } L^2(\Omega, dx dy), \quad u_n \rightarrow u \text{ in } L^2(\Omega, g(x,y) dx dy).$$

We endow the space $H^{1,0}$ with the norm

$$\|u\| = \|\nabla_{x,y} u\|_{L^2(\Omega)}.$$

Let us recall that the space $H^{1,0}(\Omega)$ is a Hilbert space (see [1]). For later purpose, we need to ensure that the functions having compact support in the x variable (at least near the surface Σ) are dense in $H^{1,0}(\Omega)$. By regularizing the function η (see Remark 3.7) it is easy to see that there exists $\eta_* \in C_b^\infty(\mathbf{R}^d)$ such that $\eta - \frac{h}{20} > \eta_*$ and

$$\{(x, y) \in \mathbf{R}^d \times \mathbf{R}; \eta_*(x) < y < \eta(x)\} \subset \Omega.$$

LEMMA 3.4. The set

$$\tilde{\mathcal{D}}_0 = \left\{ u \in \mathcal{D}_0; \text{supp}(u) \cap \{(x, y); -\eta_* + \frac{h}{30} < y < \eta\} \text{ is compact} \right\}$$

is dense in $H^{1,0}(\Omega_n)$.

PROOF. Let $u \in \mathcal{D}_0$, and $\zeta \in C^\infty(\mathbf{R})$ equal to 0 for $z < 0$ and to 1 for $z > h/30$. Then according to Proposition 3.2, we have $\zeta(y - \eta_*)u \in \mathcal{D}_0$ and $(1 - \zeta(y - \eta_*))u \in \mathcal{D}_0 \cap H_0^1(\Omega)$ (where $H_0^1(\Omega)$ is the usual Sobolev space). Let $v_n \in C_0^\infty(\Omega)$ which converges to $(1 - \zeta(y - \eta_*))u$ in $H_0^1(\Omega)$ (and hence in $H^{1,0}(\Omega)$). We get that $\zeta(y - \eta_*)u + v_n \in \tilde{\mathcal{D}}_0$ and converges to u in $H^{1,0}(\Omega)$. \square

We are able now to define the Dirichlet-Neumann operator. Let $f \in H^{\frac{1}{2}}(\mathbf{R}^d)$. We first define an H^1 lifting of f in Ω . To do so let $\chi_0 \in C^\infty(\mathbf{R})$ be such that $\chi_0(z) = 1$ if $z \geq -\frac{1}{2}$ and $\chi_0(z) = 0$ if $z \leq -1$. We set

$$\psi_1(x, z) = \chi_0(z) e^{z\langle D_x \rangle} f(x), \quad x \in \mathbf{R}^d, z \leq 0.$$

By the usual property of the Poisson kernel we have

$$\|\nabla_{x,z} \psi_1\|_{L^2([-1,0] \times \mathbf{R}^d)} \leq C \|f\|_{H^{\frac{1}{2}}(\mathbf{R}^d)}.$$

Then we set

$$\underline{\psi}(x, y) = \psi_1\left(x, \frac{y - \eta(x)}{h}\right), \quad (x, y) \in \Omega.$$

This is well defined since $\Omega \subset \{(x, y) : y < \eta(x)\}$. Moreover since the bottom Γ is contained in $\{(x, y) : y < \eta(x) - h\}$, we see that $\underline{\psi}$ vanishes identically near Γ .

Now we have obviously $\underline{\psi}|_\Sigma = f$ and since $\nabla \eta \in L^\infty(\mathbf{R}^d)$, an easy computation shows that $\underline{\psi} \in H^1(\Omega)$ and

$$(3.3) \quad \|\underline{\psi}\|_{H^1(\Omega)} \leq K(1 + \|\eta\|_{W^{1,\infty}}) \|f\|_{H^{\frac{1}{2}}(\mathbf{R}^d)}.$$

Then the map

$$v \mapsto - \int_{\Omega} \nabla_{x,y} \underline{\psi} \cdot \nabla_{x,y} v \, dx dy$$

is a bounded linear form on $H^{1,0}(\Omega)$. It follows from the Riesz theorem that there exists a unique $u \in H^{1,0}(\Omega)$ such that

$$(3.4) \quad \forall v \in H^{1,0}(\Omega), \quad \int_{\Omega} \nabla_{x,y} u \cdot \nabla_{x,y} v \, dx dy = - \int_{\Omega} \nabla_{x,y} \underline{\psi} \cdot \nabla_{x,y} v \, dx dy.$$

Then u is the variational solution to the problem

$$-\Delta_{x,y}u = \Delta_{x,y}\underline{\psi} \quad \text{in } \mathcal{D}'(\Omega), \quad u|_{\Sigma} = 0, \quad \partial_n u|_{\Gamma} = 0,$$

the latter condition being justified as soon as the bottom Γ is regular enough.

LEMMA 3.5. *The function $\phi = u + \underline{\psi}$ constructed by this procedure is independent on the choice of the lifting function $\underline{\psi}$ as long as it remains bounded in $H^1(\Omega)$ and vanishes near the bottom.*

PROOF. Consider two functions constructed by this procedure, $\phi_k = u_k + \underline{\psi}_k$, $k = 1, 2$. Then, by standard density arguments, since $\underline{\psi}_1 - \underline{\psi}_2$ vanishes at the top boundary Σ and in a neighborhood of the bottom Γ , there exists a sequence of functions $\psi_n \in C_0^\infty(\Omega)$ supported in a fixed Lipschitz domain $\tilde{\Omega} \subset \Omega$ tending to $\underline{\psi}_1 - \underline{\psi}_2$ in $H_0^1(\tilde{\Omega})$ and hence also in $H^{1,0}(\Omega)$. As a consequence, $\underline{\psi}_1 - \underline{\psi}_2 \in H^{1,0}(\Omega)$ and the function $\phi = \phi_1 - \phi_2$ is the unique (trivial) solution in $H^{1,0}(\Omega)$ of the equation $\Delta_{x,y}\phi = 0$ given by the Riesz Theorem. \square

DEFINITION 3.6. *We shall say that the function $\phi = u + \underline{\psi}$ constructed by the above procedure is the variational solution of (3.1). It satisfies*

$$(3.5) \quad \int_{\Omega} |\nabla_{x,y}\phi|^2 dx dy \leq K \|f\|_{H^{\frac{1}{2}}(\mathbf{R}^d)}^2,$$

for some constant K depending only on the Lipschitz norm of η .

Formally the Dirichlet-Neumann operator is defined by

$$(3.6) \quad G(\eta)\psi = \sqrt{1 + |\nabla\eta|^2} \partial_n \phi \Big|_{y=\eta(x)} = [\partial_y \phi - \nabla\eta \cdot \nabla\phi] \Big|_{y=\eta(x)}$$

3.1.1. *Straightenning the free boundary.* We begin by straightening the boundary. In this paragraph, we fix $s > \frac{1}{2} + \frac{d}{2}$.

We shall assume here that one can find a function η_* such that

$$(3.7) \quad \begin{cases} (i) & \eta_* + \frac{h}{4} \in H^\infty(\mathbf{R}^d), \\ (ii) & \eta(x) - \eta_*(x) = \frac{h}{4} + g, \quad \|g\|_{H^{s-\frac{1}{2}}(\mathbf{R}^d)} \leq \frac{h}{5}, \\ (iii) & \Gamma \subset \{(x, y) \in \mathcal{O} : y < \eta_*(x)\}. \end{cases}$$

REMARK 3.7. Assume that we have a function depending smoothly on the time, $\eta \in C^0([0, T], H^{s+\frac{1}{2}}(\mathbf{R}^d))$, such that $\eta|_{t=0} = \eta_0$ and satisfying condition (1.2) with $\frac{h}{2}$ (such as a solution in Theorem 1.2). Then one can construct $\eta_* = \eta_*(x)$ satisfying (i), (iii) in (3.7) and for some $T' \leq T$

$$(ii)' \quad \eta(t, x) - \eta_*(x) = \frac{h}{4} + g(t, x), \quad \|g\|_{L^\infty([0, T'], H^{s-\frac{1}{2}}(\mathbf{R}^d))} \leq \frac{h}{5}.$$

Indeed set

$$\eta_*(x) = -\frac{h}{4} + e^{-\nu\langle D_x \rangle} \eta_0(x)$$

where $\nu > 0$ is chosen such that $\nu \|\eta_0\|_{H^{s+\frac{1}{2}}(\mathbf{R}^d)} \leq \frac{h}{10}$. Then chose T' such that

$$\|\eta(t, \cdot) - \eta_0\|_{H^{s-\frac{1}{2}}(\mathbf{R}^d)} \leq \frac{h}{10}, \quad \forall t \in [0, T'],$$

and write

$$\eta(t, x) - \eta_*(x) = \eta(t, x) - \eta_0(x) + \eta_0(x) - e^{-\nu\langle D_x \rangle} \eta_0 + \frac{h}{4}.$$

Then (ii)' follows from the estimate

$$\|\eta_0(x) - e^{-\nu\langle D_x \rangle} \eta_0\|_{H^{s-\frac{1}{2}}(\mathbf{R}^d)} \leq \nu \|\eta_0\|_{H^{s+\frac{1}{2}}(\mathbf{R}^d)} \leq \frac{h}{10},$$

and (iii) is a consequence of (ii)'. Indeed for $t \in [0, T']$ we have

$$\eta(t, x) - \eta_*(x) \leq \frac{h}{4} + \|g\|_{L^\infty([0, T'] \times \mathbf{R}^d)} \leq \frac{h}{4} + \|g\|_{L^\infty([0, T'], H^{s-\frac{1}{2}}(\mathbf{R}^d))} \leq \frac{h}{2},$$

therefore

$$\Gamma \subset \{(x, y) : y < \eta(t, x) - \frac{h}{2}\} \subset \{(x, y) : y < \eta_*(x)\}.$$

In what follows we shall set

$$(3.8) \quad \begin{cases} \Omega_1 = \{(x, y) : x \in \mathbf{R}^d, \eta_*(x) < y < \eta(x)\}, \\ \Omega_2 = \{(x, y) \in \mathcal{O} : y \leq \eta_*(x)\}, \\ \Omega = \Omega_1 \cup \Omega_2. \end{cases}$$

and

$$(3.9) \quad \begin{cases} \tilde{\Omega}_1 = \{(x, z) : x \in \mathbf{R}^d, z \in I\}, \quad I = (-1, 0), \\ \tilde{\Omega}_2 = \{(x, z) \in \mathbf{R}^d \times (-\infty, -1] : (x, z+1 + \eta_*(x)) \in \Omega_2\}, \\ \tilde{\Omega} = \tilde{\Omega}_1 \cup \tilde{\Omega}_2. \end{cases}$$

Following Lannes ([39]), consider the map $(x, z) \mapsto \rho(x, z)$ from $\tilde{\Omega}$ to \mathbf{R}^{d+1} defined by

$$(3.10) \quad \begin{cases} \rho(x, z) = (1+z)e^{\delta z \langle D_x \rangle} \eta(x) - z\eta_*(x) & \text{if } (x, z) \in \tilde{\Omega}_1, \\ \rho(x, z) = z+1 + \eta_*(x) & \text{if } (x, z) \in \tilde{\Omega}_2, \end{cases}$$

where δ is chosen such that

$$\delta \|\eta\|_{H^{s+\frac{1}{2}}(\mathbf{R}^d)} := \delta_0 \quad \text{is small enough.}$$

Notice that ρ is Lipschitz on $\tilde{\Omega}$. Moreover since $s > \frac{1}{2} + \frac{d}{2}$, there exists a constant $C > 0$ such that

$$(3.11) \quad \begin{aligned} \left\| \partial_z \rho - \frac{h}{4} \right\|_{L^\infty(I, H^{s-\frac{1}{2}}(\mathbf{R}^d))} &\leq C\delta \left(\|\eta\|_{H^{s+\frac{1}{2}}(\mathbf{R}^d)} + \|\eta_*\|_{H^{s+\frac{1}{2}}(\mathbf{R}^d)} \right) \\ \|\nabla_x \rho\|_{L^\infty(I, H^{s-\frac{1}{2}}(\mathbf{R}^d))} &\leq C \left(\|\eta\|_{H^{s+\frac{1}{2}}(\mathbf{R}^d)} + \|\eta_*\|_{H^{s+\frac{1}{2}}(\mathbf{R}^d)} \right) \end{aligned}$$

and

$$(3.12) \quad \begin{aligned} \left\| \partial_z \rho - \frac{h}{4} \right\|_{L^2(I, H^s(\mathbf{R}^d))} &\leq C\delta \left(\|\eta\|_{H^{s+\frac{1}{2}}(\mathbf{R}^d)} + \|\eta_*\|_{H^{s+\frac{1}{2}}(\mathbf{R}^d)} \right) \\ \|\nabla_x \rho\|_{L^2(I, H^s(\mathbf{R}^d))} &\leq C \left(\|\eta\|_{H^{s+\frac{1}{2}}(\mathbf{R}^d)} + \|\eta_*\|_{H^{s+\frac{1}{2}}(\mathbf{R}^d)} \right) \end{aligned}$$

from which, taking δ_0 small enough, we deduce

$$(3.13) \quad \begin{cases} (i) & \partial_z \rho(x, z) \geq \min(1, \frac{h}{5}), \quad \forall (x, z) \in \tilde{\Omega}, \\ (ii) & \|\nabla_{x,z} \rho\|_{L^\infty(\tilde{\Omega})} \leq \mathcal{F}(\|\eta\|_{H^{s+\frac{1}{2}}(\mathbf{R}^d)}). \end{cases}$$

It follows from (3.13) (i) that the map $(x, z) \mapsto (x, \rho(x, z))$ is a C^1 -diffeomorphism from $\tilde{\Omega}$ to Ω . We denote by κ the inverse map of ρ :

$$(x, z) \in \tilde{\Omega}, \quad (x, \rho(x, z)) = (x, y) \Leftrightarrow (x, z) = (x, \kappa(x, y)), \quad (x, y) \in \Omega.$$

We set

$$\tilde{\phi}(x, z) = \phi(x, \rho(x, z)).$$

Then we have

$$(3.14) \quad \begin{cases} (\partial_y \phi)(x, \rho(x, z)) = (\Lambda_1 \tilde{\phi})(x, z), & (\nabla_x \phi)(x, \rho(x, z)) = (\Lambda_2 \tilde{\phi})(x, z), \\ \Lambda_1 = \frac{1}{\partial_z \rho} \partial_z & \Lambda_2 = \nabla_x - \frac{\nabla_x \rho}{\partial_z \rho} \partial_z. \end{cases}$$

If ϕ is a solution of $\Delta_{x,y} \phi = 0$ in Ω then $\tilde{\phi}$ satisfies

$$(\Lambda_1^2 + \Lambda_2^2) \tilde{\phi} = 0 \quad \text{in } \tilde{\Omega}$$

This yields

$$(3.15) \quad (a \partial_z^2 + \Delta_x + b \cdot \nabla_x \partial_z - c \partial_z) \tilde{\phi} = 0,$$

where

$$(3.16) \quad a := \frac{1 + |\nabla_x \rho|^2}{(\partial_z \rho)^2}, \quad b := -2 \frac{\nabla_x \rho}{\partial_z \rho}, \quad c := \frac{1}{\partial_z \rho} (a \partial_z^2 \rho + \Delta_x \rho + b \cdot \nabla_x \partial_z \rho).$$

It will be convenient to have a constant coefficient in front of $\partial_z^2 v$. Dividing (3.15) by a we obtain

$$(3.17) \quad (\partial_z^2 + \alpha \Delta_x + \beta \cdot \nabla_x \partial_z - \gamma \partial_z) \tilde{\phi} = 0,$$

where

$$(3.18) \quad \alpha := \frac{(\partial_z \rho)^2}{1 + |\nabla_x \rho|^2}, \quad \beta := -2 \frac{\partial_z \rho \nabla_x \rho}{1 + |\nabla_x \rho|^2}, \quad \gamma := \frac{1}{\partial_z \rho} (\partial_z^2 \rho + \alpha \Delta_x \rho + \beta \cdot \nabla_x \partial_z \rho).$$

In the coordinates (x, z) , according to (3.6) we have

$$(3.19) \quad G(\eta) \psi = U|_{z=0}, \quad U = \Lambda_1 \tilde{\phi} - \nabla_x \rho \cdot \Lambda_2 \tilde{\phi}.$$

The following remark will be useful in the sequel. We have

$$(3.20) \quad \partial_z U = -\nabla_x ((\partial_z \rho) \Lambda_2 \tilde{\phi}).$$

Indeed we can write

$$\begin{aligned} \partial_z U &= \partial_z \Lambda_1 \tilde{\phi} - \nabla_x \partial_z \rho \cdot \Lambda_2 \tilde{\phi} - \nabla_x \rho \cdot \partial_z \Lambda_2 \tilde{\phi} \\ &= (\partial_z \rho) \Lambda_1^2 \tilde{\phi} - \nabla_x \partial_z \rho \cdot \Lambda_2 \tilde{\phi} + (\partial_z \rho) (\Lambda_2 - \nabla_x) \Lambda_2 \tilde{\phi} \\ &= (\partial_z \rho) (\Lambda_1^2 + \Lambda_2^2) \tilde{\phi} - \nabla_x ((\partial_z \rho) \Lambda_2 \tilde{\phi}). \end{aligned}$$

Since $(\Lambda_1^2 + \Lambda_2^2) \tilde{\phi} = 0$ we obtain (3.20).

3.1.2. Continuity of the Dirichlet Neuman operator.

THEOREM 3.8. *Let $\eta \in W^{1,\infty}(\mathbf{R}^d)$, $f \in H^{\frac{1}{2}}(\mathbf{R}^d)$. In the system of coordinates (x, z) defined above, the variational solution of (3.1), $\tilde{\phi}$, satisfies*

$$(3.21) \quad \tilde{\phi} \in C_z^0([-1, 0]; H^{\frac{1}{2}}(\mathbf{R}^d)) \cap C_z^1([-1, 0]; H^{-\frac{1}{2}}(\mathbf{R}^d)).$$

As a consequence, the map

$$(3.22) \quad \begin{aligned} \psi \in H^{\frac{1}{2}}(\mathbf{R}^d) &\mapsto G(\eta) \psi = \sqrt{1 + |\nabla \eta|^2} \partial_n \phi \Big|_{y=\eta(x)} = [\partial_y \phi - \nabla \eta \cdot \nabla \phi] \Big|_{y=\eta(x)} \\ &= ((1 + |\nabla \eta|^2) \partial_z \tilde{\phi} - \nabla \eta \cdot \nabla \tilde{\phi}) \Big|_{z=0} \end{aligned}$$

is well defined. It furthermore satisfies

$$\|G(\eta) f\|_{H^{-\frac{1}{2}}(\mathbf{R}^d)} \leq \mathcal{F}(\|\eta\|_{W^{1,\infty}}) \|f\|_{H^{\frac{1}{2}}(\mathbf{R}^d)}.$$

The Dirichlet Neumann operator is also weakly continuous. This fact will be used in Section 6 to prove existence of solutions when passing to the weak limits on weakly convergent sequences of suitably regularized systems.

THEOREM 3.9. *Assume that $(\eta_n)_{n \in \mathbf{N}}$ and $(\psi_n)_{n \in \mathbf{N}}$ are two sequences such that*

- i) the sequence $(\eta_n, \psi_n)_{n \in \mathbf{N}}$ is bounded in $W^{1,\infty}(\mathbf{R}^d) \times H^{\frac{1}{2}}(\mathbf{R}^d)$,*
- ii) there exists $\eta \in W^{1,\infty}(\mathbf{R}^d)$ such that η_n converges strongly to η in $W_{loc}^{1,\infty}(\mathbf{R}^d)$,*
- iii) there exists $\psi \in H^{\frac{1}{2}}(\mathbf{R}^d)$ such that $(\psi_n)_{n \in \mathbf{N}}$ converges weakly to ψ in $H^{\frac{1}{2}}(\mathbf{R}^d)$,*
- iv) there exists $\eta_* \in W^{1,\infty}(\mathbf{R}^d)$, $h > 0$ such that*

$$\eta(x) - \frac{h}{2} > \eta_*(x) > \eta(x) - h, \eta_n(x) - \frac{h}{2} > \eta_*(x) > \eta_n(x) - h \quad \forall x \in \mathbf{R}^d.$$

Then the sequence $(G(\eta_n)\psi_n)$ is bounded in $H^{-\frac{1}{2}}(\mathbf{R}^d)$ and converge weakly to $G(\eta)\psi$.

Let us also state the second basic strong continuity of the Dirichlet Neumann operator. Notice that the map $\eta \mapsto G(\eta)$ is strongly non linear and hence strong continuity do not imply weak continuity.

THEOREM 3.10. *There exists a non decreasing function $\mathcal{F}: \mathbf{R}^+ \rightarrow \mathbf{R}^+$ such that, for all $\eta_j \in W^{1,\infty}(\mathbf{R}^d)$, $j = 1, 2$ and all $f \in H^{\frac{1}{2}}(\mathbf{R}^d)$,*

$$\|(G(\eta_1) - G(\eta_2))f\|_{H^{-\frac{1}{2}}} \leq \mathcal{F}(\|(\eta_1, \eta_2)\|_{W^{1,\infty} \times W^{1,\infty}}) \|\eta_1 - \eta_2\|_{W^{1,\infty}} \|f\|_{H^{\frac{1}{2}}}.$$

REMARK 3.11. We shall only prove Theorems 3.9 and 3.10 as the choice $\eta_n = \eta$, $\psi_n = \psi$ in Theorem 3.9 implies Theorem 3.8. On the other hand, the fact that we can pass to the limit in $G(\eta_n)\psi_n$ under convergence assumptions on (η_n, ψ_n) which are only local in space, shows that, in a very weak sense, the Dirichlet–Neumann operator is a local operator.

PROOF OF THEOREM 3.9. Let $M_0 > 0$ be such that for all $n \in \mathbf{N}$,

$$(3.23) \quad \|\psi_n\|_{H^{\frac{1}{2}}(\mathbf{R}^d)} + \|\psi\|_{H^{\frac{1}{2}}(\mathbf{R}^d)} + \|\eta_n\|_{W^{1,\infty}(\mathbf{R}^d)} + \|\eta\|_{W^{1,\infty}(\mathbf{R}^d)} + \|\eta_*\|_{W^{1,\infty}(\mathbf{R}^d)} \leq M_0.$$

Our purpose is to prove that $G(\eta_n)\psi_n$ is well defined and bounded in $H^{-\frac{1}{2}}(\mathbf{R}^d)$ by

$$\mathcal{F}(\|\eta_n\|_{W^{1,\infty}}) \|\psi_n\|_{H^{\frac{1}{2}}(\mathbf{R}^d)}$$

(hence uniformly bounded) and converges weakly to $G(\eta)\psi$ in $H^{-\frac{1}{2}}(\mathbf{R}^d)$. We shall proceed in several steps.

Step 1: preliminaries. We start by straightening the boundaries of the domains Ω_n and Ω using the previous section. We recall that

$$\Omega_n = \{(x, y) \in \mathcal{O} : y < \eta_n(x)\}, \quad \Omega = \{(x, y) \in \mathcal{O} : y < \eta(x)\}.$$

For this purpose we use the diffeomorphisms given by ρ_n (constructed with η_n) and ρ given by (3.10) and we shall use the vector fields $\Lambda_j^n, \Lambda_j, j = 1, 2$ described in (3.14) and we set

$$\Lambda^n = (\Lambda_1^n, \Lambda_2^n) \quad \Lambda = (\Lambda_1, \Lambda_2).$$

We construct now a H^1 -extension of ψ_n . Let $\chi \in C^\infty(\mathbf{R})$, $\chi(z) = 1$ if $z \geq -\frac{1}{2}$ and $\chi(z) = 0$ if $z \leq -1$ and set

$$(3.24) \quad \tilde{\psi}_n(x, z) = \chi(z)e^{z\langle D_x \rangle} \psi_n(x), \quad \tilde{\psi}(x, z) = \chi(z)e^{z\langle D_x \rangle} \psi(x).$$

Then $\tilde{\psi}_n(x, z) \in H^1(\mathbf{R}^d \times I)$ and $\|\tilde{\psi}_n\|_{H^1(\mathbf{R}^d \times I)} \leq C\|\psi_n\|_{H^{\frac{1}{2}}(\mathbf{R}^d)} \leq CM_0$

We make the same construction for ψ . Then it is easy to see that the sequence $(\tilde{\psi}_n)$ converges in $H^1(\tilde{\Omega})$ to $\tilde{\psi}$, Then we set

$$(3.25) \quad \tilde{\phi}_n = \tilde{u}_n + \underline{\tilde{\psi}}_n, \quad \tilde{\phi} = \tilde{u} + \underline{\tilde{\psi}}.$$

According to (3.5) and the assumptions on η_n and η we see easily that this implies

$$(3.26) \quad \|\nabla_{x,z}\tilde{u}_n\|_{L^2(\tilde{\Omega})} \leq M_2, \quad \forall n \in \mathbf{N}.$$

Then (\tilde{u}_n) is a bounded sequence in $H^{1,0}(\tilde{\Omega})$ and therefore that, up to a subsequence, it converges weakly in this space to \tilde{u} .

Step 2: passing to the limit for the variational solutions. Setting $X = (x, z) \in \tilde{\Omega}$ the variational formulation for \tilde{u}_n reads

$$(3.27) \quad \int_{\tilde{\Omega}} \Lambda^n \tilde{u}_n(X) \cdot \Lambda^n \zeta(X) J_n(X) dX = \int_{\tilde{\Omega}} \Lambda^n \underline{\tilde{\psi}}_n(X) \cdot \Lambda^n \zeta(X) J_n(X) dX$$

for all $\zeta \in C_0^\infty(\tilde{\Omega})$, where $J_n(X) = |\partial_z \rho_n(X)|$. We now want to identify the limit. We have the following Lemma.

LEMMA 3.12. *For all $\zeta \in \mathcal{D}_0(\Omega)$ we have*

$$\begin{aligned} \lim_{n \rightarrow +\infty} \int_{\tilde{\Omega}} \Lambda^n \tilde{u}_n(X) \cdot \Lambda^n \zeta(X) J_n(X) dX &= \int_{\tilde{\Omega}} \Lambda \tilde{u}(X) \cdot \Lambda \zeta(X) J(X) dX, \\ \lim_{n \rightarrow +\infty} \int_{\tilde{\Omega}} \Lambda^n \underline{\tilde{\psi}}_n(X) \cdot \Lambda^n \zeta(X) J_n(X) dX &= \int_{\tilde{\Omega}} \Lambda \underline{\tilde{\psi}}(X) \cdot \Lambda \zeta(X) J(X) dX. \end{aligned}$$

COROLLARY 3.13. *The function $u(x, y) = \tilde{u}(x, \kappa(x, y))$ is the variational solution in $H^{1,0}(\Omega)$ of the problem $-\Delta_{x,y} u = \Delta_{x,y} \underline{\psi}$ and u_n converges weakly in this space to u .*

PROOF OF LEMMA 3.12. Notice that

$$(3.28) \quad \begin{cases} \Lambda^n - \Lambda = \beta_n \partial_z, & \text{supp } \beta_n \subset \{(x, z) : x \in \mathbf{R}^d, z \in (-1, 0)\} \text{ and} \\ \|\beta_n\|_{L^\infty(K)} \leq \mathcal{F}(\|\eta\|_{W^{1,\infty}(\mathbf{R}^d)}) \|\eta_n - \eta\|_{W^{1,\infty}(K)} \end{cases}$$

Then we can write

$$\begin{aligned} \Lambda^n \tilde{u}_n \cdot \Lambda^n \zeta \cdot J_n - \Lambda \tilde{u} \cdot \Lambda \zeta \cdot J &= A_1 + A_2 + A_3 + A_4, \\ A_1 &= (\Lambda^n - \Lambda) \tilde{u}^n \cdot \Lambda^n \zeta \cdot J_n, \quad A_2 = \Lambda \tilde{u}_n \cdot \Lambda^n \zeta \cdot (J_n - J), \\ A_3 &= \Lambda \tilde{u}_n \cdot (\Lambda^n - \Lambda) \zeta \cdot J, \quad A_4 = \Lambda(\tilde{u}_n - \tilde{u}) \cdot \Lambda \zeta \cdot J. \end{aligned}$$

It follows from (3.28) that we have

$$(3.29) \quad \left| \int_{\tilde{\Omega}} A_1(X) dX \right| \leq C(M_0) \|\eta_n - \eta\|_{W^{1,\infty}(K)} \|\partial_z \tilde{u}_n\|_{L^2(\tilde{\Omega})} \|\nabla_X \zeta\|_{L^2(\tilde{\Omega})}.$$

The same estimate holds for the term coming from A_3 . Moreover since $\|J_n - J\|_{L^\infty(K)} \leq \mathcal{F}(M_0) \|\eta_n - \eta\|_{W^{1,\infty}(K)}$ we have for A_2 the same estimate as (3.29).

Eventually since (\tilde{u}_n) converges to \tilde{u} in the weak topology of $H^{1,0}(\tilde{\Omega})$ we obtain

$$\lim_{n \rightarrow +\infty} \int_{\Omega} A_4(x, y) dx dy = 0.$$

□

Step 3: taking traces. Notice that we have

$$(3.30) \quad ((\Lambda_1^n)^2 + (\Lambda_2^n)^2)\tilde{u}_n = 0, \quad ((\Lambda_1)^2 + (\Lambda_2)^2)\tilde{u} = 0,$$

and

$$(3.31) \quad \begin{cases} G(\eta_n)\psi_n = (\Lambda_1^n - \nabla_x \rho_n \cdot \Lambda_2^n)\tilde{u}_n|_{z=0} =: U_n|_{z=0}, \\ G(\eta)\psi = (\Lambda_1 - \nabla_x \rho \cdot \Lambda_2)\tilde{u}|_{z=0} =: U|_{z=0}. \end{cases}$$

Since ρ_n converges to ρ in $W_{loc}^{1,\infty}(\tilde{\Omega})$ the sequence (U_n) converges weakly to U in $L^2(\tilde{\Omega})$. Now using (3.20) we obtain

$$\partial_z U_n = -\nabla_x((\partial_z \rho_n)\Lambda_2^n \tilde{u}_n).$$

By the same way we have

$$\partial_z U = -\nabla_x((\partial_z \rho)\Lambda_2 v).$$

Since $\nabla_{x,z}\rho_n \rightarrow \nabla_{x,z}\rho$ in $L^\infty(\mathbf{R}^d \times I)$ and $\tilde{u}_n \rightarrow \tilde{u}$ weakly in $H^1(\mathbf{R}^d \times I)$, the sequence $(\partial_z U_n)$ converges to $\partial_z U$ weakly in $L^2(I, H^{-1}(\mathbf{R}^d))$.

Now we use the following well known interpolation lemma.

LEMMA 3.14. *Let $I = (-1, 0)$ and consider $u \in L^2(I, L^2(\mathbf{R}^d))$ such that $\partial_z u \in L^2(I, H^{-1}(\mathbf{R}^d))$. Then $u \in C^0([-1, 0], H^{-\frac{1}{2}}(\mathbf{R}^d))$ and there exists an absolute constant $K > 0$ such that*

$$\|u\|_{C^0([-1, 0], H^{-\frac{1}{2}}(\mathbf{R}^d))} \leq K(\|u\|_{L^2(I, L^2(\mathbf{R}^d))} + \|\partial_z u\|_{L^2(I, H^{-1}(\mathbf{R}^d))}).$$

It follows from this lemma that the sequence $(U_n|_{z=0})$ is bounded in $H^{-\frac{1}{2}}(\mathbf{R}^d)$ by $\mathcal{F}(\|\eta_n\|_{W^{1,\infty}})\|\psi_n\|_{H^{\frac{1}{2}}(\mathbf{R}^d)}$ and converges weakly in $H^{-\frac{1}{2}}(\mathbf{R}^d)$ to $U|_{z=0}$, which completes the proof of Theorem 3.9. \square

PROOF OF THEOREM 3.10. We use the notations introduced in §3.2.1. Namely, for $j = 1, 2$, we introduce $\rho_j(x, z)$ and $v_j(x, z)$ defined by (3.10),

$$\begin{aligned} \rho_j(x, z) &= (1+z)(e^{\delta z(D_x)}\eta_j)(x) - z\eta_*, \quad \text{if } x \in \mathbf{R}^d, z \in I := (-1, 0) \\ \rho_j(x, z) &= z + 1 + \eta_*, \quad \text{if } (x, z) \in \tilde{\Omega}_2 \end{aligned}$$

Notice that we have the following estimates

$$(3.32) \quad \begin{cases} (i) & \partial_z \rho_j \geq \min(1, \frac{h}{5}), \quad (x, z) \in \tilde{\Omega}, \\ (ii) & \|\nabla_{x,z}\rho_j\|_{L^\infty(\tilde{\Omega})} \leq C(1 + \|\eta_i\|_{H^{s+\frac{1}{2}}(\mathbf{R}^d)}) \\ (iii) & \|\nabla_{x,z}(\rho_1 - \rho_2)\|_{L^\infty(I, L^\infty(\mathbf{R}^d))} \leq C\|\eta_1 - \eta_2\|_{W^{1,\infty}(\mathbf{R}^d)}. \end{cases}$$

Recall also that we have set

$$(3.33) \quad \Lambda_1^i = \frac{1}{\partial_z \rho_i} \partial_z, \quad \Lambda_2^i = \nabla_x - \frac{\nabla_x \rho_i}{\partial_z \rho_i} \partial_z.$$

It follows from (3.32) that for $k = 1, 2$ we have with $W^{1,\infty} = W^{1,\infty}(\mathbf{R}^d)$,

$$(3.34) \quad \begin{cases} (i) & \Lambda_k^1 - \Lambda_k^2 = \beta_j \partial_z, \quad \text{with } \text{supp } \beta_k \subset \mathbf{R}^d \times I, \text{ where } I = (-1, 0), \\ (ii) & \|\beta_k\|_{L^\infty(I \times \mathbf{R}^d)} \leq \mathcal{F}(\|(\eta_1, \eta_2)\|_{W^{1,\infty} \times W^{1,\infty}})\|\eta_1 - \eta_2\|_{W^{1,\infty}} \end{cases}$$

Then we set $\tilde{\phi}_j(x, z) = \phi_j(x, \rho_j(x, z))$ (where $\Delta_{x,y}\phi_j = 0$ in Ω_j , $\phi_j|_{\Sigma_j} = f$) and we recall (see (3.19)) that

$$(3.35) \quad G(\eta_j)f = U_j|_{z=0}, \quad U_j = \Lambda_1^j \tilde{\phi}_j - \nabla_x \rho_j \cdot \Lambda_2^j \tilde{\phi}_j.$$

LEMMA 3.15. Set $I = (-1, 0)$, $v = \tilde{\phi}_1 - \tilde{\phi}_2$, and $\Lambda^j = (\Lambda_1^j, \Lambda_2^j)$. There exists a non decreasing function $\mathcal{F} : \mathbf{R}^+ \rightarrow \mathbf{R}^+$ such that

$$(3.36) \quad \|\Lambda^j v\|_{L^2(I; L^2(\mathbf{R}^d))} \leq \mathcal{F}(\|(\eta_1, \eta_2)\|_{W^{1,\infty} \times W^{1,\infty}}) \|\eta_1 - \eta_2\|_{W^{1,\infty}} \|f\|_{H^{\frac{1}{2}}}.$$

Let us show how this Lemma implies Theorem 3.10. According to (3.35) we have

$$(3.37) \quad \begin{aligned} U_1 - U_2 &= (1) + (2) + (3) + (4) + (5) \quad \text{where} \\ (1) &= \Lambda_1^1 v, \quad (2) = (\Lambda_1^1 - \Lambda_1^2) \tilde{\phi}_2, \quad (3) = -\nabla_x(\rho_1 - \rho_2) \Lambda_2^1 \tilde{\phi}_1 \\ (4) &= -(\nabla_x \rho_2) \Lambda_2^1 v, \quad (5) = -(\nabla_x \rho_2) (\Lambda_2^1 - \Lambda_2^2) \tilde{\phi}_2. \end{aligned}$$

The $L^2(I, L^2(\mathbf{R}^d))$ norms of (1) and (4) are estimated using (3.36). Also, the $L^2(I, L^2(\mathbf{R}^d))$ norms of (2) and (5) are estimated by the right hand side of (3.36) using (3.34) and (3.5). Eventually the $L^2(I, L^2(\mathbf{R}^d))$ norm of (3) is also estimated by the right hand side of (3.36) using (3.32) (iii) and (3.5). It follows that

$$(3.38) \quad \|U_1 - U_2\|_{L^2(I, L^2)} \leq \mathcal{F}(\|(\eta_1, \eta_2)\|_{W^{1,\infty} \times W^{1,\infty}}) \|\eta_1 - \eta_2\|_{W^{1,\infty}} \|f\|_{H^{\frac{1}{2}}}.$$

Now according to (3.20) we have

$$(3.39) \quad \partial_z(U_1 - U_2) = -\nabla_x(\partial_z(\rho_1 - \rho_2) \Lambda_2^1 \tilde{\phi}_1 + (\partial_z \rho_2)(\Lambda_2^1 - \Lambda_2^2) \tilde{\phi}_1 + (\partial_z \rho_2) \Lambda_2^2 v).$$

Therefore using the same estimates as above we see easily that

$$(3.40) \quad \|\partial_z(U_1 - U_2)\|_{L^2(I, H^{-1})} \leq \mathcal{F}(\|(\eta_1, \eta_2)\|_{W^{1,\infty} \times W^{1,\infty}}) \|\eta_1 - \eta_2\|_{W^{1,\infty}} \|f\|_{H^{\frac{1}{2}}}.$$

Then Theorem 3.10 follows from (3.38), (3.40) and Lemma 3.14. \square

PROOF OF LEMMA 3.15. We use the variational characterization of the solutions u_i . First of all we notice that $\tilde{\phi}_1 - \tilde{\phi}_2 = \tilde{u}_1 - \tilde{u}_2 =: v$. Now setting $X = (x, z)$ we have

$$(3.41) \quad \int_{\tilde{\Omega}} \Lambda^i \tilde{u}_i \cdot \Lambda^i \theta J_i dX = - \int_{\tilde{\Omega}} \Lambda^i \underline{f} \cdot \Lambda^i \theta J_i dX$$

for all $\theta \in H^{1,0}(\tilde{\Omega})$, where $J_i = |\partial_z \rho_i|$.

