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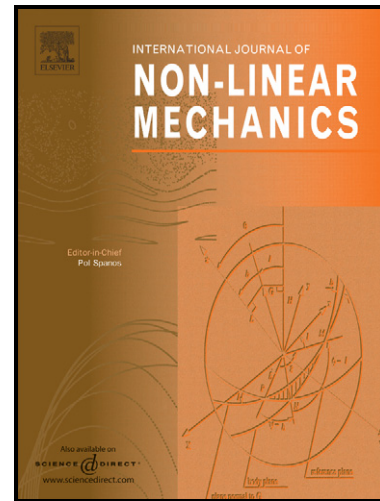
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Pressure-Driven Flow of a Rate Type Fluid with Stress Threshold in an Infinite Channel

Lorenzo Fusi Angiolo Farina

*Università degli Studi di Firenze
Dipartimento di Matematica "Ulisse Dini"
Viale Morgagni 67/A, I-50134 Firenze, Italy*

Abstract

In this paper we extend some of our previous works on continua with stress threshold. In particular here we propose a mathematical model for a continuum which behaves as a nonlinear upper convected Maxwell fluid if the stress is above a certain threshold and as a Oldroyd-B type fluid if the stress is below such a threshold. We derive the constitutive equations for each phase exploiting the theory of natural configurations (introduced by Rajagopal and co-workers) and the criterion of the maximization of the rate of dissipation. We state the mathematical problem for a one-dimensional flow driven by a constant pressure gradient and study two peculiar cases in which the velocity of the inner part of the fluid is spatially homogeneous.

Keywords:

Implicit Constitutive Relations, Natural Configurations, Rate-Type Fluids, Free Boundary problems

1. Introduction

A large variety of materials such as food products, polymers, paints, oils and foams cannot be described by the classical linear viscous model. For this reason a large class of non-Newtonian models have been developed with the aim of explaining non linear behaviors such as shear thinning/thickening, non linear creep and stress relaxation.

In his celebrated work⁷ Maxwell developed the first rate type fluid model capable of describing stress relaxation, while later Burgers¹ developed a rate type model for describing some geomaterials which included the classical rate type model due to Oldroyd⁸, namely the Oldroyd-B type model. Oldroyd was the first to develop a consistent framework for the rheology of rate type viscoelastic fluids, focussing on the importance of the frame invariance and introducing some kinds of derivatives to obtain proper frame indifferent constitutive equations.

Since these seminal works, a plethora of models for viscoelastic response have been developed and numerous frame-invariant time derivatives have been introduced. Rajagopal and co-workers have developed in¹⁰ a proper thermodynamical framework from which most of the viscoelastic constitutive relations can be derived.

The laminar flow of rate-type fluids have been extensively studied both in planar and cylindrical geometries. Waters and King¹⁶ studied the pressure driven flow of an Oldroyd-B fluid in a straight cylindrical pipe, obtaining exact solutions by means of Laplace transform method. Rahaman and Ramkissoon⁹ have studied the non stationary flow of a Maxwell fluid in a pipe. Steady solutions due to oscillating cylindrical boundaries for second grade and Oldroyd-B type fluids have been obtained by Rajagopal¹² and Rajagopal and Bhatnagar¹³.

In this paper we investigate the behavior of a non-Newtonian

incompressible rate type fluid which switches from an Oldroyd-B behavior to a non linear Maxwell behavior depending on whether the stress is larger or smaller than a certain threshold. A typical example of a continuum that changes its behavior depending on the value of some function of the stress is the so-called Bingham fluid, which is a Newtonian viscous fluid that exhibits a threshold (the so called yield stress) below which the strain rate is zero (so that no deformations occur).

In previous works we have studied a series of extensions of this simple model and we have investigated the corresponding mathematical problems in one dimensional settings. The first extension was to the case in which the region where the stress is below the threshold behaves like a Neo-Hookean elastic solid (see²) and we have subsequently extended this case to the one in which the same region behaves like a visco-elastic Maxwell fluid³.

We have then studied the case of an elastic material such that no deformation occurs above a certain threshold⁴ and we have investigated the case in which the transition from rigid to elastic occurs when the stress becomes greater than the threshold (see^{5,6}). The methodology developed in all these papers can be used to formulate a variety of models for continuum with stress threshold.

All the models were obtained in the framework of the theory of natural configurations developed by Rajagopal and co-workers¹¹. Depending on the constitutive equations for the phases constituting the continuum, we have obtained mathematical formulations of various complexity. In general these formulations consist in a free boundary problems involving hyperbolic and parabolic equations.

In this paper we study the case of a material in which one "phase" behaves as a Nonlinear Maxwell fluid while the other as a Jeffreys fluid. For the sake of simplicity we will choose for

the latter the Oldroyd-B fluid model while for the former the upper convected Maxwell model. Minor changes allow to consider other models, like corotational Maxwell model or interpolated Maxwell model for the Maxwell fluid, and generalized Jeffreys models for the rate-type fluid (e.g. Oldroyd A).

The general constitutive equation for a nonlinear Maxwell fluid is given by

$$\lambda \frac{D\mathbb{T}}{Dt} + \mathbb{T} = 2\eta\mathbb{D}, \quad (1)$$

where λ is a positive parameter, \mathbb{T} is the stress tensor, η is the viscosity and \mathbb{D} is the symmetric part of the velocity gradient, and where $D(\cdot)/Dt$ stands for any frame invariant derivative (upper convected, lower convected, corotational). On the other hand, the general constitutive equation for a generalized Jeffreys model is

$$\lambda_1 \frac{D\mathbb{T}}{Dt} + \mathbb{T} = 2\eta \left[\mathbb{D} + \lambda_2 \frac{D\mathbb{D}}{Dt} \right], \quad (2)$$

λ_1 and λ_2 being positive parameters. In both cases the stress is obtained by solving a differential equation involving a symmetric tensor (meaning six scalar differential equations). In the framework of the theory of natural configurations we will obtain specific forms for equations (1), (2) imposing the elastic response of the continuum, the way the body dissipates energy and requiring that the dissipation is maximum.

As we said we will focus on two specific kind of nonlinear visco-elastic fluids, but the procedure we are going to describe can be applied to any fluid whose constitutive equations are of the type (1), (2).

Throughout the paper we use the terminology “fluid with a stress threshold”, which seems to represent an oxymoron if one think to the very definition of a fluid. By the way, with this expression we intend to indicate that the switching from the Oldroyd-b behaviour to the Maxwell behaviour occurs when the second invariant of the stress tensor reaches a fixed threshold value.

2. Kinematical results

We consider a material domain $\Omega \in \mathcal{R}^3$ and we assume that there exists a surface Γ that divides the domain in two regions Ω_1 and Ω_2 such that $\Omega = \Omega_1 \cup \Omega_2$. We suppose that the material that occupies the region Ω_1 behaves as a upper convected Maxwell fluid, while the material within the region Ω_2 as an Oldroyd-B fluid. We assume that Ω evolves in time so that at time $t > 0$ the domain is $\Omega_t = \Omega_{1t} \cup \Omega_{2t}$, we define Γ_t as the sharp interface separating the domains Ω_{1t} and Ω_{2t} (see Fig. 1). The motion of a particle $\vec{x} \in \Omega$ is given by

$$\vec{x} = \Theta(\vec{X}, t) \quad (3)$$

where $\vec{X} \in \Omega$ is the Lagrangian coordinate and \vec{x} is the Eulerian coordinate, that is the position of particle \vec{X} at time t . We assume that Θ is invertible. The velocity, in the Eulerian coordinate system, is given by

$$\vec{v}(\vec{x}, t) = \left. \frac{\partial \Theta(\vec{X}, t)}{\partial t} \right|_{\vec{X}=\Theta^{-1}(\vec{x}, t)}, \quad (4)$$

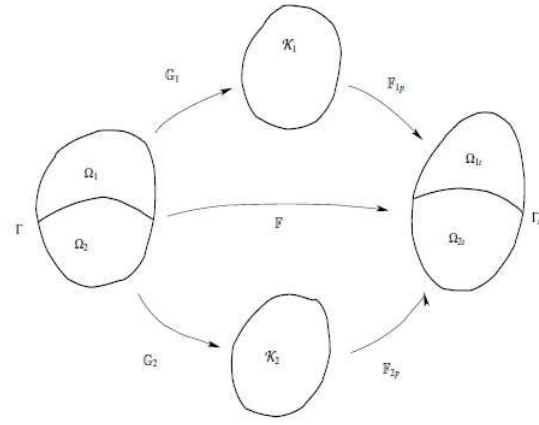


Figure 1: Schematic diagram illustrating the evolution the natural configurations of “phases” Ω_1 and Ω_2 .

and the acceleration by

$$\vec{a}(\vec{x}, t) = \left. \frac{\partial^2 \Theta(\vec{X}, t)}{\partial t^2} \right|_{\vec{X}=\Theta^{-1}(\vec{x}, t)}. \quad (5)$$

