



# Minor theory for surfaces and divides of maximal signature

Sebastian Baader, Pierre Dehornoy

## ► To cite this version:

Sebastian Baader, Pierre Dehornoy. Minor theory for surfaces and divides of maximal signature. 2012.  
hal-00759428v1

**HAL Id: hal-00759428**

**<https://hal.science/hal-00759428v1>**

Preprint submitted on 30 Nov 2012 (v1), last revised 4 Jan 2019 (v2)

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# MINOR THEORY FOR SURFACES AND DIVIDES OF MAXIMAL SIGNATURE

SEBASTIAN BAADER AND PIERRE DEHORNOY

**ABSTRACT.** We prove that the restriction of surface minority to fiber surfaces of divides is a well-quasi-order. Here surface minority is the partial order on isotopy classes of surfaces embedded in  $\mathbb{R}^3$  associated with incompressible subsurfaces. The proof relies on a refinement of the Robertson-Seymour Theorem that involves colored graphs embedded into the plane. Our result implies that every property of fiber surfaces of divides that is preserved by surface minority is characterized by a finite number of prohibited minors. For the signature to be equal to the first Betti number is such a property. We explicitly determine the corresponding prohibited minors. As an application we establish a correspondance between divide links of maximal signature and Dynkin diagrams.

This article combines techniques of low-dimensional topology and graph theory in order to investigate a partial order on isotopy classes of embedded surfaces in  $\mathbb{R}^3$  called *surface minority*. A surface  $\Sigma$  is a *surface minor* of another surface  $\Sigma'$  if  $\Sigma$  is isotopic in  $\mathbb{R}^3$  to an incompressible subsurface of  $\Sigma'$ . We then write  $\Sigma \leq^{surf} \Sigma'$ . This relation is clearly reflexive and transitive. It turns out to be also antisymmetric, hence it induces a partial order on isotopy classes of embedded surfaces.

A natural question is whether the above surface minority order is a *well-quasi-order*, that is, it has no infinite descending chain and no infinite antichain. One easily shows that  $\leq^{surf}$  has no infinite descending chain, but, if  $A_n$  denotes an annulus whose core is a trivial knot and whose writhe—the linking number between the two boundary components—is  $n$ , then the family  $\{A_n\}_{n \in \mathbb{Z}}$  forms an infinite antichain, that is, elements are pairwise incomparable. So  $\leq^{surf}$  is not a well-quasi-order in general.

The aim of this article is to show that, nevertheless, there exists a large and natural class of surfaces such that the restriction of  $\leq^{surf}$  to it is a well-quasi-order. Introduced by N. A'Campo [1, 2] for studying the topology of singularities of complex curves, *divides* are diagrams in the disc that encode certain fibered links in  $\mathbb{S}^3$ . Every divide comes associated with a preferred fiber surface—the Milnor fiber in the context of singularities. We prove

**Theorem A.** *Let  $\mathcal{SDiv}$  be the family of all isotopy classes of fiber surfaces of divides. Then the restriction of  $\leq^{surf}$  to  $\mathcal{SDiv}$  is a well-quasi-order. Equivalently, if  $\Sigma_1, \Sigma_2, \dots$  is an infinite sequence in  $\mathcal{SDiv}$ , there exist  $i < j$  such that  $\Sigma_i$  is a surface minor of  $\Sigma_j$ .*

The proof relies on a result of graph theory which is of independent interest. We consider graphs whose vertices are colored with a set  $C$  which is a well-quasi-order. We shall say that a colored graph  $G$  embedded into the plane is a *planar minor* of another embedded colored graph  $G'$ , denoted  $G \leq_C^{plan} G'$ , if a graph isomorphic to  $G$  can be obtained from  $G'$  by decreasing the colors of some vertices, and deleting or contracting some edges in the plane. The difference with the standard minor notion for abstract graphs is that the planar minority order takes the embedding

---

2000 *Mathematics Subject Classification.* Primary 57M15, 05C83; Secondary 57M27.

*Key words and phrases.* Graph minor, embedded surface, incompressible surface, knot, signature, divide.

in the plane into account. The definition immediately implies that the planar minority order for embedded graphs refines the minority order for the underlying abstract graphs. A famous theorem of N. Robertson and P. Seymour [15] says that minority is a well-quasi-order for abstract colored graphs. Here we show that the result extends to planar minority.

**Proposition B.** *For  $C$  a well-quasi-ordered set, the order  $\leq_C^{plan}$  on isotopy classes of colored graphs embedded into the plane is a well-quasi-order.*

Since trees are particular graphs, this result is also a generalization of the Kruskal Tree Theorem [13] that states that the family of colored trees embedded into the plane is a well-quasi-order.

An important consequence of being a well-quasi-order is that every property preserved under passing to smaller elements is characterized by a finite number of prohibited minors. For instance, the Kuratowski Theorem, which says that a graph is planar if and only if it does not contain the graph  $K_5$  nor the graph  $K_{3,3}$  as a graph minor, appears as an explicit instance of the Robertson-Seymour Theorem. Returning to fiber surfaces of divides, we shall similarly illustrate the situation of Theorem A with a Kuratowski-type result involving signatures of divide links.

The signature  $\text{sign}(\Sigma)$  of a surface  $\Sigma$  in  $\mathbb{S}^3$  is an isotopy invariant defined as the signature of a quadratic form, the *Seifert form*, on  $H_1(\Sigma, \mathbb{R})$  (we will recall the precise definitions in Section 3.b). As the signature of a quadratic form is not larger than the rank of the underlying vector space, we always have  $-b_1(\Sigma) \leq \text{sign}(\Sigma) \leq b_1(\Sigma)$ . A surface  $\Sigma$  is said to have *maximal signature* if  $\text{sign}(\Sigma) = b_1(\Sigma)$  holds. Having a maximal signature is a property that is preserved under going to a surface minor, since every restriction of a positive definite quadratic form is positive definite. Hence Theorem A implies that the set of fiber surfaces of divides whose signature is maximal must be characterized by a finite number of prohibited minors. Here, we determine these minors, which turn out to be the fiber surfaces of the two divides depicted in Figure 0.1. Actually, we prove a stronger result, namely we give a complete list of the fiber surfaces of divides with maximal signature.

**Theorem C.** *The fiber surface  $\Sigma(P)$  associated with a divide  $P$  has maximal signature if and only if it contains neither  $\Sigma(Q)$  nor  $\Sigma(X)$  as a surface minor, where  $Q$  and  $X$  are as in Figure 0.1. When this holds, the divide link associated with  $P$  is isotopic to the link of one of the divides depicted in Figure 0.2. These divides are described by the undirected Dynkin diagrams, except  $F_4$ .*

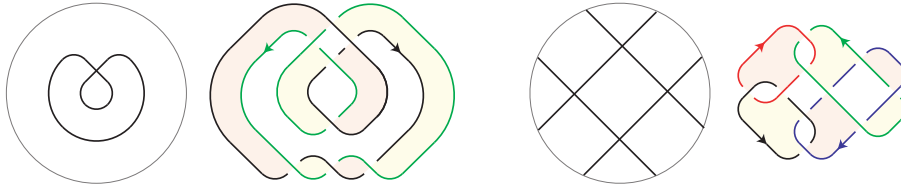


FIGURE 0.1. The divides  $Q$  and  $X$ , and the associated links:  $L(Q)$  is an oriented 2-component link (whose unoriented version is called L6a2 in Thistlethwaite's table), and  $L(X)$  is an oriented 4-component link (L8a21). The canonical fiber surfaces  $\Sigma(Q)$  and  $\Sigma(X)$  are obtained by applying the Seifert algorithm to the depicted diagrams.

Theorem C can be compared with the results of [4], that consist of a similar description of surfaces with maximal signature for another natural class, namely all fiber surfaces of positive

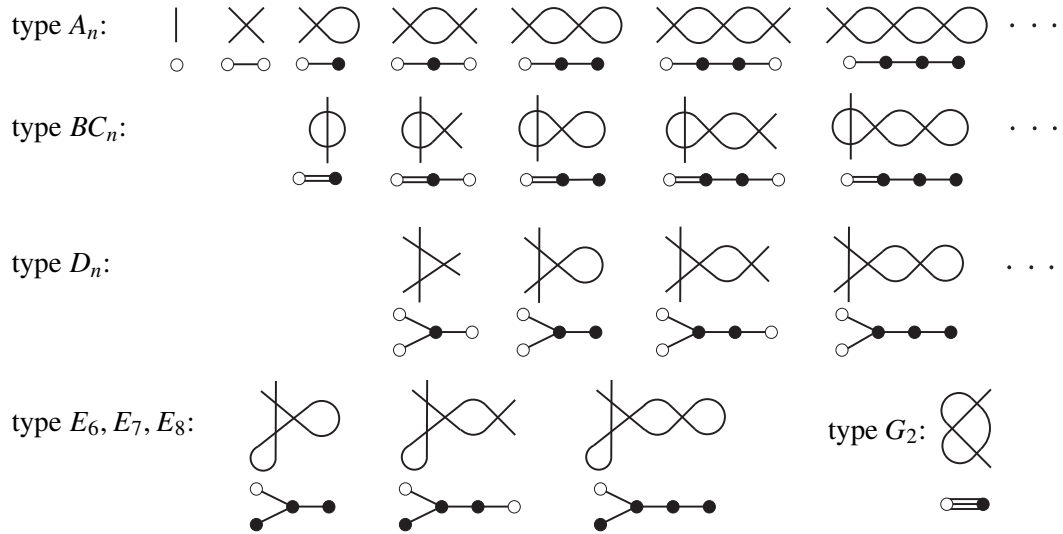


FIGURE 0.2. The list of all connected divide links whose signature is maximal. Their codes (see Section 1.d) correspond to unoriented finite Dynkin diagrams, except  $F_4$ . The coding is not injective, since  $BC_{2n}$  and  $D_{2n}$  are connected by a sequence of Reidemeister III moves which do not alter the associated link. The same holds for  $E_6$  and  $G_2$ . Except for  $BC_{2n}$ , this list corresponds to simple germs of holomorphic functions (see Arnold [3]).

braids. The statements are very similar to Theorem C, with the difference that here we have two prohibited minors instead of four (this can be informally explained by the absence of an analog to  $Q$  in the world of positive braids), and we have slightly more fiber surfaces with maximal signature: the list of Figure 0.2 is the list of [4], enriched with the family  $BC_{2n+1}$  (the divides  $BC_{2n}$  do not count, as they describe the same link as  $D_{2n}$ ).