Taking the difference between the two equations (3.41), using (3.32) and setting $\theta = v = \tilde{u}_1 - \tilde{u}_2$ one can find a positive constant C such that

$$\int_{\tilde{\Omega}} |\Lambda^1 v|^2 dX \leq C \sum_{k=1}^6 A_k$$

where

$$\begin{cases} A_1 = \int_{\tilde{\Omega}} |(\Lambda^1 - \Lambda^2) \tilde{u}_2| |\Lambda^1 v| J_1 dX, & A_2 = \int_{\tilde{\Omega}} |(\Lambda^1 - \Lambda^2) v| |\Lambda^2 \tilde{u}_2| J_1 dX, \\ A_3 = \int_{\tilde{\Omega}} |\Lambda^2 \tilde{u}_2| |\Lambda^2 v| |J_1 - J_2| dX, & A_4 = \int_{\tilde{\Omega}} |(\Lambda^1 - \Lambda^2) \underline{f}| |\Lambda^1 v| J_1 dX, \\ A_5 = \int_{\tilde{\Omega}} |(\Lambda^1 - \Lambda^2) v| |\Lambda^2 \underline{f}| J_1 dX, & A_6 = \int_{\tilde{\Omega}} |\Lambda^2 \tilde{f}| |\Lambda^2 v| |J_1 - J_2| dX. \end{cases}$$

Using (3.34), (3.5), (3.32) we can write

$$(3.42) \quad \begin{aligned} |A_1| &\leq \|\beta\|_{L^\infty(I \times \mathbf{R}^d)} \|J_1\|_{L^\infty(I \times \mathbf{R}^d)} \|\partial_z \tilde{u}_2\|_{L^2(I \times \mathbf{R}^d)} \|\Lambda^1 v\|_{L^2(\tilde{\Omega})} \\ &\leq \mathcal{F}(\|(\eta_1, \eta_2)\|_{W^{1,\infty} \times W^{1,\infty}}) \|\eta_1 - \eta_2\|_{W^{1,\infty}} \|f\|_{H^{\frac{1}{2}}} \|\Lambda^1 v\|_{L^2(\tilde{\Omega})}. \end{aligned}$$

Since $\Lambda_j^1 - \Lambda_j^2 = \frac{\beta_j}{\partial_z \rho_1} \Lambda_1^1$ the term A_2 can be bounded by the right hand side of (3.42).

Now we have $\|J_1 - J_2\|_{L^\infty(I \times \mathbf{R}^d)} \leq C \|\eta_1 - \eta_2\|_{W^{1,\infty}(\mathbf{R}^d)}$ and

$$\|\Lambda^2 v\|_{L^2(\tilde{\Omega})} \leq \mathcal{F}(\|(\eta_1, \eta_2)\|_{W^{1,\infty} \times W^{1,\infty}}) \|\Lambda^1 v\|_{L^2(\tilde{\Omega})}.$$

So using (3.5) we see that the term A_3 can be also estimated by the right hand side of (3.42). To estimate the terms A_4 to A_6 we use the same arguments and also (3.3). This completes the proof. \square

Let us finish this definition section by recalling also the following result which is a consequence of [1, Lemma 2.9].

LEMMA 3.16. *Assume that $-\frac{1}{2} \leq a < b \leq -\frac{1}{5}$ then the strip $S_{a,b} = \{(x, y) \in \mathbf{R}^{d+1} : ah < y - \eta(x) < bh\}$ is included in Ω and for any $k \geq 1$, there exists $C > 0$ such that*

$$\|\phi\|_{H^k(S_{a,b})} \leq C \|\psi\|_{H^{\frac{1}{2}}(\mathbf{R}^d)}.$$

3.2. Paralinearization of the Dirichlet-Neumann operator. In the case of smooth domains, it is known that, modulo a smoothing operator, $G(\eta)$ is a pseudo-differential operator with principal symbol given by

$$(3.43) \quad \lambda(x, \xi) := \sqrt{(1 + |\nabla \eta(x)|^2) |\xi|^2 - (\nabla \eta(x) \cdot \xi)^2}.$$

Notice that λ is well-defined for any C^1 function η . The main result of this section allow to compare $G(\eta)$ to the paradifferential operator T_λ when η has limited regularity. Namely we want to estimate the operator

$$R(\eta) = G(\eta) - T_\lambda.$$

Such an analysis was at the heart of our previous work [5] [1, Proposition 3.14] for “smooth domains” ($\eta \in H^{s+\frac{1}{2}}, s > 2 + \frac{d}{2}$). Here we are able to lower the regularity thresholds and precise the dependence of the constants (though this dependence will not be essential in the present paper; it is important in [3]).

THEOREM 3.17. *Let $d \geq 1$ and consider $s, r \in \mathbf{R}$ such that*

$$s > \frac{3}{4} + \frac{d}{2}, \quad r > 1.$$

Consider $\eta \in H^{s+\frac{1}{2}}(\mathbf{R}^d) \cap C^{\frac{3}{2}}(\mathbf{R}^d)$ and $f \in H^s(\mathbf{R}^d) \cap C_^r(\mathbf{R}^d)$, then*

$$R(\eta)f \in H^{s-\frac{1}{2}}(\mathbf{R}^d).$$

Moreover

$$(3.44) \quad \|R(\eta)f\|_{H^{s-\frac{1}{2}}} \leq \mathcal{F}(\|\eta\|_{H^{s+\frac{1}{2}}}, \|f\|_{H^s}) \left\{ 1 + \|\eta\|_{C^{\frac{3}{2}}} + \|f\|_{C_*^r} \right\},$$

for some non-decreasing function $\mathcal{F}: (\mathbf{R}^+)^2 \rightarrow \mathbf{R}^+$ depending only on s and r .

The proof of Theorem 3.17 is given in Section 3.3.4. where we also state two corollaries of the method used to prove Theorem 3.17. The following result, which we think is of independent interest will be proved in §3.3.2. It complements previous estimates about the Dirichlet-Neumann operator by Craig-Schanz-Sulem [28], Beyer–Günther [13], Wu [54, 55], Lannes [39].

THEOREM 3.18. *Let $d \geq 1$, $s > \frac{1}{2} + \frac{d}{2}$ and $\frac{1}{2} \leq \sigma \leq s + \frac{1}{2}$. Then there exists a non-decreasing function $\mathcal{F}: \mathbf{R}^+ \rightarrow \mathbf{R}^+$ such that, for all $\eta \in H^{s+\frac{1}{2}}(\mathbf{R}^d)$ and all $f \in H^\sigma(\mathbf{R}^d)$, we have $G(\eta)f \in H^{\sigma-1}(\mathbf{R}^d)$, together with the estimate*

$$(3.45) \quad \|G(\eta)f\|_{H^{\sigma-1}(\mathbf{R}^d)} \leq \mathcal{F}(\|\eta\|_{H^{s+\frac{1}{2}}(\mathbf{R}^d)}) \|f\|_{H^\sigma(\mathbf{R}^d)}.$$

We also prove error estimates (see §3.3.3).

PROPOSITION 3.19. *Let $d \geq 1$ and $s > \frac{1}{2} + \frac{d}{2}$. For any $\frac{1}{2} \leq \sigma \leq s$ and any*

$$0 < \varepsilon \leq \frac{1}{2}, \quad \varepsilon < s - \frac{1}{2} - \frac{d}{2},$$

there exists a non-decreasing function $\mathcal{F}: \mathbf{R}^+ \rightarrow \mathbf{R}^+$ such that $R(\eta)f := G(\eta)f - T_\lambda f$ satisfies

$$\|R(\eta)f\|_{H^{\sigma-1+\varepsilon}(\mathbf{R}^d)} \leq \mathcal{F}(\|\eta\|_{H^{s+\frac{1}{2}}(\mathbf{R}^d)}) \|f\|_{H^\sigma(\mathbf{R}^d)}.$$

3.2.1. *Flattening of the free boundary.* We shall use the diffeomorphism ρ defined by (3.10) which satisfies the estimates (3.13). Also we have

$$v|_{z=0} = \phi|_{y=\eta(x)} = f,$$

and

$$G(\eta)f = (\Lambda_1 v - \nabla \rho \cdot \Lambda_2 v)|_{z=0} = \left(\frac{1 + |\nabla \rho|^2}{\partial_z \rho} \partial_z v - \nabla \rho \cdot \nabla v \right) \Big|_{z=0}.$$

3.2.2. *Elliptic regularity in Sobolev spaces.* In this paragraph we state elliptic estimates for the solution v of (3.17) with boundary data f on $z = 0$. For later purpose, we will consider the non-homogeneous case. This yields no new difficulty and will be useful later to estimate the pressure (see Section 4.3). We thus consider the problem

$$(3.46) \quad \partial_z^2 v + \alpha \Delta v + \beta \cdot \nabla \partial_z v - \gamma \partial_z v = F_0, \quad v|_{z=0} = f,$$

where $f = f(x)$ and $F_0 = F_0(x, z)$ are given functions. Recall that for $\mu \in \mathbf{R}$, the spaces $X^\mu(I), Y^\mu(I)$ are defined by (see (2.45)):

$$\begin{aligned} X^\mu(I) &= C_z^0(I; H^\mu(\mathbf{R}^d)) \cap L_z^2(I; H^{\mu+\frac{1}{2}}(\mathbf{R}^d)), \\ Y^\mu(I) &= L_z^1(I; H^\mu(\mathbf{R}^d)) + L_z^2(I; H^{\mu-\frac{1}{2}}(\mathbf{R}^d)). \end{aligned}$$

LEMMA 3.20. *Assume that $\sigma > \frac{d}{2}$. Then the space $X^\sigma(I)$ is an algebra. Moreover if $F: \mathbf{C}^N \rightarrow \mathbf{C}$ is a C^∞ -bounded function such that $F(0) = 0$ one can find non decreasing functions $\mathcal{F}, \mathcal{F}_1$ from \mathbf{R}^+ to \mathbf{R}^+ such that*

$$\|F(U)\|_{X^\sigma(I)} \leq \mathcal{F}(\|U\|_{L^\infty(I \times \mathbf{R}^d)}) \|U\|_{X^\sigma(I)} \leq \mathcal{F}_1(\|U\|_{X^\sigma(I)}).$$

PROOF. It is known that $H^\sigma(\mathbf{R}^d)$ is an algebra and so is $C^0(I; H^\sigma(\mathbf{R}^d))$. On the other hand, according to (2.4) and (2.9), we have

$$\|uv\|_{H^{\sigma+\frac{1}{2}}(\mathbf{R}^d)} \lesssim \|u\|_{L^\infty(\mathbf{R}^d)} \|v\|_{H^{\sigma+\frac{1}{2}}(\mathbf{R}^d)} + \|v\|_{L^\infty(\mathbf{R}^d)} \|u\|_{H^{\sigma+\frac{1}{2}}(\mathbf{R}^d)}.$$

□

With these notations, we want to estimate the X^σ -norm of $\nabla_{x,z} v$ in terms of the $H^{\sigma+1}$ -norm of the data and the Y^σ -norm of the source term. An important point is that we need to consider the case of rough coefficients. In this section we only assume that $\eta \in H^{s+\frac{1}{2}}(\mathbf{R}^d)$ for some $s > 1/2 + d/2$. An interesting point is that we shall prove elliptic estimates as well as elliptic regularity (in other words, we do not prove only *a priori* estimates). Our only assumption is that v is given by a variational problem, so that one has an estimate for the H^1 -norm of v , and consequently

$$(3.47) \quad \|\nabla_{x,z} v\|_{X^{-\frac{1}{2}}([-1,0])} < +\infty.$$

REMARK 3.21. In the case where $v(x, z) = \phi(x, \rho(x, z))$ with ϕ the variational solution of

$$\Delta_{x,y}\phi = 0, \quad \phi|_{y=\eta} = f, \quad \partial_n\phi = 0 \text{ on } \Gamma,$$

then (3.21) shows that v satisfies this assumption (3.47).

PROPOSITION 3.22. *Let $d \geq 1$ and*

$$s > \frac{1}{2} + \frac{d}{2}, \quad -\frac{1}{2} \leq \sigma \leq s - \frac{1}{2}.$$

Consider $f \in H^{\sigma+1}(\mathbf{R}^d)$, $F_0 \in Y^\sigma([-1, 0])$ and v satisfying the assumption (3.47) solution to (3.46). Then for any $z_0 \in (-1, 0)$, $\nabla_{x,z}v \in X^\sigma([z_0, 0])$, and

$$\|\nabla_{x,z}v\|_{X^\sigma([z_0, 0])} \leq \mathcal{F}(\|\eta\|_{H^{s+\frac{1}{2}}}) \left\{ \|f\|_{H^{\sigma+1}} + \|F_0\|_{Y^\sigma([-1, 0])} + \|\nabla_{x,z}v\|_{X^{-\frac{1}{2}}([-1, 0])} \right\},$$

for some non-decreasing function $\mathcal{F}: \mathbf{R}^+ \rightarrow \mathbf{R}^+$ depending only on σ .

To prove Proposition 3.22 we shall proceed by induction on the regularity σ .

DEFINITION 3.23. *Given σ such that $-1/2 \leq \sigma \leq s-1/2$, we say that the property \mathcal{H}_σ is satisfied if for any interval $I \Subset (-1, 0)$,*

(3.48)

$$\|\nabla_{x,z}v\|_{X^\sigma(I)} \leq \mathcal{F}(\|\eta\|_{H^{s+\frac{1}{2}}}) \left\{ \|f\|_{H^{\sigma+1}} + \|F_0\|_{Y^\sigma([-1, 0])} + \|\nabla_{x,z}v\|_{X^{-\frac{1}{2}}([-1, 0])} \right\},$$

for some non-decreasing function $\mathcal{F}: \mathbf{R}^+ \rightarrow \mathbf{R}^+$ depending only on I and σ .

With this definition, note that Assumption (3.47) means that property $\mathcal{H}_{-1/2}$ is satisfied. Consequently, Proposition 3.22 is an immediate consequence of the following proposition which will be proved in Sections 3.3 and 3.3.1.

PROPOSITION 3.24. *Let $s > \frac{1}{2} + \frac{d}{2}$. For any ε such that*

$$(3.49) \quad 0 < \varepsilon \leq \frac{1}{2}, \quad \varepsilon < s - \frac{1}{2} - \frac{d}{2},$$

if \mathcal{H}_σ is satisfied for some $-1/2 \leq \sigma \leq s - 1/2 - \varepsilon$, then $\mathcal{H}_{\sigma+\varepsilon}$ is satisfied.

3.3. Nonlinear estimates. Let us fix ε satisfying (3.49), σ such that

$$-\frac{1}{2} \leq \sigma \leq s - \frac{1}{2} - \varepsilon$$

and assume that \mathcal{H}_σ is satisfied. We begin by estimating the coefficients α, β, γ in (3.18) in terms of $\|\eta\|_{H^{s+\frac{1}{2}}}$,

LEMMA 3.25. *Let $J = [-1, 0]$ and $s > \frac{1}{2} + \frac{d}{2}$. We have*

$$(3.50) \quad \left\| \alpha - \frac{h^2}{16} \right\|_{X^{s-\frac{1}{2}}(J)} + \|\beta\|_{X^{s-\frac{1}{2}}(J)} + \|\gamma\|_{X^{s-\frac{3}{2}}(J)} \leq \mathcal{F}(\|\eta\|_{H^{s+\frac{1}{2}}}).$$

PROOF. According to (3.11), (3.12) we can write

$$(\partial_z \rho)^2 = \frac{h^2}{16} + G \quad \text{with } \|G\|_{X^{s-\frac{1}{2}}(I)} \leq \mathcal{F}(\|\eta\|_{H^{s+\frac{1}{2}}(\mathbf{R}^d)}).$$

Noticing that $\frac{1}{1+|\nabla\rho|^2} = 1 - \frac{|\nabla\rho|^2}{1+|\nabla\rho|^2}$ we obtain

$$\alpha - \frac{h^2}{16} = -\left(\frac{h^2}{16}\right) \frac{|\nabla\rho|^2}{1+|\nabla\rho|^2} + G - G \frac{|\nabla\rho|^2}{1+|\nabla\rho|^2}$$

and we use Lemma 3.20 with $\sigma = s - \frac{1}{2}$ together with (3.11), (3.12). The estimates for β and γ are proved along the same lines. \square

LEMMA 3.26. *There exists a constant K such that for all $I \subset [-1, 0]$,*

$$(3.51) \quad \|F_1\|_{Y^{\sigma+\varepsilon}(I)} \leq K \|\gamma\|_{X^{s-\frac{3}{2}}(I)} \|\partial_z v\|_{X^\sigma(I)},$$

where $F_1 = \gamma \partial_z v$.

PROOF. We shall prove that, on the one hand, if $-1/2 \leq \sigma \leq s-1-\varepsilon$ then

$$(3.52) \quad \|\gamma \partial_z v\|_{L^1(I; H^{\sigma+\varepsilon})} \lesssim \|\gamma\|_{L^2(I; H^{s-1})} \|\partial_z v\|_{L^2(I; H^{\sigma+\frac{1}{2}})},$$

and on the other hand, if $-\varepsilon \leq \sigma \leq s-\frac{1}{2}-\varepsilon$ then

$$(3.53) \quad \|\gamma \partial_z v\|_{L^2(I; H^{\sigma-\frac{1}{2}+\varepsilon})} \lesssim \|\gamma\|_{L^2(I; H^{s-1})} \|\partial_z v\|_{L^\infty(I; H^\sigma)}.$$

Since $s > \varepsilon + \frac{1}{2} + d/2$, if $-1/2 \leq \sigma \leq s-1-\varepsilon$ then

$$s-1+\sigma+\frac{1}{2} > 0, \quad \sigma+\varepsilon \leq \sigma+\frac{1}{2}, \quad \sigma+\varepsilon \leq s-1, \quad \sigma+\varepsilon < s-1+\sigma+\frac{1}{2}-\frac{d}{2}.$$

and hence the product rule in Sobolev spaces (2.16) implies that

$$\|\gamma(z) \partial_z v(z)\|_{H^{\sigma+\varepsilon}} \lesssim \|\gamma(z)\|_{H^{s-1}} \|\partial_z v(z)\|_{H^{\sigma+\frac{1}{2}}}.$$

Integrating in z and using the Cauchy-Schwarz inequality, we obtain (3.52). On the other hand, if $-\varepsilon \leq \sigma \leq s-\frac{1}{2}-\varepsilon$ then one easily checks that

$$s-1+\sigma > 0, \quad \sigma-\frac{1}{2}+\varepsilon \leq \sigma, \quad \sigma-\frac{1}{2}+\varepsilon \leq s-1, \quad \sigma-\frac{1}{2}+\varepsilon < s-1+\sigma-\frac{d}{2},$$

and hence the product rule (2.16) implies that

$$\|\gamma(z) \partial_z v(z)\|_{H^{\sigma-\frac{1}{2}+\varepsilon}} \lesssim \|\gamma(z)\|_{H^{s-1}} \|\partial_z v(z)\|_{H^\sigma}.$$

Taking the L^2 -norm in z , we obtain (3.53). \square

Our next step is to replace the multiplication by α (resp. β) by the paramultiplication by T_α (resp. T_β).

LEMMA 3.27. *There exists a constant K such that for all $I \subset [-1, 0]$, v satisfies the paradifferential equation*

$$(3.54) \quad \partial_z^2 v + T_\alpha \Delta v + T_\beta \cdot \nabla \partial_z v = F_0 + F_1 + F_2,$$

for some remainder

$$(3.55) \quad F_2 = (T_\alpha - \alpha) \Delta v + (T_\beta - \beta) \cdot \nabla \partial_z v$$

satisfying

$$(3.56) \quad \|F_2\|_{Y^{\sigma+\varepsilon}(I)} \leq K \left\{ 1 + \left\| \alpha - \frac{h^2}{16} \right\|_{X^{s-\frac{1}{2}}([-1,0])} + \|\beta\|_{X^{s-\frac{1}{2}}([-1,0])} \right\} \|\nabla_{x,z} v\|_{X^\sigma(I)}.$$

PROOF. According to Proposition 2.11, we have

$$\|au - T_a u\|_{H^\gamma} \lesssim \|a\|_{H^r} \|u\|_{H^\mu},$$

provided that $r, \mu, \gamma \in \mathbf{R}$ satisfy

$$(3.57) \quad r + \mu > 0, \quad \gamma \leq r \quad \text{and} \quad \gamma < r + \mu - \frac{d}{2}.$$

Since $s > \varepsilon + 1/2 + d/2$, if $-1/2 \leq \sigma \leq s-\frac{1}{2}-\varepsilon$ then

$$s + \sigma - \frac{1}{2} > 0, \quad \sigma + \varepsilon \leq s, \quad \sigma + \varepsilon < s + \sigma - \frac{1}{2} - \frac{d}{2},$$

and hence (3.57) applies with

$$\gamma = \sigma + \varepsilon, \quad r = s, \quad \mu = \sigma - \frac{1}{2}.$$

This implies that if $-1/2 \leq \sigma \leq s - \frac{1}{2} - \varepsilon$ then

$$(3.58) \quad \begin{aligned} \|(T_\alpha - \alpha)\Delta v\|_{L^1(I; H^{\sigma+\varepsilon})} &\lesssim \left(1 + \left\| \alpha - \frac{h^2}{16} \right\|_{L^2(I; H^s)}\right) \|\Delta v\|_{L^2(I; H^{\sigma-\frac{1}{2}})}, \\ \|(T_\beta - \beta)\nabla\partial_z v\|_{L^1(I; H^{\sigma+\varepsilon})} &\lesssim \|\beta\|_{L^2(I; H^s)} \|\nabla\partial_z v\|_{L^2(I; H^{\sigma-\frac{1}{2}})}, \end{aligned}$$

which yields

$$\|F_2\|_{Y^{\sigma+\varepsilon}(I)} \leq \|F_2\|_{L^1(I; H^{\sigma+\varepsilon})} \lesssim \left\{1 + \left\| \alpha - \frac{h^2}{16} \right\|_{X^{s-\frac{1}{2}}(I)} + \|\beta\|_{X^{s-\frac{1}{2}}(I)}\right\} \|\nabla_{x,z} v\|_{X^\sigma(I)}.$$

This concludes the proof. \square

Our next task is to perform a decoupling into a forward and a backward parabolic evolution equations. Recall that by assumption $\eta \in H^{s+\frac{1}{2}}(\mathbf{R}^d)$ with $s > \varepsilon + 1/2 + d/2$. In particular, $\eta \in C_*^{1+\varepsilon}(\mathbf{R}^d)$.

LEMMA 3.28. *There exist two symbols a, A in $\Gamma_\varepsilon^1(\mathbf{R}^d \times [-1, 0])$ and a remainder F_3 such that,*

$$(3.59) \quad (\partial_z - T_a)(\partial_z - T_A)v = F_0 + F_1 + F_2 + F_3,$$

with

$$(3.60) \quad \mathcal{M}_\varepsilon^1(a) + \mathcal{M}_\varepsilon^1(A) \leq \mathcal{F}(\|\eta\|_{H^{s+\frac{1}{2}}}),$$

and

$$\|F_3\|_{L^2(I; H^{\sigma-\frac{1}{2}+\varepsilon})} \leq \mathcal{F}(\|\eta\|_{H^{s+\frac{1}{2}}}) \|\nabla_{x,z} v\|_{L^2(I; H^{\sigma+\frac{1}{2}})},$$

for some non-decreasing function $\mathcal{F}: \mathbf{R}^+ \rightarrow \mathbf{R}^+$.

PROOF. We seek a, A satisfying

$$a(z; x, \xi)A(z; x, \xi) = -\alpha(x, z)|\xi|^2, \quad a(z; x, \xi) + A(z; x, \xi) = -i\beta(x, z) \cdot \xi.$$

We thus set

$$(3.61) \quad a = \frac{1}{2}(-i\beta \cdot \xi - \sqrt{4\alpha|\xi|^2 - (\beta \cdot \xi)^2}), \quad A = \frac{1}{2}(-i\beta \cdot \xi + \sqrt{4\alpha|\xi|^2 - (\beta \cdot \xi)^2}).$$

Directly from the definition of α, β (3.18), note that

$$\exists c > 0; \quad \sqrt{4\alpha|\xi|^2 - (\beta \cdot \xi)^2} \geq c|\xi|.$$

According to (3.50) the symbols a, A belong to $\Gamma_\varepsilon^1(\mathbf{R}^d \times [-1, 0])$ and they satisfy the bound (3.60). Therefore, we have

$$(3.62) \quad (\partial_z - T_a)(\partial_z - T_A)v = \partial_z^2 v - T_\beta \nabla \partial_z v + T_\alpha \Delta v + F_3, \quad F_3 = R_0 v + R_1 v,$$

where

$$R_0(z) := T_{a(z)}T_{A(z)} - T_\alpha \Delta, \quad R_1(z) := -T_{\partial_z A(z)}.$$

According to Theorem 2.7, applied with $\rho = \varepsilon$, $R_0(z)$ is of order $2 - \varepsilon$, uniformly in $z \in [-1, 0]$. On the other hand, since

$$\partial_z \rho \in L^\infty((-1, 0); H^{s-\frac{1}{2}}), \quad \partial_z^2 \rho \in L^\infty((-1, 0); H^{s-\frac{3}{2}}),$$

according to (2.16) we have

$$\partial_z \alpha, \partial_z \beta \in L^\infty((-1, 0); H^{s-\frac{3}{2}}) \subset L^\infty((-1, 0); C_*^{\varepsilon-1}).$$

Therefore $\partial_z A \in \Gamma_{\varepsilon-1}^1(\mathbf{R}^d \times [-1, 0])$. As a consequence, using Proposition 2.13, we get that $R_1(z)$ is also of order $2 - \varepsilon$. We end up with

$$\sup_{z \in [-1, 0]} \|R_0(z)\|_{H^{\mu+2-\varepsilon} \rightarrow H^\mu} + \|R_1(z)\|_{H^{\mu+2-\varepsilon} \rightarrow H^\mu} \leq \mathcal{F}(\|\eta\|_{H^{s+\frac{1}{2}}}).$$

Now we notice that, given any symbol p and any function u , by definition of paradiifferential operators we have $T_p u = T_p(1 - \Psi(D_x))u$ for any Fourier multiplier $(I - \Psi(D_x))$ such that $\Psi(\xi) = 0$ for $|\xi| \geq 1/2$. This means that we can replace $\|v(z)\|_{H^{\sigma+\frac{3}{2}}}$ by $\|\nabla v(z)\|_{H^{\sigma+\frac{1}{2}}}$. We thus obtain the desired result from Lemma 3.27. \square

3.3.1. *Proof of Proposition 3.24.* We shall apply Proposition 2.18 twice. At first we apply it to the *forward* parabolic evolution equation $\partial_z u - T_a u = F$ (by definition $\operatorname{Re}(-a) \geq c|\xi|$). This requires an initial data on $z = -1$ that might be chosen to be 0 by using a cut-off function, up to shrinking the interval I . Next we apply it to the *backward* parabolic evolution equation $\partial_z u - T_A u = F$ (by definition $\operatorname{Re} A \geq c|\xi|$). This requires an initial data on $z = 0$ (which is given by our assumption on f) and this requires also an estimate for the remainder term F which is given by means of the first step.

Suppose that \mathcal{H}_σ is satisfied. Let $I_0 = [\zeta_0, 0]$ such that

$$\|\nabla_{x,z} v\|_{X^\sigma(I_0)} \leq \mathcal{F}(\|\eta\|_{H^{s+\frac{1}{2}}}) \left\{ \|f\|_{H^{\sigma+1}} + \|F_0\|_{Y^\sigma(I_0)} + \|\nabla_{x,z} v\|_{X^{-\frac{1}{2}}([-1,0])} \right\}.$$

We shall prove that, for any $\zeta_1 > \zeta_0$,

$$(3.63) \quad \|\nabla_{x,z} v\|_{X^{\sigma+\varepsilon}([\zeta_1,0])} \leq \mathcal{F}(\|\eta\|_{H^{s+\frac{1}{2}}}) \left\{ \|f\|_{H^{\sigma+1+\varepsilon}} + \|F_0\|_{Y^{\sigma+\varepsilon}(I_0)} + \|\nabla_{x,z} v\|_{X^{-\frac{1}{2}}([-1,0])} \right\}.$$

Introduce a cutoff function χ such that

$$\chi(\zeta_0) = 0, \quad \chi(z) = 1 \quad \text{for } z \geq \zeta_1.$$

Set $w := \chi(z)(\partial_z - T_A)v$. It follows from (3.59) for v that

$$\partial_z w - T_a w = F',$$

where

$$F' = \chi(z)(F_0 + F_1 + F_2 + F_3) + \chi'(z)(\partial_z - T_A)v.$$

We have already estimated F_1, F_2, F_3 and F_0 is given. We now turn to an estimate for $(\partial_z - T_A)v$. According to (2.4) and (3.60), we have

$$\|T_A v\|_{L^2(I_0; H^{\sigma+\frac{1}{2}})} \leq \mathcal{F}(\|\eta\|_{H^{s+\frac{1}{2}}}) \|\nabla v\|_{L^2(I_0; H^{\sigma+\frac{1}{2}})} \leq \mathcal{F}(\|\eta\|_{H^{s+\frac{1}{2}}}) \|\nabla_{x,z} v\|_{X^\sigma(I_0)},$$

and similarly

$$\|T_A v\|_{L^\infty(I_0; H^\sigma)} \leq \mathcal{F}(\|\eta\|_{H^{s+\frac{1}{2}}}) \|\nabla v\|_{L^\infty(I_0; H^\sigma)} \leq \mathcal{F}(\|\eta\|_{H^{s+\frac{1}{2}}}) \|\nabla_{x,z} v\|_{X^\sigma(I_0)},$$

Consequently

$$\|(\partial_z - T_A)v\|_{X^\sigma(I_0)} \leq \mathcal{F}(\|\eta\|_{H^{s+\frac{1}{2}}}) \|\nabla_{x,z} v\|_{X^\sigma(I_0)}.$$

This implies that

$$(3.64) \quad \|w\|_{X^\sigma(I_0)} \leq \mathcal{F}(\|\eta\|_{H^{s+\frac{1}{2}}}) \|\nabla_{x,z} v\|_{X^\sigma(I_0)},$$

$$(3.65) \quad \|F'\|_{Y^{\sigma+\varepsilon}(I_0)} \leq \mathcal{F}(\|\eta\|_{H^{s+\frac{1}{2}}}) \|\nabla_{x,z} v\|_{X^\sigma(I_0)} + \|F_0\|_{Y^{\sigma+\varepsilon}(I_0)}.$$

Since $w(x, z_0) = 0$ and since $a \in \Gamma_\varepsilon^1$ satisfies $\operatorname{Re}(-a(x, \xi)) \geq c|\xi|$, by using Proposition 2.18 applied with $J = I_0$, $\rho = \varepsilon$ and $r = \sigma + \varepsilon$, we have

$$\|w\|_{X^{\sigma+\varepsilon}(I_0)} \leq \mathcal{F}(\|\eta\|_{H^{s+\frac{1}{2}}}) \left\{ \|F'\|_{Y^{\sigma+\varepsilon}(I_0)} + \|w\|_{X^{\sigma+\varepsilon-\frac{1}{2}}(I_0)} \right\},$$

and hence, using (3.64) and (3.65)

$$(3.66) \quad \|w\|_{X^{\sigma+\varepsilon}(I_0)} \leq \mathcal{F}(\|\eta\|_{H^{s+\frac{1}{2}}}) \{ \|\nabla_{x,z}v\|_{X^\sigma(I_0)} + \|F_0\|_{Y^{\sigma+\varepsilon}(I_0)} \}.$$

Now notice that on $I_1 := [\zeta_1, 0]$ we have $\chi = 1$ so that

$$\partial_z v - T_A v = w \quad \text{for } z \in I_1.$$

Therefore the function \tilde{v} defined by $\tilde{v}(x, z) = v(x, -z)$ satisfies

$$\partial_z \tilde{v} + T_{\tilde{A}} \tilde{v} = -\tilde{w} \quad \text{for } z \in \tilde{I}_1 = [0, -\zeta_1],$$

with obvious notations for \tilde{w} and \tilde{A} . By using Proposition 2.18 with $J = \tilde{I}_1$, noticing that $\tilde{v}|_{z=0} = v|_{z=0} = f$, we obtain that

$$\|\tilde{v}\|_{X^{\sigma+1+\varepsilon}(\tilde{I}_1)} \leq \mathcal{F}(\|\eta\|_{H^{s+\frac{1}{2}}}) \left\{ \|f\|_{H^{\sigma+1+\varepsilon}} + \|\tilde{w}\|_{Y^{\sigma+1+\varepsilon}(\tilde{I}_1)} + \|\tilde{v}\|_{X^{\sigma-\frac{1}{2}}(\tilde{I}_1)} \right\}.$$

Using the obvious estimate

$$\|\tilde{w}\|_{Y^{\sigma+1+\varepsilon}(\tilde{I}_1)} = \|w\|_{Y^{\sigma+1+\varepsilon}(I_1)} \leq \|w\|_{L_z^2(I_1; H^{\sigma+\frac{1}{2}+\varepsilon})} \leq \|w\|_{X^{\sigma+\varepsilon}(I_1)},$$

it follows from (3.66) that

$$(3.67) \quad \|v\|_{X^{\sigma+1+\varepsilon}(I_1)} \leq \mathcal{F}(\|\eta\|_{H^{s+\frac{1}{2}}}) (\|f\|_{H^{\sigma+1+\varepsilon}} + \|\nabla_{x,z}v\|_{X^\sigma(I_0)} + \|F_0\|_{Y^{\sigma+\varepsilon}(I_0)} + \|v\|_{X^{\sigma-\frac{1}{2}}(I_0)}).$$

We easily estimate $\partial_z v$ directly from $\partial_z v = T_A v + w$ (by using (3.66) and the fact that T_A is an operator of order 1). This completes the proof of (3.63). This proves that if \mathcal{H}_σ is satisfied then $\mathcal{H}_{\sigma+\varepsilon}$ is satisfied and hence concludes the proof of Proposition 3.24 (and hence the proof of Proposition 3.22).

3.3.2. *Proof of Theorem 3.18.* Let v be the solution of (3.17) with data $v|_{z=0} = f$. By definition of the Dirichlet–Neumann operator we have

$$(3.68) \quad G(\eta)f = \frac{1 + |\nabla\rho|^2}{\partial_z \rho} \partial_z v - \nabla\rho \cdot \nabla v \Big|_{z=0}.$$

Now, by applying Proposition 3.22 with $F_0 = 0$ and Remark 3.21, we find that if v solves (3.17), then for any $I \Subset (-1, 0]$,

$$(3.69) \quad \|\nabla_{x,z}v\|_{X^{\sigma-1}(I)} \leq \mathcal{F}(\|\eta\|_{H^{s+\frac{1}{2}}}) \|f\|_{H^\sigma}.$$

According to (3.11) and (2.17), we obtain that

$$\left\| \frac{1 + |\nabla\rho|^2}{\partial_z \rho} \partial_z v - \nabla\rho \cdot \nabla v \Big|_{C^0([z_0, 0]; H^{\sigma-1})} \right\| \leq \mathcal{F}(\|\eta\|_{H^{s+\frac{1}{2}}}) \|f\|_{H^\sigma}.$$

As a result, taking the trace on $z = 0$ immediately implies the desired result (3.45).

3.3.3. *Proof of Proposition 3.19.* Let $1/2 \leq \sigma_0 \leq s$. It follows from (3.66) applied with $\sigma = \sigma_0 - 1$ and $F_0 = 0$ that

$$\|\chi(z)(\partial_z v - T_A v)\|_{X^{\sigma_0-1+\varepsilon}(I_0)} \leq \mathcal{F}(\|\eta\|_{H^{s+\frac{1}{2}}}) \{ \|\nabla_{x,z}v\|_{X^{\sigma_0-1}(I_0)} \},$$

for some cut-off function χ such that $\chi(0) = 1$. By using Proposition 3.22, we thus obtain

$$(3.70) \quad \|\partial_z v - T_A v|_{z=0}\|_{H^{\sigma_0-1+\varepsilon}} \leq \mathcal{F}(\|\eta\|_{H^{s+\frac{1}{2}}}) \|f\|_{H^{\sigma_0}}.$$

The previous estimate allows us to express the “normal” derivative $\partial_z v$ in terms of the tangential derivatives. Which is the main step to parilinearize the Dirichlet–Neumann operator.