The deformation gradient of the motion is

$$\mathbb{F} =: \text{grad } \Theta(\vec{X}, t), \quad (6)$$

where *grad* denotes the gradient with respect to the Lagrangean coordinates. The velocity gradient is

$$\mathbb{L}(\vec{x}, t) =: \nabla \vec{v}(\vec{x}, t), \quad (7)$$

where ∇ is the gradient taken with respect to the Eulerian coordinates. The symmetric part of the velocity gradient is given by

$$\mathbb{D}(\vec{x}, t) =: \frac{1}{2} (\mathbb{L} + \mathbb{L}^T). \quad (8)$$

We introduce also the left and right Cauchy-Green tensors

$$\mathbb{B} = \mathbb{F}\mathbb{F}^T, \quad \mathbb{C} = \mathbb{F}^T\mathbb{F}. \quad (9)$$

The material time derivative (differentiation along a particle path) will be indicated by the superposed dot, so that for any scalar quantity \mathcal{F}

$$\dot{\mathcal{F}} = \frac{\partial \mathcal{F}}{\partial t} + \nabla \mathcal{F} \cdot \vec{v}. \quad (10)$$

Following standard results we can prove that

$$\mathbb{L} = \dot{\mathbb{F}}\mathbb{F}^{-1} \quad (11)$$

and that

$$\dot{\mathbb{B}} = \mathbb{L}\mathbb{B} + \mathbb{B}\mathbb{L} - 2\mathbb{F}\mathbb{D}\mathbb{F}^T. \quad (12)$$

The frame indifferent upper convected time derivative is given by

$$\overset{\nabla}{\mathbb{B}} = \dot{\mathbb{B}} - \mathbb{L}\mathbb{B} - \mathbb{B}\mathbb{L}^T. \quad (13)$$

Following¹¹ we introduce, for each “phase”, the relative natural configuration. In practice we consider the decomposition

$$\mathbb{F}_1 = \mathbb{F}_{1p}\mathbb{G}_1, \quad \mathbb{F}_2 = \mathbb{F}_{2p}\mathbb{G}_2 \quad (14)$$

where \mathbb{F}_{ip} is the gradient of the mapping¹ from the reference configuration Ω_i to the natural configuration \mathcal{K}_i and \mathbb{G}_i is the gradient of the mapping from the natural configuration \mathcal{K}_i to the configuration Ω_{ii} (see Fig. 1). The tensors \mathbb{F}_{ip} and \mathbb{G}_i can be viewed as mappings between the tangent vectors spaces of the various configurations. We recall (see¹¹) that the tensors \mathbb{F}_{ip} are defined up to rigid rotations, so that, supposing that the material is isotropic, we can assume that the tensors \mathbb{F}_{ip} are positive definite symmetric tensors, i.e. $\mathbb{F}_{ip} = \mathbb{V}_{ip}$, where \mathbb{V}_{ip} is the right stretch tensor in the polar decomposition. For each “phase” we define

$$\mathbb{L}_{iG} =: \dot{\mathbb{G}}_i \dot{\mathbb{G}}_i^{-1}, \quad \mathbb{D}_{iG} =: \frac{1}{2}(\mathbb{L}_{iG} + \mathbb{L}_{iG}^T), \quad (15)$$

$$\mathbb{B}_{ip} =: \mathbb{F}_{ip} \mathbb{F}_{ip}^T, \quad \mathbb{C}_{ip} =: \mathbb{F}_{ip}^T \mathbb{F}_{ip}. \quad (16)$$

It can be proved that

$$\text{tr}(\mathbb{B}_{ip}) = \mathbb{I} \cdot \dot{\mathbb{B}}_{ip} = 2\mathbb{B}_{ip} \cdot (\mathbb{D} - \mathbb{D}_{iG}), \quad (17)$$

so that, by (13), we have

$$\overset{\nabla}{\mathbb{B}}_{ip} = \dot{\mathbb{B}}_{ip} - \mathbb{L}\mathbb{B}_{ip} - \mathbb{B}_{ip}\mathbb{L}^T. \quad (18)$$

Recalling that \mathbb{F}_{ip} is symmetric, we get

$$\overset{\nabla}{\mathbb{B}}_{ip} = -2\mathbb{F}_{ip}\mathbb{D}_{iG}\mathbb{F}_{ip}, \quad (19)$$

with

$$\overset{\nabla}{\mathbb{I}} = -2\mathbb{D}. \quad (20)$$

We also recall that

$$\overset{\cdot}{\det \mathbb{F}_{ip}} = \det(\mathbb{F}_{ip}) \text{tr}(\dot{\mathbb{F}}_{ip} \mathbb{F}_{ip}^{-1}). \quad (21)$$

3. Derivation of constitutive relations

In order to derive the constitutive equations for each phase, we consider the reduced-dissipation equation for isothermal conditions (see¹⁴)

$$\mathbb{T} \cdot \mathbb{L} - \rho \dot{\psi} = \rho \theta \zeta = \xi \geq 0, \quad (22)$$

where ψ is the Helmholtz potential (internal energy per unit mass), ζ is the rate of entropy production, θ is absolute temperature and ξ is the rate of dissipation. For simplicity we rewrite equation (22) as

$$\mathbb{T} \cdot \mathbb{D} - \dot{W} = \xi, \quad (23)$$

where we have exploited the fact that $\mathbb{T} \cdot \mathbb{D} = \mathbb{T} \cdot \mathbb{L}$ and where we have introduced $W = \rho \psi$. The basic assumption in our model is that

$$\xi = \hat{\xi}(\mathbb{B}_{ip}, \mathbb{D}_{iG}, \mathbb{D}), \quad W = \hat{W}(\text{tr} \mathbb{B}_{ip}, \text{tr} \mathbb{B}_{ip}^2), \quad i = 1, 2. \quad (24)$$

In doing so we are assuming that the elastic response, which is the one from the natural to the actual configuration, is the one of an elastic material, while the dissipative response is the one of a visco-elastic fluid. We shall impose specific forms for the quantities appearing in (24), without giving a specific representation of the stress tensor. The latter will be obtained through the process of maximization of the rate of dissipation. This procedure consists in assuming that, among all the possible evolutions of the natural configuration of each phase, we choose the one which maximizes the dissipation function. In practice we determine the tensor \mathbb{D}_{iG} that maximizes $\hat{\xi}$ with \mathbb{B}_{ip} and \mathbb{D} kept fixed. Depending on the constitutive forms we use to represent dissipation and the response from the natural to the actual configuration, we can model various continua like visco-elastic fluids, solids etc.

We define a parameter a (related to the principal invariants of the stress) and a threshold τ_o (related the stress threshold), such that if $a \geq \tau_o$ the body behaves like an upper convected Maxwell fluid, while if $a \leq \tau_o$ as a Oldroyd-B type fluid.

We define

$$a = \sqrt{II(\tilde{\mathbb{T}})}, \quad (25)$$

where $\tilde{\mathbb{T}}$ is the traceless part of the stress tensor

$$\tilde{\mathbb{T}} = \mathbb{T} - \frac{1}{3}(\text{tr} \mathbb{T})\mathbb{I}, \quad (26)$$

and $II(\tilde{\mathbb{T}})$ is

$$II(\tilde{\mathbb{T}}) = \frac{1}{2} \text{tr}(\tilde{\mathbb{T}}^2). \quad (27)$$

Therefore the following conditions must be fulfilled for every time $t \geq 0$

$$\sqrt{II(\tilde{\mathbb{T}})} \geq \tau_o, \quad \text{in } \Omega_{1t}, \quad (28)$$

$$\sqrt{II(\tilde{\mathbb{T}})} \leq \tau_o, \quad \text{in } \Omega_{2t}. \quad (29)$$

We define the dissipation rate as

$$\begin{aligned} \hat{\xi} = & H(\tau_o - a) \left[\eta_2 \mathbb{D}_{2G} \cdot \mathbb{B}_{2p} \mathbb{D}_{2G} + \hat{\eta}_2 \mathbb{D} \cdot \mathbb{D} \right] + \\ & + H(a - \tau_o) \eta_1 \mathbb{D}_{1G} \cdot \mathbb{B}_{1p} \mathbb{D}_{1G}, \end{aligned} \quad (30)$$

where H is the Heaviside function and η_2 , $\hat{\eta}_2$, η_1 have the dimension of a viscosity. We also define the stored energy function as

$$\begin{aligned} W = & H(\tau_o - a) \left[\frac{\mu_2}{2} (\mathbb{I} \cdot \mathbb{B}_{2p} - 3) \right] + \\ & + H(a - \tau_o) \left[\frac{\mu_1}{2} (\mathbb{I} \cdot \mathbb{B}_{1p} - 3) \right], \end{aligned} \quad (31)$$

where μ_1 and μ_2 have the dimension of a pressure. We assume that in both phases the material is incompressible.

¹The word “mapping” is actually incorrect, since it may be possible that such a mapping does not exist.