The idea for reducing Theorem A to Proposition B is to encode fiber surfaces of divides using some specific colored graphs embedded into the plane, called *discal graphs*. The key observation is that surface minority for the fiber surfaces of divides refines planar minority for the corresponding discal graphs.

For proving Proposition B, we introduce a faithful coding of graphs embedded into the plane by binary trees whose leaves are labelled with 3-connected planar graphs. By a theorem of Whitney [17], 3-connected planar graphs admit a unique embedding into the sphere, up to orientation. We then combine the instance of the Robertson-Seymour Theorem that involves planar graphs [14] with a variant of the Kruskal Tree Theorem, called Lemma on Trees [14].

For proving Theorem C, we make a case-by-case analysis and show that there is a list of six divides such that every divide not in the list of Figure 0.2 contains one of them as a subdivide. We then show that the fiber surfaces associated with these six divides all contain  $\Sigma(Q)$  or  $\Sigma(X)$  as a surface minor, and that these two surfaces have non-maximal signature.

The plan of the paper follows the above scheme: in Section 1, we introduce divides and their fiber surfaces, and we explain how to encode them using colored graphs embedded into the disc so as to reduce the topological problem to a pure question of graph theory. The latter is solved in Section 2 (Proposition B). Finally, signatures are investigated in Section 3, where Theorem C is

established. Let us mention that Sections 1 and 3 (about divides) and 2 (about graph minors) are largely independent and can be read in either order.

We thank Frédéric Mazoit: he explained to us in detail the proof of the Robertson-Seymour Theorem for planar graphs and provided us several references [7, 11]. We also thank Filip Misev for several remarks on a first version of this paper.

## 1. FIBER SURFACES FOR DIVIDES AND SUBSURFACES

Here we introduce divides, the associated links and their canonical fiber surfaces, and we define surface minority, the partial order on isotopy types of surfaces that is our main object of investigation. Next, we show how to encode the fiber surfaces of divide links into specific colored graphs. Finally, we connect the surface minority partial order to another partial order called discal minority which, under the above coding, is a simple purely graph-theoretic counterpart.

**1.a. Divides.** In the whole article,  $D$  is the unit disc in  $\mathbb{R}^2$  and  $\partial D$  is its boundary. A *divide* is a generic immersion of a one-dimensional compact manifold in  $D$ . Formally, this corresponds to

**Definition 1.1** (A'Campo [2]). A *divide* is the image of an immersion of a finite number of intervals and circles into  $D$  such that: (i) there is no triple point, (ii) there is no self-tangency, (iii) all intersections are transverse, (iv) both endpoints of every immersed segment lie on  $\partial D$  and the endpoints of all segments are distinct, (v) immersed circles do not intersect  $\partial D$ .

We denote by  $\mathcal{Div}$  the family of all isotopy classes of connected divides.

Two connected divides are displayed in Figure 0.1:  $Q$  consists of the immersion of one circle with winding number 2, and  $X$  consists of the immersion of four segments. Further examples appear in Figure 0.2 (with the disc  $D$  omitted).

Every divide gives rise to a link. Following A'Campo [2], we define the tangent bundle  $T(D)$  to  $D$  to be the product  $D \times \mathbb{R}^2$ , and the unit sphere  $ST(D)$  in  $T(D)$  to be

$$ST(D) = \{(x_1, x_2, u_1, u_2) \in D \times \mathbb{R}^2 \mid x_1^2 + x_2^2 + u_1^2 + u_2^2 = 1\}.$$

Then  $ST(D)$  is homeomorphic to the solid torus  $D \times \mathbb{S}^1$ , with the boundary 2-torus  $\partial D \times \mathbb{S}^1$  collapsed into a circle  $\partial D \times \{*\}$ . Now, for  $P$  a divide and  $(x_1, x_2)$  a point on  $P$ , denote by  $T_{(x_1, x_2)}P$  the set of vectors based at  $(x_1, x_2)$  that are tangent to  $P$ . Then, the *link*  $L(P)$  of  $P$  is the intersection of the set of tangent vectors to  $P$  with  $ST(D)$ , namely

$$L(P) = \{(x_1, x_2, u_1, u_2) \in ST(D) \mid (x_1, x_2) \in P, (u_1, u_2) \in T_{(x_1, x_2)}P\}.$$

The hypotheses in Definition 1.1 imply that  $L(P)$  is indeed an oriented link, that is, the embedding of a finite number of oriented circles into  $ST(D)$ . It consists of two points in the fiber of every regular point of  $P$ , of four points in the fiber of every double point, and of the collapsed fibers of the points of  $\partial D$  that are the extremities of the segments of  $P$ . The link  $L(P)$  has one component per segment in  $P$  and two components per circle. Divide links form a large family that in particular includes algebraic links. In general, drawing the link  $L(P)$  from a picture of  $P$  is not obvious. Let us mention that there exist at least two algorithms, one by Chmutov [5] and one by Hirasawa [9]—the one used in Figure 0.1.

**1.b. The fiber surface of a divide.** The central objects in this paper are specific Seifert surfaces associated with divide links. We describe them below. Usually, the definition appeals to Morse functions but, here, as we only need a combinatorial description a simplified version will be sufficient. We then describe a basis for the first homology group of the fiber surface.

If  $P$  is a divide, the regions of  $D \setminus P$  can be colored with two colors, say black and white, so that adjacent regions have different colors. Up to inverting the colors of all regions simultaneously, this coloring is unique (and using either version leads to equivalent results).

**Definition 1.2.** (See Figure 1.3.) Assume that  $P$  is a divide whose complement in  $D$  is colored in black and white. Then the *fiber surface*  $\Sigma(P)$  is the surface that consists of those vectors in  $ST(D)$  that are based at points of  $P$  and are directed into a black region or are tangent to  $P$ , plus all vectors based at all double points of  $P$ .

We denote by  $S\mathcal{D}iv$  the family of all isotopy classes of fiber surfaces of connected divides.

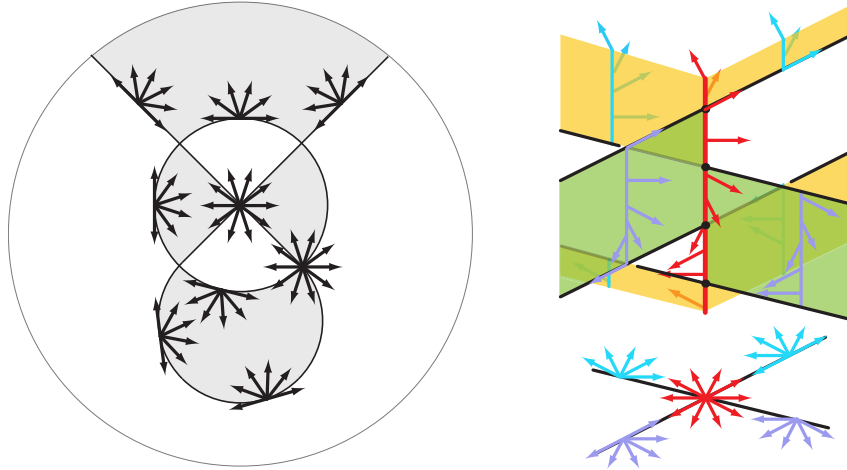


FIGURE 1.3. On the left, a divide whose complement is colored in black and white. The corresponding fiber surface is the collection of all displayed tangent vectors in the tangent bundle. On the right, the fiber surface around the fiber of a double point. The vertical direction corresponds to the direction of the fiber  $\mathbb{S}^1$  (the top has to be identified with the bottom).

The fiber surface of a divide  $P$  consists of one segment in the fiber of every regular point of  $P$ , plus the whole fiber of every double point, plus the collapsed fibers of the extremities of the segments of  $P$ . It is a topological surface; it is not smooth along the fibers of the double points of  $P$ , but this is of no importance for our purpose. Its boundary is the link  $L(P)$ .

It follows from the definition that if two divides are isotopic as divides, the corresponding fiber surfaces are isotopic in  $\mathbb{S}^3$ , so that the map  $P \mapsto \Sigma(P)$  induces a map from  $\mathcal{D}iv$  to  $S\mathcal{D}iv$ , also denoted by  $\Sigma$ . The latter map is not injective, as for example two divides that are related by a Reidemeister move of type III are not isotopic as divides, but the associated fiber surfaces are isotopic in  $\mathbb{S}^3$  (see examples in the proof of Theorem C below).

Considering only connected divides will not reduce the range of our result, but will help to avoid technicalities. In particular it permits to speak of first homology group and Betti number,

which is useful for checking incompressibility. In our context, connectivity of the fiber surface corresponds to connectivity of the underlying divide. For  $P$  a divide, we shall write  $X(P)$  for the set of its double points and  $O(P)$  for the set of its *inner faces*, that is, the regions in  $D \setminus P$  that do not touch  $\partial D$ .

**Lemma 1.3.** *With the above notation, the first Betti number  $b_1(\Sigma(P))$  of the fiber surface of a connected divide  $P$  is  $|X(P)| + |O(P)|$ .*

For example, for  $P$  the divide depicted in Figure 1.3, we have  $|X(P)| = 5$  and  $|O(P)| = 5$ , therefore  $b_1(\Sigma(P)) = 10$ .