Now, as mentioned above, by definition of v ,

$$G(\eta)f = \frac{1 + |\nabla\rho|^2}{\partial_z\rho} \partial_z v - \nabla\rho \cdot \nabla v \Big|_{z=0}.$$

Set

$$\zeta_1 := \frac{1 + |\nabla\rho|^2}{\partial_z\rho}, \quad \zeta_2 := \nabla\rho.$$

According to (3.11),

$$(3.71) \quad \left\| \zeta_1 - \frac{4}{h} \right\|_{C_z^0([-1,0]; H_x^{s-\frac{1}{2}})} + \|\zeta_2\|_{C_z^0([-1,0]; H_x^{s-\frac{1}{2}})} \leq \mathcal{F}(\|\eta\|_{H^{s+\frac{1}{2}}}).$$

Let

$$R' = \zeta_1 \partial_z v - \zeta_2 \cdot \nabla v - (T_{\zeta_1} \partial_z v - T_{\zeta_2} \nabla v).$$

Since $\varepsilon \leq \frac{1}{2}$ and $\varepsilon < s - \frac{1}{2} - \frac{d}{2}$, we verify that Proposition 2.11 applies with

$$\gamma = \sigma_0 - 1 + \varepsilon, \quad r = s - \frac{1}{2}, \quad \mu = \sigma_0 - 1,$$

which, according to (3.71) and (3.69), implies

$$\|R'\|_{C^0(I; H^{\sigma_0-1+\varepsilon})} \leq \mathcal{F}(\|\eta\|_{H^{s+\frac{1}{2}}}) \|f\|_{H^{\sigma_0}}.$$

Furthermore, according to (3.70) and (3.71), we obtain

$$T_{\zeta_1} \partial_z v - T_{\zeta_2} \nabla v \Big|_{z=0} - (T_{\zeta_1} T_A v - T_{i\zeta_2 \cdot \xi} v \Big|_{z=0}) = R'',$$

with

$$\|R''\|_{H^{\sigma_0-1+\varepsilon}} \leq \mathcal{F}(\|\eta\|_{H^{s+\frac{1}{2}}}) \|f\|_{H^{\sigma_0}}.$$

Finally, thanks to (2.5), (3.71) and (3.60), we have

$$\|T_{\zeta_1(z)} T_A(z) - T_{\zeta_1(z)A(z)}\|_{H^{\sigma_0} \rightarrow H^{\sigma_0-\frac{1}{2}}} \lesssim \|\zeta_1(z)\|_{W^{\varepsilon, \infty}} \mathcal{M}_\varepsilon^1(A) \leq \mathcal{F}(\|\eta\|_{H^{s+\frac{1}{2}}}),$$

and hence

$$G(\eta)f = T_{\zeta_1 A} v - T_{i\zeta_2 \cdot \xi} v \Big|_{z=0} + R(\eta)f$$

where

$$\|R(\eta)f\|_{H^{\sigma_0-1+\varepsilon}} \leq \mathcal{F}(\|\eta\|_{H^{s+\frac{1}{2}}}) \|f\|_{H^{\sigma_0}}.$$

Let

$$\lambda = \frac{1 + |\nabla\rho|^2}{\partial_z\rho} A - i\nabla\rho \cdot \xi \Big|_{z=0} = \sqrt{(1 + |\nabla\eta(x)|^2)|\xi|^2 - (\nabla\eta(x) \cdot \xi)^2}.$$

Then

$$G(\eta)f = T_\lambda f + R(\eta)f,$$

which concludes the proof of Proposition 3.19.

3.3.4. Tame estimates for the Dirichlet-Neumann operator. This section is devoted to the proof of Theorem 3.17. We thus consider the problem

$$(3.72) \quad \partial_z^2 v + \alpha \Delta v + \beta \cdot \nabla \partial_z v - \gamma \partial_z v = 0, \quad v|_{z=0} = f,$$

where $f = f(x)$ is a given function. In the sequel we fix indexes $\delta, s, r \in \mathbf{R}$ such that

$$(3.73) \quad 0 < \delta < \frac{1}{4}, \quad s > 1 + \frac{d}{2} - \delta, \quad r > 1, \quad \frac{1}{4} < \varepsilon = \frac{1}{2} - \delta < \min\left(\frac{1}{2}, s - \frac{1}{2} - \frac{d}{2}\right).$$

Let $z_0 \in (-1, 0)$. It follows from Proposition 3.22 and the Sobolev embedding that we have

$$\|\nabla_{x,z} v\|_{C^0([z_0,0]; C_*^{s-1-d/2})} \leq \mathcal{F}(\|\eta\|_{H^{s+\frac{1}{2}}}) \|f\|_{H^s},$$

for some non-decreasing function \mathcal{F} . However, since we only assume that $s > 3/4 + d/2$, this is not enough to control the L^∞ norm of $\nabla_{x,z} v$. The purpose of the next

result is to prove such bounds under the additional assumption that f belongs to C_*^r for some $r > 1$.

PROPOSITION 3.29. *Let $r > 1$ and $s > 3/4 + d/2$. For any $z_0 < z_1 < 0$, we have*

$$\|\nabla_{x,z}v\|_{C^0 \cap L^\infty([z_1,0] \times \mathbf{R}^d)} \leq \mathcal{F}(\|\eta\|_{H^{s+\frac{1}{2}}}) \left\{ \|f\|_{H^s} + \|f\|_{C_*^r} \right\},$$

for some non-decreasing function $\mathcal{F}: \mathbf{R}^+ \rightarrow \mathbf{R}^+$.

REMARK 3.30. Since $v|_{z=0} = f \in L^\infty(\mathbf{R}^d)$, we have also

$$\begin{aligned} \|v\|_{L^\infty(\mathbf{R}^d \times [z_1,0])} &\leq \|f\|_{L^\infty(\mathbf{R}^d)} + |z_1| \|\partial_z v\|_{L^\infty(\mathbf{R}^d \times [z_1,0])} \\ &\leq \mathcal{F}(\|\eta\|_{H^{s+\frac{1}{2}}}) \left\{ \|f\|_{H^s} + \|f\|_{C_*^r} \right\}. \end{aligned}$$

PROOF. We have already proved that there exists $z_0 \in [-1, -1/2]$ such that

$$(3.74) \quad \|\nabla_{x,z}v\|_{X^{s-1}([z_0,0])} \leq \mathcal{F}(\|\eta\|_{H^{s+\frac{1}{2}}}) \|f\|_{H^s},$$

and using again that $v|_{z=0} = f$,

$$(3.75) \quad \|v\|_{X^{s-1}([z_0,0])} \leq \mathcal{F}(\|\eta\|_{H^{s+\frac{1}{2}}}) \|f\|_{H^s}.$$

As above, introduce a cutoff function χ such that

$$\chi(z_0) = 0, \quad \chi(z) = 1 \quad \text{for } z \geq z_1 := \frac{1}{2}(z_0 - \frac{1}{2}),$$

and set $w := \chi(z)(\partial_z - T_A)v$. Since it is convenient to work with forward evolution equation, define the function \tilde{v} by $\tilde{v}(x, z) = v(x, -z)$, so that

$$\partial_z \tilde{v} + T_{\tilde{A}} \tilde{v} = -\tilde{w} \quad \text{for } z \in \tilde{I}_1 := [0, -z_1].$$

We split \tilde{v} as $\tilde{v} = \tilde{v}_1 + \tilde{v}_2$ where \tilde{v}_1 is the solution to the system

$$\partial_z \tilde{v}_1 + T_{\tilde{A}} \tilde{v}_1 = 0 \quad \text{for } z \in \tilde{I}_1, \quad \tilde{v}_1(0) = \tilde{v}(0) = f$$

given by Proposition 2.18, while $\tilde{v}_2 = \tilde{v} - \tilde{v}_1$ satisfies

$$\partial_z \tilde{v}_2 + T_{\tilde{A}} \tilde{v}_2 = -\tilde{w} \quad \text{for } z \in \tilde{I}_1, \quad \tilde{v}_2(0) = 0.$$

According to (3.66) with $\sigma = s - 1$ and ε defined in (3.73), we obtain, using (3.74)

$$\|w\|_{X^{s-1+\frac{1}{2}-\delta}(I_0)} \leq \mathcal{F}(\|\eta\|_{H^{s+\frac{1}{2}}}) \|\nabla_{x,z}v\|_{X^{s-1}(I_0)} \leq \mathcal{F}(\|\eta\|_{H^{s+\frac{1}{2}}}) \|f\|_{H^s}.$$

which in turn implies, according to Proposition 2.18,

$$(3.76) \quad \|\tilde{v}_2\|_{X^{s+\varepsilon}(\tilde{I}_1)} \leq \mathcal{F}(\|\eta\|_{H^{s+\frac{1}{2}}}) (\|\nabla_{x,z}v\|_{X^{s-1}(I_0)} + \|\tilde{v}_2\|_{X^s(\tilde{I}_1)}).$$

Using Proposition 2.19 with $r = 1, r_0 = -1$, we obtain (with $I = [z_0, 0]$)

$$\|\tilde{v}_1\|_{C^0(\tilde{I}; C_*^r)} \leq K \times (\|f\|_{C_*^r} + \|\tilde{v}_1\|_{C^0(\tilde{I}; C_*^{-1})}).$$

But, according to (3.74), (3.75) and (3.76)

$$\|\tilde{v}_1\|_{C^0(\tilde{I}; C_*^{-1})} \leq C (\|\tilde{v}\|_{C^0(\tilde{I}; H^{s-1})} + \|\tilde{v}_2\|_{C^0(\tilde{I}; H^{s-1})}) \leq \mathcal{F}(\|\eta\|_{H^{s+\frac{1}{2}}}) \|f\|_{H^s}$$

which since $\partial_z \tilde{v}_1 = T_{\tilde{A}} \tilde{v}_1$, implies also

$$\|\nabla_{x,z} \tilde{v}_1\|_{C^0(\tilde{I}_1; C_*^{r-1})} \leq \mathcal{F}(\|\eta\|_{H^{s+\frac{1}{2}}}) (\|f\|_{H^s} + \|f\|_{C_*^r}).$$

On the other hand, according to (3.66) with $\sigma = s - 1$ and ε defined in (3.73), we obtain, using (3.74)

$$\|w\|_{X^{s-1+\frac{1}{2}-\delta}(I_0)} \leq \mathcal{F}(\|\eta\|_{H^{s+\frac{1}{2}}}) \|\nabla_{x,z}v\|_{X^{s-1}(I_0)} \leq \mathcal{F}(\|\eta\|_{H^{s+\frac{1}{2}}}) \|f\|_{H^s}.$$

Let us go back to the study of \tilde{v}_2 . According to (3.76), and since $\partial_z \tilde{v}_2 = -T_{\tilde{A}} \tilde{v}_2 - \tilde{w}$ we get

$$(3.77) \quad \|\nabla_{x,z} \tilde{v}_2\|_{X^{s-1+\varepsilon}(\tilde{I}_1)} \leq \mathcal{F}(\|\eta\|_{H^{s+\frac{1}{2}}}) \|\nabla_{x,z} v\|_{X^{s-1}(I_0)}.$$

Set

$$m = s - 1 + \varepsilon - \frac{d}{2}.$$

By the Sobolev embedding we have

$$\|\nabla_{x,z} \tilde{v}_2\|_{L^\infty(\tilde{I}_1; C_*^m)} \lesssim \|\nabla_{x,z} \tilde{v}_2\|_{X^{s-1+\varepsilon}(\tilde{I}_1)}.$$

Using (3.77) and our previously established estimate for $\|\nabla_{x,z} v\|_{X^{s-1}(I_0)}$, this yields

$$\|\nabla_{x,z} \tilde{v}_2\|_{L^\infty(\tilde{I}_1; C_*^m)} \leq \mathcal{F}(\|\eta\|_{H^{s+\frac{1}{2}}}) \|f\|_{H^s}.$$

By assumption $s > 1 + d/2 - \delta$ with $\delta < 1/4$, so that

$$m = s - 1 + \varepsilon - \frac{d}{2} = s - 1 + \frac{1}{2} - \delta - \frac{d}{2} > \frac{1}{2} - 2\delta > 0,$$

and hence

$$\|\nabla_{x,z} \tilde{v}_2\|_{L^\infty(\mathbf{R}^d \times [0, -z_1])} \leq \mathcal{F}(\|\eta\|_{H^{s+\frac{1}{2}}}) \|f\|_{H^s}.$$

This completes the proof of Proposition 3.29. \square

3.3.5. *Proof of Theorem 3.17.* As already mentioned, by definition of v , we have

$$G(\eta)f = \frac{1 + |\nabla\rho|^2}{\partial_z \rho} \partial_z v - \nabla\rho \cdot \nabla v \Big|_{z=0}.$$

To perform the parilinearization of the Dirichlet-Neumann operator, we are going to revisit Section 3.3.3, using tame estimate at each step. Let us begin by recalling our notations. We have

$$(\partial_z - T_a)(\partial_z - T_A)v = F_1 + F_2 + F_3,$$

where a, A are given by (3.61) and F_1, F_2, F_3 by (3.51), (3.55) and (3.62). Recall that according to Propositions 3.22 and 3.29, for any $z_0 \in (-1, 0)$, we have

$$(3.78) \quad \begin{aligned} \|\nabla_{x,z} v\|_{C^0([z_0, 0]; H^{s-1}) \cap L^2((z_0, 0); H^{s-\frac{1}{2}})} &\leq \mathcal{F}(\|\eta\|_{H^{s+\frac{1}{2}}}) \|f\|_{H^s}, \\ \|\nabla_{x,z} v\|_{C^0 \cap L^\infty(\mathbf{R}^d \times (z_0, 0))} &\leq \mathcal{F}(\|\eta\|_{H^{s+\frac{1}{2}}}) \left\{ \|f\|_{H^s} + \|f\|_{C_*^s} \right\}. \end{aligned}$$

The end of the proof of Theorem 3.17 is in four steps.

Step 1. Tame estimates in Zygmund spaces.

LEMMA 3.31. *The coefficients α, β, γ defined in (3.18) satisfy*

$$(3.79) \quad \|(\alpha, \beta)\|_{C^0([-1, 0]; C_*^{1/2})} + \|\gamma\|_{C^0([-1, 0]; C_*^{-1/2})} \leq \mathcal{F}(\|\eta\|_{H^{s+1/2}}) (\|\eta\|_{C_*^{3/2}} + 1).$$

PROOF. It follows from (3.10) that

$$(3.80) \quad \|\nabla_x \rho\|_{C^0([-1, 0]; C_*^{1/2})} \lesssim \|\eta\|_{C_*^{3/2}}.$$

Moreover using (3.7) we get

$$(3.81) \quad \left\| \partial_z \rho - \frac{h}{4} \right\|_{C^0([-1, 0]; C_*^{1/2})} + \|\partial_z^2 \rho\|_{C^0([-1, 0]; C_*^{-1/2})} \leq C(1 + \|\eta\|_{C_*^{3/2}}).$$

On the other hand recall that we have, according to (3.10)

$$(3.82) \quad \begin{aligned} \|\nabla_{x,z}\rho\|_{C^0([-1,0];H^{s-\frac{1}{2}}(\mathbf{R}^d))} &\leq K \|\eta\|_{H^{s+\frac{1}{2}}} \\ &\Rightarrow \|\nabla_{x,z}\rho\|_{C^0([-1,0];H^{s-\frac{1}{2}}(\mathbf{R}^d))} \leq K(1 + \|\eta\|_{H^{s+\frac{1}{2}}}), \end{aligned}$$

and consequently by Sobolev embeddings,

$$\|\nabla_{x,z}\rho\|_{L^\infty((-1,0)\times\mathbf{R}^d)} \leq K(1 + \|\eta\|_{H^{s+\frac{1}{2}}}).$$

Using $H^{s+1/2} \subset C_*^{s+1/2-d/2} \subset C_*^{5/4}$, we deduce the estimates for $\alpha - 1$ and β from the tame product estimate in Zygmund spaces (see (2.18), (2.22)). The estimate for γ follows from (2.20), (2.22). \square

Step 2. Estimates for the source terms.

LEMMA 3.32. *For any $z_0 \in (-1, 0)$, and any $j = 1; 2; 3$ we have,*

$$(3.83) \quad \|F_j\|_{L^2_z((z_0,0);H^{s-1})} \leq \mathcal{F}(\|\eta\|_{H^{s+\frac{1}{2}}}, \|f\|_{H^s}) \left\{ 1 + \|\eta\|_{C^{\frac{3}{2}}} + \|f\|_{C_*^r} \right\}.$$

PROOF. By using the product rule (see (2.19))

$$\|u_1 u_2\|_{H^{s-1}} \lesssim \|u_1\|_{C_*^{-1/2}} \|u_2\|_{H^{s-\frac{1}{2}}} + \|u_2\|_{L^\infty} \|u_1\|_{H^{s-1}}$$

we find that $F_1 = \gamma \partial_z v$ satisfies

$$\begin{aligned} \|F_1\|_{L^2((z_0,0);H^{s-1})} &\lesssim \|\partial_z v\|_{C^0([z_0,0]\times\mathbf{R}^d)} \|\gamma\|_{L^2((z_0,0);H^{s-1})} \\ &\quad + \|\gamma\|_{C^0([z_0,0];C_*^{-1/2})} \|\partial_z v\|_{L^2((z_0,0);H^{s-\frac{1}{2}})}. \end{aligned}$$

The desired estimate for F_1 follows from Lemma (3.31), (3.50) and (3.78). Let us now study

$$F_2 = (T_\alpha - \alpha)\Delta v + (T_\beta - \beta) \cdot \nabla \partial_z v = -(T_{\Delta v} \alpha + R(\alpha, \Delta v) + T_{\nabla \partial_z v} \cdot \beta + R(\beta, \nabla \partial_z v)).$$

According to (2.12), we obtain

$$\begin{aligned} \|T_{\Delta v(z)} \alpha(z)\|_{H^{s-1}} &\lesssim \|\Delta v(z)\|_{C_*^{-1}} \|\alpha(z)\|_{H^s}, \\ \|T_{\nabla \partial_z v(z)} \cdot \beta(z)\|_{H^{s-1}} &\lesssim \|\nabla \partial_z v(z)\|_{C_*^{-1}} \|\beta(z)\|_{H^s}. \end{aligned}$$

On the other hand, since $s - 1 > 0$ we can apply (2.11) to obtain

$$\begin{aligned} \|R(\alpha, \Delta v)(z)\|_{H^{s-1}} &\lesssim \|\Delta v(z)\|_{C_*^{-1}} \|\alpha(z)\|_{H^s}, \\ \|R(\beta, \nabla \partial_z v)(z)\|_{H^{s-1}} &\lesssim \|\nabla \partial_z v(z)\|_{C_*^{-1}} \|\beta(z)\|_{H^s}. \end{aligned}$$

Consequently we have proved

$$(3.84) \quad \begin{aligned} \|F_2\|_{L^2([z_0,0];H^{s-1})} &\lesssim \|\Delta v\|_{C^0([z_0,0];C_*^{-1})} \|\alpha\|_{L^2([z_0,0];H^s)} \\ &\quad + \|\nabla \partial_z v\|_{C^0([z_0,0];C_*^{-1})} \|\beta\|_{L^2([z_0,0];H^s)}. \end{aligned}$$

Notice that

$$\|\Delta v\|_{C_*^{-1}} \lesssim \|\nabla v\|_{C_*^0} \lesssim \|\nabla v\|_{L^\infty}, \quad \|\nabla \partial_z v\|_{C_*^{-1}} \lesssim \|\partial_z v\|_{C_*^0} \lesssim \|\partial_z v\|_{L^\infty}$$

and consequently, according to (3.78) and Lemma 3.25, we conclude the proof of the claim (3.83) for $j = 2$. It remains to estimate F_3 . In light of (3.78) it is enough to prove that

$$(3.85) \quad \|F_3\|_{L^2(I;H^{s-1})} \leq \mathcal{F}(\|\eta\|_{H^{s+\frac{1}{2}}}) \|\eta\|_{C_*^{3/2}} \|\nabla_{x,z} v\|_{L^2(I;H^{s-1})},$$

for some non-decreasing function. Directly from the definition of α and β , by using the tame estimates in Hölder spaces (2.18), we verify that the symbols a, A belong to $\Gamma_{1/2}^1(\mathbf{R}^d \times I)$ and that they satisfy

$$(3.86) \quad \mathcal{M}_{1/2}^1(a) + \mathcal{M}_{1/2}^1(A) \leq \mathcal{F}(\|\eta\|_{H^{s+\frac{1}{2}}})\|\eta\|_{C_*^{3/2}}.$$

Notice for alter purpose that we use here only $s > \frac{1}{2} + \frac{d}{2}$. Moreover, according to (3.86) we have

$$\mathcal{M}_{\epsilon-1}^1(\partial_z A) \leq \mathcal{F}(\|\eta\|_{H^{s+\frac{1}{2}}}),$$

and by definition (see the proof of Lemma 3.28),

$$R_0(z) := T_{a(z)}T_{A(z)} - T_\alpha\Delta, \quad R_1(z) := -T_{\partial_z A}.$$

As in the proof of 3.28, we deduce, using Theorem 2.7, (ii) applied with $\rho = 1/2$, and Proposition 2.13 with $\rho = \epsilon - 1$, that

$$(3.87) \quad \sup_{z \in [-1, 0]} \|R_0(z)\|_{H^{\mu+\frac{3}{2}} \rightarrow H^\mu} + \|R_1(z)\|_{H^{\mu+\frac{3}{2}} \rightarrow H^\mu} \leq \mathcal{F}(\|\eta\|_{H^{s+\frac{1}{2}}})\|\eta\|_{C_*^{\frac{3}{2}}},$$

which implies (3.85). This completes the proof of Lemma 3.32. \square

Step 3 : elliptic estimates. Introduce a cutoff function $\kappa = \kappa(z)$, $z \in [-1, 0]$ such that $\kappa(z) = 1$ near $z = 0$ and such that $\kappa(z_1) = 0$ (recall that $I_1 = [z_1, 0]$). Set

$$(3.88) \quad W := \kappa(z)(\partial_z - T_A)v.$$

Now it follows from the paradifferential equation (3.59) for v that

$$\partial_z W - T_a W = F',$$

where

$$F' = \kappa(z)(F_1 + F_2 + F_3) + \kappa'(z)(\partial_z - T_A)v.$$

Our goal is to prove that

$$(3.89) \quad \|W\|_{L^\infty(I_1; H^{s-\frac{1}{2}})} \leq \mathcal{F}(\|\eta\|_{H^{s+\frac{1}{2}}}, \|f\|_{H^s}) \left\{ 1 + \|\eta\|_{C_*^{\frac{3}{2}}} + \|f\|_{C_*^r} \right\}.$$

We have already proved that

$$\|F'\|_{L_z^2(I_1; H^{s-1})} \leq \mathcal{F}(\|\eta\|_{H^{s+\frac{1}{2}}}, \|f\|_{H^s}) \left\{ 1 + \|\eta\|_{C_*^{\frac{3}{2}}} + \|f\|_{C_*^r} \right\}.$$

We now turn to an estimate for $(\partial_z - T_A)v$. To do that we estimate separately $\partial_z v$ and $T_A v$. Clearly, by definition of the space X^{s-1} , noting that $I_1 \subset I_0$, we have

$$\|\partial_z v\|_{L^2(I_1; H^{s-\frac{1}{2}})} \leq \|\partial_z v\|_{L^2(I_0; H^{s-\frac{1}{2}})} \leq \|\nabla_{x,z} v\|_{X^{s-1}(I_0)}.$$

On the other hand, as in the previous step, since $\mathcal{M}_0^1(A) \leq C(\|\eta\|_{H^{s+\frac{1}{2}}})$, we have

$$\begin{aligned} \|T_A v\|_{L^2(I_1; H^{s-\frac{1}{2}})} &\leq \mathcal{F}(\|\eta\|_{H^{s+\frac{1}{2}}}) \|\nabla v\|_{L^2(I_1; H^{s-\frac{1}{2}})} \\ &\leq \mathcal{F}(\|\eta\|_{H^{s+\frac{1}{2}}}) \|\nabla_{x,z} v\|_{X^{s-1}(I_0)}. \end{aligned}$$

Now recall from Proposition 3.22 that

$$\|\nabla_{x,z} v\|_{X^{s-1}(I_0)} \leq \mathcal{F}(\|\eta\|_{H^{s+\frac{1}{2}}}) \|f\|_{H^s}.$$

Therefore,

$$\|W\|_{L^2(I_1; H^{s-\frac{1}{2}})} \leq \|(\partial_z - T_A)v\|_{L^2(I_0; H^{s-\frac{1}{2}})} \leq \mathcal{F}(\|\eta\|_{H^{s+\frac{1}{2}}}) \|f\|_{H^s}.$$

We are now in position to estimate $W = \kappa(z)(\partial_z - T_A)v$ as well as to estimate the last term in the definition of F' . We end up with

$$\begin{aligned}\|W\|_{L^2(I_1; H^{s-\frac{1}{2}})} &\leq \mathcal{F}(\|\eta\|_{H^{s+\frac{1}{2}}}) \|f\|_{H^s}, \\ \|F'\|_{L^2(I_1; H^{s-1})} &\leq \mathcal{F}(\|\eta\|_{H^{s+\frac{1}{2}}}, \|f\|_{H^s}) \left\{ 1 + \|\eta\|_{C^{\frac{3}{2}}} + \|f\|_{C_*^r} \right\}.\end{aligned}$$

Since $W(x, z_1) = 0$ (by definition of the cutoff function κ) and since $a \in \Gamma_\varepsilon^1$ satisfies $\operatorname{Re}(-a(x, \xi)) \geq c|\xi|$, by using Proposition 2.18 applied with $J = I_1$, $\rho = \varepsilon$ and $r = s - 1/2$, we have

$$\|W\|_{X^{s-\frac{1}{2}}(I_1)} \leq \mathcal{F}(\|\eta\|_{H^{s+\frac{1}{2}}}) \left\{ \|F'\|_{Y^{s-\frac{1}{2}}(I_1)} + \|W\|_{L^2(I_1; H^{s-\frac{1}{2}})} \right\}.$$

By definitions, we have

$$\|W\|_{L^\infty(I_1; H^{s-\frac{1}{2}})} \leq \|W\|_{X^{s-\frac{1}{2}}(I_1)} \quad \text{and} \quad \|F'\|_{Y^{s-1}(I_1)} \leq \|F'\|_{L^2(I_1; H^{s-1})}.$$

We thus conclude that W satisfies the desired estimate (3.89).

Step 4 : parilinearization of the Dirichlet-Neumann. We conclude the proof by mimicking the analysis in §3.3.3. Again, we shall only use the following obvious consequence of (3.89): $\|W|_{z=0}\|_{H^{s-\frac{1}{2}}}$ is estimated by the right-hand side of (3.89). Since $W|_{z=0} = \partial_z v - T_A v|_{z=0}$, we thus have proved that

$$(3.90) \quad \|\partial_z v - T_A v|_{z=0}\|_{H^{s-\frac{1}{2}}} \leq \mathcal{F}(\|\eta\|_{H^{s+\frac{1}{2}}}, \|f\|_{H^s}) \left\{ 1 + \|\eta\|_{C^{\frac{3}{2}}} + \|f\|_{C_*^r} \right\}.$$

Now, recall from §3.3.3 that we have

$$G(\eta)f = \frac{1 + |\nabla\rho|^2}{\partial_z\rho} \partial_z v - \nabla\rho \cdot \nabla v \Big|_{z=0},$$

and

$$\zeta_1 := \frac{1 + |\nabla\rho|^2}{\partial_z\rho}, \quad \zeta_2 := \nabla\rho.$$

As for the coefficients α, β , we have

$$(3.91) \quad \left\| \zeta_1 - \frac{4}{h} \right\|_{L^\infty([-1,0]; C_*^{1/2})} + \|\zeta_2\|_{L^\infty([-1,0]; C_*^{1/2})} \leq \mathcal{F}(\|\eta\|_{H^{s+\frac{1}{2}}}) \|\eta\|_{C_*^{3/2}},$$

$$(3.92) \quad \left\| \zeta_1 - \frac{4}{h} \right\|_{L^\infty([-1,0]; H^{s-1/2})} + \|\zeta_2\|_{L^\infty([-1,0]; H^{s-1/2})} \leq \mathcal{F}(\|\eta\|_{H^{s+\frac{1}{2}}}).$$

Write

$$\zeta_1 \partial_z v - \zeta_2 \cdot \nabla v = T_{\zeta_1} \partial_z v - T_{\zeta_2} \nabla v + R',$$

with

$$R' = T_{\partial_z v} \zeta_1 - T_{\nabla v} \cdot \zeta_2 + R(\zeta_1, \partial_z v) - R(\zeta_2, \nabla v).$$

According to Proposition 3.29 and (3.92), we obtain

$$\|T_{\partial_z v} \zeta_1 - T_{\nabla v} \cdot \zeta_2\|_{L^\infty(I_1; H^{s-\frac{1}{2}})} \leq \mathcal{F}(\|\eta\|_{H^{s+\frac{1}{2}}}, \|f\|_{H^s}) \left\{ 1 + \|\eta\|_{C^{\frac{3}{2}}} + \|f\|_{C_*^r} \right\}.$$

To estimate the last two terms, we use the rule (2.11), (3.91), the Sobolev estimates on $\partial_{x,z} v$ proved in Proposition 3.22, and the remark that $R(\zeta_1, \partial_z v) = R(\zeta_1 - \frac{3}{2h}, \partial_z v)$ (because constants count only for low frequencies). By so doing (and using that $s > \frac{1}{2} + \frac{d}{2}$), we find that

$$\|R'\|_{L^\infty(I_1; H^{s-\frac{1}{2}})} \leq \mathcal{F}(\|\eta\|_{H^{s+\frac{1}{2}}}, \|f\|_{H^s}) \left\{ 1 + \|\eta\|_{C^{\frac{3}{2}}} + \|f\|_{C_*^r} \right\}.$$

Furthermore, (3.90) implies that

$$T_{\zeta_1} \partial_z v - T_{\zeta_2} \nabla v \Big|_{z=0} = T_{\zeta_1} T_A v - T_{i\zeta_2 \cdot \xi} v \Big|_{z=0} + R'',$$

where $\|R''\|_{H^{s-\frac{1}{2}}}$ satisfies the same estimate as $\|R'\|_{L^\infty(I_1; H^{s-\frac{1}{2}})}$ does.

Thanks to (2.5) we have

$$\begin{aligned} \|T_{\zeta_1} T_A - T_{\zeta_1 A}\|_{H^s \rightarrow H^{s-\frac{1}{2}}} &\lesssim \|\zeta_1\|_{L^\infty} \mathcal{M}_{1/2}^1(A) + \|\zeta_1\|_{C_*^{1/2}} \mathcal{M}_0^1(A) \\ &\leq \mathcal{F}(\|\eta\|_{H^{s+\frac{1}{2}}}) \|\eta\|_{C_*^{3/2}}, \end{aligned}$$

and hence

$$G(\eta)f = T_{\zeta_1 A} v - T_{i\zeta_2 \cdot \xi} v \Big|_{z=0} + R(\eta)f$$

where

$$\|R(\eta)f\|_{H^{s-\frac{1}{2}}} \leq \mathcal{F}(\|\eta\|_{H^{s+\frac{1}{2}}}) \|\eta\|_{C_*^{3/2}} \|f\|_{H^s}.$$

Since $G(\eta)f = T_\lambda f + R(\eta)f$, this concludes the proof of Theorem 3.17.

4. A priori estimates in Sobolev spaces

Recall that the system reads

$$(4.1) \quad \begin{cases} \partial_t \eta - G(\eta)\psi = 0, \\ \partial_t \psi + g\eta + \frac{1}{2} |\nabla \psi|^2 - \frac{1}{2} \frac{(\nabla \eta \cdot \nabla \psi + G(\eta)\psi)^2}{1 + |\nabla \eta|^2} = 0. \end{cases}$$

As already mentioned, we work with the unknowns $B = B(\eta, \psi)$ and $V = V(\eta, \psi)$ defined by

$$B := \frac{\nabla \eta \cdot \nabla \psi + G(\eta)\psi}{1 + |\nabla \eta|^2}, \quad V := \nabla \psi - B \nabla \eta.$$

It follows from Theorem 3.18 that, for all $s > 1/2 + d/2$ and all $(\eta, \psi) \in H^{s+\frac{1}{2}}$, B and V are well defined and belong to $H^{s-\frac{1}{2}}$. Moreover, we shall prove that if they belong initially to H^s then this regularity is propagated by the equation. We shall prove estimates in terms of

$$(4.2) \quad \begin{aligned} M_s(T) &:= \sup_{\tau \in [0, T]} \|(\psi(\tau), B(\tau), V(\tau), \eta(\tau))\|_{H^{s+\frac{1}{2}} \times H^s \times H^s \times H^{s+\frac{1}{2}}}, \\ M_{s,0} &:= \|(\psi(0), B(0), V(0), \eta(0))\|_{H^{s+\frac{1}{2}} \times H^s \times H^s \times H^{s+\frac{1}{2}}}, \\ Z_r(T) &:= \left(\int_0^T \left\{ \|\eta(\tau)\|_{C_*^{r+\frac{1}{2}}}^2 + \|(B(\tau), V(\tau))\|_{C_*^r \times C_*^r}^2 \right\} d\tau \right)^{\frac{1}{2}}. \end{aligned}$$

The main result of this section is the following proposition.

PROPOSITION 4.1. *Let $d \geq 1$ and consider $s, r \in]1, +\infty[$ such that*

$$(4.3) \quad s > \frac{3}{4} + \frac{d}{2}, \quad s + \frac{1}{4} - \frac{d}{2} > r > 1.$$

Consider a fluid domain such that, there exists $h > 0$ such that for all $t \in [0, T]$,

$$(4.4) \quad \left\{ (x, y) \in \mathbf{R}^d \times \mathbf{R} : \eta(t, x) - h < y < \eta(t, x) \right\} \subset \Omega(t).$$

Assume that for any $t \in [0, T]$,

$$a(t, x) \geq c_0,$$

for some given positive constant c_0 . Then, there exists a non-decreasing function $\mathcal{F}: \mathbf{R}^+ \rightarrow \mathbf{R}^+$ such that, for all $T \in (0, 1]$ and all smooth solution (η, ψ) of (4.1) defined on the time interval $[0, T]$, there holds

$$(4.5) \quad M_s(T) \leq \mathcal{F}(\mathcal{F}(M_{s,0}) + T\mathcal{F}(M_s(T) + Z_r(T))).$$

REMARK 4.2. (i) If $s > 1 + d/2$ then we have $Z_r(T) \lesssim M_s(T)$ for $r = s - d/2$.

Then, the estimate (4.5) is an *a priori* estimate in Sobolev spaces.

(ii) For $s < 1 + d/2$, to obtain a closed system of inequalities, one need an *a priori* estimate for Z_r . This will be done in the upcoming paper [3].

(iii) The assumption (4.4) holds provided that it holds initially at time 0 and

$$\|\eta - \eta|_{t=0}\|_{H^{s+\frac{1}{2}}} \leq \epsilon,$$

for some small enough positive constant ϵ .

4.1. Notations. From now on, we consider a fixed time $0 < T \leq 1$, indexes s, r satisfying (4.3) and a smooth solution $(\eta, \psi) \in C^\infty([0, T]; H^\infty(\mathbf{R}^d))$ of (1.6).

Given $I \subset \mathbf{R}$, and two functions $f: I \rightarrow X$ and $g: I \rightarrow Y$, we use the notation $\|f\|_X \leq \|g\|_Y$ to say that $\|f(t)\|_X \leq \|g(t)\|_Y$ for all $t \in I$.

Hereafter, \mathcal{F} always refer to a non-decreasing function $\mathcal{F}: \mathbf{R}^+ \rightarrow \mathbf{R}^+$ depending only on s, d and h, c_0 (being of course independent of T and the unknowns).

4.2. A new formulation. Since we consider low regularity solutions, various cancellations have to be used. We found that these cancellations are most easily seen by working with the incompressible Euler equation directly, and hence we do not use the Zakharov formulation. This means that we begin with a new formulation of the water waves system which involves the following unknowns

$$(4.6) \quad \zeta = \nabla\eta, \quad B = \partial_y\phi|_{y=\eta}, \quad V = \nabla_x\phi|_{y=\eta}, \quad a = -\partial_y P|_{y=\eta},$$

where recall that ϕ is the velocity potential and the pressure $P = P(t, x, y)$ is given by

$$(4.7) \quad -P = \partial_t\phi + \frac{1}{2}|\nabla_{x,y}\phi|^2 + gy.$$

PROPOSITION 4.3. *We have*

$$(4.8) \quad (\partial_t + V \cdot \nabla)B = a - g,$$

$$(4.9) \quad (\partial_t + V \cdot \nabla)V + a\zeta = 0,$$

$$(4.10) \quad (\partial_t + V \cdot \nabla)\zeta = G(\eta)V + \zeta G(\eta)B + \gamma,$$

where the remainder term $\gamma = \gamma(\eta, \psi, V)$ satisfies the following estimate : if $s > \frac{1}{2} + \frac{d}{2}$ then

$$(4.11) \quad \|\gamma\|_{H^{s-\frac{1}{2}}} \leq \mathcal{F}(\|(\eta, \psi, V)\|_{H^{s+\frac{1}{2}} \times H^s \times H^s}).$$

REMARK 4.4. In the case $\Gamma = \emptyset$, one can see that at least formally $\gamma = 0$.