4. Phase 1: Upper convected Maxwell fluid

The evolution of the “phase” Ω_1 is obtained following the criterion of maximization of the rate of dissipation. We skip some technical details for which we refer the reader to³. In this phase $a \geq \tau_o$, thus

$$\hat{\xi} = \eta_1 \mathbb{D}_{1G} \cdot \mathbb{B}_{1p} \mathbb{D}_{1G}, \quad W = \left[\frac{\mu_1}{2} (\mathbb{I} \cdot \mathbb{B}_{1p} - 3) \right]. \quad (32)$$

By virtue of (23) and (17) we get

$$(\mathbb{T} - \mu_1 \mathbb{B}_{1p}) \cdot \mathbb{D} = \eta_1 \mathbb{D}_{1G} \cdot \mathbb{B}_{1p} \mathbb{D}_{1G} - \mu_1 \mathbb{B}_{1p} \cdot \mathbb{D}_{1G}. \quad (33)$$

The right hand side does not depend on \mathbb{D} so that

$$\mathbb{T} = -P\mathbb{I} + \mu_1 \mathbb{B}_{1p} \quad (34)$$

where P is the Lagrange multiplier due to the incompressibility constraint. Using the maximization of the rate of dissipation criterion we look for the tensor \mathbb{D}_{1G} which maximizes the energy dissipated³. Thus we solve the problem

$$\begin{cases} \hat{\xi} - (\mathbb{T} + P\mathbb{I}) \cdot \mathbb{D}_{1G} = 0 \\ \mathbb{I} \cdot \mathbb{D}_{1G} = 0 \\ \frac{\partial \hat{\xi}}{\partial \mathbb{D}_{1G}} + \kappa_1 \left(\frac{\partial \hat{\xi}}{\partial \mathbb{D}_{1G}} - \mathbb{T} \right) + \kappa_2 \mathbb{I} = 0 \end{cases} \quad (35)$$

where κ_1 and κ_2 are unknown constant to be determined (Lagrange multipliers). We know (see appendix A in¹⁰) that the tensors \mathbb{B}_{1p} and \mathbb{D}_{1G} commute, since they have the same eigenvectors. Thus, recalling the first of (32) it is easy to see that

$$\mathbb{T} = 2\eta_1 \left(1 + \frac{1}{k_1} \right) \mathbb{D}_{1G} \mathbb{B}_{1p} + \frac{\kappa_2}{\kappa_1} \mathbb{I}. \quad (36)$$

Substituting (36) into (35)₁ we get $\kappa_1 = -2$. Thus

$$\mathbb{T} = \eta_1 \mathbb{D}_{1G} \mathbb{B}_{1p} - \frac{\kappa_2}{2} \mathbb{I}. \quad (37)$$

Subtracting (37) from (34) and multiplying by \mathbb{B}_{1p}^{-1} we get

$$\frac{\eta_1}{\mu_1} \mathbb{D}_{1G} = \frac{1}{\mu_1} \left(\frac{\kappa_2}{2} - P \right) \mathbb{B}_{1p}^{-1} + \mathbb{I}. \quad (38)$$

Applying the trace operator to (38) we get

$$\kappa_2 = 2P - \frac{6\mu_1}{\text{tr} \mathbb{B}_{1p}^{-1}}. \quad (39)$$

Recalling that \mathbb{F}_{1p} is symmetric and recalling (19), from (38), we get

$$\gamma_1 \overset{\nabla}{\mathbb{B}}_{1p} + \mathbb{B}_{1p} = \lambda_1 \mathbb{I}, \quad (40)$$

where

$$\gamma_1 = \frac{\eta_1}{2\mu_1}, \quad \lambda_1 = \frac{3}{\text{tr} \mathbb{B}_{1p}^{-1}}. \quad (41)$$

Thus the stress is given by

$$\begin{cases} \mathbb{T} = -P\mathbb{I} + \mu_1 \mathbb{B}_{1p}, \\ \gamma_1 \overset{\nabla}{\mathbb{B}}_{1p} + \mathbb{B}_{1p} = \lambda_1 \mathbb{I}. \end{cases} \quad (42)$$

The constitutive equation (40), although similar to the one of a upper convected Maxwell model, lacks the term containing the symmetric tensor \mathbb{D} (see¹⁵). Using the indeterminacy in the stress tensor due to the incompressibility we can rewrite the stress tensor in the form

$$\mathbb{T} = -\hat{P}\mathbb{I} + \mu_1 \hat{\mathbb{B}}_{1p} \quad (43)$$

where $\hat{P} = P - \mu_1 \lambda_1$ and $\hat{\mathbb{B}}_{1p} = \mathbb{B}_{1p} - \lambda_1 \mathbb{I}$ and equation (42) becomes

$$\begin{cases} \mathbb{T} = -\hat{P}\mathbb{I} + \mu_1 \hat{\mathbb{B}}_{1p} \\ \gamma_1 \overset{\nabla}{\hat{\mathbb{B}}}_{1p} + \hat{\mathbb{B}}_{1p} = 2\gamma_1 \lambda_1 \mathbb{D}, \end{cases} \quad (44)$$

which is the constitutive equation of the upper convected Maxwell model (see again¹⁵). Thus equations (42) and (44) are interchangeable. We will use (42) to represent the upper convected Maxwell fluid in the domain Ω_1 .

5. Phase 2: Oldroyd–B fluid

In the domain Ω_2 the body behaves as an Oldroyd-B viscoelastic fluid. Also in this case we will obtain the evolution equation applying the criterion of the maximization of the rate of dissipation. We set

$$\hat{\xi} = \eta_2 \mathbb{D}_{2G} \cdot \mathbb{B}_{2p} \mathbb{D}_{2G} + \hat{\eta}_2 \mathbb{D} \cdot \mathbb{D}, \quad W = \left[\frac{\mu_2}{2} (\mathbb{I} \cdot \mathbb{B}_{2p} - 3) \right]. \quad (45)$$

Proceeding as in the previous section we get

$$\hat{\xi} = \mathbb{T} \cdot \mathbb{D} - \mu_2 \mathbb{B}_{2p} \cdot (\mathbb{D} - \mathbb{D}_{2G}), \quad (46)$$

that is

$$\eta_2 \mathbb{D}_{2G} \cdot \mathbb{B}_{2p} \mathbb{D}_{2G} - \mu_2 \mathbb{B}_{2p} \cdot \mathbb{D}_{2G} = (\mathbb{T} - \mu_2 \mathbb{B}_{2p} - \hat{\eta}_2 \mathbb{D} + P\mathbb{I}) \cdot \mathbb{D}, \quad (47)$$

where P is once again the Lagrange multiplier due to the incompressibility constraint. Since the left hand side does not depend on \mathbb{D} we get

$$\mathbb{T} = -P\mathbb{I} + \mu_2 \mathbb{B}_{2p} + \hat{\eta}_2 \mathbb{D}. \quad (48)$$

Eliminating \mathbb{B}_{2p} from (48) and (46) we get

$$\hat{\xi} = (\mathbb{T} - \hat{\eta}_2 \mathbb{D}) \cdot \mathbb{D}_{2G} + \hat{\eta}_2 \mathbb{D} \cdot \mathbb{D}. \quad (49)$$

Applying the criterion of the maximization of the rate of dissipation we obtain

$$\begin{cases} \hat{\xi} - (\mathbb{T} - \hat{\eta}_2 \mathbb{D}) \cdot \mathbb{D}_{2G} - \hat{\eta}_2 \mathbb{D} \cdot \mathbb{D} = 0, \\ \text{tr} \mathbb{D}_{2G} = 0, \\ \frac{\partial \hat{\xi}}{\partial \mathbb{D}_{2G}} + \kappa_1 \left(\frac{\partial \hat{\xi}}{\partial \mathbb{D}_{2G}} - \mathbb{T} + \hat{\eta}_2 \mathbb{D} \right) + \kappa_2 \mathbb{I} = 0. \end{cases} \quad (50)$$

From (50)₁ and (50)₃

$$\mathbb{T} = 2\eta_2 \left(1 + \frac{1}{\kappa_1} \right) \mathbb{D}_{2G} \mathbb{B}_{2p} + \hat{\eta}_2 \mathbb{D} + \frac{\kappa_2}{\kappa_1} \mathbb{I}. \quad (51)$$

Substituting (51) into (50)₁ we obtain $\kappa_1 = -2$, so that

$$\mathbb{T} = \eta_2 \mathbb{D}_{2G} \mathbb{B}_{2p} + \hat{\eta}_2 \mathbb{D} - \frac{\kappa_2}{2} \mathbb{I}. \quad (52)$$

Subtracting (52) from (48) and multiplying by \mathbb{B}_{2p}^{-1} , we have

$$\frac{\eta_2}{\mu_2} \mathbb{D}_{2G} = \frac{1}{\mu_2} \left(\frac{\kappa_2}{2} - P \right) \mathbb{B}_{2p}^{-1} + \mathbb{I}, \quad (53)$$

and, applying the trace operator to both sides of (53), we find

$$\kappa_2 = 2P - \frac{6\mu_2}{\text{tr} \mathbb{B}_{2p}^{-1}}. \quad (54)$$

Proceeding as in the previous section, from (53), we get

$$\gamma_2 \overset{\nabla}{\mathbb{B}}_{2p} + \mathbb{B}_{2p} = \lambda_2 \mathbb{I}, \quad (55)$$

where

$$\gamma_2 = \frac{\eta_2}{2\mu_2}, \quad \lambda_2 = \frac{3}{\text{tr} \mathbb{B}_{2p}^{-1}}. \quad (56)$$