*Proof.* Consider  $P$  as a planar graph whose vertices are double points of  $P$  and points of  $P \cap \partial D$ . We may suppose that no segment of  $P$  connects two points of  $P \cap \partial D$ , since in this case the associated connected component of  $\Sigma(P)$  is a disc, and does not contribute to the first Betti number. Write  $e_P$  for the number of segments that connect two double points of  $P$ , and  $e'_P$  for the number of segments that connect one double point to one point of  $P \cap \partial D$  (for example,  $e_P = 9$  and  $e'_P = 2$  in Figure 1.3). Then the surface  $\Sigma(P)$  consists of  $e_P$  rectangles and  $e'_P$  triangles, all glued along the fibers of the double points of  $P$  as in Figure 1.3. The contribution of every rectangle to the Euler characteristic of  $\Sigma(P)$  is  $-1$  and the contribution of every triangle is  $0$ , yielding  $\chi(\Sigma(P)) = -e_P$ .

Now, the divide  $P$ , seen as a planar graph, has  $|X(P)| + e'_P$  vertices,  $e_P + e'_P$  edges, and  $|O(P)|$  inner faces. Since it is a planar graph, its Euler characteristic is  $1$ , yielding  $(|X(P)| + e'_P) - (e_P + e'_P) + |O(P)| = 1$ . We then get  $e_P = |X(P)| + |O(P)| - 1$ . Since the first Betti number of a surface with boundary is  $1$  minus its Euler characteristic, we deduce  $b_1(\Sigma(P)) = |X(P)| + |O(P)|$ .  $\square$

For computing the signature of the fiber surface of a divide in Section 3, we will need an explicit basis of its first homology group. Lemma 1.3 suggests that there might exist a natural basis in terms of double points and inner faces of the divide. As shown by Ishikawa [10], this is indeed the case. For  $P$  a divide and  $p$  a double point of  $P$ , denote by  $S_p$  the curve that describes the oriented fiber of  $p$  in  $ST(D)$  (the red curve in Figure 1.3 right). We recall that the regions of the complement of  $P$  are assumed to be colored in black and white. For  $q$  an inner face of  $P$ , let  $C_q$  be a lift of the boundary of  $q$ , travelled clockwise, that always points inside  $q$  if  $q$  is black and always points outside if  $q$  is white (see Figure 1.4). By definition, these  $|X(P)| + |O(P)|$  curves lie in the fiber surface  $\Sigma(P)$ . A *positive Hopf band* is an annulus in  $\mathbb{S}^3$  with writhe  $+2$  (see Figure 1.4 right).

**Proposition 1.4** (Ishikawa). *Assume that  $P$  is a connected divide. Then its fiber surface  $\Sigma(P)$  is the iterated plumbing of  $|X(P)| + |O(P)|$  positive Hopf bands. The cores of the bands are the curves  $S_p$  for  $p$  in  $X(P)$  and  $C_q$  for  $q$  in  $O(P)$ .*

We refer to Ishikawa [10] for the definitions of plumbing and for the proof. It is then easy to derive the expected basis for the homology of the fiber surface of a divide.

**Corollary 1.5.** *With the above notations, the classes  $\{[S_p]\}_{p \in X(P)} \cup \{[C_q]\}_{q \in O(P)}$  form a basis of  $H_1(\Sigma(P), \mathbb{Z})$ .*

**1.c. Subsurfaces and subdivides.** We now introduce two partial orders  $\leq^{surf}$  and  $\leq^{div}$  on the sets  $\mathcal{SDiv}$  and  $\mathcal{Div}$  and compare them. A subsurface  $\Sigma$  in a surface  $\Sigma'$  is *incompressible in  $\Sigma'$*  if it is  $\pi_1$ -injective. In particular, this means that no boundary component of  $\Sigma$  bounds a disc in  $\Sigma' \setminus \Sigma$ .

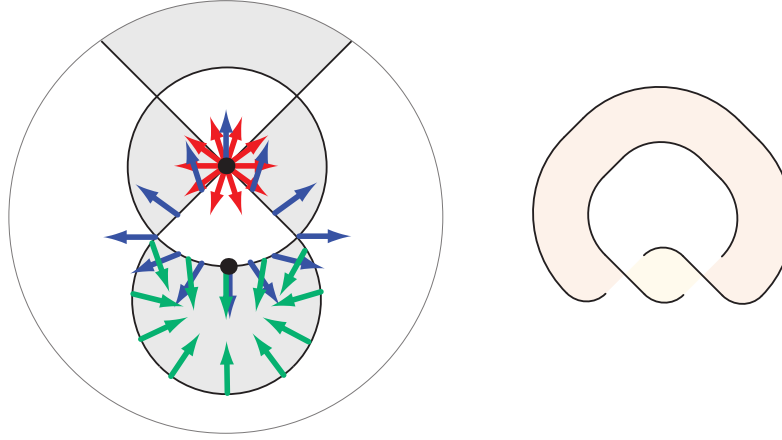


FIGURE 1.4. On the left, the curves  $S_p$  (red) and  $C_q$  (blue and green) on the fiber surface of a divide. In Lemma 3.3, we will use the fact that the lifts of the blue and green curves intersect once every adjacent red curve, and that the lifts of a blue curve and of a green curve intersect once per common side of the corresponding regions of  $D \setminus P$ . On the right, a positive Hopf band.

**Definition 1.6.** We say that (the isotopy class of) a surface  $\Sigma$  embedded in  $\mathbb{R}^3$  is a *surface minor* of (the isotopy class of) another surface  $\Sigma'$  if  $\Sigma$  is isotopic to an incompressible subsurface of  $\Sigma'$ . We then write  $\Sigma \leq^{surf} \Sigma'$ .

Surface minority is clearly a reflexive and transitive relation. The incompressibility condition implies that, if  $\Sigma \leq^{surf} \Sigma'$  holds, then the first Betti number  $b_1(\Sigma)$  is not larger than  $b_1(\Sigma')$ , implying that the relation is also antisymmetric. Hence  $\leq^{surf}$  is a partial order on isotopy classes of embedded surfaces. However, it is a weak relation, in the sense that most surfaces are incomparable, like the annuli  $A_n$  mentioned in the introduction.

From now on, our aim is to investigate the restriction of the partial order  $\leq^{surf}$  to the family  $\mathcal{SDiv}$ . We shall exploit the peculiarities of divides. The first step consists in introducing a new partial order on divides based on desingularization of double points.

Given a connected divide  $P$  and a double point  $p$  of  $P$ , there are two natural ways of desingularizing  $P$  around  $p$ , that consist in replacing the ends of the four segments of  $P$  ending at  $p$  by two arcs that connect them, see Figure 1.5. Then, two of the faces adjacent to  $p$  and that are of the same color are merged into a single one. We denote by  $P_p^\bullet$  the divide obtained by joining the two black faces adjacent to  $p$  and by  $P_p^\circ$  the divide obtained by joining the two white faces. If  $p$  is a separating vertex in  $P$ , then either  $P_p^\bullet$  or  $P_p^\circ$  is not connected. In this case, the desingularization is allowed only if one of the two components consists of one single segment, and  $P_p^\bullet$  or  $P_p^\circ$  denotes the other connected component.

**Definition 1.7.** We say that a connected divide  $P'$  is a *subdivide* of  $P$ , if  $P'$  can be obtained from  $P$  by several desingularizations and isotopies. We then write  $P' \leq^{div} P$ .

We consider the relation induced by  $\leq^{div}$  on the set  $\mathcal{Div}$ . It is clearly reflexive and transitive, and, since the number of double-points decreases when one of them is desingularized, it is also



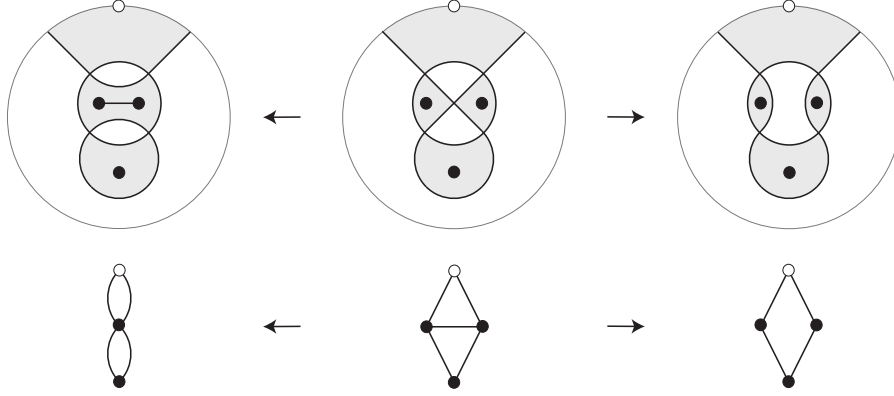


FIGURE 1.5. The two ways of desingularizing a double point of a divide  $P$ , and the corresponding operations on discal graphs (Section 1.d). The divide  $P_p^\bullet$  and its code are depicted on the left, whereas  $P_p^\circ$  and its code are on the right.

antisymmetric. Hence it induces an order on  $\mathcal{Div}$ . The connection between  $\leq^{surf}$  and  $\leq^{plan}$  comes from the following

**Lemma 1.8.** *Assume that a divide  $P'$  is a subdivide of a divide  $P$ . Then the surface  $\Sigma(P')$  is a surface minor of  $\Sigma(P)$ .*

*Proof.* Let  $p$  be a non-separating double point of  $P$ . Then the fiber surface  $\Sigma(P_p^\bullet)$  associated with  $P_p^\bullet$  is obtained from  $\Sigma(P)$  by cutting two segments in the fiber of  $p$  (see Figure 1.6). On the other hand,  $P_p^\bullet$  has one double-point and one inner region less than  $P$ , so that we have  $b_1(\Sigma(P_p^\bullet)) = b_1(\Sigma(P)) - 2$ . Then, the two cut segments are essential in  $\Sigma(P)$ , and therefore  $\Sigma(P_p^\bullet)$  is incompressible in  $\Sigma(P)$ . If  $p$  separates  $P$ , then the second cut disconnects  $\Sigma(P)$ , so that only the first cut is essential. But in this case, we have  $b_1(\Sigma(P_p^\bullet)) = b_1(\Sigma(P)) - 1$ , so that  $\Sigma(P_p^\bullet)$  is also incompressible in  $\Sigma(P)$ . By the same arguments,  $\Sigma(P_p^\circ)$  is incompressible in  $\Sigma(P)$ .  $\square$

Lemma 1.8 means that the order  $\leq^{surf}$  on fiber surfaces is a refinement of the order  $\leq^{div}$  on divides. This refinement is strict, as for example two divides that are related by a Reidemeister move of type III are incomparable under  $\leq^{div}$  although the associated fiber surfaces are isotopic.