PROOF. For any function $f = f(t, x, y)$, by using the chain rule, we check that, with $\nabla = \nabla_x$,

$$\begin{aligned} (\partial_t + V \cdot \nabla)(f|_{y=\eta(t,x)}) &= (\partial_t + V \cdot \nabla)f(t, x, \eta(t, x)) \\ &= [\partial_t f + \nabla\phi \cdot \nabla f + \partial_y f(\partial_t\eta + V \cdot \nabla\eta)]|_{y=\eta(t,x)} \\ &= [(\partial_t + \nabla_{x,y}\phi \cdot \nabla_{x,y})f]|_{y=\eta(t,x)}. \end{aligned}$$

since $\partial_t \eta + V \cdot \nabla \eta = B$ (see (3.22)). Applying ∂_y to (4.7), this identity yields (4.8). On the other hand, applying ∂_{x_k} to (4.7), the previous identity gives

$$(\partial_t + V \cdot \nabla)V + (\nabla P)|_{y=\eta} = 0.$$

Since $P|_{y=\eta} = 0$, we have

$$0 = \nabla(P|_{y=\eta}) = (\nabla P)|_{y=\eta} + (\partial_y P)|_{y=\eta} \nabla \eta,$$

which yields (4.9).

To derive equation (4.10) on $\zeta := \nabla \eta$ we start from

$$\partial_t \eta = B - V \cdot \nabla \eta$$

Differentiating with respect to x_i (for $i = 1, \dots, d$) we find that $\partial_i \eta = \partial_{x_i} \eta$ satisfies

$$(4.12) \quad (\partial_t + V \cdot \nabla) \partial_i \eta = \partial_i B - \sum_{j=1}^d \partial_i V_j \partial_j \eta,$$

Starting from the definitions of B and V ($B = \partial_y \phi|_{y=\eta}$, $V = \nabla \phi|_{y=\eta}$), and using the chain rule, we compute that

$$\begin{aligned} \partial_i B - \sum_{j=1}^d \partial_i V_j \partial_j \eta &= [\partial_i \partial_y \phi + \partial_i \eta \partial_y^2 \phi] |_{y=\eta} - \sum_{j=1}^d \partial_j \eta [\partial_i \partial_j \phi + \partial_i \eta \partial_j \partial_y \phi] |_{y=\eta} \\ &= [\partial_y \partial_i \phi - \sum_{j=1}^d \partial_j \eta \partial_i \partial_j \phi] |_{y=\eta} + \partial_i \eta [\partial_y^2 \phi - \sum_{j=1}^d \partial_j \eta \partial_j \partial_y \phi] |_{y=\eta}. \end{aligned}$$

Therefore

$$(4.13) \quad (\partial_t + V \cdot \nabla) \partial_i \eta = [\partial_y \partial_i \phi - \nabla \eta \cdot \nabla \partial_i \phi] |_{y=\eta} + \partial_i \eta [\partial_y (\partial_y \phi) - \nabla \eta \cdot \nabla \partial_y \phi] |_{y=\eta}.$$

Let now θ_i be the variational solution of the problem

$$\Delta_{x,y} \theta_i = 0 \text{ in } \Omega, \quad \theta_i|_{\Sigma} = V_i, \quad \partial_n \theta_i = 0 \text{ on } \Gamma.$$

Then

$$G(\eta) V_i = \sqrt{1 + |\nabla \eta|^2} \frac{\partial \theta_i}{\partial n} \Big|_{\Sigma} = (\partial_y \theta_i - \nabla \eta \cdot \nabla \theta_i)|_{\Sigma}.$$

Then we write

$$(4.14) \quad (\partial_y - \nabla \eta \cdot \nabla) \partial_i \phi = G(\eta) V_i + R_i, \quad \text{where } R_i = (\partial_y - \nabla \eta \cdot \nabla) (\partial_i \phi - \theta_i)|_{\Sigma}.$$

Due to the presence of the bottom we have to localize the problem near Σ .

Let $\chi_0 \in C^\infty(\mathbf{R})$, $\eta_1 \in H^\infty(\mathbf{R}^d)$ be such that $\chi_0(z) = 1$ if $z \geq 0$, $\chi_0(z) = 0$ if $z \leq -\frac{1}{4}$ and

$$\eta(x) - \frac{h}{4} \leq \eta_1(x) \leq \eta(x) - \frac{h}{5}.$$

Set

$$U_i(x, y) = \chi_0\left(\frac{y - \eta_1(x)}{h}\right) (\partial_i \phi - \theta_i)(x, y).$$

We see easily that

$$R_i = (\partial_y - \nabla \eta \cdot \nabla) U_i|_{\Sigma}.$$

Moreover U_i satisfies the equation

$$(4.15) \quad \Delta_{x,y} U_i = \left[\Delta_{x,y} \chi_0\left(\frac{y - \eta_1(x)}{h}\right) \right] (\partial_i \phi - \theta_i) := F_i$$

and, with a slight change of notation, we have

$$(4.16) \quad \text{supp } F_i \subset S_{\frac{1}{2}, \frac{1}{5}} := \left\{ (x, y) : x \in \mathbf{R}^d, \eta(x) - \frac{h}{2} \leq y \leq \eta(x) - \frac{h}{5} \right\}.$$

Moreover by ellipticity (see Lemma 3.16) we have for all $\alpha \in \mathbf{N}^{d+1}$,

$$(4.17) \quad \|D_{x,y}^\alpha F_i\|_{L^\infty(S_{\frac{1}{2},\frac{1}{5}}) \cap L^2(S_{\frac{1}{2},\frac{1}{5}})} \leq C_\alpha \|(V, B)\|_{H^{\frac{1}{2}} \times H^{\frac{1}{2}}}.$$

Now we change variables. We set $x = x, y = \rho(x, z) = (1+z)e^{\delta z \langle D_x \rangle} \eta(x) - z\eta_*(x)$ and $\tilde{g}_i(x, z) = g_i(x, \rho(x, z))$. Since we have taken $\delta \|\eta\|_{H^{s+\frac{1}{2}}} \leq \frac{1}{2}$ it is easy to see that on the image of $S_{\frac{1}{2},\frac{1}{5}}$ one has $-h \leq z \leq -\frac{h}{10}$. Now, according to section 3.2.1, \tilde{U}_i is a solution of the problem

$$(\partial_z^2 + \alpha \Delta + \beta \cdot \nabla \partial_z - \gamma \partial_z) \tilde{U}_i = \frac{(\partial_z \rho)^2}{1 + |\nabla \rho|^2} \tilde{F}_i.$$

Due to the exponential smoothing and to (4.17), on the support of \tilde{F}_i the right hand side of the above equation belongs in fact to $C_z^0((-h, 0); H^\infty(\mathbf{R}^d))$. In particular we can apply Proposition 3.22 with $f = 0$. It follows that

$$\|\nabla_{x,z} \tilde{U}_i\|_{C^0([z_0, 0]; H^{s-\frac{1}{2}}(\mathbf{R}^d))} \leq \mathcal{F}(\|\eta\|_{H^{s+\frac{1}{2}}}) (\|\tilde{F}_i\|_{Y^\sigma([-1, 0])} + \|\nabla_{x,z} \tilde{U}_i\|_{X^{-\frac{1}{2}}([-1, 0])}).$$

Notice that according to the constructions of variational solutions and (3.21), the norm of \tilde{U}_i in $X^{-\frac{1}{2}}([-1, 0])$ is bounded by

$$\mathcal{F}(\|\eta\|_{H^{s+\frac{1}{2}}}) (\|\psi\|_{H^{\frac{1}{2}}} + \|V_i\|_{H^{\frac{1}{2}}}).$$

Since

$$R_i = \left[\left(\frac{1 + |\nabla \eta|^2}{1 + \delta \langle D_x \rangle \eta} \partial_z - \nabla \eta \cdot \nabla \right) \tilde{U}_i \right] \Big|_{z=0},$$

we deduce that

$$\|R_i\|_{H^{s-\frac{1}{2}}} \leq \mathcal{F}(\|\eta\|_{H^{s+\frac{1}{2}}}) (\|\psi\|_{H^{\frac{1}{2}}} + \|V_i\|_{H^{\frac{1}{2}}}) \leq \mathcal{F}(\|\eta\|_{H^{s+\frac{1}{2}}}, \|\psi\|_{H^s}, \|V\|_{H^s}),$$

since $s > \frac{1}{2} + \frac{d}{2}$. We use exactly the same argument to show that

$$(4.18) \quad (\partial_y - \nabla \eta \cdot \nabla) \partial_y \phi = G(\eta)B + R_0,$$

where R_0 satisfies the same estimate as R_i . This completes the proof of Proposition 4.3. \square

Following the same lines, we have the following relation between V and B .

PROPOSITION 4.5. *Let $s > \frac{1}{2} + \frac{d}{2}$. Then we have $G(\eta)B = -\operatorname{div} V + \gamma$ where*

$$\|\gamma\|_{H^{s-\frac{1}{2}}} \leq \mathcal{F}(\|(\eta, V, B)\|_{H^{s+\frac{1}{2}} \times H^{\frac{1}{2}} \times H^{\frac{1}{2}}}).$$

PROOF. Recall that, by definition, $B = \partial_y \phi|_{y=\eta}$ and $V = \nabla \phi|_{y=\eta}$. Let θ be the variational solution to the problem

$$\Delta_{x,y} \theta = 0, \quad \theta|_{y=\eta} = B, \quad \partial_n \theta|_\Gamma = 0.$$

Then $G(\eta)B = (\partial_y \theta - \nabla \eta \cdot \nabla \theta)|_{y=\eta}$. Now let $\tilde{\theta} = \partial_y \phi$. We claim that

$$(\partial_y \tilde{\theta} - \nabla \eta \cdot \nabla \tilde{\theta})|_{y=\eta} = -\operatorname{div} V.$$

Indeed, on the one hand we have

$$(\partial_y \tilde{\theta} - \nabla \eta \cdot \nabla \tilde{\theta}) = \partial_y^2 \phi - \nabla \eta \cdot \nabla \partial_y \phi,$$

and on the other hand

$$\operatorname{div} V = \sum_{1 \leq i \leq d} \partial_{x_i} V = \left(\sum_{1 \leq i \leq d} \partial_i^2 \phi + \nabla \eta \cdot \partial_y \phi \right) \Big|_{y=\eta}.$$

Then our claim follows from the fact that $\Delta_{x,y}\phi = 0$. Now we have

$$\Delta_{x,y}(\theta - \tilde{\theta}) = 0, \quad (\theta - \tilde{\theta})|_{y=\eta} = 0,$$

so, as in the proof of Proposition 4.3, we deduce from Proposition 3.22 that

$$\left\| (\partial_y - \nabla\eta \cdot \nabla)(\theta - \tilde{\theta}) \right\|_{H^{s-\frac{1}{2}}} \leq \mathcal{F}(\|(\eta, V, B)\|_{H^{s+\frac{1}{2}} \times H^{\frac{1}{2}} \times H^{\frac{1}{2}}}),$$

which is the desired result. \square

4.3. Estimates for the Taylor coefficient. In this paragraph, we prove several estimates for the Taylor coefficient.

PROPOSITION 4.6. *Let $d \geq 1$ and consider s, r such that*

$$s > \frac{3}{4} + \frac{d}{2}, \quad r > 1.$$

For any $0 < \varepsilon < \min(r - 1, s - \frac{3}{4} - \frac{d}{2})$, there exists a non-decreasing function \mathcal{F} such that, for all $t \in [0, T]$,

$$(4.19) \quad \|a(t) - g\|_{H^{s-\frac{1}{2}}} \leq \mathcal{F}\left(\|(\eta, \psi, V, B)(t)\|_{H^{s+\frac{1}{2}} \times H^{s+\frac{1}{2}} \times H^s \times H^s}\right),$$

and

$$(4.20) \quad \|a(t)\|_{C^{\frac{1}{2}+\varepsilon}} + \|(\partial_t a + V \cdot \nabla a)(t)\|_{C^\varepsilon} \\ \leq \mathcal{F}\left(\|(\eta, \psi, V, B)(t)\|_{H^{s+\frac{1}{2}} \times H^{s+\frac{1}{2}} \times H^s \times H^s}\right) \left\{1 + \|(\eta, V, B)(t)\|_{C_*^{r+\frac{1}{2}} \times C_*^r \times C_*^r}\right\}.$$

We shall see below that the estimate of $\|a - g\|_{H^{s-\frac{1}{2}}}$ follows directly from the elliptic estimates (see Proposition 3.22). The estimate of the C^ε -norm of $\partial_t a + V \cdot \nabla a$ is the easiest one. The main new difficulty here is to prove a tame estimate for $\|a\|_{C^{\frac{1}{2}+\varepsilon}}$. Indeed, there are several further complications which appear in the analysis in Hölder spaces. To prove the above estimates on a , we use that

$$a = -\partial_y P|_{y=\eta},$$

where

$$P = P(t, x, y) = -\left(\partial_t \phi + \frac{1}{2} |\nabla_x \phi|^2 + \frac{1}{2} (\partial_y \phi)^2 + gy\right).$$

The basic idea is that one should be able to easily estimate P since it satisfies an elliptic equation. Indeed, since $\Delta_{x,y}\phi = 0$, we have

$$\Delta_{x,y} P = -|\nabla_{x,y}^2 \phi|^2.$$

Moreover, by assumption we have $P = 0$ on the free surface $\{y = \eta(t, x)\}$. Yet, this requires some preparation because, as we shall see, the regularity of P is *not* given by the right-hand side in the elliptic equation above. Instead the regularity of P is limited by the regularity of the domain (i.e. the regularity of the function η).

Hereafter, since the time variable is fixed, we shall skip it. We use the change of variables $(x, z) \mapsto (x, \rho(x, z))$ introduced in §3.2.1. Introduce φ and \wp given by

$$\varphi(x, z) = \phi(x, \rho(x, z)), \quad \wp(x, z) = P(x, \rho(x, z)) + g\rho(x, z),$$

and notice that

$$a - g = -\frac{1}{\partial_z \rho} \partial_z \wp |_{z=0}.$$

The first elementary step is to compute the equation satisfied by the new unknown v in $\{z < 0\}$ as well as the boundary conditions on $\{z = 0\}$. Set (see (3.14))

$$\Lambda = (\Lambda_1, \Lambda_2), \quad \Lambda_1 = \frac{1}{\partial_z \rho} \partial_z, \quad \Lambda_2 = \nabla - \frac{\nabla \rho}{\partial_z \rho} \partial_z.$$

We find that

$$\begin{aligned} (\Lambda_1^2 + \Lambda_2^2)\varphi &= 0 \quad \text{in } -1 < z < 0, \\ (\Lambda_1^2 + \Lambda_2^2)\wp &= -|\Lambda^2\varphi|^2 \quad \text{in } -1 < z < 0, \\ (\Lambda_1^2 + \Lambda_2^2)\rho &= 0 \quad \text{in } z < 0, \end{aligned}$$

together with the boundary conditions

$$\begin{aligned} \wp &= g\eta, \quad \Lambda_1\wp = g - a \quad \text{on } z = 0, \\ \Lambda_2\wp &= V, \quad \Lambda_1\varphi = B, \quad \text{on } z = 0. \end{aligned}$$

According to (3.5) and Remark 3.21, we have the *a priori estimate*

$$\|\nabla_{x,z}\varphi\|_{X^{-\frac{1}{2}}(-\frac{h}{2}, 0)} \leq \mathcal{F}(\|\eta\|_{H^{s+\frac{1}{2}}})\|\psi\|_{H^{\frac{1}{2}}},$$

while according to Proposition 3.22

$$(4.21) \quad \|\nabla_{x,z}\wp\|_{X^{-\frac{1}{2}}(-1, 0)} \leq \mathcal{F}\left(\|\mathcal{R}\|_{X^{\frac{1}{2}}(-1, 0)} + \|\|\nabla\varphi\|^2\|_{X^{\frac{1}{2}}(-1, 0)}\right) \leq \mathcal{F}(\|(\eta, \psi)\|_{H^{s+\frac{1}{2}}}).$$

where $\mathcal{R}(x, z) = R(x, \rho(x, z))$ and R is defined in Definition 1.5.

Expanding $\Lambda_1^2 + \Lambda_2^2$, we thus find that \wp solves

$$(4.22) \quad \begin{aligned} \partial_z^2\wp + \alpha\Delta\wp + \beta \cdot \nabla\partial_z\wp - \gamma\partial_z\wp &= F_0(x, z) \quad \text{for } z < 0, \\ \wp &= 0 \quad \text{on } z = 0, \end{aligned}$$

where α, β, γ are as above (see (3.18)) and where

$$(4.23) \quad F_0 = -\alpha|\Lambda^2\varphi|^2.$$

Our first task is to estimate the source term F_0 .

LEMMA 4.7. *Let $d \geq 1$ and consider $s, r \in]1, +\infty[$ such that*

$$s > \frac{3}{4} + \frac{d}{2}, \quad s + \frac{1}{4} - \frac{d}{2} > r > 1.$$

Then there exists $z_0 < 0$ such that

$$\|F_0\|_{L^1([z_0, 0]; H^{s-\frac{1}{2}})} + \|F_0\|_{L^2([z_0, 0]; C_*^{s+\frac{1}{4}-\frac{d}{2}})} \leq \mathcal{F}(\|(\eta, \psi, V, B)\|_{H^{s+\frac{1}{2}} \times H^{s+\frac{1}{2}} \times H^s \times H^s}).$$

PROOF. Since $[\Lambda_1, \Lambda_2] = 0$ we have

$$(\Lambda_1^2 + \Lambda_2^2)\Lambda_2\varphi = 0, \quad (\Lambda_1^2 + \Lambda_2^2)\Lambda_1\varphi = 0.$$

Since $\Lambda_2\varphi|_{z=0} = V$ and $\Lambda_1\varphi|_{z=0} = B$, it follows from Proposition 3.22 (and Theorem 3.8 which guarantees that $\nabla_{x,z}\varphi \in X^{-\frac{1}{2}}(z_0, 0)$) that

$$\|\nabla_{x,z}\Lambda_j\varphi\|_{X^{s-1}([z_0, 0])} \leq \mathcal{F}(\|(\eta, \psi, V, B)\|_{H^{s+\frac{1}{2}} \times H^{s+\frac{1}{2}} \times H^s \times H^s}).$$

By using the easy estimate (3.11)

$$\|\nabla_x\rho\|_{C^0([z_0, 0]; H^{s-\frac{1}{2}})} + \left\| \partial_z\rho - \frac{h}{4} \right\|_{C^0([z_0, 0]; H^{s-\frac{1}{2}})} \lesssim \|\eta\|_{H^{s+\frac{1}{2}}},$$

and the product rule in Sobolev spaces, we obtain

$$(4.24) \quad \|\Lambda_j \Lambda_k \varphi\|_{C^0([z_0, 0]; H^{s-1})} + \|\Lambda_j \Lambda_k \varphi\|_{L^2([z_0, 0]; H^{s-\frac{1}{2}})} \\ \leq \mathcal{F}(\|(\eta, \psi, V, B)\|_{H^{s+\frac{1}{2}} \times H^{s+\frac{1}{2}} \times H^s \times H^s}).$$

Since $H^{s-\frac{1}{2}}$ is an algebra, according to Lemma 3.25, we obtain

$$(4.25) \quad \|F_0\|_{L^1([z_0, 0]; H^{s-\frac{1}{2}})} \lesssim \left(1 + \left\| \alpha - \frac{h^2}{16} \right\|_{C^0([z_0, 0]; H^{s-\frac{1}{2}})}\right) \|\Lambda_j \Lambda_k \varphi\|_{L^2([z_0, 0]; H^{s-\frac{1}{2}})}^2 \\ \leq \mathcal{F}(\|(\eta, \psi, V, B)\|_{H^{s+\frac{1}{2}} \times H^{s+\frac{1}{2}} \times H^s \times H^s}).$$

By interpolation, (4.24) also implies that

$$\|\Lambda_j \Lambda_k \varphi\|_{L^4([z_0, 0]; H^{s-\frac{3}{4}})} \leq \mathcal{F}(\|(\eta, \psi, V, B)\|_{H^{s+\frac{1}{2}} \times H^{s+\frac{1}{2}} \times H^s \times H^s}).$$

Since $s > 3/4 + d/2$, the Sobolev space $H^{s-\frac{3}{4}}$ is an algebra and hence, according to Lemma 3.25,

$$\left\| \alpha |\Lambda_j \Lambda_k \varphi|^2 \right\|_{L^2([z_0, 0]; H^{s-\frac{3}{4}})} \lesssim \left(1 + \left\| \alpha - \frac{h^2}{16} \right\|_{L^\infty([z_0, 0]; H^{s-\frac{3}{4}})}\right) \|\Lambda_j \Lambda_k \varphi\|_{L^4([z_0, 0]; H^{s-\frac{3}{4}})}^2.$$

The Sobolev embedding $H^{s-\frac{3}{4}} \subset C_*^{s-\frac{3}{4}-\frac{d}{2}}$ then yields

$$(4.26) \quad \|F_0\|_{L^2([z_0, 0]; C_*^{s-\frac{3}{4}-\frac{d}{2}})} \leq \mathcal{F}(\|(\eta, \psi, V, B)\|_{H^{s+\frac{1}{2}} \times H^{s+\frac{1}{2}} \times H^s \times H^s}).$$

This completes the proof. \square

It follows from Lemma 4.7 and Proposition 3.22 applied with $\sigma = s - 1/2$ that there exists z_0 such that

$$(4.27) \quad \|\nabla_{x,z} \varrho\|_{X^{s-\frac{1}{2}}([z_0, 0])} \leq \mathcal{F}(\|(\eta, \psi)\|_{H^{s+\frac{1}{2}}}, \|F_0\|_{L^1([z_0, 0]; H^{s-\frac{1}{2}})}).$$

where we used the estimate (4.21). According to (4.25), this implies that

$$(4.28) \quad \|\nabla_{x,z} \varrho\|_{X^{s-\frac{1}{2}}([z_0, 0])} \leq \mathcal{F}(\|(\eta, \psi, V, B)\|_{H^{s+\frac{1}{2}} \times H^{s+\frac{1}{2}} \times H^s \times H^s}).$$

which in turn implies that $\|a - g\|_{H^{s-\frac{1}{2}}}$ is bounded by a constant depending only on $\|(\eta, \psi)\|_{H^{s+\frac{1}{2}}}$ and $\|(V, B)\|_{H^s}$.

We shall now deduce the estimate of $\|a\|_{C^{1/2+\epsilon}}$, which is the most delicate one, from the following result.

PROPOSITION 4.8. *Let $d \geq 1$ and consider $(s, r, r') \in \mathbf{R}^3$ such that*

$$s > \frac{3}{4} + \frac{d}{2}, \quad s + \frac{1}{4} - \frac{d}{2} \geq r > r' > 1.$$

Then there exists $z_0 < 0$ such that

$$(4.29) \quad \|\nabla_{x,z} \varrho\|_{C^0([z_0, 0]; C_*^{r'-\frac{1}{2}})} \leq \mathcal{F}(\|(\eta, \psi, V, B)\|_{H^{s+\frac{1}{2}} \times H^{s+\frac{1}{2}} \times H^s \times H^s}) \left\{1 + \|\eta\|_{C_*^{r+\frac{1}{2}}}\right\}.$$

for some non-decreasing function \mathcal{F} depending only on s, r, r' .

PROOF. It follows from (4.28) and the Sobolev embedding $H^{s-\frac{1}{2}}(\mathbf{R}^d) \subset C_*^{\frac{1}{4}}(\mathbf{R}^d)$ that

$$(4.30) \quad \|\nabla_{x,z} \varrho\|_{L^\infty([z_0, 0]; C^{1/4})} \leq \mathcal{F}(\|(\eta, \psi, V, B)\|_{H^{s+\frac{1}{2}} \times H^{s+\frac{1}{2}} \times H^s \times H^s}).$$

To prove (4.29), let us revisit the proof of Theorem 3.17. With the notations of Section 3.3 (see (3.88)), $W = \kappa(z)(\partial_z - T_A)\varphi$ satisfies a parabolic evolution equation of the form

$$\partial_z W - T_a W = F_0 + F_1 + F_2 + F_3 + F_4,$$

where the symbols a and A are as defined in Lemma 3.28, F_0 is given by (4.23) and

$$\begin{aligned} F_1 &= \gamma \partial_z \varphi, \\ F_2 &= -(T_{\Delta} \varphi \alpha + R(\alpha, \Delta \varphi) + T_{\nabla \partial_z \varphi} \cdot \beta + R(\beta, \nabla \partial_z \varphi)), \\ F_3 &= R_0 \varphi + R_1 \varphi = (T_a T_A - T_\alpha \Delta) \varphi + T_{\partial_z A} \varphi, \\ F_4 &= \kappa'(z)(\partial_z - T_A) \varphi. \end{aligned}$$

In light of (4.30) and Proposition 2.19 (applied with $r_0 = 1/4$, $q = +\infty$), one can reduce the proof of (4.29) to proving that, for some $\delta > 0$, the $L^\infty([z_0, 0]; C_*^{r' - \frac{1}{2} + \delta})$ norm of W is bounded by the right-hand side of (4.29). Again, by using Proposition 2.19 (with $q = 2$), the former estimate will result from the following lemma.

LEMMA 4.9. *There exists $z_0 < 0$ and $\epsilon > 0$ such that, for $i \in \{0, \dots, 4\}$,*

$$\|F_i\|_{L^2([z_0, 0]; C_*^{r'-1+\epsilon})} \leq \mathcal{F}(\|(\eta, \psi, V, B)\|_{H^{s+\frac{1}{2}} \times H^{s+\frac{1}{2}} \times H^s \times H^s}) \left\{ 1 + \|\eta\|_{C^{r+\frac{1}{2}}} \right\}.$$

To prove Lemma 4.9 we begin by recording two easy refinements of previous bounds. First of all, since $r-1 \leq s-1/2-d/2$ we have $H^{s-\frac{1}{2}} \subset C_*^{r-1}$ and hence it immediately follows from (4.28) that

$$(4.31) \quad \|\nabla_{x,z} \varphi\|_{C^0([z_1, 0]; C_*^{r-1})} \lesssim \|\nabla_{x,z} \varphi\|_{X^{s-\frac{1}{2}}([z_0, 0])} \leq \mathcal{F}(\|(\eta, \psi, V, B)\|_{H^{s+\frac{1}{2}} \times H^{s+\frac{1}{2}} \times H^s \times H^s}).$$

The key point is that we now have an L^∞ -estimate for $\nabla_{x,z} \varphi$ which does not depend on the higher order norms (compare with Proposition 3.29).

Our second observation concerns the estimates in Hölder spaces for the coefficients α, β, γ . Of course, since we now allow estimates to depend on $\|\eta\|_{C^{r+\frac{1}{2}}}$ instead of $\|\eta\|_{C^{\frac{3}{2}}}$, we have the following variant of Lemma 3.31:

$$\|\alpha\|_{C^0([z_0, 0]; C_*^{r-\frac{1}{2}})} + \|\beta\|_{C^0([z_0, 0]; C_*^{r-\frac{1}{2}})} + \|\gamma\|_{C^0([z_0, 0]; C_*^{r-\frac{3}{2}})} \leq \mathcal{F}(\|\eta\|_{H^{s+\frac{1}{2}}}) (\|\eta\|_{C^{r+\frac{1}{2}}} + 1).$$

However, for later purpose, we need estimates for $\alpha(z), \beta(z)$ in C_*^r instead of $C_*^{r-\frac{1}{2}}$ and $\gamma(z)$ in C_*^{r-1} instead of $C_*^{r-\frac{3}{2}}$. Similarly, it is not enough to control $\gamma(z)$ in $C_*^{r-\frac{3}{2}}$ since $r - \frac{3}{2}$ might be negative. We have such estimates to the prize of a lack of uniformity in z . Namely, we have the following estimate:

$$(4.32) \quad \|\alpha\|_{L^2([z_0, 0]; C_*^r)} + \|\beta\|_{L^2([z_0, 0]; C_*^r)} + \|\gamma\|_{L^2([z_0, 0]; C_*^{r-1})} \leq \mathcal{F}(\|\eta\|_{H^{s+\frac{1}{2}}}) (\|\eta\|_{C^{r+\frac{1}{2}}} + 1).$$

Again, such estimates follow from the definition of ρ (by means of the Poisson kernel), the Sobolev embedding $H^{s+\frac{1}{2}} \subset W^{1, \infty}$ and tame estimates in Hölder spaces (2.22). We are now ready to conclude the proof of Lemma 4.9

Estimate of F_0 . Similarly, since $C_*^{s-3/4-d/2} \subset C_*^{r-1}$, (4.26) implies that

$$\|F_0\|_{L^2([z_0, 0]; C_*^{r-1})} \leq \mathcal{F}(\|(\eta, \psi, V, B)\|_{H^{s+\frac{1}{2}} \times H^{s+\frac{1}{2}} \times H^s \times H^s}).$$

Estimate of F_1 . The term $F_1 = \gamma \partial_z \varphi$ is estimated by means of

$$\|F_1\|_{L_z^2([z_0, 0]; C_*^{r-1})} \leq K \|\partial_z \varphi\|_{L_z^\infty([z_0, 0]; C_*^{r-1})} \|\gamma\|_{L_z^2([z_0, 0]; C_*^{r-1})}.$$

The desired estimate then follows from (4.31) and (4.32).

Estimate of F_2 . According to (2.13) with $m = 1, s = r$, we obtain

$$\begin{aligned} \|T_{\Delta\wp(z)}\alpha(z)\|_{C_*^{r-1}} &\lesssim \|\Delta\wp(z)\|_{C_*^{-1}} \|\alpha(z)\|_{C_*^r}, \\ \|T_{\nabla\partial_z\wp(z)}\cdot\beta(z)\|_{C_*^{r-1}} &\lesssim \|\nabla\partial_z\wp(z)\|_{C_*^{-1}} \|\beta(z)\|_{C_*^r}. \end{aligned}$$

On the other hand, since $r > 1$ we can apply (2.10) to obtain

$$\begin{aligned} \|R(\alpha, \Delta\wp)(z)\|_{C_*^{r-1}} &\lesssim \|\Delta\wp(z)\|_{C_*^{-1}} \|\alpha(z)\|_{C_*^r}, \\ \|R(\beta, \nabla\partial_z\wp)(z)\|_{C_*^{r-1}} &\lesssim \|\nabla\partial_z\wp(z)\|_{C_*^{-1}} \|\beta(z)\|_{C_*^r}. \end{aligned}$$

By using (4.31) and (4.32), we conclude the proof of the claim in Lemma 4.9 for $i = 2$.

Estimate of F_3 . Using (3.87) with $\mu = s - 1/2$ we find

$$\|F_3(z)\|_{C_*^{r-1}} \lesssim \|F_3(z)\|_{H^{s-1/2}} \leq \mathcal{F}(\|\eta\|_{H^{s+1/2}}) \|\eta\|_{C_*^{3/2}} \|\nabla_{x,z}\wp(z)\|_{H^s}.$$

and hence

$$\|F_3\|_{L^2([z_0, 0]; C_*^{r-1})} \leq \mathcal{F}(\|\eta\|_{H^{s+1/2}}) \|\eta\|_{C_*^{3/2}} \|\nabla_{x,z}\wp\|_{X^{s-1/2}([z_0, 0])},$$

by definition of $X^{s-1/2}([z_0, 0])$. The desired estimate follows from (4.28).

Estimate of F_4 . This follows from (4.31).

This completes the proof of Lemma 4.9 and hence the proof of Proposition 4.8. \square

4.4. Paralinearization of the system. Introduce

$$(4.33) \quad U = V + T_\zeta B.$$

To clarify notations, let us mention that the i th component ($i = 1, \dots, d$) of this vector valued unknown satisfies $U_i = V_i + T_{\partial_i\eta}B$. The new unknown U is related to what is called the good-unknown of Alinhac in [5, 1, 6, 8].

To estimate (U, ζ) in Sobolev spaces, we want to estimate $(\langle D_x \rangle^s U, \langle D_x \rangle^{s-1/2} \zeta)$ in $L^\infty([0, T]; L^2 \times L^2)$ where $\langle D_x \rangle := (I - \Delta)^{1/2}$. However, for technical reasons, instead of working with $(\langle D_x \rangle^s U, \langle D_x \rangle^{s-1/2} \zeta)$, it is more convenient to work with

$$(4.34) \quad \begin{aligned} U_s &:= \langle D_x \rangle^s V + T_\zeta \langle D_x \rangle^s B, \\ \zeta_s &:= \langle D_x \rangle^s \zeta. \end{aligned}$$

PROPOSITION 4.10. *Under the assumptions of Proposition 4.1, there exists a non decreasing function \mathcal{F} such that*

$$(4.35) \quad (\partial_t + T_V \cdot \nabla)U_s + T_a \zeta_s = f_1,$$

$$(4.36) \quad (\partial_t + T_V \cdot \nabla)\zeta_s = T_\lambda U_s + f_2,$$

where recall that λ is the symbol

$$\lambda(t; x, \xi) := \sqrt{(1 + |\nabla\eta(t, x)|^2) |\xi|^2 - (\nabla\eta(t, x) \cdot \xi)^2},$$

and where, for each time $t \in [0, T]$,

$$(4.37) \quad \begin{aligned} &\|(f_1(t), f_2(t))\|_{L^2 \times H^{-1/2}} \\ &\leq \mathcal{F}(\|\eta(t)\|_{H^{s+1/2}}, \|(V, B)(t)\|_{H^s}) \left\{ 1 + \|\eta(t)\|_{C_*^{r+1/2}} + \|(V, B)(t)\|_{C_*^r} \right\}. \end{aligned}$$

PROOF. The proof is based on our main result about the parilinearization of the Dirichlet-Neumann operator (see Theorem 3.17), the Bony's parilinearization formula for a product, some simple computations and the commutator estimate proved in Section 2.4.

STEP 1: Parilinearization of the equation

$$(\partial_t + V \cdot \nabla)V + a\zeta = 0.$$

We begin by proving

LEMMA 4.11. *We have*

$$(4.38) \quad (\partial_t + T_V \cdot \nabla)V + T_a\zeta + T_\zeta(\partial_t + T_V \cdot \nabla)B = h_1$$

for some remainder h_1 satisfying

$$\|h_1\|_{H^s} \leq \mathcal{F}(\|\eta\|_{H^{s+\frac{1}{2}}}, \|(V, B)\|_{H^s}) \left\{ 1 + \|\eta\|_{C_*^{r+\frac{1}{2}}} + \|(V, B)\|_{C_*^r} \right\}.$$

PROOF. Using (2.11) and (2.4) we have $V \cdot \nabla V = T_V \cdot \nabla V + A_1$ where $A_1 = \sum_j T_{\partial_j V} V_j + R(\partial_j V, V_j)$ satisfies

$$\|A_1\|_{H^s} \lesssim \|\nabla V\|_{L^\infty} \|V\|_{H^s}.$$

Similarly, $(a - g)\zeta = T_{a-g}\zeta + T_\zeta(a - g) + R(\zeta, a - g)$ where

$$(4.39) \quad \|R(\zeta, a)\|_{H^s} \lesssim \|\zeta\|_{H^{s-1/2}} \|a\|_{C^{1/2}}.$$

and where $\|a\|_{C^{1/2}}$ is estimated by means of (4.20).

Since $T_\zeta g = 0$, by replacing a by $g + (\partial_t B + V \cdot \nabla B)$ we obtain

$$\begin{aligned} T_\zeta a &= T_\zeta(\partial_t B + V \cdot \nabla B) \\ &= T_\zeta(\partial_t B + T_V \cdot \nabla B) + T_\zeta(V - T_V) \cdot \nabla B. \end{aligned}$$

As in the analysis of A_1 above, we have

$$\|(V - T_V) \cdot \nabla B\|_{H^s} \lesssim \|\nabla B\|_{L^\infty} \|V\|_{H^s}.$$

Now we use $\|T_\zeta\|_{H^s \rightarrow H^s} \lesssim \|\zeta\|_{L^\infty} \lesssim \|\eta\|_{H^{s+1/2}}$ (since $s + 1/2 > 1 + d/2$) to obtain

$$\|T_\zeta(V - T_V) \cdot \nabla B\|_{H^s} \lesssim \|\eta\|_{H^{s+\frac{1}{2}}} \|\nabla B\|_{L^\infty} \|V\|_{H^s}.$$

This proves (4.38). \square

STEP 2. We now commute (4.38) with $\langle D_x \rangle^s = (I - \Delta)^{s/2}$. The paradifferential rule (2.5) implies that

$$\begin{aligned} \|[T_a, \langle D_x \rangle^s]\|_{H^{s-1/2} \rightarrow L^2} &\lesssim \|a\|_{W^{1/2, \infty}}, \\ \|[T_\zeta, \langle D_x \rangle^s]\|_{H^{s-1/2} \rightarrow L^2} &\lesssim \|\zeta\|_{W^{1/2, \infty}}, \\ \|[T_V \cdot \nabla, \langle D_x \rangle^s]\|_{H^s \rightarrow L^2} &\lesssim \|V\|_{W^{1, \infty}}. \end{aligned}$$

Consequently, it easily follows from (4.20) and (4.38) that

$$(\partial_t + T_V \cdot \nabla)\langle D_x \rangle^s V + T_a\langle D_x \rangle^s \zeta + T_\zeta(\partial_t + T_V \cdot \nabla)\langle D_x \rangle^s B = h_2$$

for some remainder h_2 satisfying

$$\|h_2\|_{L^2} \leq \mathcal{F}(\|\eta\|_{H^{s+\frac{1}{2}}}, \|(V, B)\|_{H^s}) \left\{ 1 + \|\eta\|_{C_*^{r+\frac{1}{2}}} + \|(V, B)\|_{C_*^r} \right\}.$$

On the other hand, Lemma 2.16 implies that

$$\begin{aligned} & \| [T_\zeta, \partial_t + T_V \cdot \nabla] \langle D_x \rangle^s B(t) \|_{L^2} \\ & \leq \mathcal{F}(\|\eta(t)\|_{H^{s+\frac{1}{2}}}, \|(V, B)(t)\|_{H^s}) \left\{ 1 + \|\eta(t)\|_{C_*^{r+\frac{1}{2}}} + \|(V, B)(t)\|_{C_*^r} \right\}. \end{aligned}$$

Here we have used the fact that the L^∞ norm of $\partial_t \zeta + V \cdot \nabla \zeta$ is, since $s > \frac{1}{2} + \frac{d}{2}$, estimated by means of the identity (4.12):

$$\begin{aligned} \|\partial_t \zeta + V \cdot \nabla \zeta\|_{L^\infty} & \lesssim \|\nabla B\|_{L^\infty} + \|\zeta\|_{L^\infty} \|\nabla V\|_{L^\infty} \\ & \lesssim \|\nabla B\|_{L^\infty} + \|\eta\|_{H^{s+\frac{1}{2}}} \|\nabla V\|_{L^\infty}. \end{aligned}$$

By combining the previous results we obtain

$$(\partial_t + T_V \cdot \nabla)(\langle D_x \rangle^s V + T_\zeta \langle D_x \rangle^s B) + T_a \langle D_x \rangle^s \zeta = f_1$$

where f_1 satisfies the desired estimate (4.37).