Therefore, the constitutive equation in Ω_2 can be written as

$$\begin{cases} \mathbb{T} = -P\mathbb{I} + \mathbb{S}, \\ \mathbb{S} = \mu_2 \mathbb{B}_{2p} + \hat{\eta}_2 \mathbb{D}, \\ \mathbb{S} + \gamma_2 \overset{\nabla}{\mathbb{S}} = \hat{\eta}_2 (\mathbb{D} + \gamma_2 \overset{\nabla}{\mathbb{D}}) + \lambda_2 \mu_2 \mathbb{I}. \end{cases} \quad (57)$$

Equation (57)₃ is very closely related to the Oldroyd-B type fluid¹⁵, but contains the extra term $\lambda_2 \mu_2 \mathbb{I}$ and has the same retardation and relaxation time γ_2 . To obtain the exact form of the Oldroyd-B type differential equation we rewrite \mathbb{T} as

$$\mathbb{T} = -\hat{P}\mathbb{I} + \hat{\mathbb{S}}, \quad (58)$$

where

$$\hat{P} = P - \lambda_2 \mu_2, \quad \hat{\mathbb{S}} = \mathbb{S} - \lambda_2 \mu_2 \mathbb{I}. \quad (59)$$

It is easy to see that

$$\hat{\mathbb{S}} + \gamma_2 \overset{\nabla}{\hat{\mathbb{S}}} = \gamma_3 (\mathbb{D} + \gamma_4 \mathbb{D}), \quad (60)$$

where

$$\gamma_3 = \hat{\eta}_2 \gamma_2, \quad \gamma_4 = \frac{\hat{\eta}_2}{\hat{\eta}_2 \gamma_2 + \eta_2 \lambda_2}. \quad (61)$$

Equation (60) is the constitutive equation of the Oldroyd-B type fluid model. Proceeding as in the previous section we make use of (57) to represent the Oldroyd-B type fluid in the domain Ω_2 .

Remark 1. From (57)₂ and (57)₃ we see that

$$\mathbb{B}_{2p} + \gamma_2 \overset{\nabla}{\mathbb{B}}_{2p} = \lambda_2 \mathbb{I}, \quad (62)$$

thus (57) is equivalent to

$$\begin{cases} \mathbb{T} = -P\mathbb{I} + \mu_2 \mathbb{B}_{2p} + \hat{\eta}_2 \mathbb{D}, \\ \mathbb{B}_{2p} + \gamma_2 \overset{\nabla}{\mathbb{B}}_{2p} = \lambda_2 \mathbb{I}. \end{cases} \quad (63)$$

It is thus clear that the only difference between the Maxwell fluid and the Oldroyd fluid is the extra viscous term $\hat{\eta}_2 \mathbb{D}$.

6. Linearized elastic response

From the constraint $\mathbb{I} \cdot \mathbb{L}_{iG} = \mathbb{I} \cdot \mathbb{D}_{iG} = 0$ we have that $\text{tr}(\dot{\mathbb{G}}_i \mathbb{G}_i^{-1}) = 0$. Recalling that $\mathbb{G}_i = \mathbb{F}_{ip}^{-1} \mathbb{F}_i$ we get

$$\dot{\mathbb{G}}_i \mathbb{G}_i^{-1} = (\dot{\mathbb{F}}_{ip}^{-1} \mathbb{F}_i + \mathbb{F}_{ip}^{-1} \dot{\mathbb{F}}_i) \mathbb{F}_i^{-1} \mathbb{F}_{ip} = \dot{\mathbb{F}}_{ip}^{-1} \mathbb{F}_{ip} + \mathbb{F}_{ip}^{-1} \mathbb{L}_i \mathbb{F}_{ip}, \quad (64)$$

where $\mathbb{L}_i = \dot{\mathbb{F}}_i \mathbb{F}_i^{-1}$ is the restriction of the velocity gradient to the i^{th} “phase”. Therefore

$$\text{tr}(\dot{\mathbb{G}}_i \mathbb{G}_i^{-1}) = \text{tr}(\dot{\mathbb{F}}_{ip}^{-1} \mathbb{F}_{ip}) + \text{tr}(\mathbb{L}_i) = 0. \quad (65)$$

Recalling that the continuum is incompressible we have $\text{tr}(\mathbb{L}_i) = 0$ and consequently $\text{tr}(\dot{\mathbb{F}}_{ip}^{-1} \mathbb{F}_{ip}) = 0$. From (21)

$$\overline{\det \mathbb{F}_{ip}} = 0 \quad \Rightarrow \quad \det \mathbb{F}_{ip} = \det \mathbb{B}_{ip} = 1. \quad (66)$$

Now we assume that the elastic response from the natural to the actual configuration is linear and we write

$$(\mathbb{B}_{ip})_{rk} = \delta_{rk} + B_{rk}^i, \quad i = 1, 2, \quad r, k = 1, 2, 3. \quad (67)$$

where $|B_{rk}^i| = O(\epsilon)$, with $\epsilon \ll 1$. Recalling that $\det \mathbb{B}_{ip} = 1$ it is easy to see that

$$1 = 1 + \sum_{r=1}^3 B_{rr}^i + O(\epsilon^2), \quad \Rightarrow \quad \sum_{r=1}^3 B_{rr}^i = O(\epsilon^2), \quad (68)$$

and

$$\text{tr} \mathbb{B}_{ip} = 3 + O(\epsilon^2). \quad (69)$$

In an analogous way we can also prove that

$$\text{tr} \mathbb{B}_{ip}^{-1} = 3 + O(\epsilon^2). \quad (70)$$

Assuming that the elastic response is linear means that we are neglecting the second order terms in (70). Therefore in the case of linear elastic response for both phases we have $\lambda_1 = \lambda_2 = 1$.

7. The one dimensional case

We now consider a channel flow driven by a pressure gradient. We assume that the flow occurs along the x -axis between two parallel plates placed at distance $2L$. We denote by y the coordinate axis perpendicular to the flow of the fluid. We assume that the domain Ω_2 (where the body behaves as an Oldroyd-B fluid) is confined between the moving surfaces $y = \pm s(t)$, while the domain Ω_1 (where the body behaves as a Maxwell fluid) is confined in the regions $s(t) \leq y \leq L$ and $-L \leq y \leq -s(t)$ (see Fig. 2).

For symmetry reasons we only consider the upper part of the layer $0 \leq y \leq L$. We define the velocity in the domains Ω_1 and Ω_2 respectively as

$$\vec{v}(y, t) = v(y, t) \vec{i}, \quad s(t) \leq y \leq L, \quad (71)$$

$$\vec{u}(y, t) = u(y, t) \vec{i}, \quad 0 \leq y \leq s(t). \quad (72)$$

In practice we are assuming that the spatial dependence is only upon the y coordinate, i.e. we are supposing that the plates

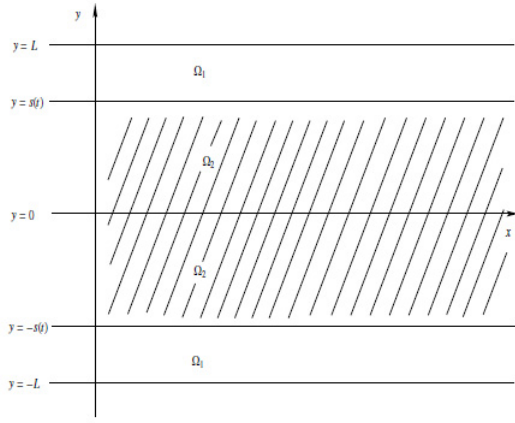


Figure 2: One dimensional setting of the problem. The filled part $-s(t) < y < s(t)$ represents the domain where the fluid behaves as an Oldroyd-B fluid.

$y = \pm L$ are sufficiently large so that the dependence upon the x and z coordinates can be safely neglected. For simplicity we will derive the equation of motion for the domain Ω_2 , since the one for the domain Ω_1 is obtained in the same way setting the extra viscosity $\hat{\eta}_2 = 0$ (recall that the constitutive equations in the two phases differ only for the extra viscosity term $\hat{\eta}_2 \mathbb{D}$, see Remark 1).