**1.d. Coding divides by embedded graphs.** We now explain how to encode a divide using a (colored) graph embedded in the disc  $D$ . This encoding will be the main tool for establishing that the surface minor order is a well-quasi-order.

**Definition 1.9.** A *discal graph* is a graph embedded into  $D$  such that the edges intersect  $\partial D$  in vertices only, and whose vertices are colored as follows: the vertices lying in the interior of  $D$  are black, those lying on  $\partial D$  are white. If there is exactly one vertex on  $\partial D$ , we can also draw a star instead of coloring in white.

We denote by  $\mathcal{G}^D$  the family of all isotopy classes of discal graphs.

**Definition 1.10.** Assume that  $P$  is a connected divide. Then the *code* of  $P$  is the discal graph  $G(P)$  obtained by taking one vertex in every black region of  $D \setminus P$  and by putting at every double point of  $P$  an edge that connects the two vertices associated with the two adjacent black regions. If  $P$

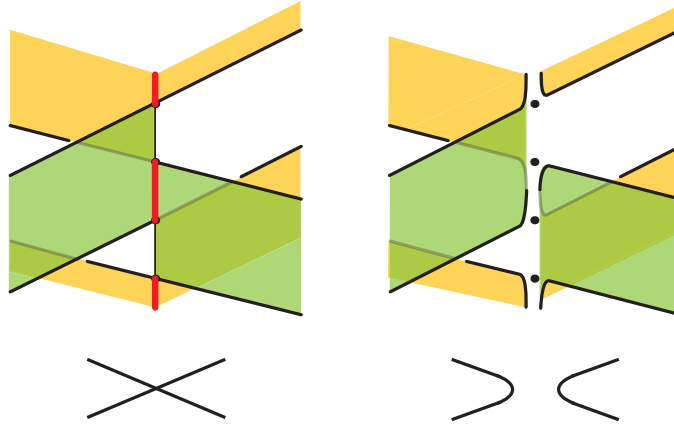


FIGURE 1.6. Desingularizing a double point in a divide amounts to cutting the fiber surface along two segments.

does not intersect  $\partial D$  at all, and if the region of  $D \setminus P$  touching  $\partial D$  is black, then we put only one vertex on  $\partial D$ , and color it with a star. For  $p$  a double point of  $P$ , the corresponding edge is denoted by  $e(p)$ .

Several examples of codes of divides are depicted in Figures 1.5, 0.2, 3.10 and 3.9. It is easily seen that exchanging the colors of the regions of  $D \setminus P$  replaces the code by its dual in the disc. It follows that every divide has two codes which are dual planar graphs.

The interesting property of the coding is that it behaves nicely with respect to divide minors. Let  $G$  be a discal graph and  $e$  be an edge of  $G$ . If  $e$  does not disconnect  $G$  or has one end which is a degree one white vertex, then we denote by  $G - e$  the discal graph obtained from  $G$  by deleting  $e$  (see the bottom right picture in Figure 1.5). If  $e$  is not a loop in  $G$ , then we denote by  $G/e$  the graph obtained by contracting  $e$  in  $G$  (bottom left picture). In the latter case, the vertex that corresponds to the collapsed edge  $e$  is black if the two ends of  $e$  were black, white if at least one of the ends was white, and a star if at least one of the ends was a star.

**Lemma 1.11.** *Assume that  $P$  a connected divide, and that  $G(P)$  is a code for  $P$ . Let  $p$  be a double point of  $P$ . If  $P_p^\circ$  is defined, then so is the discal graph  $G(P) - e(p)$ , and it is the code of  $P_p^\circ$ . Similarly, if  $P_p^\bullet$  is defined, then so is  $G(P)/e(p)$ , and it is the code of  $P_p^\bullet$ .*

*Proof.* For  $P_p^\circ$ , the two adjacent black faces are not any more adjacent at  $p$ , so that  $e(p)$  should be removed. The rest of  $G$  is unchanged. For  $P_p^\bullet$ , the two adjacent black faces have been united, so that the two corresponding vertices of  $G(P)$  should also be united. If one of the two faces is external, then their union is external; if there was only one external face before, there is also only one after, hence the coloring.  $\square$

In the line of the previous lemma, we now introduce a partial order on discal graphs that corresponds to divide minority, hence refines surface minority.

**Definition 1.12.** For  $G, G'$  two discal graphs, we say that  $G$  is a *discal minor* of  $G'$  if  $G$  can be obtained from  $G'$  by repeatedly choosing one edge  $e$  and replacing  $G$  by  $G - e$  or by  $G/e$ , if allowed. We then write  $G \leq^D G'$ .

We consider this relation for discal graphs only, which means that the white vertices should always lie on the exterior face, and that there can be a star only if there is no white vertex. It induces a partial order on  $\mathcal{G}^D$ . The restrictions in the definition of discal minority actually imply that the order  $\leq^D$  on  $\mathcal{G}^D$  is the exact counterpart of the order  $\leq^{div}$  on  $\mathcal{Div}$ , but we do not need this result. What we need is the following straightforward consequence of Lemmas 1.8 and 1.11 which reduces the proof of Theorem A to Proposition B.

**Corollary 1.13.** *Suppose that  $P, P'$  are connected divides with codes  $G(P), G(P')$ . If  $G(P')$  is a discal minor of  $G(P)$ , then the fiber surface  $\Sigma(P')$  is a surface minor of  $\Sigma(P)$ .*

## 2. A ROBERTSON-SEYMOUR TYPE RESULT FOR GRAPHS EMBEDDED INTO THE PLANE

The aim of this graph-theoretic section is to show that planar minority for colored planar graphs is a well-quasi-order (Proposition B). Merging this result with those of Section 1, we will then easily complete the proof of Theorem A.

**2.a. Planar minors.** An *embedded graph colored with a set  $C$*  is a graph embedded into the plane  $\mathbb{R}^2$ , whose vertices are labelled with elements of  $C$ . We denote by  $\mathcal{G}_C^{\mathbb{R}^2}$  the set of isotopy classes of embedded graphs colored with  $C$ . For  $G$  an embedded graph, we denote by  $G^-$  the associated abstract graph.

**Definition 2.1.** Assume that  $G$  and  $H$  are embedded colored graphs and that  $C$  is an ordered set. We say that  $H$  is a *planar minor* of  $G$ , written  $H \leq_C^{plan} G$ , if  $H$  can be obtained from  $G$  by repeatedly applying the following operations:

- (i) decreasing the color of a vertex,
- (ii) deleting an edge,
- (iii) contracting an edge whose ends have the same color,
- (iv) deleting an isolated vertex.

The relation  $\leq_C^{plan}$  is a partial order on (isotopy classes of) embedded colored graphs. The difference between the classical *graph minority* relation and the current planar minority relation lies in that, in the latter case, graphs are equipped with a specific embedding in the plane and that the minor relation has to preserve this embedding: if an embedded graph  $H$  is a planar minor of  $G$ , then  $H^-$  is a graph minor of  $G^-$ , but the converse is not true in general.

Two particular cases of Proposition B are already known. The first one is the case of embedded trees, which is known as the Kruskal Tree Theorem [13]. The second one is the case of 3-connected graphs. Indeed, by a theorem of Whitney [17], these graphs embed uniquely into the plane (up to a choice of orientation and of exterior face), so that Proposition B is nothing but the Robertson-Seymour Theorem for (non-embedded) planar 3-connected colored graphs. The latter is not as difficult as the general theorem involving arbitrary graphs; it is the subject of the four first papers [14] of the celebrated series *Graph minors* of 23 articles [15]. It has also been rewritten and simplified by several authors [6, 7].

The idea for proving Proposition B will be to combine the two previously known cases by using a decomposition result of Tutte [16] stating that every planar graph can be cut along vertices or pairs of vertices into finitely many 3-connected graphs, so that the gluing pattern can be described by a certain tree. Moreover, in the case of an embedded graph, the embedding into the plane is governed by the embedding of the corresponding tree into the plane. Using this approach,

we associate with every embedded graph an embedded tree whose leaves are colored with 3-connected graphs. Now, the Robertson-Seymour Theorem says that the colors of the leaves, that is, 3-connected planar graphs, are quasi-well-ordered, and a variant of the Kruskal Tree Theorem then implies that the above considered embedded trees are well-quasi-ordered, implying in turn that the embedded graphs we consider are well-quasi-ordered.

The rest of the section details the above arguments.

**2.b. Tutte-Mazoit trees.** Following Mazoit [11], we associate with every embedded colored graph a tree that encodes its embedding into the plane. For  $G$  a colored graph embedded in the plane or in a disc, the *planar decoration*  $G^*$  of  $G$  is the graph obtained from  $G$  by adding a special oriented label on the edges of  $G$  which are adjacent to the exterior face, so that all exterior edges are oriented in the clockwise direction. A graph is *k-connected* if removing any set of less than  $k$  vertices, and the adjacent edges, yields a graph that is still connected.

