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Legendre functions and the theory of characteristics

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Abstract

We devise a framework encompassing the classical theory of characteristics and the theory valid in the convex case recently obtained by R.T. Rockafellar and P. Wolenski. It relies on a notion of transform introduced by I. Ekeland. It involves a class of functions called Ekeland functions which is large enough to encompass convex functions, concave functions and linear-quadratic functions, as well as the class of classical Legendre functions. We also introduce a class of functions called (generalized) Legendre functions which is not as large as the class of Ekeland functions but has better reciprocity properties. It is obtained by an extension procedure and it enables one to recover the usual Fenchel-Legendre transform of convex functions.

Key words: characteristics, Ekeland function, Ekeland transform, Hamilton-Jacobi equations, Legendre function, Legendre transform, subdifferential

Mathematics Subject Classification: 49J52, 49J53, 90C26, 53D05, 37K25

Dedicated to Ivar Ekeland on his sixtieth birthday

1 Introduction

Many transforms are known in mathematics such as the Fourier transform, the Laplace transform, the Fenchel transform, the Radon transform. They enable to change a given problem into a related one which may be more tractable and help in the solution of the initial problem. In the present paper we use an extension of the Legendre transform.

The aim of the paper consists in an attempt to encompass in a single framework two theories of characteristics for first order partial differential equations: the local, classical one which assumes differentiability properties (see [8], [11], [14], [21], [46]...) and the global, convex theory of [41], [42]. For that purpose, we introduce a class of functions which are nonsmooth and nonconvex but retain the main duality property of the Legendre-Fenchel transform. This class appears in a natural way when using the Legendre duality introduced by Ekeland [12], [13]. It contains the classes of closed proper convex functions, of classical Legendre functions

and of quadratic functions defined by a nondegenerated bilinear form and it applies to integral functionals ([12], [13], [32]). However, in order to encompass the full generality of the conjugacy theory of convex analysis, we have to extend the process of [12], [13] and use a closure device. We perform such an extension by using a result in [29] akin to the Brønsted-Rockafellar theorem; it relies on the Ekeland variational principle and enables one to approach any element of the graph of a closed proper convex function by elements corresponding to points at which the function is subdifferentiable. Among the reasons justifying such an extension is the well known fact that the domain of the subdifferential of a closed proper convex function is nonconvex but its closure and interior are convex. Thus, this new class of functions we call generalized Legendre functions, instead of being more special than the class of convex functions as in [37, Chapter 26], [7], [5] is a larger class. Using the methods of nonsmooth analysis, we also introduce Ekeland subsets and Legendre subsets of normed vector spaces (n.v.s.). On the other hand, we avoid the geometrical framework of Lagrangian submanifolds and contact manifolds present in [1], [12], [15], [22], [47]. We do not consider either questions of multi-valuedness of the transform which are dealt with in [12], [31]. In fact, the notions we adopt impose single-valuedness of the transform.

Our approach can be seen as an instance of the methods of nonsmooth analysis which strive to encompass in a general framework both differential calculus and convex analysis. In our views, the most important feature of these developments is the "Copernician revolution" consisting in considering in a unified framework functions, sets and multimaps (also called multifunctions, correspondences, relations):

passage from sets to functions: to a subset S of a n.v.s. one associates its distance function $d_S(\cdot) = \inf_{s\in S} d(\cdot, s)$ or its indicator function ι_S given by $\iota_S(x) = 0$ if $x \in S$, $+\infty$ else.

passage from functions to sets: to a function f one associates its epigraph $E_f := \mathrm{epi} f$.

passage from multimaps to sets: to a multimap $F : X \rightrightarrows Y$ one associates its graph $G(F) \subset X \times Y$; conversely, to any subset G of $X \times Y$ one can associate a multimap $F : X \rightrightarrows Y$ whose graph is G.

The main tool is the notion of subdifferential which generalizes in a one-sided way the notion of derivative. But the transform we study is not limited to the use of a subdifferential. One can also use a generalized derivative taylored to the situation at hand. It is not our purpose to present here a general definition of subdifferential. We adopt the properties listed in [20] and [33]; see also [3], [18], [19], [30] and their references for axiomatic studies of subdifferentials (which may vary with the purposes). Thus, when speaking of a subdifferential ∂, we assume that ∂ is a map which assigns to any member f of a class $\mathcal{F}(X)$ of extended-real valued functions on X and $x \in \mathrm{dom}\, f$ a subset $\partial f(x)$ of the topological dual $Y := X^*$ of X. We require that $0 \in \partial f(x)$ when f attains its infimum at x and that ∂ coincides with the Moreau-Rockafellar subdifferential of f when f is a convex function, we assume that $\partial(f + c)(x) = \partial f(x)$ and $\partial(\lambda f)(x) = \lambda \partial f(x)$ for all $c \in \mathbb{R}$, $\lambda \in \mathbb{P} := (0 + \infty)$, $x \in X$; we also assume that for $p : W \to \overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, +\infty\}$ defined by $p(w) := \inf_{x\in X} j(w, x)$, where $j : W \times X \to \overline{\mathbb{R}}$, one has $(w^*, 0) \in \partial j(w, x)$ whenever $w^* \in \partial p(w)$ and $j(w, x) = p(w)$. Following A.D. Ioffe, we say that a subdifferential ∂ is *elementary* if for any pair of functions g, h finite at x one has

$\partial g(x) + \partial h(x) \subset \partial(g+h)(x)$.

We will mainly use the *firm* (or *Fréchet*) *subdifferential* ∂_F given by

$$x^* \in \partial_F f(x) \text{ iff for any } \varepsilon > 0,\ 0 \text{ is a local minimizer of } w \mapsto f(x+w) - \langle x^*, w\rangle + \varepsilon \|w\|$$

and the *directional* (or *Dini-Hadamard, or Hadamard* or *contingent) subdifferential* ∂_D given by

$$x^* \in \partial_D f(x) \text{ iff for any } \varepsilon > 0,\ u \in X\backslash\{0\},\ (0,u) \text{ is a local minimizer on } \mathbb{R}_+ \times X$$

$$\text{of the function } (t,v) \mapsto f(x+tv) - \langle x^*, tv\rangle + \varepsilon \|tv\|.$$

We also mention the proximal subdifferential ∂_P, the subdifferential associated to a bornology, the Clarke subdifferential ∂_C and the *limiting subdifferential* ∂_L defined through a limiting procedure from a subdifferential ∂ as follows: $x^* \in \partial_L f(x)$ if there exists sequences $(x_n) \to x$, $(x_n^*) \xrightarrow{*} x^*$ (i.e. for the weak* topology) such that $(f(x_n)) \to f(x)$.