STEP 3. Parilinearization of the equation

$$(\partial_t + V \cdot \nabla)\zeta = G(\eta)V + \zeta G(\eta)B + \gamma.$$

Writing $(V - T_V) \cdot \nabla \zeta = T_{\nabla \zeta} \cdot V + \sum_{j=1}^d R(\partial_j \zeta, V_j)$ and using (2.11) and (2.12), we obtain

$$(4.40) \quad \|(V - T_V) \cdot \nabla \zeta\|_{H^{s-\frac{1}{2}}} \lesssim \|\nabla \zeta\|_{C_*^{-\frac{1}{2}}} \|V\|_{H^s} \lesssim \|\eta\|_{C_*^{\frac{3}{2}}} \|V\|_{H^s}.$$

The key step is to parilinearize $G(\eta)V + \zeta G(\eta)B$. This is where we use the analysis performed in Section 3.3.4. By definition of $R(\eta) = G(\eta) - T_\lambda$ we have

$$G(\eta)V + \zeta G(\eta)B = T_\lambda U + F_2(\eta, V, B),$$

where

$$(4.41) \quad F_2 = [T_\zeta, T_\lambda]B + R(\eta)V + \zeta R(\eta)B + (\zeta - T_\zeta)T_\lambda B.$$

The commutator $[T_\zeta, T_\lambda]B$ is estimated by means of (2.5) which implies that

$$\|[T_\zeta, T_\lambda]B\|_{H^{s-\frac{1}{2}}} \lesssim \left\{ M_0^0(\zeta)M_{1/2}^1(\lambda) + M_{1/2}^0(\zeta)M_0^1(\lambda) \right\} \|B\|_{H^s}.$$

Since

$$M_{1/2}^0(\zeta) + M_{1/2}^1(\lambda) \leq \mathcal{K}(\|\eta\|_{H^{s+\frac{1}{2}}}) \|\eta\|_{C_*^{\frac{3}{2}}}, \text{ and } M_0^0(\zeta) + M_0^1(\lambda) \leq \mathcal{K}(\|\eta\|_{H^{s+\frac{1}{2}}})$$

we conclude that

$$\|[T_\zeta, T_\lambda]B\|_{H^{s-\frac{1}{2}}} \leq \mathcal{K}(\|\eta\|_{H^{s+\frac{1}{2}}}, \|B\|_{H^s}) \|\eta\|_{C_*^{\frac{3}{2}}}.$$

Moving to the estimate of the second and third terms in the right-hand side of (4.41), we use Theorem 3.17 to obtain that the $H^{s-\frac{1}{2}}$ -norm of $R(\eta)V$ and $R(\eta)B$ satisfy

$$\begin{aligned} & \|R(\eta(t))V(t)\|_{H^{s-\frac{1}{2}}} + \|R(\eta(t))B(t)\|_{H^{s-\frac{1}{2}}} \\ & \leq \mathcal{F}(\|\eta(t)\|_{H^{s+\frac{1}{2}}}, \|(V, B)(t)\|_{H^s}) \left\{ 1 + \|\eta(t)\|_{C_*^{\frac{3}{2}}} + \|(V, B)(t)\|_{C_*^r} \right\}. \end{aligned}$$

Since $H^{s-\frac{1}{2}}$ is an algebra, the term $\zeta R(\eta)B$ satisfies the same estimate as $R(\eta)B$ does. It remains only to estimate $(\zeta - T_\zeta)T_\lambda B$. To do so we write

$$(\zeta - T_\zeta)T_\lambda B = T_{T_\lambda B} \zeta + R(\zeta, T_\lambda B).$$

Thus (2.11) (applied with $\alpha = 0$ and $\beta = s - 1/2$) implies that

$$\|(\zeta - T_\zeta)T_\lambda B\|_{H^{s-\frac{1}{2}}} \lesssim \|T_\lambda B\|_{C_*^0} \|\zeta\|_{H^{s-\frac{1}{2}}}.$$

Using (2.4) this yields

$$\|(\zeta - T_\zeta)T_\lambda B\|_{H^{s-\frac{1}{2}}} \lesssim M_0^1(\lambda) \|B\|_{C_*^1} \|\zeta\|_{H^{s-\frac{1}{2}}}.$$

We thus end up with

$$\|F_2\|_{H^{s-\frac{1}{2}}} \leq \mathcal{F}(\|\eta\|_{H^{s+\frac{1}{2}}}, \|(V, B)\|_{H^s}) \left\{ 1 + \|\eta\|_{C^{\frac{3}{2}}} + \|(V, B)\|_{C_*^r} \right\}.$$

By combining the previous results, we obtain

$$(4.42) \quad (\partial_t + T_V \cdot \nabla)\zeta = T_\lambda U + h_3,$$

where

$$\|h_3\|_{H^{s-\frac{1}{2}}} \leq \mathcal{F}(\|\eta\|_{H^{s+\frac{1}{2}}}, \|(V, B)\|_{H^s}) \left\{ 1 + \|\eta\|_{C^{\frac{3}{2}}} + \|(V, B)\|_{C_*^r} \right\}.$$

As in the second step, by commuting the equation (4.42) with $\langle D_x \rangle^s$ we obtain the desired result (4.36), which concludes the proof. \square

4.5. Symmetrization of the equations. Recall that

$$U_s := \langle D_x \rangle^s V + T_\zeta \langle D_x \rangle^s B, \quad \zeta_s := \langle D_x \rangle^s \zeta,$$

satisfy the system

$$(4.43) \quad \begin{cases} (\partial_t + T_V \cdot \nabla)U_s + T_a \zeta_s = f_1, \\ (\partial_t + T_V \cdot \nabla)\zeta_s = T_\lambda U_s + f_2, \end{cases}$$

where, for each time $t \in [0, T]$,

$$\begin{aligned} & \| (f_1(t), f_2(t)) \|_{L^2 \times H^{-\frac{1}{2}}} \\ & \leq \mathcal{F}(\|\eta(t)\|_{H^{s+\frac{1}{2}}}, \|(V, B)(t)\|_{H^s}) \left\{ 1 + \|\eta(t)\|_{C_*^{r+\frac{1}{2}}} + \|(V, B)(t)\|_{C_*^r} \right\}. \end{aligned}$$

To prove an L^2 estimate for System (4.43), we begin by performing a symmetrization of the non-diagonal part. Here we use in an essential way the fact that the Taylor coefficient a is a positive function. Again, let us mention that this assumption is automatically satisfied for infinitely deep fluid domain: this result was first proved by Wu (see [54, 55]) and one can check that the proof remains valid for any $C^{1,\alpha}$ -domain, with $0 < \alpha < 1$.

PROPOSITION 4.12. *Introduce the symbols*

$$\gamma = \sqrt{a\lambda}, \quad q = \sqrt{\frac{a}{\lambda}},$$

and set $\theta_s = T_q \zeta_s$. Then

$$(4.44) \quad \partial_t U_s + T_V \cdot \nabla U_s + T_\gamma \theta_s = F_1,$$

$$(4.45) \quad \partial_t \theta_s + T_V \cdot \nabla \theta_s - T_\gamma U_s = F_2,$$

for some source terms F_1, F_2 satisfying

$$\begin{aligned} & \| (F_1(t), F_2(t)) \|_{L^2 \times L^2} \\ & \leq \mathcal{F}(\|\eta(t)\|_{H^{s+\frac{1}{2}}}, \|(V, B)(t)\|_{H^s}) \left\{ 1 + \|\eta(t)\|_{C_*^{r+\frac{1}{2}}} + \|(V, B)(t)\|_{C_*^r} \right\}. \end{aligned}$$

PROOF. Directly from (4.43), we obtain (4.44)–(4.45) with

$$\begin{aligned} F_1 &:= f_1 + (T_\gamma T_q - T_a)\zeta_s, \\ F_2 &:= T_q f_2 + (T_q T_\lambda - T_\gamma)U_s - [T_q, \partial_t + T_V \cdot \nabla]\zeta_s. \end{aligned}$$

The commutator between T_q and $\partial_t + T_V \cdot \nabla$ is estimated by means of Lemma 2.16:

$$(4.46) \quad \begin{aligned} & \|[T_q, \partial_t + T_V \cdot \nabla]\zeta_s\|_{L^2(\mathbf{R}^d)} \\ & \leq K \left\{ \mathcal{M}_0^{-\frac{1}{2}}(q) \|V\|_{C_*^{1+\varepsilon}} + \mathcal{M}_0^{-\frac{1}{2}}(\partial_t q + V \cdot \nabla q) \|V\|_{L^\infty}(t) \right\} \times \|\zeta_s\|_{H^{-\frac{1}{2}}(\mathbf{R}^d)}. \end{aligned}$$

$T_q f_2$ is estimated by means of (2.4). The key point is to estimate $(T_\gamma T_q - T_a)\zeta_s$ and $(T_q T_\lambda - T_\gamma)U_s$. Since $\gamma q = a$, the operator $T_\gamma T_q - T_a$ is of order $-1/2$ since γ is a symbol of order $1/2$, q is of order $-1/2$, and since these symbols are $C^{1/2}$ in x . Similarly, since $q\lambda = \gamma$, the operator $T_q T_\lambda - T_\gamma$ is of order 0. More precisely, by using the tame estimate for symbolic calculus (see (2.5)), we obtain

$$\begin{aligned} \|T_\gamma T_q - T_a\|_{H^{-\frac{1}{2}} \rightarrow L^2} &\lesssim M_{1/2}^{1/2}(\gamma) M_0^{-1/2}(q) + M_0^{1/2}(\gamma) M_{1/2}^{-1/2}(q), \\ \|T_q T_\lambda - T_\gamma\|_{L^2 \rightarrow L^2} &\lesssim M_{1/2}^{-1/2}(q) M_0^1(\lambda) + M_0^{-1/2}(q) M_{1/2}^1(\lambda). \end{aligned}$$

The above semi-norms are easily estimated by means of the $C^{1/2}$ norms of $\zeta = \nabla \eta$ and a (given by Proposition 4.6). \square

We are now in position to prove an L^2 estimate for (U_s, θ_s) .

LEMMA 4.13. *There exists a non-decreasing function \mathcal{F} such that*

$$(4.47) \quad \|U_s\|_{L^\infty([0,T];L^2)} + \|\theta_s\|_{L^\infty([0,T];L^2)} \leq \mathcal{F}(M_{s,0}) + \sqrt{T} \mathcal{F}(M_s(T)) Z_r(T).$$

REMARK 4.14. The fact that this implies corresponding estimates for the Sobolev norms of η, ψ, V, B is explained below in §4.6.

PROOF. Multiply (4.44) by U_s and (4.45) by θ_s and integrate in space to obtain

$$\frac{d}{dt} \left\{ \|U_s(t)\|_{L^2}^2 + \|\theta_s(t)\|_{L^2}^2 \right\} + (I) + (II) = (III),$$

where

$$\begin{aligned} (I) &:= \langle T_{V(t)} \cdot \nabla U_s(t), U_s(t) \rangle + \langle T_{V(t)} \cdot \nabla \theta_s(t), \theta_s(t) \rangle, \\ (II) &:= \langle T_{\gamma(t)} \theta_s(t), U_s(t) \rangle - \langle T_{\gamma(t)} U_s(t), \theta_s(t) \rangle, \\ (III) &:= \langle F_1, U_s \rangle + \langle F_2, \theta_s \rangle. \end{aligned}$$

Then the key points are that (see point (iii) in Theorem 2.7)

$$\|(T_{V(t)} \cdot \nabla)^* + T_{V(t)} \cdot \nabla\|_{L^2 \rightarrow L^2} \lesssim \|V(t)\|_{W^{1,\infty}},$$

and

$$\|T_{\gamma(t)} - (T_{\gamma(t)})^*\|_{L^2 \rightarrow L^2} \lesssim M_{1/2}^{1/2}(\gamma(t)).$$

We then easily obtain (4.47). \square

4.6. Back to estimates for the original unknowns. Up to now, we only estimated (U_s, θ_s) in $L^\infty([0, T]; L^2 \times L^2)$. In this section, we shall show how we can recover estimates for the original unknowns (η, ψ, V, B) in $L^\infty([0, T]; H^{s+\frac{1}{2}} \times H^{s+\frac{1}{2}} \times H^s \times H^s)$. Recall that the functions U_s and θ_s are obtained from (η, V, B) through:

$$\begin{aligned} U_s &:= \langle D_x \rangle^s V + T_\zeta \langle D_x \rangle^s B, \\ \theta_s &:= T_{\sqrt{a/\lambda}} \langle D_x \rangle^s \nabla \eta. \end{aligned}$$

The analysis is in four steps:

- (i) We first prove some estimates for (B, V, η) and the Taylor coefficient a in some low order norms.
- (ii) Then, by using the previous estimate of θ_s , we show how to recover an estimate of the $L^\infty([0, T], H^{s+\frac{1}{2}})$ -norm of η .
- (iii) Once η is estimated in $L^\infty([0, T], H^{s+\frac{1}{2}})$, by using the estimate for U_s , we estimate (B, V) in $L^\infty([0, T]; H^s)$. Here we make an essential use of our first result on the parilinearization of the Dirichlet-Neumann operator (see Proposition 3.19). Namely, we use the fact that one can parilinearize the Dirichlet-Neumann operator for any domain whose boundary is in H^μ for some $\mu > 1 + d/2$.
- (iv) The desired estimate for ψ follows directly from the previous estimates for η, V, B , the identity $\nabla \psi = V + B \nabla \eta$ and the fact that one easily obtain an $L^\infty([0, T]; L^2)$ -estimate for ψ .

We begin with the following lemma.

LEMMA 4.15. *There exists a non-decreasing function \mathcal{F} such that for any $r > 0$,*

$$(4.48) \quad \|\eta\|_{L^\infty([0, T]; H^s \cap C_x^r)} + \|(B, V)\|_{L^\infty([0, T]; C_x^{\frac{1}{2}} \cap H^{s-\frac{1}{2}})} \leq \mathcal{F}(M_{s,0}) + \sqrt{T} \mathcal{F}(M_s(T))(Z_r(T) + 1),$$

and, for any $1 < r' < r$,

$$(4.49) \quad \|a\|_{L^\infty([0, T]; C_*^{r'-1})} \leq \mathcal{F}(M_{s,0}) + \sqrt{T} \mathcal{F}(M_s(T))(Z_r(T) + 1).$$

PROOF. The proof is based on the fact that it is easy to estimate the solution w of a transport equation of the form

$$\partial_t w + V \cdot \nabla w = F.$$

Indeed, by using the estimates (4.19)–(4.20) for a , tame product rules in Sobolev or Hölder spaces and the identity $\partial_t \eta + V \cdot \nabla \eta = B$, we readily obtain that there exists a non-decreasing function \mathcal{C} (depending only on parameters that are considered fixed) such that

$$\begin{aligned} \|a - g\|_{C_*^{\frac{1}{2}} \cap H^{s-\frac{1}{2}}} &= \|\partial_t B + V \cdot \nabla B\|_{C_*^{\frac{1}{2}} \cap H^{s-\frac{1}{2}}} \leq \mathcal{C}(t), \\ \|a\zeta\|_{C_*^{\frac{1}{2}} \cap H^{s-\frac{1}{2}}} &= \|\partial_t V + V \cdot \nabla V\|_{C_*^{\frac{1}{2}} \cap H^{s-\frac{1}{2}}} \leq \mathcal{C}(t), \\ \|\partial_t \eta + V \cdot \nabla \eta\|_{C_*^r \cap H^s} &\leq \mathcal{C}(t), \\ \|\partial_t a + V \cdot \nabla a\|_{C_*^{r'-1}} &\leq \mathcal{C}(t), \end{aligned}$$

where

$$\mathcal{C}(t) = C(\|\eta(t)\|_{H^{s+\frac{1}{2}}}, \|(V, B)(t)\|_{H^s}) \left\{ 1 + \|\eta(t)\|_{C_*^{r+\frac{1}{2}}} + \|(V, B)(t)\|_{C_*^r} \right\}.$$

Let us come back to the proof of Lemma 4.15. To fix matters, we prove the estimate for V only (the proofs of the estimates for B , η and a are similar) and we begin by proving the Sobolev estimate. Using the obvious estimate

$$\|h\|_{L^1([0,T])} \leq \sqrt{T} \|h\|_{L^2([0,T])},$$

note that $F_V := \partial_t V + V \cdot \nabla V$ satisfies

$$\|F_V\|_{L^1([0,T]; C_*^{\frac{1}{2}} \cap H^{s-\frac{1}{2}})} \leq \sqrt{T} \mathcal{F}(M_s(T)) Z_r(T).$$

Using Proposition 2.15 with $\sigma = s - \frac{1}{2}$, we estimate V in $L^\infty([0, T]; H^{s-\frac{1}{2}})$.

Let us now estimate the $L^\infty([0, T]; C_*^{1/2})$ -norm of V . Let $V_{\frac{1}{2}} = \langle D_x^{\frac{1}{2}} \rangle V$. We have

$$\partial_t \dot{V}_{\frac{1}{2}} + V \cdot \nabla \dot{V}_{\frac{1}{2}} = \langle D_x^{\frac{1}{2}} \rangle F'_V + [V, \langle D_x^{\frac{1}{2}} \rangle] \nabla_x V.$$

where according to Proposition 2.14[iii], we have

$$\|[V, \langle D_x^{\frac{1}{2}} \rangle] \nabla_x V\|_{L^\infty} \leq \mathcal{F}(\|V\|_{H^s}) \|\nabla_x V\|_{L^\infty} \leq \mathcal{F}(\|V\|_{H^s}) \|V\|_{C_*^r}.$$

As a consequence, we get

$$\left\| (\partial_t + V \cdot \nabla) \langle D_x \rangle^{1/2} V \right\|_{L^1([0,T]; L^\infty)} \leq \sqrt{T} \mathcal{F}(M_s(T)) Z_r(T).$$

We next use the usual L^∞ estimate for transport equation (see (2.27)). This gives an estimate for $\langle D_x \rangle^{1/2} V$ in $L^\infty([0, T]; L^\infty)$ and hence the desired estimate for V in $L^\infty([0, T]; C_*^{1/2})$. \square

LEMMA 4.16. *There exists a non-decreasing function \mathcal{F} such that*

$$(4.50) \quad \|\eta\|_{L^\infty([0,T]; H^{s+\frac{1}{2}})} \leq \mathcal{F}(\mathcal{F}(M_{s,0}) + T\mathcal{F}(M_s(T) + Z_r(T))).$$

PROOF. Chose ε and an integer N such that

$$0 < \varepsilon < r - 1, \quad (N + 1)\varepsilon > \frac{1}{2}.$$

Set $R = I - T_{1/q} T_q$ to obtain

$$\zeta_s = T_{1/q} T_q \zeta_s + R \zeta_s,$$

where recall that $\zeta_s = \langle D_x \rangle^s \zeta$. Consequently,

$$\zeta_s = (I + R + \cdots + R^N) T_{1/q} T_q \zeta_s + R^{N+1} \zeta_s.$$

By definition of $q = \sqrt{a/\lambda}$, Theorem 2.7 implies that, for all $\mu \in \mathbf{R}$, there exists a non-decreasing function \mathcal{F} depending only on ε and $\inf_{(t,x) \in [0,T] \times \mathbf{R}^d} a(t,x) > 0$ such that,

$$\|R(t)\|_{H^\mu \rightarrow H^{\mu+\varepsilon}} \leq \mathcal{F}(\|a(t)\|_{C_*^\varepsilon}, \|\eta(t)\|_{C_*^{1+\varepsilon}}),$$

and

$$\|T_{1/q}(t)\|_{H^{\mu+1/2} \rightarrow H^\mu} \leq \mathcal{F}(\|\eta(t)\|_{W^{1,\infty}}).$$

Therefore

$$\|\nabla \eta\|_{H^{s-\frac{1}{2}}} = \|\zeta_s\|_{H^{-\frac{1}{2}}} \leq \mathcal{F}(\|a\|_{C_*^\varepsilon}, \|\eta\|_{C_*^{1+\varepsilon}}) \{ \|T_q \zeta_s\|_{L^2} + \|\zeta_s\|_{H^{-1}} \}.$$

Now it follows from Lemma 4.15 that

$$\begin{aligned} \|a\|_{L^\infty([0,T]; C_*^\varepsilon)} + \|\eta\|_{L^\infty([0,T]; C_*^{1+\varepsilon})} + \|\zeta_s\|_{L^\infty([0,T]; H^{-1})} \\ \leq \mathcal{F}(M_{s,0}) + \sqrt{T} \mathcal{F}(M_s(T)) Z_r(T). \end{aligned}$$

On the other hand, it follows from Lemma 4.13 that

$$\|T_q \zeta_s\|_{L^\infty([0,T];L^2)} \leq \mathcal{F}(M_{s,0}) + \sqrt{T} \mathcal{F}(M_s(T)) Z_r(T).$$

This implies the desired result. \square

It remains only to estimate (V, B) .

LEMMA 4.17. *There exists a non-decreasing function \mathcal{F} such that*

$$(4.51) \quad \|(V, B)\|_{L^\infty([0,T];H^s)} \leq \mathcal{F}(\mathcal{F}(M_{s,0}) + T\mathcal{F}(M_s(T) + Z_r(T))).$$

PROOF. The proof is based on the relation between V and B given by Proposition 4.5.

STEP 1. Recall that $U = V + T_\zeta B$. We begin by proving that there exists a non-decreasing function \mathcal{F} such that

$$(4.52) \quad \|U\|_{L^\infty([0,T];H^{s-\frac{1}{4}})} \leq \mathcal{F}(\mathcal{F}(M_{s,0}) + T\mathcal{F}(M_s(T) + Z_r(T))).$$

To see this, write

$$\langle D_x \rangle^{s-\frac{1}{4}} U = \langle D_x \rangle^{-\frac{1}{4}} \{U_s + [\langle D_x \rangle^s, T_\zeta] B\}$$

and use Theorem 2.7 to obtain

$$\|[\langle D_x \rangle^s, T_\zeta] B\|_{H^{-\frac{1}{4}}} \lesssim \|\zeta\|_{C_*^{\frac{1}{4}}} \|B\|_{H^{s-\frac{1}{2}}}.$$

Since, by assumption, $s > 3/4 + d/2$ we have

$$\|\zeta\|_{C_*^{\frac{1}{4}}} \lesssim \|\zeta\|_{H^{s-\frac{1}{2}}} \leq \|\eta\|_{H^{s+\frac{1}{2}}}$$

and hence

$$\|U\|_{H^{s-\frac{1}{4}}} \lesssim \|U_s\|_{H^{-\frac{1}{4}}} + \|\eta\|_{H^{s+\frac{1}{2}}} \|B\|_{H^{s-\frac{1}{2}}}.$$

The three terms in the right-hand side of the above inequalities have been already estimated (see Lemma 4.13 for U_s , Lemma 4.15 for B and Lemma 4.16 for η). This proves (4.52).

STEP 2. Taking the divergence in $U = V + T_\zeta B$, we get according to Proposition 4.5, Lemma 4.15 and Lemma 4.16:

$$\begin{aligned} \operatorname{div} U &= \operatorname{div} V + \operatorname{div} T_\zeta B = \operatorname{div} V + T_{\operatorname{div} \zeta} B + T_\zeta \cdot \nabla B \\ &= -G(\eta)B + T_{i\zeta \cdot \xi + \operatorname{div} \zeta} B + \gamma \\ &= -T_\lambda B + R(\eta)B + T_{i\zeta \cdot \xi + \operatorname{div} \zeta} B + \gamma \\ &= T_q B + R(\eta)B + T_{\operatorname{div} \zeta} B + \gamma \end{aligned}$$

where, by notation,

$$q := -\lambda + i\zeta \cdot \xi,$$

and

$$\|\gamma\|_{H^{s-\frac{1}{2}}} \leq \mathcal{F}(\|(\eta, V, B)\|_{H^{s+\frac{1}{2}} \times H^{\frac{1}{2}} \times H^{\frac{1}{2}}}).$$

According to Proposition 3.19 (with $\mu = s - \frac{1}{2}$) and Lemma 4.15, we deduce

$$(4.53) \quad T_q B = \operatorname{div} U - T_{\operatorname{div} \zeta} B - R(\eta)B - \gamma.$$

Now write

$$B = T_{\frac{1}{q}} T_q B + \left(I - T_{\frac{1}{q}} T_q\right) B$$

to obtain from (4.53)

$$B = T_{\frac{1}{q}} \operatorname{div} U - T_{\frac{1}{q}} \gamma + R_\epsilon B$$

where

$$(4.54) \quad R_{-\epsilon} := T_{\frac{1}{q}} \left(-T_{\text{div} \zeta} - R(\eta) \right) + \left(I - T_{\frac{1}{q}} T_q \right).$$

Notice now that according to Lemma 4.16, we control $\text{div} \zeta = \Delta \eta$ in $H^{s-\frac{3}{2}}$, and since $s > \frac{1}{2} + \frac{d}{2}$, there exists $\epsilon > 0$ (here one can fix $\epsilon = 1/4$) such that $T_{\text{div} \zeta}$ is an operator of order (both Sobolev and Hölder) $1 - \epsilon$. Finally, $q = -\lambda + i\zeta \cdot \xi \in \Gamma_{\epsilon}^1$ with $M_{\epsilon}^1(q) \leq C(\|\eta\|_{H^{s+\frac{1}{2}}})$ if $s > \frac{1}{2} + \frac{1}{2} + \epsilon$. Moreover, q^{-1} is of order -1 and we have

$$M_{\epsilon}^1(q^{-1}) \leq C(\|\eta\|_{H^{s+\frac{1}{2}}}).$$

Consequently, according to (2.4) and (2.5), the operator $R_{-\epsilon}$ given by (4.54) is of order $-\epsilon$. applying $T_{(-\lambda+i\zeta \cdot \xi)^{-1}}$ to (4.53), we get

$$B = W + R_{-\epsilon} B$$

where $W := T_{\frac{1}{q}} \text{div} U - T_{\frac{1}{q}} \gamma$ satisfies

$$\|W\|_{H^s} \leq \mathcal{F}(\mathcal{F}(M_{s,0}) + T\mathcal{F}(M_s(T) + Z_r(T))),$$

and $R_{-\epsilon}$ is an operator of order $-\epsilon$. Since we have already estimated the $H^{s-\frac{1}{2}}$ -norm of B (see Lemma 4.15), we conclude by a classical bootstrap argument that

$$\|B\|_{H^s} \leq \mathcal{F}(\mathcal{F}(M_{s,0}) + T\mathcal{F}(M_s(T) + Z_r(T))),$$

and coming back to the relation $U = V + T_{\zeta} B$ we get that V satisfies the same estimate. \square

LEMMA 4.18. *There exists a non-decreasing function \mathcal{F} such that*

$$\|\psi\|_{L^{\infty}([0,T]; H^{s+\frac{1}{2}})} \leq \mathcal{F}(\mathcal{F}(M_{s,0}) + T\mathcal{F}(M_s(T) + Z_r(T))).$$

PROOF. Since $\nabla \psi = V + B \nabla \eta$ and since the $L^{\infty}([0, T]; H^{s-\frac{1}{2}})$ -norm of $(\nabla \eta, V, B)$ has been previously estimated, it remains only to estimate $\|\psi\|_{L^{\infty}([0,T]; L^2)}$.

Since

$$B := \frac{\nabla \eta \cdot \nabla \psi + G(\eta) \psi}{1 + |\nabla \eta|^2},$$

the equation for ψ can be written under the form

$$\partial_t \psi + g\eta + \frac{1}{2} |\nabla \psi|^2 - \frac{1}{2} (1 + |\nabla \eta|^2) B^2 = 0.$$

Therefore, since $V = \nabla \psi - B \nabla \eta$,

$$(4.55) \quad \begin{aligned} \partial_t \psi + V \cdot \nabla \psi &= \partial_t \psi + |\nabla \psi|^2 - B \nabla \eta \cdot \nabla \psi \\ &= -g\eta + \frac{1}{2} |\nabla \psi|^2 + \frac{1}{2} (1 + |\nabla \eta|^2) B^2 - B \nabla \eta \cdot \nabla \psi \\ &= -g\eta + \frac{1}{2} |\nabla \psi - B \nabla \eta|^2 - \frac{1}{2} B^2 |\nabla \eta|^2 + \frac{1}{2} (1 + |\nabla \eta|^2) B^2 \\ &= -g\eta + \frac{1}{2} V^2 + \frac{1}{2} B^2. \end{aligned}$$

The desired L^2 estimate then follows from classical results (see Proposition 2.15). \square

5. Contraction

In this section we prove the uniqueness of solutions.

THEOREM 5.1. *Let (η_j, ψ_j) , $j = 1, 2$, be two solutions of (1.6) such that*

$$(\eta_j, \psi_j, V_j, B_j) \in C^0([0, T_0]; H^{s+\frac{1}{2}} \times H^{s+\frac{1}{2}} \times H^s \times H^s),$$

for some fixed $T_0 > 0$, $d \geq 1$ and $s > 1 + d/2$. We also assume that the condition (1.2) holds for $0 \leq t \leq T_0$ and that there exists a positive constant c such that for all $0 \leq t \leq T_0$ and for all $x \in \mathbf{R}^d$, we have $a_j(t, x) \geq c$ for $j = 1, 2$, $t \in [0, T]$. Set

$$M_j := \sup_{t \in [0, T]} \|(\eta_j, \psi_j, V_j, B_j)(t)\|_{H^{s+\frac{1}{2}} \times H^{s+\frac{1}{2}} \times H^s \times H^s},$$

$$\eta := \eta_1 - \eta_2, \quad \psi = \psi_1 - \psi_2, \quad V := V_1 - V_2, \quad B = B_1 - B_2.$$

Then we have

$$(5.1) \quad \|(\eta, \psi, V, B)\|_{L^\infty((0, T); H^{s-\frac{1}{2}} \times H^{s-\frac{1}{2}} \times H^{s-1} \times H^{s-1})} \leq \mathcal{K}(M_1, M_2) \|(\eta, \psi, V, B)|_{t=0}\|_{H^{s-\frac{1}{2}} \times H^{s-\frac{1}{2}} \times H^{s-1} \times H^{s-1}}.$$

Let us recall that

$$(5.2) \quad \begin{cases} (\partial_t + V_j \cdot \nabla) B_j = a_j - g, \\ (\partial_t + V_j \cdot \nabla) V_j + a_j \zeta_j = 0, \\ (\partial_t + V_j \cdot \nabla) \zeta_j = G(\eta_j) V_j + \zeta_j G(\eta_j) B_j + \gamma_j, \quad \zeta_j = \nabla \eta_j, \end{cases}$$

where γ_j is the remainder term given by (4.10). Let

$$N(T) := \sup_{t \in [0, T]} \|(\eta, \psi, V, B)(t)\|_{H^{s-\frac{1}{2}} \times H^{s-\frac{1}{2}} \times H^{s-1} \times H^{s-1}}.$$

Our goal is to prove an estimate of the form

$$(5.3) \quad N(T) \leq \mathcal{K}(M_1, M_2) N(0) + T \mathcal{K}(M_1, M_2) N(T),$$

for some non-decreasing function \mathcal{K} depending only on s and d . Then, by choosing T small enough, this implies $N(T) \leq 2\mathcal{K}(M_1, M_2) N(0)$ for T_1 smaller than the minimum of T_0 and $1/2\mathcal{K}(M_1, M_2)$, and iterating the estimate between $[T_1, 2T_1], \dots, [T - T_1, T_1]$ implies Theorem 5.1.

REMARK 5.2. Notice that we prove a Lipschitz property in weak norms. This is a general fact related to the fact that the flow map of a quasi-linear equation is not expected to be Lipschitz in the highest norms (this means that one does not expect to control the difference (η, ψ, V, B) in $L^\infty([0, T_0]; H^{s+\frac{1}{2}} \times H^{s+\frac{1}{2}} \times H^s \times H^s)$).

5.1. Contraction for the Dirichlet-Neumann. The first step in the proof of Theorem 5.1 is to prove a Lipschitz property for the Dirichlet-Neumann operator. This was already achieved in a very weak norm in Theorem 3.10, and here we used elliptic theory to improve the result.

THEOREM 5.3. *Assume that $s > 1 + \frac{d}{2}$. There exists a non-decreasing function \mathcal{F} such that, for all $\eta_1, \eta_2 \in H^{s+\frac{1}{2}}$ and all $f \in H^s$, we have*

$$(5.4) \quad \|[G(\eta_1) - G(\eta_2)]f\|_{H^{s-\frac{3}{2}}} \leq \mathcal{F}(\|(\eta_1, \eta_2)\|_{H^{s+\frac{1}{2}}}) \|\eta_1 - \eta_2\|_{H^{s-\frac{1}{2}}} \|f\|_{H^s}.$$

REMARK 5.4. In the right-hand side, $\|\eta_1 - \eta_2\|_{H^{s-\frac{1}{2}}}$ controls only the $C^{\frac{1}{2}}$ -norm of $\eta_1 - \eta_2$. This is the source of some serious difficulties in the proof below.

PROOF. The proof follows closely that of Theorem 3.10 and we keep the notations $\rho_j, \tilde{\phi}_j, v = \tilde{\phi}_1 - \tilde{\phi}_2, \Lambda^j$ introduced there.

Notice that, using the smoothing property of the Poisson kernel, we have

$$(5.5) \quad \begin{cases} (i) & \Lambda_k^1 - \Lambda_k^2 = \beta_k \partial_z, \quad \text{with } \text{supp } \beta_k \subset \mathbf{R}^d \times J, \text{ where } J = [-1, 0], \\ (ii) & \|\beta_k\|_{L^2(J, H^{s-1}(\mathbf{R}^d))} \leq \mathcal{F}(\|(\eta_1, \eta_2)\|_{H^{s+\frac{1}{2}} \times H^{s+\frac{1}{2}}}) \|\eta_1 - \eta_2\|_{H^{s-\frac{1}{2}}(\mathbf{R}^d)}. \end{cases}$$

Recall that

$$(5.6) \quad G(\eta_j)f = U_j|_{z=0}, \quad U_j = \Lambda_1^j \tilde{\phi}_j - \nabla_x \rho_j \cdot \Lambda_2^j \tilde{\phi}_j.$$

Let us set $U = U_1 - U_2$. According to (3.14), Theorem 5.3 will follow from the following estimate

$$(5.7) \quad \|U\|_{L^2(J, H^{s-1})} + \|\partial_z U\|_{L^2(J, H^{s-2})} \leq \mathcal{F}(\|(\eta_1, \eta_2)\|_{H^{s+\frac{1}{2}} \times H^{s+\frac{1}{2}}}) \|f\|_{H^s} \|\eta_1 - \eta_2\|_{H^{s-\frac{1}{2}}}.$$

According to (3.37) and (3.39) the estimate (5.7) will be a consequence of the following one

$$(5.8) \quad \sum_{k=1}^5 \|B_k\|_{L^2(J, H^{s-1})} \leq \mathcal{F}(\|(\eta_1, \eta_2)\|_{H^{s+\frac{1}{2}} \times H^{s+\frac{1}{2}}}) \|f\|_{H^s} \|\eta_1 - \eta_2\|_{H^{s-\frac{1}{2}}} \quad \text{where}$$

$$B_1 = \Lambda_1^1 v, \quad B_2 = (\nabla_{x,z} \rho_2) \Lambda_2^2 v, \quad B_3 = (\Lambda_1^1 - \Lambda_1^2) \tilde{\phi}_2, \quad B_4 = \nabla_{x,z}(\rho_1 - \rho_2) \Lambda_2^1 \tilde{\phi}_1,$$

$$B_5 = (\nabla_{x,z} \rho_2)(\Lambda_1^1 - \Lambda_1^2) \tilde{\phi}_1.$$

Since $\tilde{\phi}_j$ is a variational solution, Proposition 3.22 with $\sigma = s - 1$ show that

$$\|\nabla_{x,z} \tilde{\phi}_j\|_{L^\infty(J, H^{s-1})} + \|\Lambda_k^j \tilde{\phi}_j\|_{L^\infty(I, H^{s-1})} \leq \mathcal{F}(\|\eta_j\|_{H^{s+\frac{1}{2}}}) \|f\|_{H^s}.$$

Since $s > 1 + \frac{d}{2}$, it follows from (3.34) that

$$(5.9) \quad \sum_{l=3}^5 \|B_l\|_{L^2(J, H^{s-1})} \leq \mathcal{F}(\|(\eta_1, \eta_2)\|_{H^{s+\frac{1}{2}} \times H^{s+\frac{1}{2}}}) \|\eta_1 - \eta_2\|_{H^{s-\frac{1}{2}}} \|f\|_{H^s}.$$

Since

$$\|B_1\|_{L^2(J, H^{s-1})} + \|B_2\|_{L^2(J, H^{s-1})} \leq \mathcal{F}(\|(\eta_1, \eta_2)\|_{H^{s+\frac{1}{2}} \times H^{s+\frac{1}{2}}}) \|\nabla_{x,z} v\|_{L^2(I, H^{s-1})}$$

using the estimate (5.9), we see that (5.8) will be a consequence of the following Lemma. Therefore Theorem 5.3 will be proved if we prove the following result.