**Definition 2.2.** (See Figure 2.7.) Let  $G$  be an embedded colored graph. A *Tutte-Mazoit tree* for  $G$  is an abstract binary tree  $T_G$ , such that every node  $v$  is labelled with (a) a disc  $D(v)$  in the plane whose boundary  $\partial D(v)$  intersects  $G$  in some vertices only, and (b) the planar decoration  $G(v)^*$  of the restriction  $G(v)$  of  $G$  to  $D(v)$ , and such that

- (i) for every leaf  $v$  of  $T_G$ , the graph  $G(v)$  is 3-connected,
- (ii) for every internal node  $v$  with sons  $v_1, v_2$ , the disc  $D(v)$  is the union of  $D(v_1)$  and  $D(v_2)$ ,
- (iii) denoting by  $r(T_G)$  the root of  $T_G$ , the graph  $G(r(T_G))$  equals the whole graph  $G$  (or, equivalently, the disc  $D(r(T_G))$  contains all of  $G$ ).

Given an embedded planar graph, it is easy to construct an associated Tutte-Mazoit tree by repeatedly cutting along separating vertices and pairs of vertices, until the remaining blocks are 3-connected. Note that a graph can admit different Tutte-Mazoit trees, which differ by the choice of the discs and the order in which they are glued. The next result states that planar minors behave nicely with respect to Tutte-Mazoit trees.

**Lemma 2.3.** Assume that  $G$  and  $H$  are two embedded colored graphs, that  $T_G$  and  $T_H$  are two associated Tutte-Mazoit trees, that  $u$  is an internal node in  $T_G$  with sons  $u_1, u_2$ , and that  $v$  is an internal node in  $T_H$  with sons  $v_1, v_2$ . If  $H(v_1) \leq_C^{plan} G(u_1)$  and  $H(v_2) \leq_C^{plan} G(u_2)$  hold, then so does  $H(v) \leq_C^{plan} G(u)$ .

*Proof.* Perform the reductions of  $G(u_1)$  to  $H(v_1)$  inside  $D(u_1)$  and of  $G(u_2)$  to  $H(v_2)$  inside  $D(u_2)$ , keeping on  $\partial D(u_1)$  (resp.  $\partial D(u_2)$ ) the vertices that belong to  $\partial D(u_1)$  (resp.  $\partial D(u_2)$ ). By gluing the two discs  $D(u_1)$  and  $D(u_2)$ , and identifying the vertices on the boundaries that are in  $G(u_1) \cap G(u_2)$ , we obtain a reduction from  $G(u_1) \cup G(u_2)$  to  $H(v_1) \cup H(v_2)$ , implying that  $H(v)$  is a planar minor of  $G(u)$ .  $\square$

**2.c. Proof of Proposition B.** The following result is the variant of the Kruskal Tree Theorem that we need. Roughly speaking, it states that, in a sequence of finite trees with an order relation on the nodes that is compatible with the tree structure, if there exists an infinite antichain, then there exists a minimal one, that is, an antichain  $X$  such that every set above  $X$  is not an antichain.

**Lemma 2.4** (Lemma on trees [14]). Assume that  $T_1, T_2, \dots$  is an infinite sequence of rooted trees. Assume that  $\leq$  is a partial order on the set  $W = V(T_1) \cup V(T_2) \cup \dots$  of all nodes, such that for two nodes  $u$  in  $V(T_i)$  and  $v$  in  $V(T_j)$  with  $i < j$ , if  $u \leq v$  holds, then  $u \leq w$  holds for every node  $w$  in  $V(T_j)$

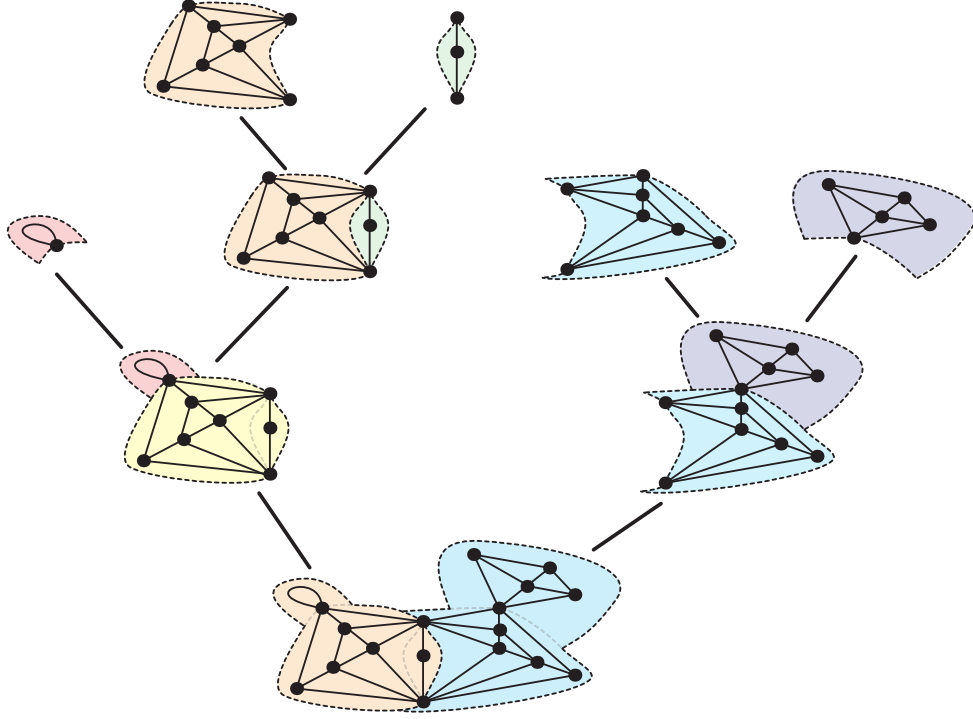


FIGURE 2.7. A Tutte-Mazoit tree.

that lies on the path between  $v$  and  $r(T_j)$ . Finally assume that the set of roots  $r(T_1), r(T_2), \dots$  is an infinite antichain for  $\preceq$ . Then there exists an infinite antichain  $X$  in  $W$  for  $\preceq$ , with  $|X \cap V(T_i)| \leq 1$  for all  $i$ , such that the set  $Y$  of all sons of elements of  $X$  contains no infinite antichain for  $\preceq$ .

We refer to the article [14] by Robertson and Seymour for the proof; it follows the scheme of Nash-Williams' proof of the Kruskal Tree Theorem [13]. We can now complete the argument.

*Proof of Proposition B.* The proof is by contradiction. Suppose that  $G_1, G_2, \dots$  is an infinite antichain for the planar minor order. Let  $T_{G_1}, T_{G_2}, \dots$  be associated Tutte-Mazoit trees. Let  $\preceq_C^{plan}$  be the partial order on the set  $V(T_{G_1}) \cup V(T_{G_2}) \cup \dots$  defined as follows: for  $u$  a node of  $T_{G_i}$  and  $v$  a node of  $T_{G_j}$ ,  $u \preceq_C^{plan} v$  holds if and only if  $G_i(u)$  is a planar minor of  $G_j(v)$ . By hypothesis, the set of roots  $r(T_{G_1}), r(T_{G_2}), \dots$  is an infinite antichain for  $\preceq_C^{plan}$ . Also, for  $w$  an ancestor of  $v$  in the corresponding tree,  $u \preceq_C^{plan} v$  implies  $u \preceq_C^{plan} w$ . Thus, the hypotheses of Lemma 2.4 are satisfied. Therefore, there exists an infinite set  $X$  in  $V(T_1) \cup V(T_2) \cup \dots$  which is an antichain and such that the set  $Y$  of all sons of the elements of  $X$  contains no infinite antichain.

Suppose first that  $Y$  is finite. Then infinitely many elements of  $X$  are leaves in their respective trees. By the Robertson-Seymour Theorem [14], there exist  $i < j$ ,  $u$  a leaf of  $T_{G_i}$  and  $v$  a leaf of  $T_{G_j}$  such that  $G_i(u)^-$  is a graph minor of  $G_j(v)^-$ . Since  $u$  and  $v$  are leaves,  $G_i(u)^-$  and  $G_j(v)^-$  are 3-connected planar graphs. Therefore, by Whitney's Theorem [17], their embeddings as  $G_i(u)$  and  $G_j(v)$  are fixed by the decorations  $G_i(u)^*$  and  $G_j(v)^*$ . Then we also have  $G_i(u) \preceq_C^{plan} G_j(v)$ , a contradiction.

Suppose now that  $Y$  is infinite. By hypothesis  $Y$  contains no infinite antichain. Let  $Y^{(2)}$  denote the set of pairs of sons of all elements in  $X$ . Then, by Higman's Lemma [8],  $Y^{(2)}$  also contains no infinite antichain. That means that there exist  $i < j$ , a node  $u$  in  $T_{G_i} \cap X$  with sons  $u_1, u_2$  and a node  $v$  in  $T_{G_j} \cap X$  with sons  $v_1, v_2$  satisfying  $u_1 \preceq_C^{plan} v_1$  and  $u_2 \preceq_C^{plan} v_2$ . In other words,  $G_i(u_1)$  is a planar minor of  $G_j(v_1)$ , and  $G_i(u_2)$  is a planar minor of  $G_j(v_2)$ . Then, by Lemma 2.3,  $G_i(u)$  is a planar minor of  $G_j(v)$ , a contradiction again.  $\square$

**2.d. Application to fiber surfaces of divides.** Returning to fiber surfaces of divides and combining Corollary 1.13 and Proposition B, we immediately deduce Theorem A, that is, the result that surface minority is a well-quasi-order on the family  $\mathcal{SDiv}$  of fiber surfaces of divides.

*Proof of Theorem A.* First, surface minority admits no infinite descending chain. Indeed, taking an incompressible strict subsurface decreases the first Betti number. Next, no infinite antichain of  $\preceq^{surf}$  may be included in  $\mathcal{SDiv}$ . Indeed, by Corollary 1.13, the codes of the involved surfaces would provide an infinite antichain in the set of discal graphs equipped with  $\preceq^D$ . Now  $\preceq^D$  is the restriction to the class  $\mathcal{G}^D$  of the order  $\preceq_C^{plan}$  on colored graphs embedded into the plane, where  $C$  is the ordered set  $\star < \circ < \bullet$ . Then, by Proposition B,  $(\mathcal{G}^D, \preceq^D)$  contains no infinite antichain.  $\square$

### 3. SIGNATURES OF DIVIDES

In this section, we prove Theorem C about the fiber surfaces of divides whose signature is maximal. Since it is easier to enumerate the minors of a graph than the minors of a surface, the proof will deal with divides and their codes rather than with fiber surfaces. This is the reason why we will first prove a version of Theorem C dealing with divides.