We also use the *normal cone* $N(E,x)$ to a subset E of X at $x \in E$ defined as $\partial \iota_E(x)$, where ι_E is the indicator function of E. It is easy to check that when ∂ is either ∂_C, ∂_D, ∂_F, or ∂_L this definition coincides with the usual geometric notions. In particular, for $\partial = \partial_D$, the normal cone $N(E,x)$ is the polar of the tangent cone $T(E,x)$, where the *tangent cone* $T(E,x)$ to a subset E of a n.v.s. X at $x \in E$ is the set of $v \in X$ such that there is a sequence $((t_n, v_n)) \to (0_+, v)$ in $\mathbb{R} \times X$ satisfying $x + t_n v_n \in E$ for each $n \in \mathbb{N}$.

After some preliminaries recalling the classical theory of characteristics and the Rockafellar-Wolenski approach in the convex case, we deal with the Legendre transform as devised by Ekeland and we specify the classes of functions we consider. We mention some examples and properties and we relate these classes to classes of sets. We display an explicit formula for the solution of a first order Hamilton-Jacobi equation with data satisfying assumptions related to these classes of functions and we end our study with a connection with the theory of characteristics.

Other approaches to the theory of characteristics are given in the recent monographs [23], [25], [43], [46].

2 Preliminaries: characteristics

Let us recall some elements of the theory of characteristics. We refer to [8], [14], [15], [21] and [46] for more information. Consider the first order partial differential equation with unknown u

$$F(w, Du(w), u(w)) = 0 \quad w \in \Omega \tag{1}$$

$$u(w) = g(w) \quad w \in \partial\Omega \tag{2}$$

where Ω is the interior of a submanifold with boundary $\overline{\Omega}$ of class C^2 of some Euclidean or Banach space W, $\partial\Omega$ is its boundary and $F : \Omega \times W^* \times \mathbb{R} \to \mathbb{R}$, $g : \partial\Omega \to \mathbb{R}$ are given C^2 functions.

In order that such an equation be solvable, some compatibility condition has to be satisfied. The classical theory of characteristics is a means to find a local solution around some point $(w_0, p_0, z_0) \in \partial\Omega \times W^* \times \mathbb{R}$ satisfying $F(w_0, p_0, z_0) = 0$, $g(w_0) = z_0$, $p_0 \mid T(\partial\Omega, w_0) = Dg(w_0)$ by using the solution to the system of ordinary differential equations

$$\widehat{w}'(s) = D_p F(\widehat{w}(s), \widehat{p}(s), \widehat{z}(s)) \tag{3}$$
$$\widehat{p}'(s) = -D_w F(\widehat{w}(s), \widehat{p}(s), \widehat{z}(s)) - D_z F(\widehat{w}(s), \widehat{p}(s), \widehat{z}(s))\widehat{p}(s) \tag{4}$$
$$\widehat{z}'(s) = D_{w^*} F(\widehat{w}(s), \widehat{p}(s), \widehat{z}(s)).\widehat{p}(s) \tag{5}$$

and the initial conditions

$$\widehat{w}(0) = w_0, \;\; \widehat{p}(0) = p_0, \;\; \widehat{z}(0) = g(w_0).$$

It is easy to show that if u is a solution of class C^2 of (1)-(2), then the solution of the system (3)-(5) satisfies $z(s) = u(w(s))$, so that the value of u at $w(s)$ is determined by $z(\cdot)$.

Suppose $D_{w^*} F(w_0, w_0^*, z_0).v \neq 0$ for some normal vector $v \in N(\partial\Omega, w_0)$. Then, performing a change of variables (x, t) instead of w one can assume that locally $\Omega = X_0 \times \mathbb{P}$ where X_0 is an open subset of an hyperplane X of W and that equations (1)-(2) are of the form

$$D_t u(x,t) + H(t, x, D_x u(x,t), u(x,t)) = 0, \qquad (x,t) \in X_0 \times \mathbb{P} \tag{6}$$
$$u(x,0) = g(x) \qquad x \in X_0. \tag{7}$$

Then the characteristic system is transformed into the following one

$$\begin{aligned}
\widehat{x}'(s) &= D_y H(s, \widehat{x}(s), \widehat{y}(s), \widehat{z}(s)) & \widehat{x}(0) &= x_0 \\
\widehat{y}'(s) &= -D_x H(s, \widehat{x}(s), \widehat{y}(s), \widehat{z}(s)) - D_z H(s, \widehat{x}(s), \widehat{y}(s), \widehat{z}(s))\widehat{y}(s) & \widehat{y}(0) &= Dg(x_0) \\
\widehat{z}'(s) &= D_y H(s, \widehat{x}(s), \widehat{y}(s), \widehat{z}(s)).\widehat{y}(s) - H(s, \widehat{x}(s), \widehat{y}(s), \widehat{z}(s)) & \widehat{z}(0) &= g(x_0).
\end{aligned}$$

We denote by $(\widehat{x}(s, x_0), \widehat{y}(s, x_0), \widehat{z}(s, x_0))$ the solution of this system. Then one has the following classical result.

Theorem 2.1 *Suppose that for some $x_0 \in X_0$ and some $t_0 > 0$ the first component $\widehat{x}(t_0, \cdot)$ of the solution to this system with initial data x_0, $y_0 := Dg(x_0)$, $z_0 = g(x_0)$ realizes a diffeomorphism of a neighborhood U of x_0 onto a neighborhood V of $\widehat{x}(t_0, x_0)$. Then the function $(x,t) \mapsto \widehat{z}(t, \widehat{x}(t, \cdot)^{-1}(x))$ is a solution of class C^2 to the Hamilton-Jacobi equation.*

This result is particularly effective when dealing with quasi-linear equations and conservation laws. In the last section of the present paper we will deal with the case when H does not depend on t, x, z; then the characteristic system takes a particularly simple form.