LEMMA 5.5. *We have*

$$\|\nabla_{x,z} v\|_{L^2(J, H^{s-1})} \leq \mathcal{F}(\|(\eta_1, \eta_2)\|_{H^{s+\frac{1}{2}} \times H^{s+\frac{1}{2}}}) \|\eta_1 - \eta_2\|_{H^{s-\frac{1}{2}}} \|f\|_{H^s}.$$

PROOF. Notice that $v = \tilde{\phi}_1 - \tilde{\phi}_2$ is a solution of the problem

$$(5.10) \quad \partial_z^2 v + \alpha_1 \Delta v + \beta_1 \cdot \nabla \partial_z v - \gamma_1 \partial_z v = F, \quad v|_{z=0} = 0$$

where

$$F = (\alpha_2 - \alpha_1) \Delta \tilde{\phi}_2 + (\beta_2 - \beta_1) \cdot \nabla \partial_z \tilde{\phi}_2 - (\gamma_2 - \gamma_1) \partial_z \tilde{\phi}_2$$

and α_j are given by (3.18). We would like to apply Proposition 3.22 with $\sigma = s - \frac{3}{2}$. To this end, according to (2.45), we shall estimate the $L^2(J, H^{s-2}(\mathbf{R}^d))$ norm of F and the $X^{-\frac{1}{2}}(J)$ norm of $\nabla_{x,z} v$.

Estimate on F: Since $s > 1 + \frac{d}{2}$ (thus $2s - 3 > 0$) we may apply (2.16) with $s_1 = s - 2, s_2 = s - 1, s_0 = s - 2$. We get

$$\begin{aligned} \left\| (\alpha_1 - \alpha_2) \Delta \tilde{\phi}_2 \right\|_{L^2(J, H^{s-2})} &\leq K \|\alpha_1 - \alpha_2\|_{L^2(J, H^{s-1})} \left\| \Delta \tilde{\phi}_2 \right\|_{L^\infty(J, H^{s-2})}, \\ \left\| (\beta_1 - \beta_2) \cdot \nabla \partial_z \tilde{\phi}_2 \right\|_{L^2(J, H^{s-2})} &\leq K \|\beta_1 - \beta_2\|_{L^2(J, H^{s-1})} \left\| \nabla \partial_z \tilde{\phi}_2 \right\|_{L^\infty(J, H^{s-2})}, \\ \left\| (\gamma_1 - \gamma_2) \partial_z \tilde{\phi}_2 \right\|_{L^2(J, H^{s-2})} &\leq K \|\gamma_1 - \gamma_2\|_{L^2(J, H^{s-2})} \left\| \partial_z \tilde{\phi}_2 \right\|_{L^\infty(J, H^{s-1})}. \end{aligned}$$

Then, using the product rule in Sobolev space (2.16), and (3.11), (3.12) we obtain

$$(5.11) \quad \begin{aligned} &\|\alpha_1 - \alpha_2\|_{L^2(J, H^{s-1})} + \|\beta_1 - \beta_2\|_{L^2(J, H^{s-1})} + \|\gamma_1 - \gamma_2\|_{L^2(J, H^{s-2})} \\ &\leq \mathcal{F}(\|(\eta_1, \eta_2)\|_{H^{s+\frac{1}{2}} \times H^{s+\frac{1}{2}}}) \|\eta_1 - \eta_2\|_{H^{s-\frac{1}{2}}}. \end{aligned}$$

Moreover from Proposition 3.22 with $\sigma = s - 1$ we have

$$\left\| \nabla_{x,z} \tilde{\phi}_j \right\|_{L^\infty(J, H^{s-1})} \leq C(\|\eta_j\|_{H^{s+\frac{1}{2}}}) \|f\|_{H^s}.$$

It follows that

$$(5.12) \quad \|F\|_{L^2_z(J, H^{s-2})} \leq \mathcal{F}(\|(\eta_1, \eta_2)\|_{H^{s+\frac{1}{2}}}) \|\eta_1 - \eta_2\|_{H^{s-\frac{1}{2}}} \|f\|_{H^s}$$

Estimate of $\|\nabla_{x,z} v\|_{X^{-\frac{1}{2}}(J)}$, $J = (-1, 0)$.

We claim that

$$(5.13) \quad \|\nabla_{x,z} v\|_{X^{-\frac{1}{2}}(J)} \leq \mathcal{F}(\|(\eta_1, \eta_2)\|_{H^{s+\frac{1}{2}} \times H^{s+\frac{1}{2}}}) \|\eta_1 - \eta_2\|_{H^{s-\frac{1}{2}}} \|f\|_{H^s}.$$

Since $\tilde{\phi}_j = \tilde{u}_j + \tilde{f}$ we have $v = \tilde{u}_1 - \tilde{u}_2$. We begin by proving the following estimate.

There exists a non decreasing function $\mathcal{F}: \mathbf{R}^+ \rightarrow \mathbf{R}^+$ such that

$$(5.14) \quad \|\nabla_{x,z} v\|_{L^2(J, L^2)} \leq \mathcal{F}(\|(\eta_1, \eta_2)\|_{H^{s+\frac{1}{2}} \times H^{s+\frac{1}{2}}}) \|\eta_1 - \eta_2\|_{H^{s-\frac{1}{2}}} \|f\|_{H^s}$$

For this purpose we use the variational characterization of the solutions u_i . Setting $X = (x, z)$ we have

$$(5.15) \quad \int_{\tilde{\Omega}} \Lambda^i \tilde{u}_i \cdot \Lambda^i \theta J_i dX = - \int_{\tilde{\Omega}} \Lambda^i \tilde{f} \cdot \Lambda^i \theta J_i dX$$

for all $\theta \in H^{1,0}(\tilde{\Omega})$, where $J_i = |\partial_z \rho_i|$.

Making the difference between the two equations (5.15), using (3.32) and taking $\theta = v = \tilde{u}_1 - \tilde{u}_2$ one can find a positive constant C such that

$$\int_{\tilde{\Omega}} |\Lambda^1 v|^2 dX \leq C(A_1 + A_2 + A_3 + A_4)$$

where

$$\begin{cases} A_1 = \int_{\tilde{\Omega}} |(\Lambda^1 - \Lambda^2) \tilde{u}_2| |\Lambda^1 v| J_1 dX, & A_2 = \int_{\tilde{\Omega}} |(\Lambda^1 - \Lambda^2) v| |\Lambda^2 \tilde{u}_2| J_1 dX, \\ A_3 = \int_{\tilde{\Omega}} |\Lambda^2 \tilde{u}_2| |\Lambda^2 v| |J_1 - J_2| dX, & A_4 = \int_{\tilde{\Omega}} |(\Lambda^1 - \Lambda^2) \tilde{f}| |\Lambda^1 \tilde{u}| J_1 dX, \\ A_5 = \int_{\tilde{\Omega}} |(\Lambda^1 - \Lambda^2) v| |\Lambda^2 \tilde{f}| J_1 dX, & A_6 = \int_{\tilde{\Omega}} |\Lambda^2 \tilde{f}| |\Lambda^2 v| |J_1 - J_2| dX \end{cases}$$

It follows from the elliptic regularity theorem that

$$\begin{aligned} A_1 &\leq \|\Lambda^1 v\|_{L^2(\tilde{\Omega})} \|\beta\|_{L^2(\tilde{\Omega})} \|\partial_z \tilde{u}_2\|_{L^\infty(J, L^\infty(\mathbf{R}^d))} \\ &\leq \|\Lambda^1 v\|_{L^2(\tilde{\Omega})} \mathcal{F}(\|\eta_1\|_{H^{s+\frac{1}{2}}(\mathbf{R}^d)}) \|\eta_2\|_{H^{s+\frac{1}{2}}(\mathbf{R}^d)}, \|\psi\|_{H^s(\mathbf{R}^d)} \|\eta_1 - \eta_2\|_{H^{\frac{1}{2}}(\mathbf{R}^d)}. \end{aligned}$$

Noticing that $\Lambda^1 - \Lambda^2 = \beta(\partial_z \rho_1) \Lambda_1^1$ where β satisfies the estimate in (3.34) we obtain

$$A_2 \leq \|\partial_z \rho_1\|_{L^\infty(\tilde{\Omega})} \|\beta\|_{L^2(\tilde{\Omega})} \|\Lambda^2 \tilde{u}_2\|_{L^\infty(\tilde{\Omega})} \|\Lambda^1 v\|_{L^2(\tilde{\Omega})}.$$

Using (3.32), (3.34) and the elliptic regularity we obtain

$$A_2 \leq \|\Lambda^1 v\|_{L^2(\tilde{\Omega})} \mathcal{F}(\|(\eta_1, \eta_2)\|_{H^{s+\frac{1}{2}} \times H^{s+\frac{1}{2}}}) \|\eta_1 - \eta_2\|_{H^{s-\frac{1}{2}}} \|f\|_{H^s}.$$

Now we estimate A_3 as follows. We have

$$A_3 \leq \|\Lambda^2 \tilde{u}_2\|_{L^\infty(\tilde{\Omega})} \|\Lambda^2 v\|_{L^2(\tilde{\Omega})} \|J_1 - J_2\|_{L^2(\tilde{\Omega})}.$$

Then we observe that

$$\begin{aligned} \|J_1 - J_2\|_{L^2(\tilde{\Omega})} &\leq C \|\eta_1 - \eta_2\|_{H^{\frac{1}{2}}(\mathbf{R}^d)} \\ \|\Lambda^2 v\|_{L^2(\tilde{\Omega})} &\leq C \|\Lambda^1 v\|_{L^2(\tilde{\Omega})} \end{aligned}$$

and we use the elliptic regularity. To estimate A_4 and A_5 we recall that $\tilde{f} = e^{z\langle D_x \rangle} f$. Then we have

$$\|\beta \partial_z \tilde{f}\|_{L^2(J \times \mathbf{R}^d)} \leq \|\beta\|_{L^2(I \times \mathbf{R}^d)} \|\partial_z \tilde{f}\|_{L^\infty(J \times \mathbf{R}^d)}.$$

Since $\|\partial_z \tilde{f}\|_{L^\infty(J \times \mathbf{R}^d)} \leq \|\partial_z \tilde{f}\|_{L^\infty(J, H^{s-1}(\mathbf{R}^d))} \leq \|f\|_{H^s(\mathbf{R}^d)}$, using (3.34) we obtain

$$A_4 + A_5 \leq \|\Lambda^1 v\|_{L^2(\tilde{\Omega})} \mathcal{F}(\|(\eta_1, \eta_2)\|_{H^{s+\frac{1}{2}}}) \|\eta_1 - \eta_2\|_{H^{s-\frac{1}{2}}} \|f\|_{H^s}.$$

The term A_6 is estimated like A_3 . Since $\frac{1}{2} < s - \frac{1}{2}$ this proves (5.14).

To complete the proof of (5.13) we have to estimate $\|\nabla_{x,z} v\|_{L^\infty(I, H^{-\frac{1}{2}})}$. The estimate of $\|\nabla_x v\|_{L^\infty(J, H^{-\frac{1}{2}})}$ follows from (5.14) and from Lemma 3.14. To estimate $\|\partial_z v\|_{L^\infty(J, H^{-\frac{1}{2}})}$ we have to use (5.14) and the equation satisfied by v . If we prove that

$$(5.16) \quad \|\partial_z^2 v\|_{L^2(J, H^{-1})} \leq \mathcal{F}(\|(\eta_1, \eta_2)\|_{H^{s+\frac{1}{2}} \times H^{s+\frac{1}{2}}}) \|\eta_1 - \eta_2\|_{H^{s-\frac{1}{2}}} \|f\|_{H^s}.$$

the result will follow again from Lemma 3.14. Recall that v satisfies the equation (5.10).

It follows that we have

$$(5.17) \quad \begin{aligned} \|\partial_z^2 v\|_{L^2(J, H^{-1})} &\leq \|\alpha_1 \Delta v\|_{L^2(J, H^{-1})} + \|\beta_1 \cdot \nabla \partial_z v\|_{L^2(J, H^{-1})} \\ &\quad + \|\gamma_1 \partial_z v\|_{L^2(J, H^{-1})} + \|F\|_{L^2(J, H^{-1})}. \end{aligned}$$

Since $-1 < s - 2$ (5.12) yields

$$\|F\|_{L^2(J, H^{-1})} \leq \|F\|_{L^2(J, H^{s-2})} \leq \mathcal{F}(\|(\eta_1, \eta_2)\|_{H^{s+\frac{1}{2}} \times H^{s+\frac{1}{2}}}) \|\eta_1 - \eta_2\|_{H^{s-\frac{1}{2}}} \|f\|_{H^s}.$$

On the other hand, since $s - \frac{1}{2} - 1 > 0$ and $-1 < s - \frac{1}{2} - 1 - \frac{d}{2}$ (2.16) show that we have

$$\begin{aligned} \|\alpha_1 \Delta v\|_{L^2(J, H^{-1})} &\leq \|\alpha_1\|_{L^\infty(J, H^{s-\frac{1}{2}})} \|\nabla_x v\|_{L^2(J, L^2)} \\ \|\beta_1 \cdot \nabla \partial_z v\|_{L^2(J, H^{-1})} &\leq \|\beta_1\|_{L^\infty(J, H^{s-\frac{1}{2}})} \|\partial_z v\|_{L^2(J, L^2)} \\ \|\gamma_1 \partial_z v\|_{L^2(J, H^{-1})} &\leq \|\gamma_1\|_{L^\infty(J, H^{s-\frac{3}{2}})} \|\partial_z v\|_{L^2(J, L^2)}. \end{aligned}$$

Using Lemma 3.25 and (5.14) we obtain eventually (5.16).

Now Lemma 5.5 follows from (5.12), (5.13) and Proposition 3.22 with $\sigma = s - \frac{3}{2}$. \square

Finally Lemma 5.5 together with (5.8) prove (5.7) which in turn proves Proposition 5.3. \square

5.2. Parilinearization of the equations. We begin by noticing that, as in the proof of Lemma 4.18, it is enough to estimate η, B, V . Indeed, the estimate of the $L^\infty([0, T]; H_x^{s-1/2})$ -norm of ψ is in two elementary steps. Firstly, since $V_j = \nabla\psi_j - B_j\nabla\eta_j$, one can estimate the $L^\infty([0, T]; H^{s-3/2})$ -norm of $\nabla\psi$ from the identity

$$\nabla\psi = V + B\nabla\eta_1 + B_2\nabla\eta.$$

On the other hand, the estimate of the $L^\infty([0, T]; L_x^2)$ -norm of ψ follows from the equation (4.55).

An elementary calculation shows that the functions

$$\zeta = \zeta_1 - \zeta_2, \quad V = V_1 - V_2, \quad B = B_1 - B_2$$

satisfy the system of equations

$$(5.18) \quad \begin{cases} \partial_t B + V_1 \cdot \nabla B + V \cdot \nabla B_2 = a, \\ \partial_t V + V_1 \cdot \nabla V + V \cdot \nabla V_2 + a_2 \zeta + a \zeta_1 = 0, \\ \partial_t \zeta + V_2 \cdot \nabla \zeta + V \cdot \nabla \zeta_1 = G(\eta_1)V + \zeta_1 G(\eta_1)B + \zeta G(\eta_2)B_2 + R + \gamma, \end{cases}$$

where

$$(5.19) \quad R = [G(\eta_1) - G(\eta_2)]V_2 + \zeta_1 [G(\eta_1) - G(\eta_2)]B_2,$$

and $\gamma = \gamma_1 - \gamma_2$, γ_j are given by (4.10)

LEMMA 5.6. *The differences ζ, B, V satisfy a system of the form*

$$(5.20) \quad \begin{cases} (\partial_t + V_1 \cdot \nabla)(V + \zeta_1 B) + a_2 \zeta = f_1, \\ (\partial_t + V_2 \cdot \nabla)\zeta - G(\eta_1)V - \zeta_1 G(\eta_1)B = f_2, \end{cases}$$

for some remainders such that

$$\|(f_1, f_2)\|_{L^\infty([0, T]; H^{s-1} \times H^{s-\frac{3}{2}})} \leq \mathcal{K}(M_1, M_2)N(T).$$

PROOF. We begin by rewriting System (5.18) under the form

$$\begin{cases} \partial_t B + V_1 \cdot \nabla B = a + R_1, \\ \partial_t V + V_1 \cdot \nabla V + a_2 \zeta + a \zeta_1 = R_2, \\ \partial_t \zeta + V_2 \cdot \nabla \zeta = G(\eta_1)V + \zeta_1 G(\eta_1)B + R + \gamma + R_3, \end{cases}$$

where R is given by (5.19), $\gamma = \gamma_1 - \gamma_2$ and

$$R_1 = -V \cdot \nabla B_2, \quad R_2 = -V \cdot \nabla V_2, \quad R_3 = V \cdot \nabla \zeta_1 + \zeta G(\eta_2)B_2.$$

From Theorem 5.3 one has

$$\|R\|_{L^\infty(0, T; H^{s-\frac{3}{2}})} \leq \mathcal{K}(M_1, M_2)N(T).$$

Similarly, proceeding as in the end of the proof of Proposition 4.3, we have

$$\|\gamma\|_{L^\infty(0, T; H^{s-\frac{3}{2}})} \leq \mathcal{K}(M_1, M_2)N(T).$$

On the other hand, since $s-1 > d/2$, H^{s-1} is an algebra and

$$\|V \cdot \nabla B_2\|_{H^{s-1}} \leq K \|V\|_{H^{s-1}} \|\nabla B_2\|_{H^{s-1}} \leq K \|V\|_{H^{s-1}} \|B_2\|_{H^s}$$

and similarly

$$\|V \cdot \nabla V_2\|_{H^{s-1}} \leq K \|V\|_{H^{s-1}} \|V_2\|_{H^s}.$$

On the other hand, according to Theorem 3.18 we have

$$\|G(\eta_2)B_2\|_{H^{s-1}} \leq C(\|\eta_2\|_{H^{s+1/2}}) \|B_2\|_{H^s},$$

and hence

$$\|\zeta G(\eta_2)B_2\|_{H^{s-\frac{3}{2}}} \leq C(\|\eta_2\|_{H^{s+1/2}}) \|B_2\|_{H^s} \|\zeta\|_{H^{s-\frac{3}{2}}}.$$

To estimate $V \cdot \nabla \zeta_1$ we use the product rule (2.16) to deduce

$$\|V \cdot \nabla \zeta_1\|_{H^{s-\frac{3}{2}}} \leq K \|V\|_{H^{s-1}} \|\nabla \zeta_1\|_{H^{s-\frac{3}{2}}} \leq K \|V\|_{H^{s-1}} \|\eta_1\|_{H^{s+\frac{1}{2}}}.$$

Therefore we have,

$$\|R_1\|_{H^{s-1}} + \|R_2\|_{H^{s-1}} + \|R_3\|_{H^{s-\frac{3}{2}}} \leq C \left\{ \|\eta\|_{H^{s-\frac{1}{2}}} + \|B\|_{H^{s-1}} + \|V\|_{H^{s-1}} \right\},$$

for some constant C depending only on $\|\eta_j\|_{H^{s+\frac{1}{2}}}$, $\|B_j\|_{H^s}$, $\|V_j\|_{H^s}$. The next step consists in transforming again the equation. We want to replace $a\zeta_1$ in the second equation by

$$(\partial_t B + V_1 \cdot \nabla B - R_1)\zeta_1.$$

The idea is that this allows to factor out the convective derivative $\partial_t + V_1 \cdot \nabla$. Writing

$$(\partial_t B + V_1 \cdot \nabla B)\zeta_1 = (\partial_t + V_1 \cdot \nabla)(B\zeta_1) - B(\partial_t + V_1 \cdot \nabla)\zeta_1$$

we thus end up with

$$(5.21) \quad (\partial_t + V_1 \cdot \nabla)(V + \zeta_1 B) + a_2 \zeta = R_1 \zeta_1 + B(\partial_t + V_1 \cdot \nabla)\zeta_1 + R_2.$$

Since

$$(\partial_t + V_1 \cdot \nabla)\zeta_1 = G(\eta_1)V_1 + \zeta_1 G(\eta_1)B_1 + \gamma_1,$$

we have

$$\|(\partial_t + V_1 \cdot \nabla)\zeta_1\|_{H^{s-1}} \leq \mathcal{F}(\|(\eta_1, B_1, V_1)\|_{H^{s+\frac{1}{2}} \times H^s \times H^s}).$$

By using this estimate and our previous bounds for R_1, R_2 , we find

$$\|R_1 \zeta_1 + B(\partial_t + V_1 \cdot \nabla)\zeta_1 + R_2\|_{H^{s-1}} \leq C \left\{ \|\eta\|_{H^{s-\frac{1}{2}}} + \|B\|_{H^{s-1}} + \|V\|_{H^{s-1}} \right\},$$

for some constant C depending only on $\|\eta_j\|_{H^{s+\frac{1}{2}}}$, $\|B_j\|_{H^s}$, $\|V_j\|_{H^s}$. Notice that here, as we used the equation satisfied by ζ_1 , it was important to have $(\partial_t + V_1 \cdot \nabla)$ in the l.h.s. of (5.21) and not $(\partial_t + V_2 \cdot \nabla)$, and this algebraic reduction required some care in the previous step. \square

5.3. Estimates for the good unknown. We now symmetrize System (5.20). We set $I = [0, T]$.

LEMMA 5.7. *Set*

$$\ell := \sqrt{\lambda_1 a_2}, \quad \varphi := T_{\sqrt{\lambda_1}}(V + \zeta_1 B), \quad \vartheta := T_{\sqrt{a_2}}\zeta.$$

Then

$$(5.22) \quad (\partial_t + T_{V_1} \cdot \nabla)\varphi + T_\ell \vartheta = g_1,$$

$$(5.23) \quad (\partial_t + T_{V_2} \cdot \nabla)\vartheta - T_\ell \varphi = g_2,$$

where

$$\|(g_1, g_2)\|_{L^\infty(I; H^{s-\frac{3}{2}} \times H^{s-\frac{3}{2}})} \leq \mathcal{K}(M_1, M_2)N(T).$$

PROOF. We start from Lemma 5.6. By using Proposition 2.11, one can replace $V_1 \cdot \nabla$ by $T_{V_1} \cdot \nabla$ and $a_2 \zeta$ by $T_{a_2} \zeta$, modulo admissible remainders. It is found that

$$(5.24) \quad (\partial_t + T_{V_1} \cdot \nabla)(V + \zeta_1 B) + T_{a_2} \zeta = f'_1,$$

for some remainder f'_1 such that

$$\|f'_1\|_{L^\infty(I; H^{s-1})} \leq \mathcal{K}(M_1, M_2)N(T).$$

Similarly, one can replace $V_2 \cdot \nabla$ by $T_{V_2} \cdot \nabla$. According to Proposition 3.19, with $\varepsilon = \frac{1}{2}$, we have

$$\|G(\eta_1)V - T_{\lambda_1}V\|_{L^\infty(I; H^{s-\frac{3}{2}})} + \|G(\eta_1)B - T_{\lambda_1}B\|_{L^\infty(I; H^{s-\frac{3}{2}})} \leq \mathcal{K}(M_1)N(T),$$

and according to Proposition 2.11, with $\gamma = r = s - \frac{3}{2}$, $\mu = s - \frac{1}{2}$,

$$\|\zeta_1 G(\eta_1)B - T_{\zeta_1}T_{\lambda_1}B\|_{L^\infty(I; H^{s-\frac{3}{2}})} \leq \mathcal{K}(M_1)N(T).$$

We deduce

$$(5.25) \quad (\partial_t + T_{V_2} \cdot \nabla)\zeta - T_{\lambda_1}V - T_{\zeta_1}T_{\lambda_1}B = f'_2,$$

where

$$\lambda_1 := \sqrt{(1 + |\nabla\eta_1|^2)|\xi|^2 - (\nabla\eta_1 \cdot \xi)^2},$$

and

$$\|f'_2\|_{L^\infty(I; H^{s-\frac{3}{2}})} \leq \mathcal{K}(M_1, M_2)N(T).$$

Now, according to Lemma 2.17, (3.43) and (4.10) and we find that

$$(5.26) \quad \|[T_{\sqrt{\lambda_1}}, (\partial_t + T_{V_1} \cdot \nabla)]\|_{H^{s-1} \rightarrow H^{s-\frac{3}{2}}} \leq \mathcal{K}(M_1)(\mathcal{M}_0^{\frac{1}{2}}(\sqrt{\lambda_1}) + \mathcal{M}_0^{\frac{1}{2}}((\partial_t + V_1 \cdot \nabla)\sqrt{\lambda_1})) \leq \mathcal{K}'(M_1)$$

and similarly, according to Lemma 2.17 and (4.20),

$$(5.27) \quad \|[T_{\sqrt{a_2}}, (\partial_t + T_{V_2} \cdot \nabla)]\|_{H^{s-1} \rightarrow H^{s-\frac{3}{2}}} \leq \mathcal{K}(M_2)(\mathcal{M}_0^0(\sqrt{a_2}) + \mathcal{M}_0^0((\partial_t + V_2 \cdot \nabla)\sqrt{a_2})) \leq \mathcal{K}'(M_2),$$

which implies

$$(5.28) \quad (\partial_t + T_{V_1} \cdot \nabla)T_{\sqrt{\lambda_1}}(V + \zeta_1 B) + T_{\sqrt{\lambda_1}}T_{a_2}\zeta = f''_1,$$

$$(5.29) \quad (\partial_t + T_{V_2} \cdot \nabla)T_{\sqrt{a_2}}\zeta - T_{\sqrt{a_2}}(T_{\lambda_1}V - T_{\zeta_1}T_{\lambda_1}B) = f''_2,$$

where

$$\|(f''_1, f''_2)\|_{L^\infty(I; H^{s-\frac{3}{2}})} \leq \mathcal{K}(M_1, M_2)N(T).$$

According to (2.5), (3.43) and (4.19), since $s > 1 + \frac{d}{2}$,

$$T_{\sqrt{\lambda_1}}T_{a_2} - T_{\sqrt{\lambda_1 a_2}}T_{\sqrt{a_2}} \quad \text{is of order 0,}$$

which implies (5.22). On the other hand, according to (2.5) and (3.43) the operators $T_{\zeta_1}T_{\lambda_1} - T_{\lambda_1\zeta_1}$ and $T_{\lambda_1}T_{\zeta_1} - T_{\lambda_1\zeta_1}$ are of order 1/2 (with norm controlled by $\mathcal{K}(M_1)$, which allows to commute $T_{\sqrt{\lambda_1}}$ and T_{a_2} in (5.29)). Now, according to Proposition 2.11 (with $\gamma = r = s - \frac{1}{2}$, $\mu = s - 1$)

$$\|T_{\zeta_1}B - \zeta_1 B\|_{H^{s-\frac{1}{2}}} \leq \mathcal{K}(M_1)\|B\|_{H^{s-1}}.$$

Which implies (5.23) (using again (2.5)). \square

Recall that we have set

$$(5.30) \quad N(T) := \sup_{t \in I} \|(\eta, \psi, V, B)(t)\|_{H^{s-\frac{1}{2}} \times H^{s-\frac{1}{2}} \times H^{s-1} \times H^{s-1}}.$$

LEMMA 5.8. *Set*

$$N'(T) := \sup_{t \in I} \{ \|\vartheta(t)\|_{H^{s-\frac{3}{2}}} + \|\varphi(t)\|_{H^{s-\frac{3}{2}}} \}.$$

We have

$$(5.31) \quad N'(T) \leq \mathcal{K}(M_1, M_2)(N(0) + TN(T)).$$

PROOF. We first prove that

$$(5.32) \quad N'(T) \leq \mathcal{K}(M_1, M_2)(N'(0) + TN(T) + TN'(T)).$$

The desired estimate (5.31) then follows from the fact that

$$(5.33) \quad N'(0) \leq \mathcal{K}(M_1, M_2)N(0), \quad N'(T) \leq \mathcal{K}(M_1, M_2)N(T),$$

which follows from the continuity of paradifferential operators in the Sobolev spaces (see Theorem (2.7)) and the fact that $H^{s-1}(\mathbf{R}^d)$ is an algebra since $s > 1 + \frac{d}{2}$.

The proof of (5.32) is based on a classical argument : we commute $\langle D_x \rangle^{s-3/2}$ to (5.22)–(5.23) and perform an L^2 estimate. Then the key points are that (see point (iii) in Theorem 2.7)

$$(5.34) \quad \begin{aligned} \|(T_{V_j} \cdot \nabla)^* + T_{V_j} \cdot \nabla\|_{L^2 \rightarrow L^2} &\lesssim \|V_j\|_{W^{1,\infty}}, \\ \|T_\ell - (T_\ell)^*\|_{L^2 \rightarrow L^2} &\leq \mathcal{F}(\|(\eta_1, \eta_2)\|_{W^{3/2,\infty}}) \end{aligned}$$

and that the commutators $[T_{V_j} \cdot \nabla, \langle D_x \rangle^{s-3/2}]$ are, according to (2.5), of order $s - \frac{3}{2}$.

Notice that since $(\varphi, \vartheta) \in C^1([0, T_0]; H^{s-\frac{3}{2}})$, we do not need to regularize the equations. \square

Finally, let us notice that an elementary argument allows to control lower norms of (V, B) (and hence also of $V + \zeta_1 B$):

$$(5.35) \quad \|(V, B)\|_{L^\infty(I; H^{s-\frac{3}{2}})} \leq \mathcal{K}(M_1, M_2)(N(0) + TN(T)).$$

Indeed, (the proof of) Theorem 5.3 implies that (with $a = a_1 - a_2$)

$$(5.36) \quad \|a\|_{L^\infty(I; H^{s-\frac{3}{2}})} \leq \mathcal{K}(M_1, M_2)N(T).$$

Since $\partial_t B + V_1 \cdot \nabla B = a - V \cdot \nabla B_2$, we have

$$(5.37) \quad \begin{aligned} \|B\|_{L^\infty(I; H^{s-2})} &\leq \|B(0)\|_{H^{s-2}} + \int_0^T \left(\|V_1 \cdot \nabla B\|_{H^{s-2}} + \|a\|_{H^{s-2}} + \|V \cdot \nabla B_2\|_{H^{s-2}} \right) dt' \\ &\leq \|B(0)\|_{H^{s-2}} + T\mathcal{K}(M_1, M_2)N(T). \end{aligned}$$

Similarly, we have

$$(5.38) \quad \|V\|_{L^\infty(I; H^{s-2})} \leq \|V(0)\|_{H^{s-2}} + T\mathcal{K}(M_1, M_2)N(T).$$

Now we have

$$V + \zeta_1 B = T_{\sqrt{\lambda_1}^{-1}}\varphi + (\text{Id} - T_{\sqrt{\lambda_1}^{-1}}T_{\sqrt{\lambda_1}})(V + \zeta_1 B),$$

where according to (2.5), the operator $\text{Id} - T_{\sqrt{\lambda_1}^{-1}}T_{\sqrt{\lambda_1}}$ is of order $-1/2$. Hence, we deduce from (5.33), (5.31), (5.35), (5.38) and a bootstrap argument

$$(5.39) \quad \|V + \zeta_1 B\|_{L^\infty(I; H^{s-1})} \leq \mathcal{K}(M_1, M_2)\{N(0) + TN(T)\}.$$

5.4. Back to the original unknowns. Recall that $I = [0, T]$ (resp. $J = (-1, 0)$) is an interval in the t variable (resp. in the z variable).

LEMMA 5.9.

$$(5.40) \quad \|\eta\|_{L^\infty(I; H^{s-\frac{1}{2}})} \leq \mathcal{K}(M_1, M_2)\{N(0) + TN(T)\}.$$

PROOF. From the equation $\partial_t \eta_j = G(\eta_j) \psi_j$ we have,

$$\eta(t) = \eta(0) + \int_0^t G(\eta_1) \psi(t') dt' + \int_0^t (G(\eta_1) - G(\eta_2)) \psi_2(t') dt',$$

from which we deduce according to Theorem 5.3,

$$(5.41) \quad \|\eta\|_{L^\infty(I; H^{s-2})} \leq \|\eta(0)\|_{H^{s-2}} + TK(M_1, M_2) \|\eta\|_{L^\infty(I; H^{s-\frac{1}{2}})}.$$

Let $R = Id - T \frac{1}{\sqrt{a_2}} T \sqrt{a_2}$, which, according to (2.5) and (4.19) is an operator of order $-\frac{1}{2}$ (with norm estimated by $\mathcal{K}(M_2)$). We have

$$\nabla \eta = R \nabla \eta + T \frac{1}{\sqrt{a_2}} \vartheta.$$

Therefore we deduce from (5.41), (5.31) and a bootstrap argument,

$$\|\nabla \eta\|_{L^\infty(I; H^{s-\frac{3}{2}})} \leq \mathcal{K}(M_1, M_2) (N(0) + TN(T)).$$

Combining with (5.41) gives Lemma 5.9. \square

We are now ready to estimate (V, B) .

PROPOSITION 5.10.

$$(5.42) \quad \|(V, B)\|_{L^\infty(I; H^{s-1})} \leq \mathcal{K}(M_1, M_2) \{N(0) + TN(T)\}.$$

The proof will require several preliminary Lemmas. We begin by noticing that it is enough to estimate B . Indeed, if

$$\|B\|_{L^\infty(I; H^{s-1})} \leq \mathcal{K}(M_1, M_2) \{N(0) + TN(T)\}.$$

then, by using the triangle inequality, the estimate (5.39) for $V + \zeta_1 B$ implies that V satisfies the desired estimate.

Let $v = \tilde{\phi}_1 - \tilde{\phi}_2$, where $\tilde{\phi}_j$ is the harmonic extension in $\tilde{\Omega}$ of the function ψ_j and set

$$b_2 := \frac{\partial_z \tilde{\phi}_2}{\partial_z \rho_2}, \quad w = v - T_{b_2} \rho.$$

We notice that

$$(5.43) \quad w|_{z=0} = \psi - T_{B_2} \eta.$$

We first state the following result.

LEMMA 5.11. *We have*

$$(5.44) \quad \|\psi - T_{B_2} \eta\|_{L^\infty(I; H^s)} \leq \mathcal{K}(M_1, M_2) \{N(0) + TN(T)\}$$

PROOF. Indeed, the low frequencies are estimated by (5.35), while for the high frequencies, we write

$$\begin{aligned} \nabla(\psi - T_{B_2} \eta) &= \nabla \psi - T_{B_2} \nabla \eta - T_{\nabla B_2} \eta \\ &= \nabla \psi_1 - \nabla \psi_2 - T_{B_2} \nabla \eta - T_{\nabla B_2} \eta \\ &= V_1 + B_1 \nabla \eta_1 - V_2 - B_2 \nabla \eta_2 - T_{B_2} \nabla \eta - T_{\nabla B_2} \eta \\ &= V + (B_1 - B_2) \nabla \eta_1 + B_2 (\nabla \eta_1 - \nabla \eta_2) - T_{B_2} \nabla \eta - T_{\nabla B_2} \eta \\ &= V + \zeta_1 B + (B_2 - T_{B_2}) \nabla \eta - T_{\nabla B_2} \eta, \end{aligned}$$

where we used that, by definition, $\nabla \psi_j = V_j + B_j \nabla \eta_j$ and $\zeta_1 = \nabla \eta_1$.

The main term $V + \zeta_1 B$ is estimated using (5.39), while the two other terms are estimated using (5.40), the *a priori* estimate on B_2 and the product rules (2.9) and (2.12). \square

We next relate w , ρ and B .