**3.a. An explicit version of Theorem C.** Denote by  $Q$  the divide whose code consists of a cycle with a black vertex, and by  $X$  the divide whose code is a cross with four edges, one black vertex in the center, and four white vertices on the border (see Figure 0.1). For  $p, q, r$  positive integers, we denote by  $E_{p,q,r}$  the divide coded by a tree with three branches with respective lengths  $\lfloor (p+1)/2 \rfloor$ ,  $\lfloor (q+1)/2 \rfloor$  and  $\lfloor (r+1)/2 \rfloor$ , and a black or white vertex at the end depending on whether  $p$  (resp.  $q$ , resp.  $r$ ) is even or odd (see Figure 3.8). Finally, for  $p, q$  positive, we denote by  $F_{p,q}$  the divide coded by a triangle with two black and one white vertices, and two branches of respective lengths  $p-2$  and  $q-2$  (in the sense of the definition of  $E_{p,q,r}$ ) starting from the two black vertices (see Figure 3.8).

Theorem C then essentially reduces to the following result, which is the goal of this section.

**Proposition 3.1.** *Assume that  $P$  is a divide. Then the signature of  $\Sigma(P)$  is maximal if and only if  $P$  does not contain  $Q, X, E_{2,2,2}, E_{1,3,3}, E_{1,2,5}$  or  $F_{2,5}$  as a subdivide (see Figure 3.8).*

The strategy for the proof is to first show that the fiber surfaces corresponding to the given divides do not have maximal signature. In order to do this, we introduce in the next subsection a simple algorithm for computing the signature of a divide link (Lemma 3.3). Then, we list the divides that do contain none of the prohibited divides (this step requires some care, but is not difficult) and check that their signatures are maximal. This is done in the third subsection. Finally we deduce Theorem C from Proposition 3.1.

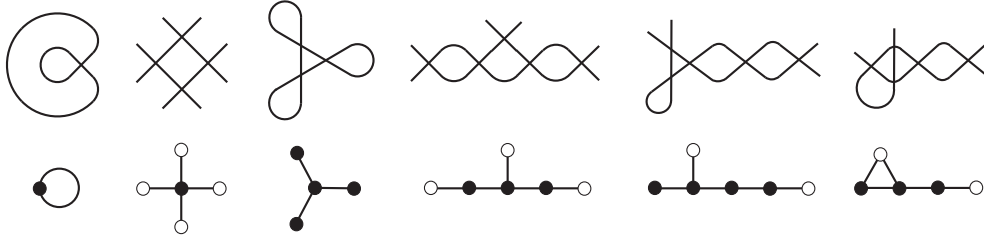


FIGURE 3.8. The prohibited divides  $Q, X, E_{2,2,2}, E_{1,3,3}, E_{1,2,5}$  and  $F_{2,5}$ , and their codes.

**3.b. Computing the signature of the fiber surface of a divide.** Here we recall the definition of the signature of a surface in  $\mathbb{R}^3$ , and give a simple algorithm for computing the signature of the fiber surface associated with any divide, starting from a code of the divide.

Let  $\Sigma$  be an oriented embedded surface in  $\mathbb{R}^3$  and  $\{b_1, \dots, b_n\}$  be a family of curves on  $\Sigma$  that induces a basis of  $H_1(\Sigma, \mathbb{Z})$ . The associated Seifert matrix is the  $n \times n$  matrix  $S$  defined by  $S_{i,j} = \text{lk}(b_i, b_j^+)$ , where  $b_j^+$  denotes a loop in  $\mathbb{R}^3$  obtained by slightly pushing  $b_j$  out of  $\Sigma$  in the positive direction, and  $\text{lk}$  denotes the usual linking number. It turns out that the signature of the symmetric matrix  $S + {}^t S$  does not depend on the family  $\{b_1, \dots, b_n\}$ , but on  $\Sigma$  only (as explained in Murasugi's book [12]), and actually on the link  $\partial\Sigma$  only.

**Definition 3.2.** For  $\Sigma$  an oriented surface in  $\mathbb{R}^3$ , the *signature* of  $\Sigma$  is the signature of any symmetrized Seifert matrix  $S + {}^t S$ .

Assume that  $G$  is a discal graph, that its black vertices are called  $v_1, \dots, v_a$ , its edges  $e_{a+1}, \dots, e_{a+b}$ , and its faces  $f_{a+b+1}, \dots, f_{a+b+c}$ . We say that a vertex and an edge are *adjacent* if the vertex is an extremity of the edge, and that they are *twice adjacent* if the edge is a loop based at the vertex. Similarly, we say that an edge and a face are adjacent (*resp.* twice adjacent) if the edge appears once (*resp.* twice) on the boundary of the face. We say that a vertex and a face are  $k$  times adjacent if the vertex appears  $k$  times on the boundary of the face. By convention, two elements that are not adjacent are 0 time adjacent. We define the *symmetrized Seifert matrix* of  $G$  as follows:  $S(G)$  is a square matrix of size  $a+b+c$  whose rows and columns correspond to the black vertices, the edges, and the faces of  $G$ ; all diagonal coefficients are equal to 2; for  $i \neq j$ , there is a  $k$  in position  $(i, j)$  if the two corresponding elements of  $G$  are  $k$  times adjacent.

**Lemma 3.3.** Assume that  $P$  is a divide and let  $G(P)$  be a code of  $P$ . Then the symmetrized Seifert matrix of  $G(P)$  coincides, up to permutation of the indices, with the symmetrized Seifert matrix associated with the basis  $\{[S_p]\}_{p \in X(P)} \cup \{[C_q]\}_{q \in O(P)}$  of  $H_1(\Sigma(P), \mathbb{Z})$  given by Corollary 1.5.

*Proof.* Remember from Proposition 1.4 that the edges of  $G(P)$  correspond to the double points of  $P$ , that is here, to loops of the form  $S_p$  for  $p$  in  $X(P)$ , and that the vertices and the faces of  $G(P)$  respectively correspond to the black and to the white faces of  $D \setminus P$ , that is here, to loops of the form  $C_q$  for  $q$  in  $O(P)$ . All these loops have a tubular neighbourhood in  $\Sigma(P)$  that is a positive Hopf band, therefore they have self-linking number 1 (Proposition 1.4 and Figure 1.4). Since we consider the symmetrized matrix, we get coefficients +2 everywhere on the diagonal.

Now, every curve that is a diameter of  $D$  lifts to a 2-sphere in  $ST(D)$ . Then, if the projections of two loops on  $D$  are disjoint, the loops are not linked, so that the only possibility for two distinct loops to yield a non-zero coefficient is to correspond to two adjacent faces  $q, q'$  in  $D \setminus P$ , or to a

face  $q$  and a double point  $p$  that is adjacent to  $q$ . For the matrix  $S(P)$ , this gives all the mentioned coefficients 0.

Next, for every double point  $p$  of  $P$ , there are at most four loops that intersect  $S_p$ , corresponding to the at most four adjacent inner faces of  $D \setminus P$ . For every such face, the corresponding loop intersects  $S_p$  in one point, and the contribution to the linking number is positive. Thus the contribution to the Seifert matrix is  $+1$  per adjacency.

Finally, for every pair of adjacent inner faces  $q, q'$  in  $D \setminus P$ , the corresponding loops intersect once for every common edge. Once again, all these intersection points contribute positively to the linking number, thus the contribution to the Seifert matrix is again  $+1$  per adjacency.  $\square$

**3.c. Prohibited divides for the maximality of the signature.** Call a divide and its associated codes *maximal* if the associated divide surface has maximal signature. Our goal is then to determine all maximal divides. For that, we first need to see which divides of the form  $E_{p,q,r}$  have maximal signature. Remember that the determinant of a surface is the invariant defined as the determinant of any of its symmetrized Seifert matrices (see for example [12]). Remember also that an  $n \times n$  matrix  $M$  has maximal signature if and only if the determinant of every minor of  $M$  (in the sense of linear algebra) is positive.

**Lemma 3.4.** *For all positive integers  $p, q, r$ , we have  $\det(\Sigma(E_{p,q,r})) = -pqr + p + q + r + 2$ .*

*Proof.* Let  $M_p$  denote the  $p \times p$ -matrix whose diagonal coefficients are equal to 2, and whose upper- and lower-diagonal coefficients are equal to 1:  $\begin{bmatrix} 2 & 1 & & \\ 1 & 2 & & \\ & 1 & 2 & \\ & & \ddots & \ddots \\ & & & 1 & 2 \end{bmatrix}$ . By induction we have  $\det(M_p) = p + 1$ . By Lemma 3.3, the symmetrized Seifert matrix of  $\Sigma(E_{p,q,r})$  is the block matrix

$$\begin{bmatrix} 2 & 1 & 0 & \dots & 1 & 0 & \dots & 1 & 0 & \dots \\ 1 & & & & & & & & & \\ 0 & M_p & & & & & & & & \\ \vdots & & & & & & & & & \\ 1 & & & & & & & & & \\ 0 & & & & M_q & & & & & \\ \vdots & & & & & & & & & \\ 1 & & & & & & & & & \\ 0 & & & & & & & & M_r & \\ \vdots & & & & & & & & & \end{bmatrix}.$$

Developping the first line, we get  $\det(\Sigma(E_{p,q,r})) = 2(p+1)(q+1)(r+1) - p(q+1)(r+1) - (p+1)q(r+1) - (p+1)(q+1)r = -pqr + p + q + r + 2$ .  $\square$