3 Characteristics in convex analysis

In two remarkable papers, [41], [42], Rockafellar and Wolenski study the Hamilton-Jacobi equation

$$\begin{aligned} D_t u(x,t) + H(x, D_x u(x,t)) = 0, \qquad & (x,t) \in X \times \mathbb{P} \\ u(x,0) = g(x) \qquad & x \in X, \end{aligned}$$

where g is a given closed proper convex function and H is a finite concave-convex function on $X \times Y$ (with X finite dimensional, $Y := X^*$) satisfying the following growth conditions for some real numbers $\alpha, \beta, \gamma, \delta$ and some finite convex functions $\varphi,\ \psi$

$$\begin{aligned} H(x,y) \leq\ & \varphi(y) + (\alpha \|y\| + \beta) \|x\| \qquad \forall (x,y) \in X \times X^*, \\ H(x,y) \geq\ & -\psi(x) - (\gamma \|x\| + \delta) \|y\| \qquad \forall (x,y) \in X \times X^*. \end{aligned}$$

They associate to it the Bolza problem

$$\text{minimize } g(w(0)) + \int_0^t L(w(s), w'(s))ds \ : \qquad w(\cdot) \in W^{1,1}([0,t], X),\ w(t) = x$$

in which $L(x,\cdot) = H(x,\cdot)^*$, where f^* (resp. h^*) denotes the Fenchel transform of a function f on X (resp. h on Y) defined by

$$f^*(y) := \sup_{x \in X} (\langle x, y \rangle - f(x)), \qquad h^*(x) := \sup_{y \in Y} (\langle x, y \rangle - h(y)).$$

Here $W^{1,1}([0,t], X)$ denotes the set of $w \in C([0,t], X)$ such that there exists an element w' of $L_1([0,t], X)$ satisfying $w(s) = w(0) + \int_0^s w'(r)dr$ for all $s \in [0,t]$. They prove that the value $v(x,t)$ of the above Bolza problem is a solution to the Hamilton-Jacobi equation in the sense that $v(\cdot, 0) = g(\cdot)$ and for all $(x,t) \in X \times (0, +\infty)$ one has

$$(p,q) \in \partial_L v(x,t) \Leftrightarrow p \in \partial v(\cdot, t)(x),\ q = -H(x,p)$$

where ∂_L is the limiting subdifferential and ∂ is either ∂_L or the contingent or Hadamard subdifferentials or the subdifferential of convex analysis. In these papers culminates the duality theory for convex Bolza problems performed in the seventies ([38]-[40]). Moreover, they give a global version of the theory of characteristics. They observe that for a given convex function $g : X \to \mathbb{R}_\infty := \mathbb{R} \cup \{+\infty\}$ its subjet

$$J^- g := \{(x,y,z) \in X \times X^* \times \mathbb{R} : y \in \partial g(x),\ z = g(x)\}$$

is a Lipschitzian submanifold of $X \times X^* \times \mathbb{R}$ and that the flow associated with the characteristic equations

$$\begin{aligned} & \widehat{x}'(t) \in \partial H(\widehat{x}(t), \cdot)(\widehat{y}(t)) \\ & \widehat{y}'(t) \in \partial(-H)(\cdot, \widehat{y}(t))(\widehat{x}(t)) \\ & \widehat{z}'(t) = \langle \widehat{x}'(t), \widehat{y}(t) \rangle - H(\widehat{x}(t), \widehat{y}(t)) \end{aligned}$$

carries the subjet J^-g of g onto the subjet $J^-v(t,\cdot)$ of $v(t,\cdot)$. Here ∂ is the subdifferential in the sense of convex analysis.

It is our purpose here to consider such questions under relaxed convexity assumptions.

4 The Ekeland and Legendre transforms

In [31], following [12], [13], given two sets X and Y paired by a coupling function $c : X \times Y \to \mathbb{R}$, we defined the *Ekeland transform* as a simple transform for multimaps $F : X \rightrightarrows Y \times \mathbb{R}$ (identified with their graphs $\mathrm{gph}(F)$ whenever there is no risk of confusion) associating to F the multimap $F^E : Y \rightrightarrows X \times \mathbb{R}$ given by

$$F^E(y) := \{(x,s) \in X \times \mathbb{R} : (y, c(x,y) - s) \in F(x)\}.$$

In terms of graphs one has

$$\mathrm{gph}(F^E) = \{(y,x,s) \in Y \times X \times \mathbb{R} : (x,y,c(x,y) - s) \in \mathrm{gph}(F)\},$$

so that $\mathrm{gph}(F^E)$ is the image of $\mathrm{gph}(F)$ by the mapping $\mathcal{L} : X \times Y \times \mathbb{R} \to Y \times X \times \mathbb{R}$ given by $\mathcal{L}(x,y,z) := (y,x,c(x,y) - z)$. Using the coupling $c^\intercal : (y,x) \mapsto c(x,y)$ and the mapping $\mathcal{L}^\intercal : Y \times X \times \mathbb{R} \to X \times Y \times \mathbb{R}$ given by $\mathcal{L}^\intercal(y,x,z) := (x,y,c^\intercal(y,x) - z)$, one can define in a similar way the Ekeland transform G^E of a multimap $G : Y \rightrightarrows X \times \mathbb{R}$ by $\mathrm{gph}(G^E) = \mathcal{L}^\intercal(\mathrm{gph}(G))$ (here we use a slight abuse of notation, using G^E instead of $G^{E^\intercal}$). Since $\mathcal{L}^\intercal \circ \mathcal{L}$ is the identity $I_{X \times Y \times \mathbb{R}}$ of $X \times Y \times \mathbb{R}$, this transform is involutive in the sense that

$$(F^E)^E = F.$$

This transform has a special interest when Y has a base point 0_Y and when $c(x, 0_Y) = 0$ for all $x \in X$.

Here we take for X a Banach space and for Y its dual space, and unless explicitely mentioned, c is the canonical pairing $\langle \cdot, \cdot \rangle$; we write either $\langle x, y \rangle$ or $\langle y, x \rangle$ for $c(x,y)$. We suppose a generalized derivative or subdifferential ∂ has been chosen. Then the Ekeland transform can be specialized to functions $f : X \to \mathbb{R}_\infty := \mathbb{R} \cup \{\infty\}$ in the class $\mathcal{F}(X)$ of functions on which ∂ is defined. For this purpose, one can associate to f its (first order) *subjet* $J^\partial f : X \rightrightarrows Y \times \mathbb{R}$ (or ∂-subjet, to be more precise) given by

$$J^\partial f(x) := \{(y,r) : \ y \in \partial f(x), \ r = f(x)\}.$$

Note that when f is of class C^k ($k \geq 2$), and ∂ is the differentiation operator, $J^\partial f = J^1 f$, the one-jet of f, is a *Lagrangian submanifold* of class C^{k-1} of $X \times X^* \times \mathbb{R}$. That means that the pull-back to $J^1 f$ of the differential form ω on $X \times X^* \times \mathbb{R}$ given by

$$\omega(x,y,z,u,v,w) = w - \langle y, u \rangle$$

is null, or in other terms, that for any $x \in X$ and any (u, v, w) in the tangent space $T(J^1 f, (x, y, z))$ to $J^1 f$ at (x, y, z) with $y := f'(x)$, $z := f(x)$ one has $\omega(x, y, z, u, v, w) = 0$. For a nonsmooth function f and for a subdifferential ∂ contained in ∂_D, one may observe that the subjet $J^\partial f$ is a *super-Lagrangian subset* of $X \times X^* \times \mathbb{R}$ in the sense that $\omega(x, y, z, u, v, w) \geq 0$ for any $(x, y, z) \in J^\partial f$ and any $(u, v, w) \in T(J^\partial f, (x, y, z))$. Note that the image $M' = \mathcal{L}(M)$ of a super-Lagrangian subset V of $X \times X^* \times \mathbb{R}$ is a *sub-Lagrangian* of $X^* \times X \times \mathbb{R}$ in the sense that $\omega(y', x', z', v', u', w') \leq 0$ for any $(y', x', z') \in M'$ and any $(v', u', w') \in T(M', (y', x', z'))$ ([31]). We shall not make use of these remarks in the sequel but we note them in order to point out links with a geometric approach.