For the specific case we are considering we observe that the tensor \mathbb{B}_{2p} depends only on y and t , that is $\mathbb{B}_{2p} = \mathbb{B}_{2p}(y, t)$. Thus we have

$$\mathbb{L} = \begin{pmatrix} 0 & u_y & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathbb{D} = \frac{1}{2} \begin{pmatrix} 0 & u_y & 0 \\ u_y & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad (73)$$

and, since we are assuming a linearized elastic response,

$$\mathbb{B}_{2p} = \begin{pmatrix} 1 + B_{11} & B_{12} & B_{13} \\ B_{12} & 1 + B_{22} & B_{23} \\ B_{13} & B_{23} & 1 + B_{33} \end{pmatrix}, \quad (74)$$

where for simplicity we have omitted the apex 2. It is easy to prove that equation (63)₂ (recall that $\lambda_2 = 1$) reduces to the six scalar differential equations

$$\begin{cases} \frac{\partial B_{11}}{\partial t} + \frac{1}{\gamma_2} B_{11} - 2B_{12} \frac{\partial u}{\partial y} = 0, \\ \frac{\partial B_{12}}{\partial t} + \frac{1}{\gamma_2} B_{12} - (1 + B_{22}) \frac{\partial u}{\partial y} = 0, \\ \frac{\partial B_{13}}{\partial t} + \frac{1}{\gamma_2} B_{13} - B_{23} \frac{\partial u}{\partial y} = 0, \\ \frac{\partial B_{22}}{\partial t} + \frac{1}{\gamma_2} B_{22} = 0, \\ \frac{\partial B_{23}}{\partial t} + \frac{1}{\gamma_2} B_{23} = 0, \\ \frac{\partial B_{33}}{\partial t} + \frac{1}{\gamma_2} B_{33} = 0, \end{cases} \quad (75)$$

where the initial conditions are $B_{rk}(y, 0) = 0$, $r, k = 1, 2, 3$. Problem (75) can be solved obtaining

$$\begin{cases} B_{11}(y, t) = 2 \int_0^t \frac{\partial u}{\partial y}(y, \tau) B_{12}(y, \tau) \exp\left(\frac{\tau - t}{\gamma_2}\right) d\tau, \\ B_{12}(y, t) = \int_0^t \frac{\partial u}{\partial y}(y, \tau) \exp\left(\frac{\tau - t}{\gamma_2}\right) d\tau, \\ B_{13}(y, t) = B_{22}(y, t) = B_{23}(y, t) = B_{33}(y, t) = 0. \end{cases} \quad (76)$$

Considering now the balance of linear momentum

$$\rho \vec{a} = \nabla \cdot \mathbb{T}, \quad (77)$$

we get

$$\begin{cases} \rho \frac{\partial u}{\partial t} = -\frac{\partial P}{\partial x} + \mu_2 \int_0^t \frac{\partial u}{\partial y}(y, \tau) \exp\left(\frac{\tau - t}{\gamma_2}\right) d\tau + \frac{\hat{\eta}_2}{2} \frac{\partial^2 u}{\partial y^2}, \\ 0 = -\frac{\partial P}{\partial y}, \\ 0 = -\frac{\partial P}{\partial z}, \end{cases} \quad (78)$$

where the last two equations imply

$$P(x, t) = -f(t)x + g(t). \quad (79)$$

Multiplying both sides of (78)₁ by $\exp(t/\gamma_2)$ and differentiating w.r.t. time we get

$$\begin{aligned} \frac{1}{c_2^2} \left(\frac{\partial^2 u}{\partial t^2} + \frac{1}{\gamma_2} \frac{\partial u}{\partial t} \right) &= \frac{1}{\mu_2} \left[\frac{1}{\gamma_2} f(t) + \frac{df(t)}{dt} \right] + \\ &+ \left(1 + \frac{\hat{\eta}_2}{\eta_2} \right) \frac{\partial^2 u}{\partial y^2} + \frac{\hat{\eta}_2}{2\mu_2} \frac{\partial^3 u}{\partial t \partial y^2}, \end{aligned} \quad (80)$$

where

$$c_2 = \sqrt{\frac{\mu_2}{\rho}}. \quad (81)$$

The equation of motion in Ω_1 can be obtained using the same procedure and setting $\hat{\eta}_2 = 0$. Therefore we have

$$\frac{1}{c_1^2} \left(\frac{\partial^2 v}{\partial t^2} + \frac{1}{\gamma_1} \frac{\partial v}{\partial t} \right) = \frac{1}{\mu_1} \left[\frac{1}{\gamma_1} f(t) + \frac{df(t)}{dt} \right] + \frac{\partial^2 v}{\partial y^2}, \quad (82)$$

where

$$c_1 = \sqrt{\frac{\mu_1}{\rho}}. \quad (83)$$

Remark 2. The pressure field is determined imposing a known applied pressure at some points $x = a$ and $x = b$, namely $P|_{x=a} = P_a(t)$ and $P|_{x=b} = P_b(t)$. In this case

$$f(t) = \frac{P_b(t) - P_a(t)}{a - b}, \quad g(t) = \frac{P_b(t)a - P_a(t)b}{a - b} \quad (84)$$

Remark 3. Notice that rescaling B_{11} and B_{12} in (76) implies

$$B_{12} = O\left(\frac{Vt_c}{L}\right), \quad B_{11} = O\left(\frac{V^2t_c^2}{L^2}\right), \quad (85)$$

where V , L and t_c are the characteristic velocity, length and time respectively. Setting $\epsilon = Vt_c^{-1}$, we have $B_{12} = O(\epsilon)$ and $B_{11} = O(\epsilon^2)$. Since we are considering a linearized elastic response we neglect second order terms and the traceless part of the stress tensor $\hat{\mathbb{T}}$ defined in (26) becomes

$$\hat{\mathbb{T}} = \begin{bmatrix} 0 & \mu_1 B_{12} & 0 \\ \mu_1 B_{12} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad (86)$$

in $s(t) \leq y \leq L$ and

$$\hat{\mathbb{T}} = \begin{bmatrix} 0 & \mu_2 B_{12} + \frac{\hat{\eta}_2}{2} \frac{\partial u}{\partial y} & 0 \\ \mu_2 B_{12} + \frac{\hat{\eta}_2}{2} \frac{\partial u}{\partial y} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad (87)$$

in $0 \leq y \leq s(t)$. Conditions (28) and (29) become

$$|\mu_1 B_{12}| \geq \tau_o, \quad \text{in } s(t) \leq y \leq L, \quad (88)$$

$$\left| \mu_2 B_{12} + \frac{\hat{\eta}_2}{2} \frac{\partial u}{\partial y} \right| \leq \tau_o, \quad \text{in } 0 \leq y \leq s(t), \quad (89)$$

while, on $y = s(t)$,

$$|\mu_1 B_{12}| = \left| \mu_2 B_{12} + \frac{\hat{\eta}_2}{2} \frac{\partial u}{\partial y} \right| = \tau_o \quad (90)$$

Remark 4. Assuming enough regularity for $u(y, t)$ and $v(y, t)$, from (78)₁, we get

$$\frac{\partial u(y, 0)}{\partial t} = \frac{f(0)}{\rho} + \frac{\hat{\eta}_2}{2\rho} \frac{\partial^2 u(y, 0)}{\partial y^2}. \quad (91)$$

Analogously, in the region $s(t) \leq y \leq L$, we have

$$\frac{\partial v(y, 0)}{\partial t} = \frac{f(0)}{\rho}. \quad (92)$$

8. Boundary conditions

We want to derive the boundary conditions at the interface $y - s(t) = 0$, which we denote by Σ . We consider the classical Rankine-Hugoniot conditions, which express the conservation law on the moving discontinuity interface Σ

$$[Q]\vec{w} \cdot \vec{n} = [\vec{q}] \cdot \vec{n}, \quad (93)$$

where Q is any scalar quantity which is C^1 on both sides of Σ , \vec{q} is the flux of Q across Σ , \vec{n} is the unit normal vector of Σ , \vec{w} is the velocity of Σ and $[Q] = Q^+ - Q^-$, $[\vec{q}] = \vec{q}^+ - \vec{q}^-$ represent the jumps of Q and \vec{q} across Σ . The positive and negative sides are defined choosing the orientation to the normal vector \vec{n} , so that Q^\pm and \vec{q}^\pm represent the limits for (x, t) tending to Σ^\pm . In

the case we are considering we have $\vec{n} = \vec{j}$ so that $\vec{w} = \dot{s}(t)\vec{j}$ and we write

$$[Q]\dot{s}(t) = [\vec{q}] \cdot \vec{j}. \quad (94)$$

When Q is density $Q = \rho$ and $\vec{q} = \rho\vec{v}$, where \vec{v} is the velocity of the medium. At $y = s(t)$ we have

$$[\rho^+ - \rho^-]\dot{s}(t) = [(\rho^+v - \rho^-u)\vec{j}] \cdot \vec{j} = 0, \quad (95)$$

implying the continuity of density across the interface. When Q represents the momentum along the x direction we have $Q^+ = \rho v$, $Q^- = \rho u$ and on $y = s(t)$

$$\vec{q}^+ = \mathbb{T}^+\vec{i} = [-P^+ + \mu_1(1 + B_{11}^+)]\vec{i} + \mu_1 B_{12}^+\vec{j}, \quad (96)$$

$$\vec{q}^- = \mathbb{T}^-\vec{i} = [-P^- + \mu_2(1 + B_{11}^-)]\vec{i} + \left(\mu_2 B_{12}^- + \frac{\hat{\eta}_2}{2} \frac{\partial u}{\partial y} \right) \vec{j}. \quad (97)$$

Therefore, from (93),

$$\rho(v - u)\dot{s}(t) = \mu_1 B_{12}^+ - \mu_2 B_{12}^- - \frac{\hat{\eta}_2}{2} \frac{\partial u}{\partial y}. \quad (98)$$