**Lemma 3.5.** *A divide of type  $E_{p,q,r}$  is maximal if and only if, up to permutation, we have  $p = q = 1$ , or  $p = 1, q = 2$  and  $r = 2, 3$  or 4.*

*Proof.* The signature of a surface is maximal if and only if the determinant of every minor is positive. Applying this observation to  $\Sigma(E_{p,q,r})$  with the canonical Seifert matrix, and using Lemma 3.4, we deduce that the divides  $E_{1,2,2}, E_{1,2,3}, E_{1,2,4}$ , and  $E_{1,1,r}$ , for every  $r$ , are maximal. Every divide not in this list contains  $E_{2,2,2}, E_{1,3,3}$ , or  $E_{1,2,5}$  as a minor. By Lemma 3.4, we have  $\det(\Sigma(E_{2,2,2})) = \det(\Sigma(E_{1,3,3})) = \det(\Sigma(E_{1,2,5})) = 0$ , so that these divides are not maximal.  $\square$

*Proof of Proposition 3.1.* First we show that  $Q, X, E_{2,2,2}, E_{1,3,3}, E_{1,2,5}$  and  $F_{2,5}$  are not maximal. Lemma 3.5 treats the case of  $E_{2,2,2}, E_{1,3,3}$  and  $E_{1,2,5}$ . By Lemma 3.3, the Seifert matrix of  $\Sigma(Q)$  is



$\begin{bmatrix} 2 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 1 & 2 \end{bmatrix}$ , which has determinant zero, and the Seifert matrix of  $\Sigma(X)$  is  $\begin{bmatrix} 2 & 1 & 1 & 1 \\ 1 & 2 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 1 & 0 & 0 & 2 \end{bmatrix}$ , which also has determinant zero. Finally, Figure 3.9 shows that  $F_{2,5}$  is obtained from  $E_{1,2,5}$  by a Reidemeister III move; this implies that the surfaces  $\Sigma(F_{2,5})$  and  $\Sigma(E_{1,2,5})$  are isotopic, so that  $F_{2,5}$  is not maximal either.

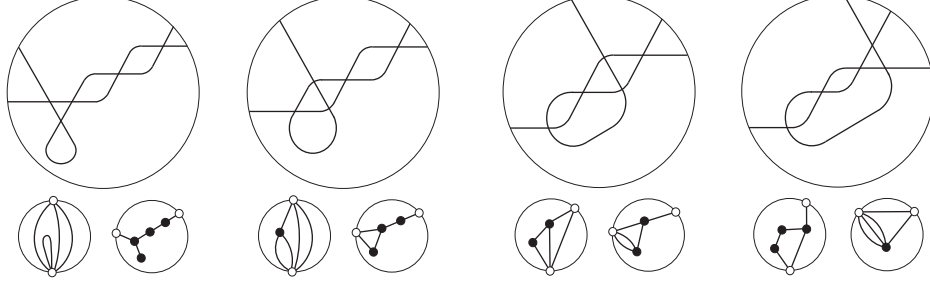


FIGURE 3.9. The divides  $E_{1,2,5}$ ,  $F_{2,5}$ , and the two divides obtained by applying Reidemeister III moves. The two latter have a code containing a cycle of length 4, hence the fiber surfaces contain  $\Sigma(X)$  as a surface minor.

We now want to show that every divide either contain one of the six above mentioned divides or is maximal. The connected sum of two surfaces has maximal signature if and only if the two summands have maximal signature, since both the signature and the first Betti number are additive under connected sum. Now, if the code of a divide has a separating white vertex, then the corresponding fiber surface is a connected sum. Therefore we can restrict our attention to codes with no separating white vertex.

Let  $P$  be a maximal divide whose fiber is not a connected sum. We will show that either  $P$  is maximal or, after possibly applying several Reidemeister moves of type III,  $P$  contains one of the minors of Figure 3.8. We use a case-by-case analysis. Let  $\Sigma$  be the fiber surface associated with  $P$ , and  $G$  be a code of  $P$ . By hypothesis,  $G$  has no separating white vertex.

Case 1:  $G$  contains a cycle all of whose vertices are black. Then contracting all edges of this cycle but one yields a  $Q$ -minor.

Case 2:  $G$  has a black vertex of degree at least 4. Then  $G$  has an  $X$ -minor.

Case 3:  $G$  has more than one black vertex of degree 3.

Case 3.1: There are two degree 3 black vertices that are connected by a path of black vertices. The contraction of this path yields a minor with a degree 4 vertex, hence an  $X$ -minor for  $P$ .

Case 3.2: There are two black vertices connected by two different paths (whose vertices need not to be black). Then the union of these two paths is a cycle of length at least 4, and the dual of  $G$  contains a black vertex of degree at least 4, hence an  $X$ -minor for  $P$ .

Case 4:  $G$  contains one black vertex of degree 3, say  $v$ .

A *branch* is a path all of whose vertices are black. Since  $v$  is the unique degree 3 vertex, a branch starting at  $v$  ends either at a degree 1 black vertex, or at a white vertex (of any degree).

Case 4.1: The three branches starting at  $v$  end at different white vertices or end at black vertices. In this case, the union of these branches is a tree  $T_v$  around  $v$  of type  $E_{p,q,r}$ . By Lemma 3.5, either  $T_v$  is maximal or it contains  $E_{2,2,2}$ ,  $E_{1,3,3}$  or  $E_{1,2,5}$ .

Case 4.1.1:  $T_v = E_{1,2,2}$  or  $E_{1,2,4}$ . Then  $T_v$  has only one white vertex. If  $G$  contains an additional edge, it must contain a loop that encircles  $T_v$ . This gives a  $Q$ -minor in the dual of  $G$ .

Case 4.1.2:  $T_v = E_{1,2,3}$ . If  $G$  has additional edges, then either there is a loop that encircles  $T_v$  as above, or there is an edge joining the two white vertices of  $T_v$ . This, in turn, gives a cycle of length 4, hence an  $X$ -minor in the dual of  $G$ .

Case 4.1.3:  $T_v = E_{1,1,r}$  for some  $r \geq 1$ . Once again, there can be some additional loops based at the white vertices. They can encircle  $T_v$ , which give a  $Q$ -minor as above, or they can be inside a face of  $G$ , in which case the dual of  $G$  contains two or more black vertices of degree 3, hence is not maximal by case 3. The only case left is if there are edges that connect the two white vertices of  $T_v$ . In this situation we can apply several Reidemeister III moves and turn  $P$  into a divide of type  $E_{1,1,r'}$  where  $r' - r$  is twice the number of additional edges. The latter has maximal signature.

Case 4.2: Two of the three branches starting at  $v$  end at the same white vertex.

Case 4.2.1: The two branches are not of length 1. Then either their lengths add up to 3 and  $G$  is  $F_{2,3}$  or  $F_{2,4}$  or it contains  $F_{2,5}$  as a minor, or the branches are longer, and  $G$  contains a cycle of length 4 as a minor. Now we applying a Reidemeister III move to  $F_{2,3}$  and  $F_{2,4}$  respectively yields  $E_{1,2,3}$  and  $E_{1,2,4}$  which are maximal.

Case 4.2.2: The two branches are of length 1. If  $G$  contains nothing but the third branch, then  $G$  contains some  $E_{1,1,r}$  as a minor of corank 1 and one checks  $\det(G) > 0$ . In this case, the signature is maximal, and  $G$  is of type  $BC_n$ .

Otherwise, the two extremal white vertices have an edge between them. If the third branch is of length 1, and there is one additional edge,  $G$  is the dual to  $F_{2,3}$  or is self-dual and  $G$  is maximal. If there is more than one edge, either all edges are on the same side and the dual of  $G$  contains  $F_{2,5}$ , or they are on different sides and the dual of  $G$  contains an inner black vertex, which contradicts the maximality of  $G$ . If the third branch is of length larger than 1, then there can be no edge between the white edges, for otherwise  $G$  would contain a 4-cycle.

Case 4.3: All three branches starting at  $v$  end at the same white vertex.

Case 4.3.1: All branches are of length 1. Then  $G$  corresponds to the Dynkin diagram  $G_2$ . Applying a Reidemeister III move, we obtain  $E_{1,2,2}$ , which is maximal.

Case 4.3.2: One of the branches is of length two. If it is the central one, then the dual of  $G$  contains a cycle of two black vertices, hence a  $Q$ -minor. If it is an exterior branch that is of length 2, then the dual of  $G$  is  $F_{2,3}$  or  $F_{2,4}$ . If there is any other additional edge, we get an  $X$ -minor. In all other cases, there is a cycle of length 4 in  $G$ , hence also an  $X$ -minor in  $P$ .

Case 5:  $G$  contains no black vertex of degree 3.

Case 5.1:  $G$  contains more than 3 white vertices. Then, since  $G$  has no separating white vertex, there is cycle of length 4, hence an  $X$ -minor in  $P$ .

Case 5.2:  $G$  contains three white vertices. If there is a black vertex, there is a cycle of length 4, hence an  $X$ -minor. Otherwise the dual of  $G$  is a tree, and these have already been analysed in case 4.1.

Case 5.3:  $G$  contains two white vertices. Then  $G$  is made of several paths connecting these two vertices, plus some loops.

Case 5.3.1: There is one path only. Then  $G$  is of type  $A_{2n}$ .

Case 5.3.2: There is more than one path. Either there are at least two black vertices, and then there is a cycle of length 4, thus an  $X$ -minor, or there is at most one black vertex. In the latter case, either it is adjacent to an exterior face and  $G$  is again the dual of a tree, or it is in the interior of  $G$ , and the dual of  $G$  contains a black cycle.

*Case 5.4:*  $G$  contains one white vertex only. Then either  $G$  is a path of type  $A_{2n+1}$  or it contains black vertices on its exterior loop only. If  $G$  has more than two black vertices, then its dual contains an  $X$ -minor. If  $G$  has exactly two black vertices, then its dual has a degree 3 black vertex, and these have been classified. If  $G$  has one black vertex, either  $G$  is a cycle with one white and one black vertex, and  $G$  corresponds to the diagram  $BC_1$ , or the dual of  $G$  contains a degree 3 black vertex.