4.1 Ekeland functions

We shall restrict our attention to a special class of functions we call Ekeland functions.

Definition 4.1 *A function $f : X \to \mathbb{R}_\infty$ is an Ekeland function (for ∂ and c) if for all y in the image $\partial f(X)$ of ∂f in $Y := X^*$ one has*

$$x_1, x_2 \in (\partial f)^{-1}(y) \Rightarrow c(x_1, y) - f(x_1) = c(x_2, y) - f(x_2). \tag{8}$$

Then the Ekeland transform of f is the function f^E given by $f^E(y) := c(x, y) - f(x)$ for $y \in \partial f(X)$, $x \in (\partial f)^{-1}(y)$, $f^E(y) := +\infty$ for $y \in Y \backslash \partial f(X)$.

Condition (8) means that the restriction of the projection mapping $(y, s) \mapsto y$ to the projection of $\left(J^\partial f\right)^E$ on $Y \times \mathbb{R}$ is injective. It can be interpreted as follows: it requires that $\left(J^\partial f\right)^E$ is an *hypergraph* in the sense that it is a subset of $Y \times X \times \mathbb{R}$ of the form

$$\bigcup_{y \in Y} \{y\} \times G(y) \times \{g(y)\}$$

for some multimap $G : Y \rightrightarrows X$ and some function $g : Y \to \mathbb{R}_\infty$ such that $\operatorname{dom} G = \operatorname{dom} g = \partial f(X)$. In fact,

$$\left(J^\partial f\right)^E = \{(y, x, z) \in Y \times X \times \mathbb{R} : y \in \partial f(X),\ y \in \partial f(x),\ z = c(x, y) - f(x)\}$$

It is natural to consider the function $y \mapsto f^E(y) := g(y)$ given by $g(y) := c(x, y) - f(x)$ for $x \in (\partial f)^{-1}(y)$ as a transform of f; we call it the Ekeland transform f^E of f. Before going any further, let us give some examples.

Example 1. Let $\mathcal{F}(X)$ be the set $\Gamma_0(X)$ of closed, proper, convex functions on X and let ∂ be the subdifferential of convex analysis or any other subdifferential coinciding with it on $\mathcal{F}(X)$, c being the usual pairing of X with it dual space $Y := X^*$. Then, for any $f \in \mathcal{F}(X)$, $y \in Y$, the relation $x \in (\partial f)^{-1}(y)$ means that the function $w \mapsto f(w) - \langle w, y \rangle$ attains its minimum at x. Thus, condition (8) is fulfilled and, for every $y \in \partial f(X)$, $g(y) := f^E(y)$ coincides with $f^*(y) := \max_{w \in X} (\langle w, y \rangle - f(w))$, the value at y of the Fenchel conjugate function of f.

Example 2. Let X and Y be arbitrary sets and let $c : X \times Y \to \mathbb{R}$ be an arbitrary coupling function. Defining the subdifferential ∂^c associated to c by

$$y \in \partial^c f(x) \Leftrightarrow (\forall w \in X \qquad f(w) - c(w, y) \geq f(x) - c(x, y)),$$

one gets that any function on X is an Ekeland function. The next example is a special case of such a general scheme; it yields Example 1 when taking $k = 0$.

Example 3. Let $\mathcal{F}(X) := \Gamma_k(X)$ be the set of *paraconvex* (or *semiconvex*) functions on X with respect to some function $k : X \to \mathbb{R}$, i.e. the set of functions f such that $f + k$ is closed, proper, convex. Let us take here the pairing $c_k : X \times Y \to \mathbb{R}$ given by $c_k(x, y) := \langle x, y\rangle - k(x)$ and the subdifferential ∂_k given by

$$y \in \partial_k f(x) \Leftrightarrow (\forall w \in X \qquad f(w) - c_k(w, y) \geq f(x) - c_k(x, y)),$$

or, equivalently, $y \in \partial(f + k)(x)$, ∂ being the subdifferential of convex analysis. Then, as easily checked, every $f \in \mathcal{F}(X)$ is an Ekeland function for ∂_k and c_k, and $f^E(y) = \sup\{c_k(x, y) - f(x) : x \in X\} = (f + k)^*(y)$ for all $y \in \partial(f + k)(X)$.

Example 4. Let $\mathcal{F}(X)$ be the set of concave functions on X with values in $\mathbb{R} \cup \{-\infty\}$ and let ∂ be either the Fréchet or the Hadamard subdifferential, c being the usual pairing of X with its dual space $Y := X^*$. Then, for any $f \in \mathcal{F}(X)$, $x \in f^{-1}(\mathbb{R})$, $y \in Y$, the relation $x \in (\partial f)^{-1}(y)$ means that f is Fréchet or Hadamard differentiable at x; then the function $w \mapsto f(w) - \langle w, y\rangle$ attains its maximum at x. Thus, condition (8) is fulfilled and $f^E(y)$ coincides with $f_*(y) := \inf_{x\in X}(\langle w, y\rangle - f(w))$, the value of the concave conjugate function of f.

Let us note another choice. It consists in setting $\partial f(x) := -\partial_{MR}(-f)(x)$, where ∂_{MR} is the Moreau-Rockafellar subdifferential. Then $f^E(y) = -(-f)^*(y)$.

Example 5. Let $\mathcal{F}(X)$ be the set of linear-quadratic functions on X, i.e. the set of functions f given by $f(x) := \frac{1}{2}\langle Ax, x\rangle - \langle b, x\rangle + c$ for some continuous symmetric linear map $A : X \to Y := X^*$, $b \in Y$, $c \in \mathbb{R}$. Let ∂ be either the Fréchet or the Hadamard subdifferential (or the corresponding differentiation operator). Then f is an Ekeland function. In fact, given $y \in Y$, $x_1, x_2 \in X$ such that $f'(x_i) = y$ for $i = 1, 2$, one has

$$\langle x_i, y\rangle - f(x_i) = \langle x_i, Ax_i - b\rangle - \frac{1}{2}\langle Ax_i, x_i\rangle + \langle b, x_i\rangle - c = \frac{1}{2}\langle Ax_i, x_i\rangle - c$$

and

$$\langle Ax_1, x_1\rangle - \langle Ax_2, x_2\rangle = \langle A(x_1 - x_2), x_1\rangle + \langle Ax_2, x_1 - x_2\rangle = 0$$

since A is symmetric and $Ax_1 = y + b = Ax_2$. Thus, for $y \in A(X) - b$, we can write $f^E(y) = \frac{1}{2}\langle y + b, A^{-1}(y + b)\rangle - c$, even if A is non invertible. When A is surjective and its kernel has a complement Z, $f^E(y) = \frac{1}{2}\langle y + b, A_Z^{-1}(y + b)\rangle - c$, where A_Z is the restriction of A to Z, so that f^E is a continuous quadratic function.