Assuming the no-slip condition $u(s, t) = v(s, t)$ and recalling (76)₂ we get

$$\begin{aligned} & \mu_1 \int_0^t \frac{\partial v(s, \tau)}{\partial y} \exp\left(\frac{\tau - t}{\gamma_1}\right) d\tau = \\ & = \mu_2 \int_0^t \frac{\partial u(s, \tau)}{\partial y} \exp\left(\frac{\tau - t}{\gamma_2}\right) d\tau + \frac{\hat{\eta}_2}{2} \frac{\partial u(s, t)}{\partial y}. \end{aligned} \quad (99)$$

Both sides of (99) represent the stress at the interface $y = s(t)$ and, from (90),

$$\mu_1 \int_0^t \frac{\partial v(s, \tau)}{\partial y} \exp\left(\frac{\tau - t}{\gamma_1}\right) d\tau = -\tau_o, \quad (100)$$

$$\mu_2 \int_0^t \frac{\partial u(s, \tau)}{\partial y} \exp\left(\frac{\tau - t}{\gamma_2}\right) d\tau + \frac{\hat{\eta}_2}{2} \frac{\partial u(s, t)}{\partial y} = -\tau_o, \quad (101)$$

where we take $-\tau_o$ since we expect $v_y(s, t)$ and $u_y(s, t)$ to be negative

If we differentiate (100) w.r.t. time and recall (78)₁ with $\hat{\eta}_2 = 0$ we get the interface condition

$$\frac{\partial v(s, t)}{\partial y} + \frac{\dot{s}}{c_1^2} \frac{\partial v(s, t)}{\partial t} = \frac{1}{\mu_1} \left[f(t)\dot{s}(t) - \frac{\tau_o}{\gamma_1} \right], \quad (102)$$

while if we differentiate (101) w.r.t. time and recall (78)₁ we get the interface condition

$$\left(1 + \frac{\hat{\eta}_2}{\eta_2} \right) \frac{\partial u(s, t)}{\partial y} + \frac{\dot{s}}{c_2^2} \frac{\partial u(s, t)}{\partial t} + \frac{\hat{\eta}_2}{2\mu_2} \frac{\partial^2 u(s, t)}{\partial t \partial y} = \frac{1}{\mu_2} \left[f(t)\dot{s}(t) - \frac{\tau_o}{\gamma_2} \right]. \quad (103)$$

If we now consider the momentum along the y direction we have $Q = 0$ and

$$\vec{q}^+ = \mathbb{T}^+\vec{j} = \mu_1 B_{12}^+\vec{i} + (-P + \mu_1)\vec{j}, \quad (104)$$

$$\vec{q}^- = \mathbb{T}^-\vec{j} = \left[\mu_2 B_{12}^- + \frac{\hat{\eta}_2}{2} \frac{\partial u(s, t)}{\partial y} \right] \vec{i} + (-P + \mu_2)\vec{j}. \quad (105)$$

From (93) we get

$$P^+ - P^- = \mu_1 - \mu_2. \quad (106)$$

so, recalling Remark 2, we know that $P^+ = P^-$, implying necessarily $\mu_1 = \mu_2$, that is the elastic modulus must be the same in both phases. Thus from now on we set $\mu_1 = \mu_2 = \mu$.

Last, if we consider the balance of energy across the interface Σ we get

$$Q^+ = \frac{1}{2}\rho v^2 + W^+, \quad Q^- = \frac{1}{2}\rho u^2 + W^-, \quad (107)$$

$$\vec{q}^+ = \mathbb{T}^+ \vec{v} = v[-P + \mu(1 + B_{11}^+)] \vec{i} + v\mu B_{12}^+ \vec{j}, \quad (108)$$

$$\begin{aligned} \vec{q}^- &= \mathbb{T}^- \vec{u} = \\ &= u[-P + \mu(1 + B_{11}^-)] \vec{i} + v \left[\mu B_{12}^+ + \frac{\hat{\eta}_2}{2} \frac{\partial u(s, t)}{\partial y} \right] \vec{j}. \end{aligned} \quad (109)$$

From (93), on $y = s(t)$, we get

$$(W^+ - W^-) \dot{s}(t) = u \left[\mu B_{12}^+ - \mu B_{12}^- - \frac{\hat{\eta}_2}{2} \frac{\partial u(s, t)}{\partial y} \right]. \quad (110)$$

The right hand side of (110) vanishes because of (99) and therefore energy is continuous across Σ and $\psi^+ = \psi^-$, meaning that the Helmholtz potential is continuous across the interface $y = s(t)$.

9. The mathematical problem

We are now able to state the mathematical formulation of the problem. Since $\mu_1 = \mu_2 = \mu$, we have

$$c =: c_1 = c_2 = \sqrt{\frac{\mu}{\rho}}, \quad \gamma_1 = \frac{\eta_1}{2\mu}, \quad \gamma_2 = \frac{\eta_2}{2\mu}. \quad (111)$$

Moreover, for the sake of simplicity, we assume that the pressure gradient is constant, that is $f(t) =: f_o > 0$. We have

$$\frac{1}{c^2} \left[\frac{\partial^2 v}{\partial t^2} + \frac{1}{\gamma_1} \frac{\partial v}{\partial t} \right] = \frac{\partial^2 v}{\partial y^2} + \frac{2f_o}{\eta_1}, \quad s(t) \leq y \leq L, \quad t \geq 0, \quad (112)$$

$$v(y, 0) = v_o(y), \quad s_o \leq y \leq L, \quad (113)$$

$$\frac{\partial v(y, 0)}{\partial t} = \frac{f_o}{\rho}, \quad s_o \leq y \leq L, \quad (114)$$

$$v(L, t) = 0, \quad t \geq 0, \quad (115)$$

$$v(s, t) = u(s, t), \quad t \geq 0, \quad (116)$$

$$\frac{\partial v(s, t)}{\partial y} + \frac{\dot{s}}{c^2} \frac{\partial v(s, t)}{\partial t} = \frac{\dot{s} f_o}{\mu} - \frac{2\tau_o}{\eta_1}, \quad t \geq 0, \quad (117)$$

$$\frac{1}{c^2} \left(\frac{\partial^2 u}{\partial t^2} + \frac{1}{\gamma_2} \frac{\partial u}{\partial t} \right) =$$

$$= \frac{2f_o}{\eta_2} + \left(1 + \frac{\hat{\eta}_2}{\eta_2} \right) \frac{\partial^2 u}{\partial y^2} + \frac{\hat{\eta}_2}{2\mu} \frac{\partial^3 u}{\partial t \partial y^2}, \quad 0 \leq y \leq s(t), \quad t \geq 0, \quad (118)$$

$$u(y, 0) = u_o(y), \quad 0 \leq y \leq s_o, \quad (119)$$

$$\frac{\partial u(y, 0)}{\partial t} = \frac{f_o}{\rho} + \frac{\hat{\eta}_2}{2\rho} u_o''(y), \quad 0 \leq y \leq s_o \quad (120)$$

$$\frac{\partial u(0, t)}{\partial y} = 0, \quad t \geq 0, \quad (121)$$

$$\begin{aligned} \frac{\partial u(s, t)}{\partial y} \left(1 + \frac{\hat{\eta}_2}{\eta_2} \right) + \frac{\dot{s}}{c^2} \frac{\partial u(s, t)}{\partial t} + \frac{\hat{\eta}_2}{2\mu} \frac{\partial^2 u(s, t)}{\partial t \partial y} &= \\ &= \frac{\dot{s} f_o}{\mu} - \frac{2\tau_o}{\eta_2}, \quad t \geq 0. \end{aligned} \quad (122)$$

We rescale the problem (112)-(122) setting

$$t = Tt^*, \quad u = Uu^*, \quad v = Uv^*, \quad y = Ly^*, \quad (123)$$

$$s = Ls^*, \quad s_o = Ls_o^*, \quad (124)$$

and introduce

$$U = \frac{2f_o L^2}{\eta_2}, \quad T = \frac{L}{c}. \quad (125)$$

$$\alpha^2 = \frac{\eta_2}{\eta_2 + \hat{\eta}_2}, \quad \beta^2 = \frac{\eta_2}{\eta_1}, \quad \text{Bn} = \frac{\tau_o}{f_o L}, \quad (126)$$

$$\text{We}_1 = \frac{\gamma_1 c}{L}, \quad \text{We}_2 = \frac{\gamma_2 c}{L}, \quad \widehat{\text{We}}_2 = \frac{\hat{\gamma}_2 c}{L} = \frac{\hat{\eta}_2 c}{2\mu L}, \quad (127)$$

$$\text{Re}_1 = \frac{2\rho c L}{\eta_1}, \quad \text{Re}_2 = \frac{2\rho c L}{\eta_2}, \quad \widehat{\text{Re}}_2 = \frac{2\rho c L}{\hat{\eta}_2}, \quad (128)$$