*Case 5.5:*  $G$  has no white vertex. Then  $G$  is a path of black vertices, and the corresponding divide is of type  $A_{2n}$ .

This completes the proof: either  $P$  is maximal or  $P$  contains one of the prohibited minors.  $\square$

We can now return to fiber surfaces and conclude.

*Proof of Theorem C.* For the list of prohibited minors, starting from Proposition 3.1, we only have to show that the surfaces  $\Sigma(E_{2,2,2})$ ,  $\Sigma(E_{1,3,3})$ ,  $\Sigma(E_{1,2,5})$ , and  $\Sigma(F_{2,5})$  all contain  $\Sigma(Q)$  or  $\Sigma(X)$  as a surface minor. For  $\Sigma(E_{2,2,2})$ , Figure 3.10 shows that, applying a Reidemeister move of type III to the divide of  $E_{2,2,2}$  yields a cycle of length 3, which contains  $Q$  as a divide minor. Since a Reidemeister III move for the divide corresponds to an isotopy for the fiber surface,  $\Sigma(E_{2,2,2})$  contains  $\Sigma(Q)$  as a surface minor.

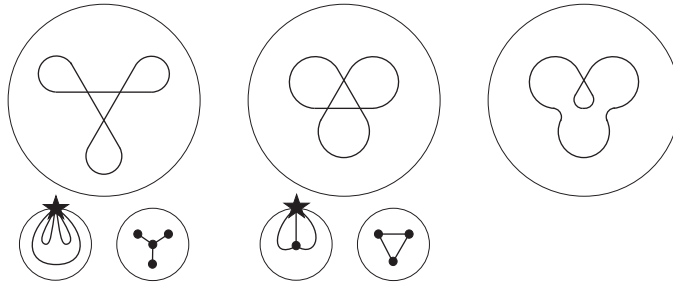


FIGURE 3.10. The divide  $E_{2,2,2}$  and its image after Reidemeister III move. One can then desingularize two double points and obtain  $Q$ .

Similarly, starting from the codes of  $\Sigma(E_{1,3,3})$ ,  $\Sigma(E_{1,2,5})$  and  $\Sigma(F_{2,5})$ , and applying Reidemeister III moves, we obtain some graphs that contain cycles of length 4 (see Figure 3.9), so that their duals contain  $X$ . Therefore, the corresponding fiber surfaces all contain  $\Sigma(X)$  as a surface minor.

The complete list of all codes whose associated fiber surface is not a connected sum and has maximal signature can now be made by following the proof of Proposition 3.1. If  $G$  has no degree 3 vertex, then the divide is of type  $A_n$ . Otherwise it has one degree 3 vertex. If case 4.1 holds, then  $G$  is of type  $D_n$  (which is also  $E_{1,1,n}$  in our notation), or it is  $E_{1,2,2}$ ,  $E_{1,2,3}$  or  $E_{1,2,4}$ . If case 4.2 holds, then  $G$  is of type  $BC_n$ , or it is  $F_{2,3}$  or  $F_{2,4}$ . If the latter holds, we can apply a Reidemeister III move and transform them into  $E_{1,2,3}$  and  $E_{1,2,4}$  respectively. If case 4.3 holds,  $G$  can be transformed into  $E_{1,2,2}$ . In case 5, considering the dual of  $G$  leads back to case 4.

The divides listed above then correspond to those depicted in Figure 0.2.  $\square$

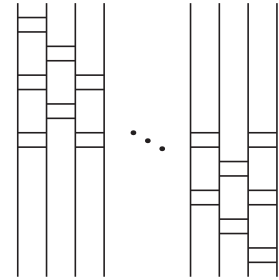
## 4. QUESTIONS

With Theorem A, we have exhibited a class of surfaces—fiber surfaces of divides—for which surface minority is a well-quasi-order. It is natural to wonder whether surface minority is a well-quasi-order for other natural classes of surfaces. An obvious candidate is the family of the canonical Seifert surfaces of positive braids, for which an analog of Theorem C is already known [4].

**Question 4.1.** Is the family of all canonical Seifert surfaces for positive braids, equipped with the surface minor order, a well-quasi-order?

If we restrict to fiber surfaces of positive braids with bounded braid width  $n$ , then the answer to Question 4.1 is positive. Indeed, such a surface is coded by a word on the alphabet  $\{\sigma_1, \dots, \sigma_{n-1}\}$  that describes the braid. Erasing a letter in this word amounts to cutting an essential ribbon in the surface, so that if a word  $w$  is a subword of  $w'$ , then the surface associated with  $w$  is a surface minor of the surface associated with  $w'$ . Then, the property just follows from the Higman Lemma [8] which states that, for words on a finite alphabet, the subword order is a well-quasi-order.

On the other hand, we conjecture that the answer to Question 4.1 with no restriction on the braid index is negative. For an infinite antichain, we propose the canonical Seifert surfaces of the following family of positive braids

$$\sigma_1^2 \sigma_2^2 \sigma_1^2 \sigma_3^2 \sigma_2^2 \sigma_1^2 \sigma_4^2 \sigma_3^2 \dots \sigma_k^2 \sigma_{k-1}^2 \dots \sigma_{n-2}^2 \sigma_{n-3}^2 \sigma_n^2 \sigma_{n-1}^2 \sigma_{n-2}^2 \sigma_n^2 \sigma_{n-1}^2 \sigma_n^2 = \dots$$


A reason why this family could be an infinite antichain is that the associated Seifert surfaces contain a natural ribbon that visits all strands and has writhe  $2n - 1$ . The presence of the  $\sigma_1^2$  and  $\sigma_n^2$  at the two ends then makes it unlikely that the surface can be embedded into another larger surface without changing the framing.

In another direction, we can look for other properties of fiber surfaces of divides eligible for Theorem A. Falling short from having maximal signature by a bounded amount is such a property: for  $c$  positive,  $\text{sign}(\Sigma) \geq b_1(\Sigma) - c$  is a property of  $\Sigma$  that is stable under going to a surface minor.

**Question 4.2.** For  $c$  a fixed integer, what are the prohibited minors for the property  $\text{sign}(\Sigma) \geq b_1(\Sigma) - c$  for fiber surfaces of divides?

For  $\Sigma = \Sigma(Q)$  and  $\Sigma(X)$ , we have  $\text{sign}(\Sigma) = b_1(\Sigma) - 1$ , whence  $\text{sign}(\Sigma) = b_1(\Sigma) - c - 1$  for all surfaces obtained by connected sum of  $c + 1$  copies of  $\Sigma(Q)$  or  $\Sigma(X)$ . So any such surface is a prohibited minor for the property  $\text{sign}(\Sigma) \geq b_1(\Sigma) - c$ . We do not know if there are any others.

## REFERENCES

- [1] N. A'CAMPO, Le groupe de monodromie du déploiement des singularités isolées de courbes planes II, *Proc. Internat. Congress of Mathematicians, vol. 1*, Vancouver (1974), 395–404.
- [2] N. A'CAMPO, Generic immersions of curves, knots, monodromy and gordian number, *Publ. Math. Inst. Hautes Études Sci.* **88**, (1998), 152–169.

- [3] V.I. ARNOLD, Critical points of smooth functions, *Proc. Internat. Congress of Mathematicians, vol. 1*, Vancouver (1974), 19–40.
- [4] S. BAADER, Positive braids of maximal signature, preprint, arxiv:1211.4824.
- [5] S. CHMUTOV, Diagrams of divide knots, *Proc. Amer. Math. Soc.* **131**, (2003), 1623–1627.
- [6] R. DIESTEL, Graph theory, *Graduate Texts in Mathematics* **173**, 4th ed. (2010), Springer, 451pp.
- [7] J.F. GEELENA, A. GERARDS, G. WHITTLED, Branch-Width and Well-Quasi-Ordering in Matroids and Graphs, *J. Combin. Theory Ser. B* **84**, (2002), 270–290.
- [8] G. HIGMAN, Ordering by divisibility in abstract algebras, *Proc. Lond. Math. Soc. (3)* **2**, (1952), 326–336.
- [9] M. HIRASAWA, Visualization of A’Campo’s Fibered Links and Unknotting Operation, *Topol. Appl.* **121**, (2002), 287–304.
- [10] M. ISHIKAWA, Plumbing constructions of connected divides and the Milnor fibers of plane curve singularities, *Indag. Mathem.* **13**, (2002), 499–514.
- [11] F. MAZOIT, Tree-width of hypergraphs and surface duality, *J. Combin. Theory Ser. B* **102**, (2012), 671–687.
- [12] K. MURASUGI, Knot Theory and Its Applications, Birkhäuser, (1993), 341 pp.
- [13] C. NASH-WILLIAMS, On well-quasi-ordering finite trees, *Proc. Cambridge Phil. Soc.* **59**, (1963), 833–835.
- [14] N. ROBERTSON, P. SEYMOUR, Graph minors. IV. Tree-width and well-quasi-ordering, *J. Combin. Theory Ser. B* **48**, (1990), 227–254.
- [15] N. ROBERTSON, P. SEYMOUR, Graph minors. XX. Wagner’s conjecture, *J. Combin. Theory Ser. B* **92**, (2004), 325–357.
- [16] W. TUTTE, Graph Theory, *Advanced Book Program*, Addison-Wesley, (1984), 333 pp.
- [17] H. WHITNEY, Congruent Graphs and the Connectivity of Graphs, *Amer. J. Math.* **54**, (1932), 150–168.

UNIVERSITÄT BERN, SIDLERSTRASSE 5, CH-3012 BERN, SWITZERLAND  
*E-mail address:* sebastian.baader@math.unibe.ch

UNIVERSITÄT BERN, SIDLERSTRASSE 5, CH-3012 BERN, SWITZERLAND  
*E-mail address:* pierre.dehornoy@math.unibe.ch