LEMMA 5.12. *We have*

$$B = \left[\frac{1}{\partial_z \rho_1} \left(\partial_z w - (b_2 - T_{b_2}) \partial_z \rho + T_{\partial_z b_2} \rho \right) \right] \Big|_{z=0}.$$

PROOF. Write

$$\begin{aligned} B_1 - B_2 &= \frac{\partial_z \tilde{\phi}_1}{\partial_z \rho_1} - \frac{\partial_z \tilde{\phi}_2}{\partial_z \rho_2} \Big|_{z=0} \\ &= \frac{1}{\partial_z \rho_1} (\partial_z \tilde{\phi}_1 - \partial_z \tilde{\phi}_2) + \left(\frac{1}{\partial_z \rho_1} - \frac{1}{\partial_z \rho_2} \right) \partial_z \tilde{\phi}_2 \Big|_{z=0} \\ &= \frac{1}{\partial_z \rho_1} \partial_z v - \frac{1}{\partial_z \rho_1} \frac{\partial_z \tilde{\phi}_2}{\partial_z \rho_2} \partial_z \rho \Big|_{z=0} \end{aligned}$$

and replace v by $w + T_{b_2} \rho$ in the last expression. \square

LEMMA 5.13. *Recall that $b_2 := \frac{\partial_z \tilde{\phi}_2}{\partial_z \rho_2}$. For $k = 0, 1, 2$, we have*

$$\left\| \partial_z^k b_2 \right\|_{C^0([-1, 0], L^\infty(I, H^{s-\frac{1}{2}-k}))} \leq C \|\psi_2\|_{H^{s+\frac{1}{2}}}.$$

for some constant C depending only on $\|\eta_2\|_{H^{s+\frac{1}{2}}}$.

PROOF. We estimate $\nabla_{x,z} \tilde{\phi}_2$ in $C^0([-1, 0], L^\infty(I, H^{s-\frac{1}{2}}))$ by using the elliptic regularity (see Proposition 3.22 and Remark 3.21). Now, using the equation satisfied by $\tilde{\phi}_2$ and the product rule in Sobolev spaces, we successively estimate $\partial_z^2 \tilde{\phi}_2$ and $\partial_z^3 \tilde{\phi}_2$. This proves the lemma since the derivatives of ρ_2 are estimated directly from the definition of ρ_2 . \square

Notice that η and hence ρ are estimated in $L^\infty(I; H^{s-\frac{1}{2}})$ (see (5.40)). Now, use Lemma 5.13 and Proposition 2.11 (applied with $s > 1 + d/2$, $\gamma = s - 1$, $r = s - 1/2$, $\mu = s - 3/2$) to obtain

$$\|(b_2 - T_{b_2}) \partial_z \rho\|_{H^{s-1}} \lesssim \|b_2\|_{H^{s-\frac{1}{2}}} \|\eta\|_{H^{s-1}}.$$

Now, (2.15) implies that

$$\|T_{\partial_z b_2} \rho\|_{H^{s-1}} \lesssim \|b_2\|_{H^{s-1/2}} \|\eta\|_{H^{s-1}},$$

and hence, to complete the proof of the Proposition 5.10, it remains only to estimate $\partial_z w|_{z=0}$ in $L^\infty(I, H^{s-1})$. This is the purpose of the following result.

LEMMA 5.14. *For $t \in [0, T]$ we have*

$$(5.45) \quad \|\nabla_{x,z} w\|_{C^0([-1, 0], H^{s-1})} \leq \mathcal{K}(M_1, M_2) \{N(0) + TN(T)\}.$$

PROOF. To prove this estimate, we are going to show that w satisfies an elliptic equation in the variables (x, z) to which we may apply the results of Proposition 3.22. We have

$$\partial_z^2 v + \alpha_1 \Delta v + \beta_1 \cdot \nabla \partial_z v - \gamma_1 \partial_z v = (\gamma_1 - \gamma_2) \partial_z \tilde{\phi}_2 + F_1,$$

where (see (5.10))

$$F_1 = (\alpha_2 - \alpha_1) \Delta \tilde{\phi}_2 + (\beta_2 - \beta_1) \cdot \nabla \partial_z \tilde{\phi}_2.$$

We claim that for $t \in [0, T]$

$$(5.46) \quad \|F_1(t, \cdot)\|_{L^2(J, H^{s-\frac{3}{2}})} \leq \mathcal{K}(M_1, M_2) \{N(0) + TN(T)\}.$$

The two terms in F_1 are estimated by the same way. We will only consider the first one. Using the product rule (2.16) with $s_0 = s - \frac{3}{2}$, $s_1 = s - 1$, $s_2 = s - \frac{3}{2}$ we can write for fixed t

$$\|(\alpha_2 - \alpha_1)\Delta\tilde{\phi}_2\|_{L^2(J, H^{s-\frac{3}{2}})} \leq C\|\alpha_2 - \alpha_1\|_{L^2(J, H^{s-1})}\|\Delta\tilde{\phi}_2\|_{L^\infty(J, H^{s-\frac{3}{2}})}.$$

Then we use (5.11), Proposition 3.22 with $\sigma = s - \frac{1}{2}$ and Lemma 5.9 to conclude that the term above is estimated by the right hand side of (5.46).

Now we introduce the operators

$$P_j := \partial_z^2 + \alpha_j \Delta + \beta_j \cdot \nabla \partial_z, \quad L_j = P_j - \gamma_j \partial_z, \quad (j = 1, 2).$$

With these notations we have $\gamma_j = \frac{1}{\partial_z \rho_j} P_j \rho_j$ and

$$(5.47) \quad L_1 v = (\gamma_1 - \gamma_2) \partial_z \tilde{\phi}_2 + F_1.$$

Moreover

$$\begin{aligned} \gamma_1 - \gamma_2 &= \frac{1}{\partial_z \rho_1} P_1 \rho_1 - \frac{1}{\partial_z \rho_2} P_2 \rho_2 = \frac{1}{\partial_z \rho_2} P_1 \rho_1 + \left(\frac{1}{\partial_z \rho_1} - \frac{1}{\partial_z \rho_2} \right) P_1 \rho_1 - \frac{1}{\partial_z \rho_2} P_2 \rho_2 \\ &= \frac{1}{\partial_z \rho_2} P_1 \rho + \frac{1}{\partial_z \rho_2} (P_1 - P_2) \rho_2 + \left(\frac{1}{\partial_z \rho_1} - \frac{1}{\partial_z \rho_2} \right) P_1 \rho_1 \\ &= \frac{1}{\partial_z \rho_2} P_1 \rho + \left(\frac{1}{\partial_z \rho_1} - \frac{1}{\partial_z \rho_2} \right) P_1 \rho_1 + F_2, \end{aligned}$$

where

$$(5.48) \quad F_2 = \frac{1}{\partial_z \rho_2} \left((\alpha_1 - \alpha_2) \Delta \rho_2 + (\beta_1 - \beta_2) \cdot \nabla \partial_z \rho_2 \right).$$

Now we observe that

$$\left(\frac{1}{\partial_z \rho_1} - \frac{1}{\partial_z \rho_2} \right) P_1 \rho_1 = -\frac{\partial_z \rho}{\partial_z \rho_2} \frac{P_1 \rho_1}{\partial_z \rho_1} = -\frac{\partial_z \rho}{\partial_z \rho_2} \gamma_1,$$

which implies

$$\gamma_1 - \gamma_2 = \frac{1}{\partial_z \rho_2} P_1 \rho - \frac{\partial_z \rho}{\partial_z \rho_2} \gamma_1 + F_2 = \frac{1}{\partial_z \rho_2} L_1 \rho + F_2.$$

Plugging this into (5.47) yields

$$(5.49) \quad L_1 v - b_2(L_1 \rho) = F_1 + (\partial_z \tilde{\phi}_2) F_2.$$

We claim that for fixed t we have

$$(5.50) \quad \left\| (\partial_z \tilde{\phi}_2) F_2(t, \cdot) \right\|_{L^2(J, H^{s-\frac{3}{2}})} \leq \mathcal{K}(M_1, M_2) \{N(0) + TN(T)\}.$$

Indeed we first use the product rule (2.16) to write

$$\left\| (\partial_z \tilde{\phi}_2) F_2(t, \cdot) \right\|_{L^2(J, H^{s-\frac{3}{2}})} \leq \left\| (\partial_z \tilde{\phi}_2)(t, \cdot) \right\|_{L^2(J, H^{s-\frac{1}{2}})} \left\| F_2(t, \cdot) \right\|_{L^2(J, H^{s-\frac{3}{2}})}.$$

By the elliptic regularity the first term in the right hand side is bounded by $\mathcal{K}(M_2)$. It is therefore sufficient to bound the second one. We have, for fixed t

$$\left\| \frac{1}{\partial_z \rho_2} (\alpha_1 - \alpha_2) \Delta \rho_2 \right\|_{L^2(J, H^{s-\frac{3}{2}})} \leq \mathcal{K}(M_2) \|\alpha_1 - \alpha_2\|_{L^2(J, H^{s-1})} \|\Delta \rho_2\|_{L^\infty(J, H^{s-\frac{3}{2}})}.$$

Using (5.11) and (3.11) we see that the right hand side is bounded by the right hand side of (5.50). The second term in F_2 is estimated by the same way.

To estimate $v - T_{b_2} \rho$ we parilinearize in writing

$$(5.51) \quad b_2(L_1 \rho) = T_{b_2} L_1 \rho + T_{L_1 \rho} b_2 + F_3.$$

We claim that for $t \in [0, T]$

$$(5.52) \quad \|F_3(t, \cdot)\|_{L^2(J, H^{s-\frac{3}{2}})} \leq \mathcal{K}(M_1, M_2)\{N(0) + TN(T)\}.$$

To prove it we shall use (2.9) with $\alpha = s - \frac{1}{2}, \beta = s - 2$. Then $\alpha + \beta - \frac{d}{2} > s - \frac{3}{2}$. It follows that, for fixed z and t we have

$$\|F_3(t, \cdot, z)\|_{H^{s-\frac{3}{2}}} \leq C\|b_2\|_{H^{s-\frac{1}{2}}}\|L_1\rho\|_{H^{s-2}}.$$

Therefore

$$\|F_3(t, \cdot)\|_{L^2(J, H^{s-\frac{3}{2}})} \leq C\|b_2\|_{L^\infty(J, H^{s-\frac{1}{2}})}\|L_1\rho\|_{L^2(J, H^{s-2})}.$$

Now as we have seen before we have $\|b_2(t, \cdot)\|_{L^\infty(J, H^{s-\frac{1}{2}})} \leq \mathcal{K}(M_2)$ and due to the smoothing of the Poisson kernel $\|L_1\rho\|_{L^2(J, H^{s-2})} \leq \mathcal{K}(M_1)\|\eta\|_{H^{s-\frac{1}{2}}}$. The estimate (5.52) thus follows from (5.40).

Setting $F_4 = T_{L_1\rho}b_2$ we claim that for fixed t we have

$$(5.53) \quad \|F_4(t, \cdot)\|_{L^2(J, H^{s-\frac{3}{2}})} \leq \mathcal{K}(M_1, M_2)\{N(0) + TN(T)\}.$$

To see this we use (2.15) with $s_0 = s - \frac{3}{2}, s_1 = s - 2, s_2 = s - \frac{1}{2}$. We get

$$\|F_4(t, \cdot)\|_{L^2(J, H^{s-\frac{3}{2}})} \leq \|L_1\rho(t, \cdot)\|_{L^2(J, H^{s-2})}\|b_2(t, \cdot)\|_{L^\infty(J, H^{s-\frac{1}{2}})}$$

and (5.53) follows from estimates used above.

Now according to (5.49), (5.51) we have

$$L_1v - T_{b_2}L_1\rho = F_1 + (\partial_z\tilde{\phi}_2)F_2 + F_3 + F_4.$$

We claim that we have

$$L_1T_{b_2}\rho = T_{b_2}L_1\rho - F_5$$

with

$$\|F_5(t, \cdot)\|_{L^2(J, H^{s-\frac{3}{2}})} \leq \mathcal{K}(M_1, M_2)\{N(0) + TN(T)\}.$$

To see this we use (5.13) and (2.15). It follows then that we have

$$L_1w = L_1(v - T_{b_2}\rho) = F_1 + (\partial_z\tilde{\phi}_2)F_2 + F_3 + F_4 + F_5 := F$$

where $\|F(t, \cdot)\|_{L^2(J, H^{s-\frac{3}{2}})}$ is bounded by the right hand side of (5.53).

Using (5.43) and Lemma 5.11 we may then apply to w Proposition 3.22 with $\sigma = s - 1$ to conclude the proof of Lemma 5.14 and thus that of Proposition 5.10. \square

6. Well-posedness of the Cauchy problem

Here we conclude the proof of Theorem 1.2 about the Cauchy theory for $s > 1 + d/2$ for the system

$$(6.1) \quad \begin{cases} \partial_t\eta + G(\eta)\psi = 0, \\ \partial_t\psi + g\eta + \frac{1}{2}|\nabla\psi|^2 - \frac{1}{2}\frac{(\nabla\eta \cdot \nabla\psi + G(\eta)\psi)^2}{1 + |\nabla\eta|^2} = 0. \end{cases}$$

We previously proved the uniqueness of solutions (see Theorem 5.1). To complete the proof of Theorem 1.2, it remains to prove the existence. We obtain solutions to the water waves system as limits of smooth solutions to approximate systems. This approach has been detailed in [1], where we considered the problem with surface tension. The analysis is actually easier without surface tension. One reason is that with surface tension, we needed in [1] to use some mollifiers with various properties

(since we need good estimates for commutators with the principal part of the operator). Here it is possible to use a simpler regularization of the equations since the reduced paradifferential system involves only operator of order less than or equal to 1.

6.1. The infinite depth case. To explain the scheme of the proof, we first consider the case without bottom ($\Gamma = \emptyset$). This case is easier since we know, from previous results (see Wu [54, 55], Lannes [39], Lindblad [41]), that the Cauchy problem is well-posed for smooth initial data. Then, one can obtain the existence of smooth approximate solutions in a straightforward way : by smoothing the initial data.

Denote by J_ε the usual Friedrichs mollifiers, defined by $J_\varepsilon = j(\varepsilon D_x)$ where $j \in C_0^\infty(\mathbf{R}^d)$, $0 \leq j \leq 1$, is such that

$$j(\xi) = 1 \quad \text{for } |\xi| \leq 1, \quad j(\xi) = 0 \quad \text{for } |\xi| \geq 2.$$

Set $\psi_0^\varepsilon = J_\varepsilon \psi_0$ and $\eta_0^\varepsilon = J_\varepsilon \eta_0$. Then $(\psi_0^\varepsilon, \eta_0^\varepsilon) \in H^\infty(\mathbf{R}^d)^2$ and the Cauchy problem for (6.1) has a unique smooth solution $(\psi^\varepsilon, \eta^\varepsilon)$ defined on some time interval $[0, T_\varepsilon^*]$. Since $s > 1 + d/2$, by applying Proposition 4.1 with $r = s - d/2$, it follows that there exists a function \mathcal{F} such that, for all $\varepsilon \in (0, 1]$ and all $T < T_\varepsilon$, we have

$$(6.2) \quad M_s^\varepsilon(T) \leq \mathcal{F}(\mathcal{F}(M_{s,0}) + T\mathcal{F}(M_s^\varepsilon(T))),$$

with obvious notations. Then by standard arguments, we infer that the lifespan of $(\eta_\varepsilon, \psi_\varepsilon)$ is bounded from below by a positive time T_0 independent of ε and that we have uniform estimates on $[0, T_0]$. The fact that one can pass to the limit in the equations follows from the previous contraction estimates (see (5.1)), which allows us to prove that $(\eta_\varepsilon, \psi_\varepsilon, B_\varepsilon, V_\varepsilon)$ is a Cauchy sequence (this argument has been explained in [1]). Notice that these estimates were proved under the assumption $a(t) > a_0/2$. This actually follows from the *a priori* bound (6.2), (4.19), (4.20) and a bootstrap method. Then, it remains to prove that the limit solution has the desired regularity properties. Again, this follows from the analysis in [1].

6.2. The general case. In the case with a general bottom, to apply the strategy explained above, the only remaining point is to prove that, for smooth enough initial data, the Cauchy problem has a smooth solution. Namely, for our purposes it is enough to prove the following weak well-posedness result (where we allow an arbitrarily large loss of N derivatives).

PROPOSITION 6.1. *For all $s_0 > 0$ there exists $N > 0$ such that the following result holds. Consider an initial data $(\eta_0, \psi_0) \in H^{s_0+N}(\mathbf{R}^d)^2$ such that, initially, the Taylor coefficient a is bounded from below by $a_0 > 0$. Then, there exists $T > 0$ and a solution $(\eta, \psi) \in C^1([0, T]; H^{s_0}(\mathbf{R}^d))^2$ to (6.1) satisfying $(\eta(0), \psi(0)) = (\eta_0, \psi_0)$ and $a(t) > a_0/2$.*

PROOF. We use a parabolic regularization of the equations and seek solutions of the Cauchy problem (6.1) as limits of solutions of the following approximate systems:

$$(6.3) \quad \begin{cases} \partial_t \eta - G(\eta)\psi = \varepsilon \Delta \eta, \\ \partial_t \psi + g\eta + \frac{1}{2} |\nabla \psi|^2 - \frac{1}{2} \frac{(\nabla \eta \cdot \nabla \psi + G(\eta)\psi)^2}{1 + |\nabla \eta|^2} = \varepsilon \Delta \psi, \\ (\eta, \psi)|_{t=0} = (J_\varepsilon \eta_0, J_\varepsilon \psi_0), \end{cases}$$

where $\varepsilon \in]0, 1]$ is a small parameter and

$$(6.4) \quad B = \frac{\nabla\eta \cdot \nabla\psi + G(\eta)\psi}{1 + |\nabla\eta|^2}.$$

Write (6.3) under the form

$$u = e^{\varepsilon t \Delta} u_0 + \int_0^t e^{\varepsilon(t-\tau)\Delta} \mathcal{A}[u(\tau)] d\tau, \quad u = (\eta, \psi).$$

To solve this Cauchy problem we use two ingredients. Firstly we have

$$\|e^{\varepsilon t \Delta} f\|_{L^2([0, T]; H^{s+1})} \leq K(\varepsilon) \|f\|_{H^s},$$

together with a dual estimate, and secondly (for s large enough)

$$\|\mathcal{A}[u]\|_{L^2([0, T]; H^{s-1})} \leq \sqrt{T} \mathcal{F}(\|u\|_{L^\infty([0, T]; H^s)}) \|u\|_{L^\infty([0, T]; H^s)},$$

which follows from Theorem 3.18 and the product rule in Sobolev spaces. Then, given $\varepsilon > 0$ and $(\eta_0, \psi_0) \in (H^s(\mathbf{R}^d))^2$ with s large enough, by applying the Banach fixed point theorem in $L^\infty([0, T]; H^s) \cap L^2([0, T]; H^{s+1})$, one gets that, for T small enough possibly depending on ε , the Cauchy problem for (6.3) has a smooth solution. Moreover, the maximal time of existence T_ε satisfies: either $T_\varepsilon = +\infty$ or

$$\limsup_{t \rightarrow T_\varepsilon} \|(\eta, \psi)(t, \cdot)\|_{H^s} = +\infty.$$

We have to prove that these solutions exist for a time interval independent of $\varepsilon \in]0, 1]$. To do so, we begin by computing the equations satisfied by B and $V = \nabla\psi - B\nabla\eta$.

We first observe that, since $G(\eta)\psi = B - V \cdot \nabla\eta$, we have

$$(6.5) \quad \partial_t \eta + V \cdot \nabla \eta = B + \varepsilon \Delta \eta.$$

On the other hand, as in (4.55), we have

$$(6.6) \quad \partial_t \psi + V \cdot \nabla \psi = -g\eta + \frac{1}{2}|V|^2 - \frac{1}{2}B^2 + \varepsilon \Delta \psi.$$

We next write that

$$\begin{aligned} \partial_t V + V \cdot \nabla V &= (\partial_t + V \cdot \nabla)(\nabla\psi - B\nabla\eta) \\ &= \nabla(\partial_t \psi + V \cdot \nabla \psi) - (\partial_t B + V \cdot \nabla B)\nabla\eta - B\nabla(\partial_t \eta + V \cdot \nabla \eta) + R \end{aligned}$$

where $R = (R_1, \dots, R_d)$ with

$$R_k = - \sum_j \partial_k V_j \partial_j \psi + B \sum_j \partial_k V_j \partial_j \eta.$$

Then, it follows from (6.5) and (6.6) that

$$\partial_t V + V \cdot \nabla V = \nabla \left(-g\eta + \frac{1}{2}|V|^2 - \frac{1}{2}B^2 + \varepsilon \Delta \psi \right) - (\partial_t B + V \cdot \nabla B)\nabla\eta + B\nabla(B + \varepsilon \Delta \eta) + R.$$

Now, observing that $R + \frac{1}{2}\nabla|V|^2 = 0$ and simplifying,

$$\partial_t V + V \cdot \nabla V + (g + \partial_t B + V \cdot \nabla B)\nabla\eta = \varepsilon \nabla \Delta \psi - \varepsilon B \nabla \Delta \eta$$

which in turn implies that

$$\partial_t V + V \cdot \nabla V + \tilde{a}\zeta = \varepsilon \Delta V + 2\varepsilon(\nabla B \cdot \nabla)\zeta,$$

where $\zeta = \nabla\eta$ and

$$(6.7) \quad \tilde{a} = g + \partial_t B + V \cdot \nabla B - \varepsilon \Delta B.$$

Now, for $s > 5/2 + d/2$, from the proof of (4.38) (using Lemma 2.17 instead of Lemma 2.16 as in the proof of Lemma 5.6), we obtain that $U = V + T_\zeta B$ satisfies

$$(6.8) \quad \partial_t U + T_V \cdot \nabla U + T_{\tilde{a}} \zeta - \varepsilon \Delta U - \varepsilon T_{\nabla B} \cdot \nabla \zeta = F_1$$

where

$$\|F_1\|_{L_t^\infty(H^s)} \leq \mathcal{F}(\|(\psi, \eta, V, B, \tilde{a})\|_{L_t^\infty(H^{s+1/2} \times H^{s+1/2} \times H^s \times H^s \times C^{1/2})}).$$

Also, as in the proof of Proposition 4.12, we find that $\theta = T_q \zeta$ (where $q = \sqrt{\tilde{a}/\lambda}$) satisfies

$$\partial_t \theta + T_V \cdot \nabla \theta - T_\gamma U - \varepsilon \Delta \theta - \varepsilon [T_q, \Delta] T_{q^{-1}} \theta = F_2,$$

where F_2 is estimated by means of Theorem 2.7 and (2.17):

$$\|F_2\|_{L_t^\infty(H_x^s)} \leq \mathcal{F}(\|(\psi, \eta, V, B, \tilde{a}, \partial_t \tilde{a} + V \cdot \nabla \tilde{a})\|_{L_t^\infty(H^{s+1/2} \times H^{s+1/2} \times H^s \times H^s \times C^{1/2} \times L^\infty)}).$$

Using the unknown θ , one can rewrite (6.8) as

$$(6.9) \quad \partial_t U + T_V \cdot \nabla U + T_{\tilde{a}} \zeta - \varepsilon \Delta U - \varepsilon T_{\nabla B} \cdot \nabla T_{q^{-1}} \theta = \tilde{F}_1$$

where $\tilde{F}_1 = F_1 + \varepsilon T_{\nabla B} \cdot \nabla (T_q T_{q^{-1}} - I) \zeta$ and it follows from Theorem 2.7 that

$$\|T_{\nabla B} \cdot \nabla (T_q T_{q^{-1}} - I) \zeta\|_{H^s} \lesssim \|\nabla B\|_{L^\infty} M_{3/2}^{1/2}(q) M_{3/2}^{-1/2}(q^{-1}) \|\eta\|_{H^{s+1/2}}.$$

Since $M_{3/2}^{1/2}(q) M_{3/2}^{-1/2}(q^{-1}) \leq C(\|\eta\|_{C^{5/2}}, \|\tilde{a}\|_{C^{3/2}})$ we obtain

$$\|\tilde{F}_1\|_{L_t^\infty(H^s)} \leq \mathcal{F}(\|(\psi, \eta, V, B, \tilde{a})\|_{L_t^\infty(H^{s+1/2} \times H^{s+1/2} \times H^s \times H^s \times C^{3/2})}).$$

Now, since the operator

$$\begin{pmatrix} \Delta & T_{\nabla B} \cdot \nabla T_{q^{-1}} \\ 0 & \Delta + [T_q, \Delta] T_{q^{-1}} \end{pmatrix}$$

is elliptic (being a perturbation of Δ of order 3/2), we estimate (U, θ) in $L_t^\infty(H^s \times H^s)$ by commuting the equation with $(I - \Delta)^{s/2}$ and performing an L^2 estimate. To do so it remains only to prove that

$$\|(\tilde{a}, \partial_t \tilde{a} + V \cdot \nabla \tilde{a})\|_{L_t^\infty(C^{3/2} \times L^\infty)} \leq \mathcal{F}(\|(\psi, \eta, V, B)\|_{L_t^\infty(H^{s+1/2} \times H^{s+1/2} \times H^s \times H^s)}).$$

Here this is much easier than in the above analysis. Indeed, since we do not need to work with critical regularity, it is enough to prove that there exist $s_2 \geq s_1 \geq 3/2 + s_0$ with $s_0 > d/2$ and $s_2 \leq s$ (possibly $s_0 \ll s_1 \ll s_2$) such that

$$(6.10) \quad \|\tilde{a} - g\|_{L_t^\infty(H^{s_1})} \leq \mathcal{F}(\|(\eta, \psi)\|_{L_t^\infty(H^{s_2} \times H^{s_2})}),$$

$$(6.11) \quad \|\partial_t \tilde{a}\|_{L_t^\infty(H^{s_0})} \leq \mathcal{F}(\|(\eta, \psi)\|_{L_t^\infty(H^{s_2} \times H^{s_2})}).$$

To prove (6.10) we express $\tilde{a} - g$ in terms of η, ψ (as noticed in [38]). To do so, we use two observations. Firstly, by definition of B (see (6.4)), we have

$$\partial_t B = \frac{1}{1 + |\nabla \eta|^2} \left(\nabla \partial_t \eta \cdot \nabla \psi + \nabla \cdot \nabla \partial_t \psi + \partial_t G(\eta) \psi \right) - \frac{2 \nabla \partial_t \eta \cdot \nabla \eta}{1 + |\nabla \eta|^2} B.$$

Secondly, we have

$$(6.12) \quad \partial_t G(\eta) \psi = G(\eta) (\partial_t \psi - B \partial_t \eta) - \operatorname{div}(V \partial_t \eta).$$

The formula (6.12) is proved by Lannes for smooth bottoms (see [39]) or infinite depth ([38]). In the case with a general bottom considered here, this follows from [4]. Then, replacing $\partial_t \psi$ and $\partial_t \eta$ by their expressions computed using System (6.3), we are in position to express $\partial_t B$ in terms of η, ψ only. Setting the result thus obtained in (6.7) we get an expression of $\tilde{a} - g$ in terms of η, ψ only. The estimate (6.10) then follows from the nonlinear estimates in Sobolev spaces (see (2.17) and (2.21)) and

repeated uses of the estimate for the Dirichlet-Neumann proved in Theorem 3.18. Repeating this reasoning, we get an expression of $\partial_t \tilde{a}$ in terms of η, ψ only and hence (6.11).

Up to now, this gives uniform estimates in ε for (U, θ) in $L^\infty([0, T]; H^s \times H^s)$, as long as $\tilde{a}(t)$ is bounded from below by $a_0/2$. Moreover, as in the proof of Lemma 4.15 and Lemma 4.16, it follows from (6.11) that one can estimate (η, B, V) in $L^\infty([0, T]; H^s \times H^{s_0} \times H^{s_0})$. Now we can recover estimates for the unknowns (η, ψ, B, V) in $L^\infty([0, T]; H^{s+\frac{1}{2}} \times H^{s+\frac{1}{2}} \times H^s \times H^s)$ by a bootstrap argument as in the proof of Lemma 4.17.

Using these uniform estimates, we get by a bootstrap argument that the solutions to (6.3) exist on a time interval independent of ε (and satisfy $a(t) > a_0/2$ according to (6.10) (6.11)) on this time interval uniform estimates. To conclude it remains to pass to the limit in (6.3). For this we use again the uniform estimates which hold as long as the Taylor coefficient $a(t)$ is bounded from below by $a_0/2$ (and the equation to bound the time derivatives) to extract subsequences converging weakly and the fact that the Dirichlet-Neumann operator, though nonlocal, passes to the limit thanks to Theorem 3.9. Eventually, we use again a bootstrap argument to control the Taylor coefficient using (4.19) and (4.20). \square

7. Three-dimensional waves in a canal

Let us recall that we consider a fluid domain which at time t is of the form

$$\Omega(t) = \{(x_1, x_2, y) \in (0, 1) \times \mathbf{R} \times \mathbf{R} : b(x) < y < \eta(t, x), x = (x_1, x_2)\},$$

for some given function b . Denote by Σ the free surface and by Γ the fixed boundary of the canal:

$$\Sigma(t) = \{(x_1, x_2, y) \in (0, 1) \times \mathbf{R} \times \mathbf{R} : y = \eta(t, x)\},$$

and we set $\Gamma = \partial\Omega(t) \setminus \Sigma(t)$ (which does not depend on time). We have

$$\Gamma = \Gamma_1 \cup \Gamma_2,$$

$$(7.1) \quad \begin{aligned} \Gamma_1 &= \{(x_1, x_2, y) \in (0, 1) \times \mathbf{T} \times \mathbf{R}; \tilde{b}(x_1, x_2) = y\} \\ \Gamma_2 &= \{(x_1, x_2, y) \in \{0, 1\} \times \mathbf{T} \times \mathbf{R}; \tilde{b}(x_1, x_2) < y < \eta(x_1, x_2)\} \end{aligned}$$

Denote by n the normal to the boundary Γ (remark that near $\Gamma_1, n = (\pm 1, 0, 0)$) and denote by ν the boundary to the free surface Σ .

We consider an initial surface η_0 and an initial velocity field v_0 in Ω satisfying

$$(7.2) \quad \operatorname{div}_{x,y} v_0 = 0, \quad \operatorname{curl}_{x,y} v_0 = 0, \quad v_0|_{\Gamma} \cdot n = 0.$$

We write $v_0 = \nabla_{x,y} \phi$ and define as before

$$V_0 = \nabla_x \phi(x, \eta_0(x)), \quad B_0 = \partial_y \phi(x, \eta_0(x)).$$

We will also need to consider v_{x_1}, v_{x_2} and v_y the components of the velocity field directed respectively in the x_1, x_2 and y direction, Let us recall that we assume

- i) there exists a positive constant h such that $\eta_0(x) \geq b(x) + h$
- ii) The Taylor sign condition (1.8) holds initially.

We begin with the following (elementary but seemingly new) observation: in the case of vertical walls, as long as the Taylor sign condition (1.8) is satisfied, for the Cauchy problem to be well posed, it is *necessary* that at the points where the free surface and the boundary of the canal meet $(\Sigma(t) \cap \Gamma)$, the scalar product between the two

normals (to the free surface and to the boundary of the canal) vanishes : $\nu \cdot n = 0$ on $\Sigma \cap \Gamma$, which means that the free surface Σ necessarily makes a right-angle with the rigid walls (see Figure 4).

PROPOSITION 7.1. *Let $(\eta, v) \in C^0([0, T]; C^{\frac{3}{2}}([0, 1] \times \mathbf{R}) \times C^1(\overline{\Omega}))$ be a solution of System (1.5) such that the Taylor coefficient a is continuous and non-vanishing. Then the angle between the free surface, $\Sigma(t)$ and the boundary of the canal Γ is a right angle:*

$$\forall t \in [0, T], \forall x \in \Sigma(t) \cap \Gamma, \quad n \cdot \nu(t, x) = 0,$$

which is equivalent to

$$(7.3) \quad \partial_{x_1} \eta(t, x_1, x') \big|_{x_1=0,1} = 0.$$

REMARK 7.2. Taking into account that

$$\partial_{x_1} \psi = \partial_{x_1} \phi(x, \eta(x)) + \partial_y \phi(x, \eta(x)) \partial_{x_1} \eta(x),$$

and using that on the vertical part of the boundary Γ , the normal velocity vanishes which implies

$$\partial_{x_1} \phi(x_1, x', y) = 0, \quad x_1 = 0, 1, b(x) < y < \eta(x),$$

and consequently $v_{x_1} \big|_{x_1=0,1} = 0$.

PROOF OF PROPOSITION 7.1. Since near the free surface, $n = (\pm 1, 0, 0)$ is constant, and the vector field $v \cdot \nabla$ is tangent to Γ , the boundary condition $\partial_n \phi = v \cdot n = 0$ implies that $[\partial_t v + (v \cdot \nabla_{x,y})v] \cdot n = 0$ on Γ_1 , and consequently also by continuity on the intersection of $\overline{\Gamma_1}$ with $\overline{\Sigma}$. It follows from the Euler equation that

$$\nabla_{x,y} P \cdot n = 0 \quad \text{on } \overline{\Gamma_1} \cap \overline{\Sigma} = \{(x_1 = 0, 1), x_2, y = \eta(t, x)\}.$$

On the other hand, by assumption, the pressure is constant on the free surface and hence $\nabla_{x,y} P$ is proportional to the normal to Σ , ν . Notice now that the non vanishing of the Taylor coefficient a reads $\partial_y P \big|_{\Sigma} \neq 0$ and consequently $\nabla_{x,y} P \big|_{\Sigma} \neq 0$. This implies that $\nu \cdot n = 0$ on $\Sigma \cap \Gamma$. \square

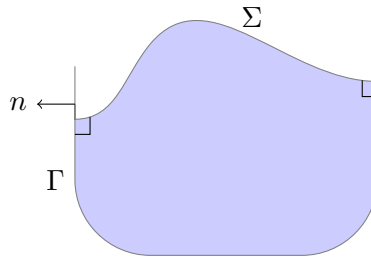


FIGURE 4. Two-dimensional section of the fluid domain, exhibiting the right-angles at the interface $\Sigma \cap \Gamma$

Here is our result about the Cauchy problem for water waves in a canal.

THEOREM 7.3. *Let $s \in (2, 3)$, $s \neq \frac{5}{2}$,*

$$\mathcal{H}^s((0, 1) \times \mathbf{R}) = H^{s+\frac{1}{2}}((0, 1) \times \mathbf{R}) \times H^{s+\frac{1}{2}}((0, 1) \times \mathbf{R}) \times H^s((0, 1) \times \mathbf{R}) \times H^s((0, 1) \times \mathbf{R}),$$

and consider initial data η_0, v_0 , satisfying

$$(\eta_0, \psi_0, V_0, B_0) \in \mathcal{H}^s((0, 1) \times \mathbf{R})$$

and

- i) $\partial_{x_1}\eta_0|_{x_1=0,1} = 0$,
- ii) $\partial_{x_1}\psi_0|_{x_1=0,1} = 0$,
- iii) the Taylor sign condition, $a_0(x) \geq c > 0$ is satisfied at time $t = 0$ and $\eta_0(x) \geq b(x) + h$ for some positive constant h .

Then there exists a time $T > 0$ and a unique solution (η, v) of the system (1.3) (1.4) such that

- i) $(\eta, \psi, V, B) \in C([0, T]; \mathcal{H}^s((0, 1) \times \mathbf{R}))$,
- ii) the Taylor sign condition is satisfied at time t and $\eta(t) \geq b + h/2$.

REMARK 7.4. Assumptions (i) and (ii) are according to Proposition 7.1 and Remark 7.2 necessary to solve the water waves system. In the case of a flat bottom (say $b(x) = -1$) we do not need assumption (iii) which is in this case always satisfied. Also, this condition is satisfied under a smallness assumption. Finally, our result excludes the case $s = \frac{5}{2}$ for technical reasons. It would be possible (but unnecessarily complicated) to include this case.

7.1. Reduction to a result for periodic data. In this section, we shall denote by $\mathbf{T} = \mathbf{R}/2\mathbf{Z}$ the torus and consider a domain

$$\Omega = \{ (t, x_1, x_2, y) \in [0, T] \times \mathbf{R} \times \mathbf{T} \times \mathbf{R} \times \mathbf{R} : (x_1, x_2, y) \in \Omega(t) \},$$

such that, for each time t , one has

$$\Omega(t) = \{ (x, y) \in \mathcal{O} : b(x_1, x_2) < y < \tilde{\eta}(t, x_1, x_2) \},$$

where b is a fixed continuous function, $\tilde{\eta}$ is an unknown function and \mathcal{O} is a given open domain which contains a fixed strip around the free surface

$$\Sigma = \{ (t, x, y) \in [0, T] \times \mathbf{T} \times \mathbf{R} \times \mathbf{R} : y = \eta(t, x) \}.$$

This implies that there exists $h > 0$ such that, for all $t \in [0, T]$,

$$(7.4) \quad b(x_1, x_2) \leq \eta(t, x_1, x_2) - h.$$

Following Boussinesq (see [15, page 37]) the strategy of proof is to perform a symmetrization process (following the process which is illustrated on Figure 3 in the introduction). The two main points in this section are to reduce the proof of Theorem 7.3 to the proof of the following result and to show how in turn Theorem 7.5 can be proved by a rather straightforward adaptation of the proof of Theorem 1.2.