Example 6. Let $X := U \times V$ be a product of two n.v.s. and let $Y := U^* \times V^*$. Let $f : X \to \mathbb{R}$ be a saddle function in the sense that for all $v \in V$, $f(\cdot, v)$ is a convex function on U and, for all $u \in U$, $f(u, \cdot)$ is a concave function on V. Then, taking the coupling c given by

$c((u,v),(u^*,v^*)) := \langle u,u^*\rangle + \langle v,v^*\rangle$ and setting $(u^*,v^*) \in \partial f(u,v)$ if, and only if $u^* \in \partial f(\cdot,v)(u)$ and $v^* \in -\partial(-f)(u,\cdot)(v)$, we see that f is an Ekeland function on X. For that, we note that since f is convex-concave, $(u^*,v^*) \in \partial f(u,v)$ iff (u,v) is a saddle point of $f - u^* \circ p_U - v^* \circ p_V$, where p_U and p_V are the canonical projections from X onto U and V respectively, so that $-f^E(u^*,v^*)$ is the saddle value of $f - u^* \circ p_U - v^* \circ p_V$. In fact, if $(u^*,v^*) \in \partial f(u,v)$, then for all $(u',v') \in X$ one has

$$\begin{aligned} f(u',v) - \langle u^*,u'\rangle - \langle v^*,v\rangle &\geq f(u,v) - \langle u^*,u\rangle - \langle v^*,v\rangle \qquad \text{as} \qquad u^* \in \partial f(\cdot,v)(u), \\ f(u,v) - \langle u^*,u\rangle - \langle v^*,v\rangle &\geq f(u,v') - \langle u^*,u\rangle - \langle v^*,v'\rangle \qquad \text{as } -v^* \in \partial(-f(u',\cdot))(v'). \end{aligned}$$

Conversely, when (u,v) is a saddle point of $f - u^* \circ p_U - v^* \circ p_V$, the preceding inequalities show that $(u^*,v^*) \in \partial f(u,v)$. Thus, the passages from f to $(v^*,u^*) \mapsto f^E(u^*,v^*)$ or from f to $-f^E$ can be considered as the appropriate Young-Fenchel transforms for saddle functions. In particular, for $(u,v) \in \mathrm{dom}\partial f$ one has $-(-f^E)^E(u,v) = f(u,v)$.

When $f(u,v) := g(u) - h(v)$, where $g : U \to \mathbb{R}$, $h : V \to \mathbb{R}$ are convex functions, one has $f^E(u^*,v^*) = g^*(u^*) - h^*(v^*)$ for $(u^*,v^*) \in \partial g(U) \times \partial h(V)$. When X is finite dimensional, the decomposition of quadratic forms on X shows that the preceding example is a special case of the present example.

Example 7. Let W be an open subset of a normed vector space X and let $f : W \to \mathbb{R}$ be differentiable and positively homogeneous. Then f is an Ekeland function since for any $x \in W$ one has $Df(x).x - f(x) = 0$ by Euler's relation.

Example 8. Let $f : X \to \mathbb{R}$ be differentiable and such that its derivative f' is injective. Then f is an Ekeland function.

Example 9. Let $(S,\mathcal{S},\mu)$ be a σ-finite complete measure space and let $f : S \times E \to \overline{\mathbb{R}}$ be a normal (nonconvex) integrand. Then the associated integral functional $I_f : L_1(S,E) \to \overline{\mathbb{R}}$ given by

$$I_f(x) := \inf\{\int_S y d\mu : \ y \in L_1(S,\mathbb{R}), \ y(\cdot) \geq f(\cdot,x(\cdot)) \text{ a.e.}\}$$

is an Ekeland function when one takes either the Fréchet subdifferential or the limiting subdifferential. This assertion stems from the result proved in [9], [32] that for $x^* \in L_\infty(S,E^*)$ one has $x^* \in \partial_F I_f(x)$ if, and only if for all $u \in L_1(S,E)$ one has $I_f(u) \geq I_f(x) + \langle x^*, u - x\rangle$.

These examples show that the domain of the Ekeland transform may be very small and that the Ekeland transform of an Ekeland function is not necessarily an Ekeland function. We will turn later to a remedy to these ailments.

The next example is a refinement of the classical notion of Legendre function of class C^k. Recall that that notion enables one to pass from the Euler-Lagrange equations of the calculus of variations to the Hamilton equations. The latter are explicit (rather than implicit) differential equations of first order (instead of second order). Let us give a precise definition in which we say that a mapping $g : U \to V$ between two metric spaces is *stable* or is *Stepanovian* if for all $\overline{u} \in U$ there exist some $r > 0$, $c \in \mathbb{R}_+$ such that for every u in the ball $B(\overline{u},r)$ of center $\overline{u}$ and radius r one has

$$d(g(u),g(\overline{u})) \leq c d(u,\overline{u}).$$

Definition 4.2 *A function $f : U \to \mathbb{R}$ on an open subset U of a Banach space X is a classical Legendre function if it is (Fréchet) differentiable, if its derivative $f' : U \to Y := X^*$ is a Stepanovian bijection onto an open subset V of Y whose inverse h is also Stepanovian.*

Then one defines the Legendre transform of f as the function $f^L : V \to \mathbb{R}$ given by

$$f^L(y) := \langle h(y), y\rangle - f(h(y)) \qquad y \in V.$$

It coincides with the Ekeland transform f^E of f associated with the derivative. Since h is just a Stepanovian function, it is surprising that f^L is in fact of class C^1 (and of class C^k when f is of class C^k).

Lemma 4.3 *If f is a classical Legendre function on U, then it is an Ekeland function and its Legendre transform f^L is of class C^1 on $V := f'(U)$ and coincides with f^E. Moreover f^L is a classical Legendre function, $\left(f^L\right)^L = f$ and*

$$v = Df(u) \Leftrightarrow u = Df^L(v) \quad \forall (u, v) \in U \times V.$$

Furthermore, when f is of class C^k, f^L is of class C^k.