where Bn is the Bingham number, $\text{We}_1, \text{We}_2, \widehat{\text{We}}_2$ are the Weissenberg numbers with respect to viscosities $\eta_1, \eta_2, \hat{\eta}_2$, while $\text{Re}_1, \text{Re}_2, \widehat{\text{Re}}_2$ are the Reynolds numbers with respect to viscosities $\eta_1, \eta_2, \hat{\eta}_2$. The adimensional problem (we omit the * to keep notation simple) becomes

$$\frac{\partial^2 v}{\partial t^2} + \frac{1}{\text{We}_1} \frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial y^2} + \beta^2, \quad y \in [s(t), 1], \quad t > 0, \quad (129)$$

$$v(y, 0) = v_o(y), \quad s_o \leq y \leq 1, \quad (130)$$

$$\frac{\partial v(y, 0)}{\partial t} = \frac{1}{\text{Re}_2}, \quad s_o \leq y \leq 1, \quad (131)$$

$$v(1, t) = 0, \quad t \geq 0, \quad (132)$$

$$v(s, t) = u(s, t), \quad t \geq 0, \quad (133)$$

$$\frac{\partial v(s, t)}{\partial y} + s \frac{\partial v(s, t)}{\partial t} = \text{We}_2 \dot{s} - \beta^2 \text{Bn}, \quad t \geq 0, \quad (134)$$

$$\begin{aligned} \frac{\partial^2 u}{\partial t^2} + \frac{1}{\text{We}_2} \frac{\partial u}{\partial t} &= 1 + \frac{1}{\alpha^2} \frac{\partial^2 u}{\partial y^2} + \\ + \widehat{\text{We}_2} \frac{\partial^3 u}{\partial t \partial y^2}, \quad y \in [0, s(t)], \quad t > 0, \end{aligned} \quad (135)$$

$$u(y, 0) = u_o(y), \quad 0 \leq y \leq s_o, \quad (136)$$

$$\frac{\partial u(y, 0)}{\partial t} = \frac{1}{\text{Re}_2} + \frac{1}{\widehat{\text{Re}_2}} u_o''(y), \quad 0 \leq y \leq s_o, \quad (137)$$

$$\frac{\partial u(0, t)}{\partial y} = 0, \quad t \geq 0, \quad (138)$$

$$\begin{aligned} \frac{1}{\alpha^2} \frac{\partial u(s, t)}{\partial y} + s \frac{\partial u(s, t)}{\partial t} + \\ + \widehat{\text{We}_2} \frac{\partial^2 u}{\partial t \partial y} = \text{We}_2 \dot{s} - \text{Bn}, \quad t > 0. \end{aligned} \quad (139)$$

Problem (129)-(139) is a free boundary problem involving a telegraph equation and a third order equation, the free boundary being the surface $y = s(t)$. Due to its high complexity we do not prove any analytical results but we focus on a particular case which is the subject of section 10.

9.1. The stationary solution

Let us study the stationary version of problem (129)-(139), that is the one in which all the time derivatives are set to zero. We have

$$\frac{\partial^2 v}{\partial y^2} = -\beta^2, \quad y \in [s, 1], \quad (140)$$

$$v(1) = 0, \quad (141)$$

$$v(s) = u(s), \quad (142)$$

$$\frac{\partial v(s)}{\partial y} = -\beta^2 \text{Bn}, \quad (143)$$

$$\frac{\partial^2 u}{\partial y^2} = -\alpha^2, \quad y \in [0, s], \quad (144)$$

$$\frac{\partial u(0)}{\partial y} = 0, \quad (145)$$

$$\frac{\partial u(s)}{\partial y} = -\alpha^2 \text{Bn}. \quad (146)$$

It is easy to see that the solution of (140)-(146) is given by

$$v(y) = \frac{\beta^2}{2}(1 - y^2), \quad (147)$$

$$u(y) = \frac{\alpha^2}{2}(s^2 - y^2) + \frac{\beta^2}{2}(1 - s^2), \quad (148)$$

$$s = \text{Bn}. \quad (149)$$

Remark 5. We notice that the stationary solution $s = \text{Bn}$ has a physical meaning if $0 < s = \text{Bn} < 1$, that is if the condition

$$f_o > \frac{\tau_o}{L} \quad (150)$$

is fulfilled.

Remark 6. In the next section we will consider the case in which $\alpha^2 \ll 1$. In view of (148) we see that in this situation the stationary solution of the problem does not depend on the spatial coordinate y . Therefore, if $\alpha^2 \ll 1$ and $\beta^2 = O(1)$, it is reasonable to assume that the initial conditions $u(y, 0)$ and $u_t(y, 0)$ are spatially uniform.

10. The reduced problem: case 1

Let us go back to problem (129)-(139) and let us first consider the particular case in which $\hat{\eta}_2 \gg \eta_2$. In particular, we assume that

$$\hat{\eta}_2 \gg \eta_2, \quad \text{We}_i = O(1), \quad \text{Re}_i = O(1), \quad i = 1, 2 \quad (151)$$

$$\text{Bn} = O(1), \quad \beta^2 = O(1). \quad (152)$$

Under assumptions (151) and recalling (126)-(127), it is easy to show that

$$\alpha^2 \ll 1, \quad \alpha^2 \widehat{\text{We}_2} \approx \text{We}_2 = O(1), \quad (153)$$

$$\widehat{\text{Re}_2} \ll \text{Re}_2 \approx \text{Re}_1 = O(1). \quad (154)$$

Thus, neglecting the terms whose coefficients are ‘‘small’’, problem (129)-(139) reduces to the following

$$\frac{\partial^2 v}{\partial t^2} + \frac{1}{\text{We}_1} \frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial y^2} + \beta^2, \quad y \in [s(t), 1], \quad t > 0, \quad (155)$$

$$v(y, 0) = v_o(y), \quad s_o \leq y \leq 1, \quad (156)$$

$$\frac{\partial v(y, 0)}{\partial t} = \frac{1}{\text{Re}_2}, \quad s_o \leq y \leq 1, \quad (157)$$

$$v(1, t) = 0, \quad t \geq 0, \quad (158)$$

$$v(s, t) = u(s, t), \quad t \geq 0, \quad (159)$$

$$\frac{\partial v(s, t)}{\partial y} + s \frac{\partial v(s, t)}{\partial t} = \text{We}_2 \dot{s} - \beta^2 \text{Bn}, \quad t \geq 0, \quad (160)$$

$$\frac{\partial^2 u}{\partial y^2} + \alpha^2 \widehat{\text{We}_2} \frac{\partial^3 u}{\partial t \partial y^2} = 0, \quad y \in [0, s(t)], \quad t > 0, \quad (161)$$

$$u(y, 0) = U_o, \quad 0 \leq y \leq s_o, \quad (162)$$

$$\frac{\partial u(y, 0)}{\partial t} = 0, \quad 0 \leq y \leq s_o, \quad (163)$$

$$\frac{\partial u(0, t)}{\partial y} = 0, \quad t \geq 0, \quad (164)$$

$$\frac{\partial u(s, t)}{\partial y} + \alpha^2 \widehat{\text{We}}_2 \frac{\partial^2 u(s, t)}{\partial t \partial y} = 0, \quad t > 0, \quad (165)$$

where, as observed in remark 6, we assume that the initial condition $u(y, 0) = U_o > 0$ is spatially uniform. Focussing on (161) and setting $\alpha^2 \widehat{\text{We}}_2 = \zeta$, we get

$$\begin{aligned} \frac{\partial^2}{\partial y^2} \left[u(y, t) + \zeta \frac{\partial u(y, t)}{\partial t} \right] &= 0 \implies \\ \implies \frac{\partial}{\partial y} \left[u(y, t) + \zeta \frac{\partial u(y, t)}{\partial t} \right] &= F(t). \end{aligned} \quad (166)$$

From (165) $F(t) = 0$, hence

$$u(y, t) + \zeta \frac{\partial u(y, t)}{\partial t} = G(t), \quad (167)$$

where, at this stage, $G(t)$ is unknown. Equation (167) can be integrated so that

$$u = \left[U_o + \int_0^t G(\tau) \exp(\zeta \tau) d\tau \right] \exp(-\zeta t). \quad (168)$$

The latter implies that the velocity field in the region $[-s, s]$ does not depend on y , that is $u = u(t)$.

We now have to determine an equation for $u(t)$ not involving $G(t)$. To this aim we consider a region V with $-s(t) < y < s(t)$ and unitary length and width (see Fig. 3).

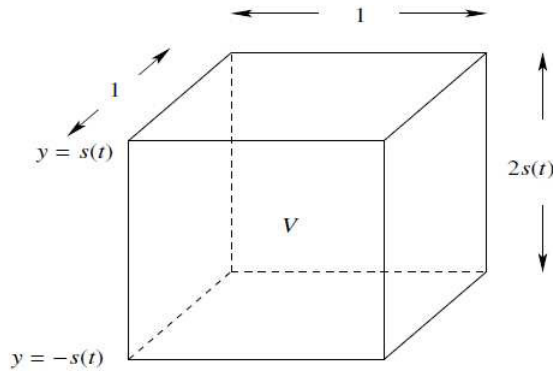


Figure 3: A portion V of the region $-s(t) < y < s(t)$ of unitary length and width.