THEOREM 7.5. *Let $s > 2$ (notice that the space dimension of our free surface is $d = 2$ and consequently $s > 1 + d/2$). Let*

$$\mathcal{H}^s(\mathbf{T} \times \mathbf{R}) = H^{s+\frac{1}{2}}(\mathbf{T} \times \mathbf{R}) \times H^{s+\frac{1}{2}}(\mathbf{T} \times \mathbf{R}) \times H^s(\mathbf{T} \times \mathbf{R}) \times H^s(\mathbf{T} \times \mathbf{R}),$$

and assume that the initial data satisfy

$$(\tilde{\eta}_0, \tilde{\psi}_0, \tilde{V}_0, \tilde{B}_0) \in \mathcal{H}^s(\mathbf{T} \times \mathbf{R})$$

are periodic with respect to the x_1 variable, the Taylor sign condition is satisfied at time $t = 0$ and $\eta_0 \geq b + h$ for some positive constant h . Then there exists a time $T > 0$ and a unique solution (η, v) of the system (1.3) (1.4) such that

- i) $(\eta_0, \psi_0, V_0, B_0) \in C([0, T]; \mathcal{H}^s(\mathbf{T} \times \mathbf{R}))$,
- ii) the Taylor sign condition is satisfied at time t and $\eta \geq b + h/2$.

Before proving this result, let us show how it implies Theorem 7.3 by a simple reflection procedure.

7.2. Reduction to a periodic setting. Without additional assumptions, the reflection procedure should yield in general a Lipschitz singularity. However, here the possible singularities are weaker according to the above physical observation about the right angles at the interface which implies that

$$\partial_{x_1}\eta(t, 0, x_2) = 0 \text{ and } \partial_{x_1}\eta(t, 1, x_2) = 0.$$

For a function v defined on $(0, +\infty)$, define v^{ev} and v^{od} to be the even and odd extensions of v to $(-\infty, +\infty)$ defined by

$$(7.5) \quad \begin{aligned} v^{\text{ev}}(y) &= \begin{cases} v(-y), & \text{if } y < 0 \\ v(y) & \text{if } y \geq 0. \end{cases} \\ v^{\text{od}}(y) &= \begin{cases} -v(-y), & \text{if } y < 0 \\ v(y) & \text{if } y \geq 0. \end{cases} \end{aligned}$$

We have the following result

PROPOSITION 7.6. *We have*

- Assume that $s \in (-\frac{1}{2}, \frac{1}{2}) \cup (\frac{1}{2}, \frac{3}{2})$. Then the map $v \mapsto v^{\text{ev}}$ is continuous from $H^s(0, +\infty)$ to $H^s(\mathbf{R})$.
- Assume that $s \in (\frac{3}{2}, \frac{5}{2}) \cup (\frac{5}{2}, \frac{7}{2})$ and let $\widehat{H}^s(0, +\infty)$ be the closed subspace $H^s(0, +\infty)$ of functions whose derivatives vanish at 0. Then the map $v \mapsto v^{\text{ev}}$ is continuous from $\widehat{H}^s(0, +\infty)$ to $H^s(\mathbf{R})$.
- Assume that $s \in (-\frac{1}{2}, \frac{1}{2})$. Then the map $v \mapsto v^{\text{od}}$ is continuous from $H^s(0, +\infty)$ to $H^s(\mathbf{R})$.
- Assume that $s \in (\frac{1}{2}, \frac{3}{2}) \cup (\frac{3}{2}, \frac{5}{2})$ and let $\widehat{H}^s(0, +\infty)$ be the closed subspace $H^s(0, +\infty)$ of functions vanishing at 0. Then the map $v \mapsto v^{\text{ev}}$ is continuous from $\widehat{H}^s(0, +\infty)$ to $H^s(\mathbf{R})$.
- Assume that $s \in (\frac{5}{2}, \frac{7}{2}) \cup (\frac{7}{2}, \frac{9}{2})$ and let $\widehat{H}^s(0, +\infty)$ be the closed subspace $H^s(0, +\infty)$ of functions vanishing at 0 such that $\partial_x^2 u|_{x=0} = 0$. Then the map $v \mapsto v^{\text{ev}}$ is continuous from $\widehat{H}^s(0, +\infty)$ to $H^s(\mathbf{R})$.

PROOF. For $s \in (-\frac{1}{2}, \frac{1}{2})$, it is well known that the space $C_0^\infty(I)$ is dense in $H^s(I)$ for any interval $I \subset \mathbf{R}$, hence

$$w \in H^s(0, +\infty) \mapsto w1_{y>0}$$

is bounded from $H^s(0, +\infty)$ to $H^s(\mathbf{R})$ and the result follows for $s \in (-\frac{1}{2}, \frac{1}{2})$. For $s \in (\frac{1}{2}, \frac{3}{2})$, the jump formula gives

$$\partial_y v^{\text{ev}} = \partial_y v^{\text{ev}} + [v^{\text{ev}}](0) \otimes \delta_{y=0} = \partial_y v^{\text{ev}}$$

where

$$[v^{\text{ev}}](0) = v^{\text{ev}}(0+) - v^{\text{ev}}(0-)$$

denotes the jump of the function v^{ev} at 0 (which exists since $s > \frac{1}{2}$ and is equal to 0 since the function v^{ev} is even. As a consequence, since

$$\|v^{\text{ev}}\|_{H^s(\mathbf{R})} \leq C(\|v^{\text{ev}}\|_{L^2} + \|\partial_y v^{\text{ev}}\|_{H^{s-1}(\mathbf{R})}),$$

and the result for $s \in (\frac{1}{2}, \frac{3}{2})$ follows directly from the result for $s \in (-\frac{1}{2}, \frac{1}{2})$, and the fact that $C_0^\infty([0, +\infty))$ is dense in $H^s(\mathbf{R})$. For $s \in (\frac{3}{2}, \frac{5}{2})$, another application of the jump formula and the fact that by assumption

$$[\partial_y v^{\text{ev}}](0) = 2\partial_y v(0) = 0,$$

gives

$$\partial_y^2 v^{\text{ev}} = \partial_y^2 v^{\text{ev}} + [\partial_y v^{\text{ev}}](0) \otimes \delta_{y=0} = \partial_y v^{\text{ev}}$$

and the result for $s \in (\frac{3}{2}, \frac{5}{2})$ follows from the result for $s \in (\frac{1}{2}, \frac{3}{2})$. A last application of the jump formula (and the fact that the function v^{ev} is even) gives the result for $s \in (\frac{5}{2}, \frac{7}{2})$. The proof for the map $v \mapsto v^{\text{od}}$ is similar. \square

Let us now for a function u on $(0, 1) \times \mathbf{R}^d$ define the even (resp odd) periodic extensions on $\mathbf{T} \times \mathbf{R}^d$, u^{ev} (resp. u^{od}), by

$$(7.6) \quad v^{\text{ev}}(x_1, x') = \begin{cases} v(-x_1, x'), & \text{if } -1 < x_1 < 0 \\ v(x_1, x'), & \text{if } 1 > x_1 \geq 0, \\ v(x_1 - 2k, x'). & \text{if } x_1 - 2k \in (-1, 1) \end{cases}$$

$$v^{\text{od}}(x_1, x') = \begin{cases} -v(-x_1, x'), & \text{if } -1 < x_1 < 0 \\ v(x_1, x'), & \text{if } 1 > x_1 \geq 0, \\ v(x_1 + 2, x'). & \text{if } x_1 - 2k \in (-1, 1) \end{cases}$$

COROLLARY 7.7. \bullet Assume that $s \in (-\frac{1}{2}, \frac{1}{2}) \cup (\frac{1}{2}, \frac{3}{2})$. Then the map $v \mapsto v^{\text{ev}}$ is continuous from $H^s((0, 1) \times \mathbf{R}^d)$ to $H^s(\mathbf{T} \times \mathbf{R}^d)$.

- \bullet Assume that $s \in (\frac{3}{2}, \frac{5}{2}) \cup (\frac{5}{2}, \frac{7}{2})$ and let $\widehat{H}^s((0, 1) \times \mathbf{R}^d)$ be the closed subspace $H^s((0, 1) \times \mathbf{R}^d)$ of functions whose derivatives vanish at 0 and 1. Then the map $v \mapsto v^{\text{ev}}$ is continuous from $\widehat{H}^s((0, 1) \times \mathbf{R}^d)$ to $H^s(\mathbf{T} \times \mathbf{R}^d)$.
- \bullet Assume that $s \in (-\frac{1}{2}, \frac{1}{2})$. Then the map $v \mapsto v^{\text{od}}$ is continuous from $H^s((0, 1) \times \mathbf{R}^d)$ to $H^s(\mathbf{T} \times \mathbf{R}^d)$.
- \bullet Assume that $s \in (\frac{1}{2}, \frac{3}{2}) \cup (\frac{3}{2}, \frac{5}{2})$ and let $\widehat{H}^s((0, 1) \times \mathbf{R}^d)$ be the closed subspace $H^s((0, 1) \times \mathbf{R}^d)$ of functions vanishing at 0. Then the map $v \mapsto v^{\text{ev}}$ is continuous from $\widehat{H}^s((0, 1) \times \mathbf{R}^d)$ to $H^s(\mathbf{T} \times \mathbf{R}^d)$.
- \bullet Assume that $s \in (\frac{5}{2}, \frac{7}{2}) \cup (\frac{7}{2}, \frac{9}{2})$ and let $\widehat{H}^s((0, 1) \times \mathbf{R}^d)$ be the closed subspace $H^s((0, 1) \times \mathbf{R}^d)$ of functions vanishing at 0 and 1 such that $\partial_x^2 u|_{x=0,1} = 0$. Then the map $v \mapsto v^{\text{ev}}$ is continuous from $\widehat{H}^s((0, 1) \times \mathbf{R}^d)$ to $H^s(\mathbf{T} \times \mathbf{R}^d)$.

Indeed, since $(D_x^\alpha v)^{\text{ev}} = D_x^\alpha (v^{\text{ev}})$, $(D_{x'}^\alpha v)^{\text{od}} = D_{x'}^\alpha (v^{\text{od}})$ the result is clearly a one dimensional result and it is enough to prove it for $d = 0$, in which case it is a direct consequence of Proposition 7.6 and a localization argument.

We can now show how Theorem 7.5 implies Theorem 7.3. Consider initial data (η_0, v_0) satisfying the assumptions in Theorem 7.3. The function η_0 is defined in the domain $(0, 1) \times \mathbf{R}$, while the function v_0 is defined on the domain

$$\Omega_0 = \{(x_1, x_2, y) \in \mathbf{R} \times (0, 1) \times \mathbf{R}; b(x_1, x_2) < y < \eta_0(x_1, x_2)\}.$$

Let $(\widetilde{\eta}_0, \widetilde{b}) = (\eta_0^{\text{ev}}, b_0^{\text{ev}})$ be the even periodic extensions of (η_0, b) to $\mathbf{T} \times \mathbf{R}$ defined as in (7.6), and $\widetilde{V} = (\widetilde{v}_{0,x_1}, \widetilde{v}_{0,x_2}, \widetilde{v}_{0,y}) = (v_{0,x_1}^{\text{od}}, v_{0,x_2}^{\text{ev}}, v_{0,y}^{\text{ev}})$. Now \widetilde{v}_0 is defined on the domain

$$\widetilde{\Omega}_0 = \{(x_1, x_2, y) \in \mathbf{R} \times \mathbf{T} \times \mathbf{R}; \widetilde{b}_0(x_1, x_2) < y < \widetilde{\eta}_0(x_1, x_2)\}.$$

Notice that the boundary of $\widetilde{\Omega}_0$ is

$$(7.7) \quad \partial \widetilde{\Omega}_0 = \widetilde{\Sigma}_0 \cup \widetilde{\Gamma}_2, \quad \widetilde{\Gamma}_2 = \{(x_1, x_2, y) \in \mathbf{R} \times \mathbf{T} \times \mathbf{R}; \widetilde{b}_0(x_1, x_2) = y\}$$

while the boundary of Ω_0 is given by (7.1). Denote by

$$\widetilde{\Sigma}_0 = \{(x_1, x_2, y) \in \mathbf{R} \times \mathbf{T} \times \mathbf{R}; y = \widetilde{\eta}_0\}.$$

Let us finally define $\widetilde{V}_0, \widetilde{B}_0$ the traces of the horizontal and vertical velocities on $\widetilde{\Sigma}_0$.

Clearly, we have

$$\tilde{V}_{0,x_1} = V_{0,x_1}^{\text{od}}, \quad \tilde{V}_{0,x'} = V_{0,x'}^{\text{ev}}, \quad \tilde{B}_0 = B_0^{\text{ev}}.$$

Let us now consider the velocity potential ϕ_0 , ψ_0 its trace on Σ_0 , $\tilde{\phi}_0$ its even periodic extension to $\tilde{\Omega}_0$, and $\tilde{\psi}_0$ the even periodic extension on ψ_0 on $\mathbf{T} \times \mathbf{R}$. Notice that the function $\tilde{\phi}_0$ is harmonic in $\tilde{\Omega}_0$. Indeed, according to the jump formula

$$\Delta \tilde{\phi}_0 = [\partial_{x_1} \tilde{\phi}_0] \otimes \delta_{\tilde{\Gamma}_2},$$

where $[\partial_{x_1} \tilde{\phi}_0] = 2\partial\phi_0$ denotes the jump of the function $\partial_{x_1} \tilde{\phi}_0$ through $\tilde{\Gamma}_2$. But since $\partial_n \phi|_{\partial\Omega_0} = 0$, this jump is vanishing.

We notice now that the following properties are satisfied for $\varepsilon = 0, 1$ (as soon as the functions are smooth enough so that the traces make sense)

(7.8)

$$\begin{aligned} \partial_{x_1}(\tilde{\psi}_0)|_{x_1=\varepsilon} &= \partial_{x_1}\phi(\varepsilon, x', \eta(\varepsilon, x')) + \partial_y\phi(\varepsilon, x', \eta(\varepsilon, x'))\partial_{x_1}\eta_0(\varepsilon, x') = 0 \\ \partial_{x_1}(\tilde{B}_0)|_{x_1=\varepsilon} &= \partial_{x_1}\partial_y\phi(\varepsilon, x', \eta(\varepsilon, x')) + \partial_y^2\phi(\varepsilon, x', \eta(\varepsilon, x'))\partial_{x_1}\eta_0(\varepsilon, x') = 0 \\ \partial_{x_1}(\tilde{V}_{0,x'})|_{x_1=\varepsilon} &= \partial_{x_1}\partial_{x'}\phi(\varepsilon, x', \eta(\varepsilon, x')) \\ &\quad + \partial_y\partial_{x'}\phi(\varepsilon, x', \eta(\varepsilon, x'))\partial_{x'}\eta_0(\varepsilon, x') = 0 \\ \tilde{V}_{0,x_1}|_{x_1=\varepsilon} &= \partial_{x_1}\phi(\varepsilon, x', \eta(\varepsilon, x')) = 0, \\ \partial_{x_1}\tilde{V}_{0,x_1}|_{x_1=\varepsilon} &= \partial_{x_1}^3\phi(\varepsilon, x', \eta(\varepsilon, x')) + \partial_y\partial_{x_1}^2\phi(\varepsilon, x', \eta(\varepsilon, x'))\partial_{x_1}\eta(\varepsilon, x') \\ &\quad + \partial_y^2\partial_{x_1}\phi(\varepsilon, x', \eta(\varepsilon, x'))(\partial_{x_1}\eta(\varepsilon, x'))^2 + \partial_y\partial_{x_1}\phi(\varepsilon, x', \eta(\varepsilon, x'))\partial_y\partial_{x_1}\eta(\varepsilon, x') = 0, \end{aligned}$$

where in the last equality, we used that since ϕ is harmonic $\partial_{x_1}^3\phi = -(\partial_{x'}^2 + \partial_y^2)\partial_{x_1}\phi$. Now we deduce from Proposition 7.1, assumptions i) ii) in Theorem 7.3 and Corollary 7.7 that the extensions we just defined satisfy

$$(\tilde{\eta}_0, \tilde{\psi}_0, \tilde{V}_0, \tilde{B}_0) \in H^{s+\frac{1}{2}}(\mathbf{T} \times \mathbf{R}) \times H^{s+\frac{1}{2}}(\mathbf{T} \times \mathbf{R}) \times H^s(\mathbf{T} \times \mathbf{R}) \times H^s(\mathbf{T} \times \mathbf{R}).$$

Let $(\tilde{\eta}, \tilde{v})$ be the solution of the free surface water waves system given by Theorem 7.5. Since the initial data $(\tilde{\eta}_0, \tilde{\psi}_0, \tilde{V}_0, \tilde{B}_0)$ are even while \tilde{V}_{0,x_1} is odd, our uniqueness result guaranties that the solution satisfies the same symmetry property (because if we consider our solution, the function obtained by symmetrization is also a solution with same initial data). As a consequence if we define v, η, P as the trace of $\tilde{v}, \tilde{\eta}, \tilde{P}$ on $(0, 1) \times \mathbf{R}$, we get that they satisfy trivially the system free boundary Euler equation

$$(7.9) \quad \begin{aligned} \partial_t v + v \cdot \nabla_{x,y} v + \nabla_{x,y} P &= -ge_y, \quad \text{div}_{x,y} v = 0 \quad \text{in } \Omega, \\ \partial_t \eta &= \sqrt{1 + |\nabla \eta|^2} v \cdot \nu \quad \text{on } \Sigma, \\ P &= 0 \quad \text{on } \Sigma, \end{aligned}$$

and to conclude on the existence point in Theorem 7.3, it only remains to check that the "solid wall condition"

$$(7.10) \quad v \cdot n = 0, \quad \text{on } \Gamma = \Gamma_1 \cup \Gamma_2$$

is satisfied. On Γ_1 it is a straightforward consequence of the condition $\tilde{v} \cdot \tilde{\Gamma} = 0$, while on Γ_2 it is simply consequence of the fact that the component of the velocity field along x_1 , \tilde{v}_{x_1} is odd and 2-periodic. To prove the uniqueness part in Theorem 7.3, starting from a a solution of (7.9), (7.10), on the time interval $[-T, T]$, if we define the function $\tilde{v}, \tilde{\eta}$ at each time t following the same procedure, we end up with a

solution of (1.3), (1.4) in the domain $\{(t, x, y); t \in (-T, T), (x, y) \in \tilde{\Omega}(t)\}$, at the same level of regularity. Indeed, the jump formula gives

$$\partial_t \tilde{v} + \tilde{v} \cdot \nabla_{x,y} \tilde{v} + \nabla_{x,y} \tilde{P} = -g e_y + [v_{x_1} \cdot \partial_{x_1} v] \otimes \delta_{\Gamma_2} = -g e_y,$$

where in the last equality we used that the component of the velocity field along x_1 vanishes on Γ_2 . The uniqueness part in Theorem 7.3 consequently follows from the uniqueness part in Theorem 7.5

7.3. Sketch of proof of Theorem 7.5. The proof of our well posedness result in the periodic setting is the same word to word as in the non periodic case, once one is able to develop a suitable para-differential calculus and show that all properties in Section 2 remain true. It is quite standard that the theory of pseudo-differential operators (or more generally para-differential operators) can be developed in the framework of manifolds. Here, since we are working in the context of $\mathbf{T} \times \mathbf{R}$, the presentation is particularly simple. The torus is endowed with the following atlas: let $\zeta_j, j = 1, 2$ be defined by

$$(7.11) \quad \begin{aligned} \kappa_1 &: (-1, 1)_{\text{mod}2} \ni \bar{x} \mapsto x \in (-1, 1) \\ \kappa_2 &: (0, 2)_{\text{mod}2} \ni \bar{x} \mapsto x \in (0, 2). \end{aligned}$$

and consider the operators on functions

$$\kappa_j^* u(x) = u(\kappa_j^{-1}(x)), \kappa_{j,*} v(\bar{x}) = v(\kappa_j(\bar{x})).$$

Let $1 = \zeta_1^2(\bar{x}) + \zeta_2^2(\bar{x})$ be a partition of unity compatible with this atlas (i.e. $\text{supp}(\zeta_1) \subset (-1, 1)_{\text{mod}2}$, and $\text{supp}(\zeta_2) \subset (0, 2)_{\text{mod}2}$, and $\tilde{\zeta}_{1,2}$ two functions supported respectively in $(-1, 1)_{\text{mod}2}$, and $(0, 2)_{\text{mod}2}$ such that $\tilde{\zeta}_j \zeta_j = \zeta_j$). We use the classical definition of pseudo-differential operators on manifolds (i.e. these operators are the operators which in the maps of the atlas are usual pseudo-differential operators). In our context, the definition is very simple

DEFINITION 7.8. Given $\rho \in [0, 1]$ and $m \in \mathbf{R}$, $\Gamma_\rho^m(\mathbf{T} \times \mathbf{R})$ denotes the space of locally bounded functions $a(\cdot; x', \xi)$ on $\mathbf{T} \times \mathbf{R} \times (\mathbf{R}^d \setminus 0)$, which are C^∞ with respect to ξ for $\xi \neq 0$ and such that, for all $\alpha \in \mathbf{N}^d$ and all $\xi \neq 0$, the function $(x, x') \mapsto \partial_\xi^\alpha a(x, x', \xi)$ belongs to $W^{\rho, \infty}(\mathbf{T} \times \mathbf{R})$ and there exists a constant C_α such that,

$$(7.12) \quad \forall |\xi| \geq \frac{1}{2}, \quad \|\partial_\xi^\alpha a(\cdot, \xi)\|_{W^{\rho, \infty}(\mathbf{T} \times \mathbf{R})} \leq C_\alpha (1 + |\xi|)^{m - |\alpha|}.$$

We shall use the notation

$$\widetilde{M}_0^m(a) := \sup_{t \in [0, T]} \sup_{|\alpha| \leq \frac{3d}{2} + 1 + \rho} \sup_{|\xi| \geq 1/2} \left\| (1 + |\xi|)^{|\alpha| - m} \partial_\xi^\alpha a(t; \cdot, \xi) \right\|_{L^\infty(\mathbf{T} \times \mathbf{R})}.$$

DEFINITION 7.9. Given a symbol $a \in \Gamma_\rho^m(\mathbf{T} \times \mathbf{R})$, we define the paradifferential operator \widetilde{T}_a associated to a and acting on functions on $\mathbf{T} \times \mathbf{R}$ by

$$(7.13) \quad \widetilde{T}_a u = \sum_{j=1,2} \kappa_{j,*} \left(\kappa_j^* (\tilde{\zeta}_j) T_{\kappa_j^* \zeta_j a} \kappa_j^* (\zeta_j u) \right),$$

For two functions, we also define the para-product

$$T_u v = \sum_{j=1,2} \kappa_{j,*} \left(\kappa_j^* (\tilde{\zeta}_j) T_{\kappa_j^* \zeta_j u} \kappa_j^* (\zeta_j v) \right),$$

$$uv = \widetilde{T}_u v + \widetilde{T}_v u + \widetilde{R}(u, v),$$

with

$$(7.14) \quad \begin{aligned} \tilde{R}(u, v) &= \sum_{j=1,2} \kappa_{j,*} \left(\kappa_j^* (\tilde{\zeta}_j) R(\kappa_j^* \zeta_j u, \kappa_j^* (\zeta_j v)) \right), \\ uv &= \tilde{T}_u v + \tilde{T}_v u + \tilde{R}(u, v). \end{aligned}$$

In the sequel, to take benefit from the very simple structure of \mathbf{T} , we shall identify functions on $\mathbf{T} \times \mathbf{R}$ with 2-periodic functions (with respect with the first variable) on \mathbf{R}^2 . As a consequence, the definition (7.15) becomes

$$(7.15) \quad \tilde{T}_a u = \sum_{k \in 2\mathbf{Z}} \tau_k \left(\sum_{j=1,2} \tilde{\zeta}_j T_{\zeta_j a} \zeta_j u \right),$$

where $\tau_k(f)(x) = f(x+k)$, $\zeta_j(x) = \zeta_j(\kappa_j^{-1}(x))$ are smooth functions supported on $(-1, 1)$ and $(0, 2)$ respectively such that

$$\sum_{k \in 2\mathbf{Z}} \tau_k(\zeta_1^2 + \zeta_2^2) = 1,$$

and the functions u, a are simply 2-periodic function (with respect to the first variable) on \mathbf{R}^2 . Notice however that in this context, the H^s norms have to be taken only on a period (say $(-1, 1) \times \mathbf{R}$ or $(0, 2) \times \mathbf{R}$).

REMARK 7.10. Though we will not need it, it is worth noticing that the definition above does not depend, modulo operators of orders $-\infty$ on the choice of the cut-off functions ζ_j and $\tilde{\zeta}_j$. Also, for any smooth function bounded as well as all its derivatives, ζ , since $T_\zeta - \zeta$ is of order $-\infty$, we get that $\tilde{T}_\zeta - \zeta$ is also of order $-\infty$.

Taking benefit of the particularly simple structure of our atlas, it is very easy to show that all the properties of the calculus of para-differential operators proved on \mathbf{R}^d remain true in this context (and are actually consequences of these properties), under the same assumptions (replacing of course \mathbf{R}^2 by $\mathbf{T} \times \mathbf{R}$). On the more general framework of an abstract compact manifold, this is still true but require the invariance of para-differential operators by diffeomorphisms, while here we use the invariance by translations.

PROPOSITION 7.11. *All the estimates and symbolic calculus of Sections 2.2, 2.3 and 2.4 remain mutatis mutandi true in the context of $\mathbf{T} \times \mathbf{R}$.*

PROOF. Estimates (2.4), (2.12) (2.13), (2.14) and (2.23) from the definition of the paradifferential operator (7.15) and the fact that these estimates hold for the usual para-differential operators on \mathbf{R}^2 . Similarly, relations (2.9), (2.10) and (2.11) follow from (7.9). Let us prove the composition rules (2.5). We have

$$(7.16) \quad \tilde{T}_a \tilde{T}_b u = \sum_{k \in 2\mathbf{Z}} \tau_k \left(\sum_{j=1,2} \tilde{\zeta}_j T_{\zeta_j a} \zeta_j \left(\sum_{q \in 2\mathbf{Z}} \tau_q \left(\sum_{i=1,2} \tilde{\zeta}_i T_{\zeta_i b} \zeta_i u \right) \right) \right)$$

Let us denote by Q_1 the contribution of $j = 1$ in the sum above. Due to the support properties of the functions ζ_1 and $\tau_q \tilde{\zeta}_i$, in the expression defining Q_1 , the sum is restricted to $q = 0$ for $(j, i) = (1, 1)$ and $q = 0, -2$ for $(i, j) = (1, 2)$. We also have according to (2.5) (on \mathbf{R}^2),

$$\|[\chi_1, T_{\zeta_j b}]\|_{H^\mu \rightarrow H^{\mu-m-m'+\rho}} \leq K M_\rho^{m'}(\zeta_j b) \leq K \tilde{M}_\rho^{m'}(b).$$

Using that $T_\zeta - \zeta$ is of order $-\infty$, we get, modulo an operator which is bounded from $H^\mu(\mathbf{T} \times \mathbf{R})$ to $H^{\mu-m-m'+\rho}(\mathbf{T} \times \mathbf{R})$ by

$$K M_0^m(\zeta_1 a) M_\rho^{m'}(\zeta_j b) \leq K \tilde{M}_0^m(a) \tilde{M}_\rho^{m'}(b),$$

$$(7.17) \quad Q_1 = \sum_{k \in 2\mathbf{Z}} \tau_k \left(\tilde{\zeta}_1 T_{\zeta_1 a} \left(T_{\zeta_1^2 b} \zeta_1 u + T_{\zeta_2^2 b} \zeta_1 u + \tau_{-2} (T_{\zeta_2^2 b} \zeta_1 u) \right) \right) \\ = \sum_{k \in 2\mathbf{Z}} \tau_k \left(\tilde{\zeta}_1 T_{\zeta_1 a} \left(T_{(\zeta_1^2 + \zeta_2^2 \tau_{-2} \zeta_2^2) b} \zeta_1 u \right) \right)$$

Applying again (2.5) (on \mathbf{R}^2), we get that modulo an operator which is bounded from $H^\mu(\mathbf{T} \times \mathbf{R})$ to $H^{\mu-m-m'+\rho}(\mathbf{T} \times \mathbf{R})$ by

$$K \left(\widetilde{M}_\rho^m(a) \widetilde{M}_0^{m'}(b) + \widetilde{M}_0^m(a) \widetilde{M}_\rho^{m'}(b) \right) \\ Q_1 = \sum_{k \in 2\mathbf{Z}} \tau_k \left(\tilde{\zeta}_1 T_{\zeta_1 \times (\zeta_1^2 + \zeta_2^2 \tau_{-2} \zeta_2^2) ab} \zeta_1 u \right)$$

Noticing now that on the support of ζ_1 , we have $(\zeta_1^2 + \zeta_2^2 \tau_{-2} \zeta_2^2) = 1$, and estimating similarly the contribution of $j = 2$ in (7.16), this concludes the proof of (2.5) for $\mathbf{T} \times \mathbf{R}$. The proof of (2.6) follows similar lines.

The most delicate part is to prove that Lemma 2.16 holds in this context. We shall actually prove that it is a consequence of the result on \mathbf{R}^2 . Let us bound the L^2 norm of

$$(7.18) \quad \sum_{k \in 2\mathbf{Z}} \tau_k \left(\sum_{j=1,2} \tilde{\zeta}_j T_{\zeta_j p} \zeta_j (\partial_t u + \sum_{q \in 2\mathbf{Z}} \tau_q \left((\tilde{\zeta}_1 T_{\zeta_1 V} \zeta_1 + \tilde{\zeta}_2 T_{\zeta_2 V} \zeta_2) \nabla u \right)) \right) \\ - \left[\partial_t \sum_{k \in 2\mathbf{Z}} \tau_k \left(\sum_{j=1,2} \tilde{\zeta}_j T_{\zeta_j p} \zeta_j u \right) \right. \\ \left. + \sum_{q \in 2\mathbf{Z}} \tau_q \left((\tilde{\zeta}_1 T_{\zeta_1 V} \zeta_1 + \tilde{\zeta}_2 T_{\zeta_2 V} \zeta_2) \nabla \sum_{k \in 2\mathbf{Z}} \tau_k \left(\sum_{j=1,2} \tilde{\zeta}_j T_{\zeta_j p} u \right) \right) \right]$$

The contributions of $j = 1, 2$ in the expression above are estimated separately and using the same method. Let us estimate

$$(7.19) \quad \sum_{k \in 2\mathbf{Z}} \tau_k \left(\tilde{\zeta}_1 T_{\zeta_1 p} \zeta_1 (\partial_t u + \sum_{q \in 2\mathbf{Z}} \tau_q \left((\tilde{\zeta}_1 T_{\zeta_1 V} \zeta_1 + \tilde{\zeta}_2 T_{\zeta_2 V} \zeta_2) \nabla u \right)) \right) \\ - \left[\partial_t \sum_{k \in 2\mathbf{Z}} \tau_k \left(\tilde{\zeta}_1 T_{\zeta_1 p} \zeta_1 u \right) + \sum_{q \in 2\mathbf{Z}} \tau_q \left((\tilde{\zeta}_1 T_{\zeta_1 V} \zeta_1 + \tilde{\zeta}_2 T_{\zeta_2 V} \zeta_2) \nabla \sum_{k \in 2\mathbf{Z}} \tau_k \left(\tilde{\zeta}_1 T_{\zeta_1 p} u \right) \right) \right]$$

We compute the L^2 -norm on $(-1, 1)$. Since for $k \neq 0$, $\tau_k(\tilde{\zeta}_1)$ is disjoint from $(-1, 1)$ and for $k \neq 0, -2$, $\tau_k(\tilde{\zeta}_2)$ is disjoint from $(-1, 1)$, while for $k \neq 0, 2$ $\tau_k(\tilde{\zeta}_1)$ has its support disjoint from ζ_2 , we need only to estimate the L^2 -norm of

$$(7.20) \quad \tilde{\zeta}_1 T_{\zeta_1 p} \zeta_1 \left[\partial_t u + \left(\tilde{\zeta}_1 T_{\zeta_1 V} \zeta_1 \nabla u + \tilde{\zeta}_2 T_{\zeta_2 V} \zeta_2 \nabla u + \tau_{-1} \left(\tilde{\zeta}_2 T_{\zeta_2 V} \zeta_2 \nabla u \right) \right) \right] \\ - \left[\partial_t \tilde{\zeta}_1 T_{\zeta_1 p} \zeta_1 u + \left(\tilde{\zeta}_1 T_{\zeta_1 V} \zeta_1 \nabla \left(\tilde{\zeta}_1 T_{\zeta_1 p} \zeta_1 u \right) \right) \right. \\ \left. + \sum_{q=0,-2} \tau_q \left(\tilde{\zeta}_2 T_{\zeta_2 V} \zeta_2 \nabla \left(\sum_{k=0,2} \tau_k \left(\tilde{\zeta}_1 T_{\zeta_1 p} \zeta_1 u \right) \right) \right) \right]$$

We shall say that a term in the expression above is admissible if its L^2 norm on $(-1, 1) \times \mathbf{R}$ is bounded by

$$K \mathcal{M}_0^m(p) \|V(t)\|_{C_*^{1+\varepsilon}} \|u(t)\|_{H^m(\mathbf{T} \times \mathbf{R})}.$$

According to Proposition 2.11, we can replace in (7.20) the multiplication by any smooth function ζ by T_ζ in the expression above, modulo admissible remainder

terms R . According to (2.5), we can also replace in (7.20) $T_\zeta T_{\zeta V}$ by $T_{\zeta^2 V}$ and we can commute T_ζ and $T_{\zeta V}$ modulo remainder terms. As a consequence, we end up estimating (using that $\zeta_j \zeta_j = \zeta_j$ and that $\tau_k V = V, \tau_k u = u$)

$$(7.21) \quad \begin{aligned} & \tilde{\zeta}_1 T_{\zeta_1 p} \left(\partial_t \zeta_1 u - \left(T_{(\zeta_1^2 + \zeta_2^2 + \tau_{-2} \zeta_2^2)} V \nabla \zeta_1 u \right) \right) \\ & - \tilde{\zeta}_1 \left[\partial_t T_{\zeta_1 p} \zeta_1 u - \left(T_{\zeta_1^2 V} \nabla \circ T_{\zeta_1 p} \zeta_1 u + \sum_{q=0, -2} T_{\tau_q \zeta_2^2 V} \nabla \circ \sum_{k=0, 2} T_{\tau_{q+k} \zeta_1 p} \tau_{q+k} (\zeta_1 u) \right) \right] \end{aligned}$$

Notice here that in the sum above we can keep only the terms such that $q + k = 0$, because modulo admissible remainder terms we have

$$(7.22) \quad \tau_q \left(\tilde{\zeta}_2 T_{\zeta_2 V} \zeta_2 \nabla \circ \tau_k (\tilde{\zeta}_1 T_{\zeta_1 p} \zeta_1 u) \right) = \tau_{q+k} \tilde{\zeta}_1 \tau_q \left(\tilde{\zeta}_2 T_{\zeta_2 V} \zeta_2 \nabla \circ \tau_k (T_{\zeta_1 p} \zeta_1 u) \right)$$

which is vanishing on $(-1, 1)$ if $q + k \neq 0$. Finally, we end up with

$$(7.23) \quad \begin{aligned} & \tilde{\zeta}_1 T_{\zeta_1 p} \left(\partial_t \zeta_1 u + \left(T_{(\zeta_1^2 + \zeta_2^2 + \tau_{-2} \zeta_2^2)} V \nabla \zeta_1 u \right) \right) \\ & - \tilde{\zeta}_1 \left(\partial_t T_{\zeta_1 p} \zeta_1 u + \left(T_{(\zeta_1^2 + \zeta_2^2 + \tau_{-2} \zeta_2^2)} V \nabla \circ T_{\zeta_1 p} \zeta_1 u \right) \right) \\ & = \tilde{\zeta}_1 \left[T_{\zeta_1 p}, \partial_t + \left(T_{(\zeta_1^2 + \zeta_2^2 + \tau_{-2} \zeta_2^2)} V \nabla \circ T_{\zeta_1 p} \right) \right] \zeta_1 u \end{aligned}$$

We can now apply Lemma 2.16 in the context of \mathbf{R}^2 to get that the L^2 norm of this expression is bounded by

$$(7.24) \quad \begin{aligned} & K \left(\mathcal{M}_0^m(\zeta_1 p) \left\| (\zeta_1^2 + \zeta_2^2 + \tau_{-2} \zeta_2^2) V(t) \right\|_{C_*^{1+\varepsilon}} \right. \\ & \left. + \mathcal{M}_0^m \left(\partial_t (\zeta_1 p) + (\zeta_1^2 + \zeta_2^2 + \tau_{-2} \zeta_2^2) V \cdot \nabla (\zeta_1 p) \right) \left\| (\zeta_1^2 + \zeta_2^2 + \tau_{-2} \zeta_2^2) V(t) \right\|_{L^\infty} \right) \\ & \left\| \zeta_1 u(t) \right\|_{H^m(\mathbf{R}^2)}. \end{aligned}$$

We now use that on the support of ζ_1 , $(\zeta_1^2 + \zeta_2^2 + \tau_{-2} \zeta_2^2) V = V$ and obtain that (7.24) is bounded by

$$(7.25) \quad K \left(\tilde{M}_0^m(p) \|V(t)\|_{C_*^{1+\varepsilon}} + \tilde{M}_0^m \left(\partial_t p + V \cdot \nabla p \right) \|V(t)\|_{L^\infty} \right) \|u(t)\|_{H^m(\mathbf{T} \times \mathbf{R})}.$$

This concludes the proof of Proposition 7.11. \square

REMARK 7.12. Let us remark that the method developed in this section (including the geometric result in Proposition 7.1) could be easily adapted to deal with the totally periodic framework $\mathbf{T} \times \mathbf{T}$ (the case of wave pools)

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