Proof. Since the derivative f' of a classical Legendre function f is injective, f is an Ekeland function by Example 8. For $v \in V$, since $h(v)$ is the unique $u \in U$ such that $f'(u) = v$, we have $f^L(v) = f^E(v)$. Let us show that f^L is differentiable. Given $u \in U$, $v := f'(u) \in V$, let $y \in V - v$, let $x := h(v + y) - h(v) \in U - u$, and let $r(x) = f(u + x) - f(u) - f'(u)x$. Then, since $h(v) = u$, $h(v + y) = u + x$, one has

$$\begin{aligned} f^L(v + y) - f^L(v) - \langle u, y\rangle &= \langle u + x, v + y\rangle - f(u + x) - \langle u, v\rangle + f(u) - \langle u, y\rangle \\ &= \langle x, v + y\rangle - f'(u)x - r(x) \\ &= \langle x, y\rangle - r(x). \end{aligned}$$

Since there exists $c \in \mathbb{R}_+$ such that $\|x\| \le c\,\|y\|$ for $\|y\|$ small enough, the last right hand side is a remainder, i.e. $\|y\|^{-1}\left(\langle x, y\rangle - r(x)\right) \to 0$ as $y \to 0$. Thus f^L is differentiable at v and $(f^L)'(v) = u = h(v)$. Moreover, f^L is a classical Legendre function, and since $f^L = f^E$ in such a case, $\left(f^L\right)^L = f$.

Suppose now that f is of class C^2. Let $g := f'$, $u \in U$, $v := g(u) \in V$, $x \in X$, $A := g'(u)$. For $t > 0$ small enough to ensure $u + tx \in U$, we set $y_t := t^{-1}(g(u + tx) - g(u))$, so that $\|tx\| = \|h(v + ty_t) - h(v)\| \le tc\,\|y_t\|$ for t small enough. Since $(y_t) \to y := A(x)$ as $t \to 0_+$, we get $\|x\| \le c\,\|A(x)\|$. Thus A is injective and its image is a complete subspace of Y, as easily seen. Let us show that this image is dense, which will prove that A is an isomorphism. Given $y \in Y$, let us set $x_t := t^{-1}(h(v + ty) - h(v))$, so that $v + ty = g(h(v) + tx_t)$ and $ty = g(u + tx_t) - v = g(u + tx_t) - g(u) = tA(x_t) + tz_t$, where $(z_t) \to 0$ as $t \to 0_+$ since (x_t) is bounded. Thus $d(y, A(X)) \le \lim_t \|z_t\| = 0$ and $y \in \mathrm{cl}(A(X)) = A(X)$ which is closed in Y.

Thus A is an isomorphism and the inverse mapping theorem shows that the inverse h of g is differentiable at v with inverse $A^{-1} := g'(u)^{-1}$. Thus h is of class C^1 and since $(f^L)' = h$, we get that f^L is of class C^2.

When f is of class C^k, $(f^L)' = h$ is of class C^{k-1} as an induction shows. Thus, f^L is of class C^k. $\square$

Remark. If $f : U \to \mathbb{R}$ is Hadamard differentiable, i.e. directionally differentiable and if f' is a bijection from U onto an open subset V of Y whese inverse h is directionally differentiable, a computation similar to the one of the preceding proof shows that f^E is directionally differentiable and $Df^E(v) = h(v)$ for all $v \in V$, so that $v = Df(u)$ is equivalent to $u = Df^E(v)$. $\square$

Remark. Suppose $f : U \to \mathbb{R}$ is an Ekeland function such that its Fréchet subdifferential $\partial_F f$ satisfies the following condition for some $u \in U$, $v \in \partial_F f(u)$:

there exist some c, $r > 0$ such that $d(u, (\partial_F f)^{-1}(w)) \leq cd(v, w)$ for all $w \in B(v, r)$.

Then, by estimates similar to the computation of the preceding proof one gets that $u \in -\partial_F(-f^E)(v)$. $\square$

One may wonder to what extent the definition of Ekeland functions depends on the choice of the subdifferential. A comparison between Examples 1 and 2 shows that the class of Ekeland functions for the Moreau type subdifferential of Example 2 is much larger than the class of Ekeland functions for the Fréchet or the Hadamard subdifferentials. We just present obvious observations in this vein.

Proposition 4.4 *(a) If a function $f : X \to \mathbb{R}_\infty$ is an Ekeland function for a subdifferential ∂, then it is an Ekeland function for any smaller subdifferential.*

(b) A function $f : X \to \mathbb{R}_\infty$ is an Ekeland function for a subdifferential ∂ if, and only if it is an Ekeland function for the limiting subdifferential ∂_L associated with ∂.

As a consequence, one gets that in an Asplund space the notions of Ekeland function for the Fréchet subdifferential ∂_F and for the limiting subdifferential ∂_L associated with ∂_F coincide. We also note that for important classes of functions such as the class of paraconvex (or semiconvex) functions, or, more generally, the class of approximately convex functions, usual subdifferentials coincide (see [26] for instance).

Preservation of the Ekeland property under composition or usual operations is not ensured in general. Such a question will be dealt with in Section 4. For the moment, we just note the following elementary result.

Proposition 4.5 *If $A : X \to Y$ is a surjective continuous linear map between Banach spaces and if $g : Y \to \mathbb{R}$ is an Ekeland function for the Hadamard, the Fréchet, the Clarke or the limiting subdifferential, then $f := g \circ A$ is an Ekeland function on X. Moreover, for all $x^* \in A^T(Y^*)$ one has $f^E(x^*) = g^E(y^*)$ for $y^* \in (A^T)^{-1}(x^*)$.*

If g is a classical Legendre function on an open subset Y_0 of Y and if A is an isomorphism, then $f := g \circ A$ is a classical Legendre function on $X_0 := A^{-1}(Y_0)$.

Proof. It is not difficult to check under the assumption that $\partial f(x) = \partial g(A(x)) \circ A$ for all $x \in X$ for $\partial = \partial_D$, ∂_F, ∂_C or ∂_L. Given $x_1, x_2 \in X$, $x^* \in \partial f(x_1) \cap \partial f(x_2)$, so that there exists $y_1^* \in \partial g(A(x_1))$, $y_2^* \in \partial g(A(x_2))$ satisfying $x^* = y_1^* \circ A$, $x^* = y_2^* \circ A$, hence $y_1^* = y_2^* = y^*$. Then, from the assumption that g is an Ekeland function, we deduce that for $y_1 := A(x_1)$, $y_2 := A(x_2)$, $y^* := y_1^* = y_2^*$, we have

$$\langle y_1, y^* \rangle - g(y_1) = \langle y_2, y^* \rangle - g(y_2)$$

hence $\langle x_1, x^* \rangle - f(x_1) = \langle y_1, y^* \rangle - g(y_1) = \langle y_2, y^* \rangle - g(y_2) = \langle x_2, x^* \rangle - f(x_2)$: f is an Ekeland function.