The non-dimensional global balance of linear momentum relative to the domain V yields

$$\begin{aligned} \int_{-s(t)}^{s(t)} \frac{du(t)}{dt} dy &= \frac{2}{\text{Re}_2} [s - \text{Bn}] \implies \\ \implies \frac{du}{dt} &= \frac{1}{\text{Re}_2} \left[1 - \frac{\text{Bn}}{s} \right]. \end{aligned} \quad (169)$$

In particular, combining (167) and (169), we get also the following expression for G

$$G(t) = u(t) + \frac{\zeta}{\text{Re}_2} \left[1 - \frac{\text{Bn}}{s(t)} \right]. \quad (170)$$

Going back to (169) we have

$$u(t) = U_o + \frac{1}{\text{Re}_2} \int_0^t \left[1 - \frac{\text{Bn}}{s(\tau)} \right] d\tau. \quad (171)$$

The problem (155)-(165) reduces to

$$\frac{\partial^2 v}{\partial t^2} + \frac{1}{\text{We}_1} \frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial y^2} + \beta^2, y \in [s, 1], \quad t > 0, \quad (172)$$

$$v(y, 0) = v_o(y), s_o \leq y \leq 1, \quad (173)$$

$$\frac{\partial v(y, 0)}{\partial t} = \frac{1}{\text{Re}_2}, s_o \leq y \leq 1, \quad (174)$$

$$v(1, t) = 0, t \geq 0, \quad (175)$$

$$v(s, t) = U_o + \frac{1}{\text{Re}_2} \int_0^t \left[1 - \frac{\text{Bn}}{s(\tau)} \right] d\tau, t \geq 0, \quad (176)$$

$$\frac{\partial v(s, t)}{\partial y} + s \frac{\partial v(s, t)}{\partial t} = \text{We}_2 s - \beta^2 \text{Bn}, t \geq 0, \quad (177)$$

which is a free boundary problem for a telegraph equation.

11. The reduced problem: case 2

Here we consider the case in which $\widehat{\text{We}}_2 \gg 1$ and $\text{We}_2 \ll 1$, that is

$$\hat{\eta}_2 \gg \frac{2\mu L}{c}, \quad \eta_2 \ll \frac{2\mu L}{c}. \quad (178)$$

Relations (178) imply

$$\hat{\eta}_2 \gg \eta_2, \quad \alpha^2 \ll 1, \quad \text{We}_2 = \frac{1}{\text{Re}_2} \ll 1, \quad (179)$$

$$\alpha^2 \widehat{\text{We}}_2 \approx \text{We}_2 \ll 1, \quad \alpha^2 \text{We}_2 \ll 1, \quad (180)$$

$$\frac{\alpha^2}{\text{We}_2} \approx \frac{1}{\widehat{\text{We}}_2} \ll 1. \quad (181)$$

Once again we assume

$$\text{We}_1 = O(1), \quad \text{Re}_1 = O(1), \quad (182)$$

$$\text{Bn} = O(1), \quad \beta^2 = O(1). \quad (183)$$

In this second case problem (129)-(139) reduces to

$$\frac{\partial^2 v}{\partial t^2} + \frac{1}{\text{We}_1} \frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial y^2} + \beta^2, y \in [s(t), 1], \quad t > 0, \quad (184)$$

$$v(y, 0) = v_o(y), \quad s_o \leq y \leq 1, \quad (185)$$

$$\frac{\partial v(y, 0)}{\partial t} = 0, \quad s_o \leq y \leq 1, \quad (186)$$

$$v(1, t) = 0, \quad t \geq 0, \quad (187)$$

$$v(s, t) = u(s, t), \quad t \geq 0, \quad (188)$$

$$\frac{\partial v(s, t)}{\partial y} + s \frac{\partial v(s, t)}{\partial t} = -\beta^2 Bn, \quad t \geq 0, \quad (189)$$

$$\frac{\partial^2 u}{\partial y^2} = 0, \quad y \in [0, s(t)], \quad t > 0, \quad (190)$$

$$u(y, 0) = u_o(y), \quad 0 \leq y \leq s_o, \quad (191)$$

$$\frac{\partial u(y, 0)}{\partial t} = \frac{1}{\text{Re}_2} u''_o(y), \quad 0 \leq y \leq s_o, \quad (192)$$

$$\frac{\partial u(0, t)}{\partial y} = 0, \quad t \geq 0, \quad (193)$$

$$\frac{\partial u(s, t)}{\partial y} = 0, \quad t \geq 0. \quad (194)$$

We observe that $u = u(t)$ and, writing the balance of linear momentum as in section 10, we get

$$\frac{du(t)}{dt} = \frac{1}{\text{Re}_2} \left[1 - \frac{Bn}{s(t)} \right] = 0, \quad (195)$$

since we are assuming $\text{Re}_2^{-1} \ll 1$. Therefore, the velocity field in $[0, s(t)]$ is uniform in space and time, that is $u(t) = U_o = v(s_o) > 0$ and we have to solve

$$\frac{\partial^2 v}{\partial t^2} + \frac{1}{\text{We}_1} \frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial y^2} + \beta^2, \quad y \in [s(t), 1], \quad t > 0, \quad (196)$$

$$v(y, 0) = v_o(y), \quad s_o \leq y \leq 1, \quad (197)$$

$$\frac{\partial v(y, 0)}{\partial t} = 0, \quad s_o \leq y \leq 1, \quad (198)$$

$$v(1, t) = 0, \quad t \geq 0, \quad (199)$$

$$v(s, t) = U_o, \quad t \geq 0, \quad (200)$$

$$\frac{\partial v(s, t)}{\partial y} + s \frac{\partial v(s, t)}{\partial t} = -\beta^2 Bn, \quad t \geq 0. \quad (201)$$

In this case the stationary solution is found solving the following

$$\frac{\partial^2 v}{\partial y^2} = -\beta^2, \quad y \in [s, 1], \quad (202)$$

$$v(1) = 0, \quad (203)$$

$$v(s) = U_o, \quad (204)$$

$$\frac{\partial v(s)}{\partial y} = -\beta^2 Bn. \quad (205)$$

From (202), (204), (205) we get

$$v(y) = -\frac{\beta^2}{2} (s - Bn - y)^2 + \frac{\beta^2 Bn^2}{2} + U_o, \quad (206)$$

and imposing (203) we find

$$(s - Bn - 1) = \pm \sqrt{Bn^2 + \frac{2U_o}{\beta^2}}. \quad (207)$$

We immediately notice that the r.h.s. of (207) must be negative if we want that the boundary s has a physical meaning. Indeed, supposing that the r.h.s. is positive we have $s > Bn + 1 > 1$, which has no physical interpretation. Thus

$$s = 1 + Bn - \sqrt{Bn^2 + \frac{2U_o}{\beta^2}}. \quad (208)$$

Next we want that $s \in [0, 1]$, thus we require

$$\sqrt{Bn^2 + \frac{2U_o}{\beta^2}} \leq 1 + Bn \leq \sqrt{Bn^2 + \frac{2U_o}{\beta^2}} + 1. \quad (209)$$

The left inequality holds if

$$U_o \leq \beta^2 \left(\frac{1}{2} + Bn \right) \quad (210)$$

while the right inequality is always satisfied. Therefore, under hypothesis (210), the stationary solution is unique and is given by

$$v(y) = -\frac{\beta^2}{2} (s - Bn - y)^2 + \frac{\beta^2 Bn^2}{2} + U_o, \quad (211)$$

$$s = 1 + Bn - \sqrt{Bn^2 + \frac{2U_o}{\beta^2}}. \quad (212)$$

Notice that when $U_o = 0$ the stationary interface $s = 1$ and $u = 0$ in $[0, 1]$, meaning that the system has come to a stop.

12. Conclusions

We have developed a mathematical model for a fluid which behaves like a upper convected Maxwell fluid if a function of the principal invariants of the stress is above a certain threshold τ_o and like a Oldroyd-B type fluid if it is below such a threshold. Our model can be easily generalized to the case in which one phase is occupied by a general nonlinear Maxwell fluid and the other by a generalized Jeffreys fluid. We have formulated the mathematical problem for a one-dimensional channel flow driven by a pressure gradient. The problem has turned out to be a free boundary problem involving a hyperbolic telegraph equation and a third order equation. We have explicitly written the stationary solution and we have studied two particular cases in which the velocity of the inner part of the layer (the one below the threshold) has uniform velocity. Due to the high complexity of the general model, no analytical results have been proven. The analytical study of problem (196)-(205) will be the subject of a forthcoming paper.

An interesting extension of this work would be the one in which one considers the full thermodynamical problem, for instance assuming that the plates have different temperatures.

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Research Highlights

The reaserch highlights can be outlined as follows:

- 1) This work provides additional insight into the response of rate type fluids with a threshold.
- 2) The paper extends results previously obtained by the authors on materials with stress threshold (see references in the paper for alist of such works). The one-dimensional formulation, which is presented in the paper, is interesting also from the mathematical point of view, since it turns out to be a very complicated free boundary problem involving a hyperbolic and a parabolic equation. Various simplified cases are studied.

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