If $A : X \to Y$ is an isomorphism and if $g : Y_0 \to \mathbb{R}$ is a classical Legendre function, then $f := g \circ A$ is such that $x \mapsto f'(x) = g'(A(x)) \circ A$ is a Stepanovian bijection from $X_0 := A^{-1}(Y_0)$ onto $A^T(g'(Y_0))$ whose inverse is also Stepanovian. Then, for $x^* \in A^T(g'(Y_0))$, $x \in X$ with $x^* = A^T(g'(Ax))$, one has $f^L(x^*) = \langle x, x^* \rangle - f(x) = \langle Ax, g'(Ax) \rangle - g(Ax) = g^L((A^T)^{-1}x^*)$. $\square$

4.2 Legendre functions and Legendre transform

The following definition stems from our wish to get a symmetric concept. It is also motivated by the convex case in which the domain of f^E is the image of ∂f which is not necessarily convex, while a natural extension of f^E is the Fenchel conjugate whose domain is convex and which enjoys nice properties (lower semicontinuity, local Lipschitz property on the interior of its domain...).

Definition 4.6 *Let X be a Banach space with dual Y. A proper function $f : X \to \mathbb{R}_\infty$ is said to be a (generalized) Legendre function for a subdifferential ∂ if it is lower semicontinuous (l.s.c.) on its domain and if there exists a function $f^L : Y \to \mathbb{R}_\infty$ which is l.s.c. on its domain and is such that*
(a) f and f^L are Ekeland functions and $f^L \mid \partial f(X) = f^E$;
(b) for any $x \in \mathrm{dom} f$ there is a sequence $(x_n, y_n, r_n)_n$ in $J^\partial f$ such that $(x_n, \langle x_n - x, y_n \rangle, r_n) \to (x, 0, f(x))$;
(b') for any $y \in \mathrm{dom} f^L$ there is a sequence $(y_n, x_n, s_n)_n$ in $J^\partial f^L$ such that $(y_n, \langle x_n, y_n - y \rangle, s_n) \to (y, 0, f^L(y))$;
(c) the relations $x \in X$, $y \in \partial f(x)$ are equivalent to $y \in Y$, $x \in \partial f^L(y)$.

The preceding definition is symmetric. Indeed, taking $(f^L)^L = f$, for any $x \in \partial f^L(Y) = \mathrm{dom}\partial f$ (by (c)), we see that $(f^L)^L(x) = f(x) = \langle x, y \rangle - f^E(y) = (f^L)^E(x)$ for $y \in \partial f(x)$.

Condition (b) ensures that f is determined by its restriction to $\mathrm{dom}\partial f$. In fact, for any $x \in \mathrm{dom} f$ one has

$$f(x) = \liminf_{x' (\in \mathrm{dom}\partial f) \to x} f(x')$$

since $f(x) \leq \liminf_{x' \to x} f(x')$ and (b) implies $f(x) = \lim_n f(x_n)$ for some sequence $(x_n) \to x$ in $\mathrm{dom}\partial f$. Similarly, f^L is determined by its restriction to $\mathrm{dom}\partial f^L = \partial f(X)$. Conditions (b') and (c) imply that f^L is determined by f.

Condition (b) can be simplified when ∂f is locally bounded, in particular when ∂f is. contained in $\partial_C f$ and f is locally Lipschitzian on its domain. In that case, condition (b) is equivalent to the simpler condition

(b_0) for any $x \in \mathrm{dom} f$ there exists a sequence $(x_n)_n$ in $\mathrm{dom}\,\partial f$ such that $((x_n, f(x_n)) \to (x, f(x))$.

A similar observation holds for condition (b'). The interest of the stringent conditions (b) and (b') is to make the extensions as close as possible to f and f^E respectively.

The following result uses the key fact of nonsmooth analysis that condition (b) of Definition 4.6 is satisfied for the Fréchet subdifferential on spaces having smooth bump functions.

Proposition 4.7 *Let X be a Banach space such that X and X^* have C^1 smooth bump functions. Let f be a lower semicontinuous Ekeland function whose Ekeland transform f^E is a lower semicontinuous Ekeland function. Then f is a (generalized) Legendre function for the Fréchet subdifferential whenever condition (c) of Definition 4.6 is satisfied.*

Let us give some examples of Legendre functions.

Proposition 4.8 *(a) Any classical Legendre function is a (generalized) Legendre function.*

(b) Any l.s.c. proper convex function is a (generalized) Legendre function.

(c) If $f : W \to \mathbb{R}$ is a continuous, Hadamard (resp. Fréchet) differentiable, concave function on an open convex subset W of X whose concave conjugate f_* is a continuous, Hadamard (resp. Fréchet) differentiable function on a convex subset of X^*, then f is a (generalized) Legendre function with $f^L = f_*$ for the Hadamard (resp. Fréchet) subdifferential.

(d) If f is a quadratic function whose Hessian A is an isomorphism, then f is a Legendre function.

Proof. The first assertion is obvious: for condition (b) of Definition 4.6 one takes $(x_n, y_n, r_n) = (x, f'(x), f(x))$ and we make a similar choice in (b'). Assertion (b) is a consequence of [29, Cor. 1.2], taking for f^L the Fenchel conjugate f^* of f. If f is as in assertion (c), and if $g := -f$, one has $f^L = f_* = -g^*(-\cdot)$, so that $y \in \partial f(x)$ if, and only if, $-y = Dg(x)$, if, and only if $x \in \partial g^*(-y) = \{Df^L(y)\} = \partial f^L(y)$. When the Hessian A is an isomorphism, assertion (d) is a special case of assertion (a) since a quadratic function whose Hessian is an isomorphism is a classical Legendre function. □

Example 10. Let $\mathcal{F}(X)$ be the set of partially quadratic functions on a Banach space X, i.e. the set of functions f given by $f(x) := \frac{1}{2}\langle A(x-a), x-a\rangle - \langle b, x-a\rangle + c$ when x belongs to some closed affine subspace $W + a$ of X (with W a closed linear subspace of X, $a \in X$) and $f(x) = +\infty$ when $x \in X \backslash (W + a)$, where $A : W \to W^*$ is some continuous symmetric linear map, $b \in W^*$, $c \in \mathbb{R}$. Let ∂ be either the Fréchet or the Hadamard subdifferential. Then f is an Ekeland function. In fact, given $y \in Y := X^*$, $x_1, x_2 \in X$ such that $y \in \partial f(x_i)$ one has

$y \mid W = A(x_i - a) - b$ and

$$\begin{aligned}\langle y, x_i\rangle - f(x_i) &= \langle y, a\rangle + \langle A(x_i - a) - b, x_i - a\rangle - \frac{1}{2}\langle A(x_i - a), x_i - a\rangle + \langle b, x_i - a\rangle - c \\ &= \frac{1}{2}\langle A(x_i - a), x_i - a\rangle + \langle y, a\rangle - c\end{aligned}$$

and

$$\langle A(x_1 - a), x_1 - a\rangle - \langle A(x_2 - a), x_2 - a\rangle = \langle A(x_1 - x_2), x_1 - a\rangle + \langle A(x_2 - a), x_1 - x_2\rangle = 0$$

since A is symmetric and $A(x_1 - a) = y \mid W + b = A(x_2 - a)$. Moreover, when A is invertible, or, more generally, when $A(W)$ is closed and the null space of A is complemented, in particular when W is finite dimensional, f^E is also a partially quadratic function. In that case, we see that $\operatorname{dom} \partial f = \operatorname{dom} f$ and that the domain of f^E is the set of $y \in Y$ such that $(y + b) \mid W \in A(W)$, so that f is a Legendre function with $f^L = f^E$. $\square$

4.3 Operations on Ekeland and Legendre functions

In the present subsection X is a n.v.s. with dual Y and ∂ is a general subdifferential. The compatibility of the usual operations with respect to the concepts we study is not as rich as what occurs for the Fenchel transform, as simple examples show for the sum. However, some simple properties can be devised, in particular for the *infimal convolution* $\square$ which is defined by

$$(g \square h)(x) := \inf\{g(u) + h(v) : u, v \in X,\ u + v = x\}$$

for two functions g, h on X. The infimal convolution $g \square h$ is said to be *exact* at $x \in X$ if there exists some $u, v \in X$ such that $u + v = x$ and $(g \square h)(x) = g(u) + h(v)$.

Proposition 4.9 *(a) If f is an Ekeland function, then for any positive real number λ the function λf is an Ekeland function and $(\lambda f)^E(y) = \lambda f^E(\lambda^{-1} y)$. Moreover, if f is a Legendre function, then λf is a Legendre function.*

(b) If $f_i : X_i \to \mathbb{R}_\infty$ is an Ekeland (resp. Legendre) function for $i = 1, ..., k$, then, f given by $f(x) := f_1(x_1) + ... + f_k(x_k)$ for $x := (x_1, ..., x_k)$ is an Ekeland (resp. Legendre) function.

Proof. (a) Let $x_i \in X$ be such that $y \in \partial(\lambda f)(x_i)$ for $i = 1, 2$. Then $\lambda^{-1} y \in \partial f(x_i)$ and

$$\langle x_i, y\rangle - \lambda f(x_i) = \lambda \left(\langle x_i, \lambda^{-1} y\rangle - f(x_i)\right) = \lambda f^E(\lambda^{-1} y) \qquad i = 1, 2,$$

so that λf is an Ekeland function and $(\lambda f)^E(y) = \lambda f^E(\lambda^{-1} y)$. When f is a Legendre function, setting $(\lambda f)^L(y) = \lambda f^L(\lambda^{-1} y)$, we easily get that λf is a Legendre function.

(b) Since f is a separable function, when $x := (x_1, ..., x_k)$, $y := (y_1, ..., y_k)$, we have $y \in \partial f(x)$ if, and only if, $y_i \in \partial f(x_i)$ for $i = 1$,...,k. The fact that f is an Ekeland function when f_i are Ekeland functions ensues and $f^E(y) = f_1^E(y_1) + ... + f_k^E(y_k)$. From this formula one sees that f is a Legendre function when $f_1, ..., f_k$ are Legendre functions as f^L is also separable. $\square$

In order to study sums and infimal convolutions, it will be useful to introduce the following definition.

Definition 4.10 *A pair of functions (g,h) on X is said to be ∂-convolable at $x \in X$ if for every $u_i, v_i \in X$, satisfying $u_i + v_i = x$, $\partial g(u_i) \cap \partial h(v_i) \neq \varnothing$ for $i = 1,2$, one has $g(u_1) + h(v_1) = g(u_2) + h(v_2)$. Then, one writes $(g \Box_\partial h)(x) = g(u) + h(v)$ for $u, v \in X$ satisfying $x = u + v$ and $\partial g(u) \cap \partial h(v) \neq \varnothing$.*

The functions g, h are said to be ∂-convolable if they are ∂-convolable at all $x \in C(g,h)$ with

$$C(g,h) := \{x \in X : \exists u, v \in X,\ u + v = x,\ \partial g(u) \cap \partial h(v) \neq \varnothing\}.$$

Then $(g \Box_\partial h)(x)$ is defined as above for $x \in C(g,h)$ and $(g \Box_\partial h)(x) = +\infty$ otherwise.

Examples 11. (a) If g and h are convex functions, then (g,h) is ∂-convolable and $g \Box_\partial h$ coincides with the infimal convolution $g \Box h$ on $C(g,h)$: given $x, u_i, v_i \in X$, $y_i \in \partial g(u_i) \cap \partial h(v_i)$, with $u_i + v_i = x$ for $i = 1,2$, one has

$$g(u_2) + h(v_2) \geq g(u_1) + \langle y_1, u_2 - u_1 \rangle + h(v_1) + \langle y_1, v_2 - v_1 \rangle = g(u_1) + h(v_1)$$

and, by symmetry, equality. Moreover, for all $w \in X$ one has

$$g(x - w) + h(w) \geq g(u_1) + \langle y_1, x - w - u_1 \rangle + h(v_1) + \langle y_1, w - v_1 \rangle = g(u_1) + h(v_1),$$

hence $(g \Box h)(x) \geq (g \Box_\partial h)(x)$, and in fact equality holds since the preceding relations are equalities for $w = v_1$. Similar properties hold if g and h are (nonconvex) integral functionals on $L_1(S,E)$ and ∂ is the firm or the limiting subdifferential as in Example 9. In fact, the result holds for arbitrary functions when ∂ is the Moreau-Rockafellar subdifferential.

(b) If g and h are concave functions, then $g \Box_\partial h$ coincides with the supremal convolution $-((-g) \Box (-h))$ since $y_i \in \partial g(u_i) \cap \partial h(v_i)$ if, and only if g and h are differentiable at u_i and v_i respectively (and thus $g - y_i$ and $h - y_i$ attain their maximum at such points).

(c) Given symmetric operators B, $C : X \to X^*$ such that $B + C$ is an isomorphism from X onto $Y := X^*$, b, $c \in Y$, β, $\gamma \in \mathbb{R}$, let g and h be given by $g(x) := (1/2)\langle Bx, x \rangle - \langle b, x \rangle + \beta$, $h(x) := (1/2)\langle Cx, x \rangle - \langle c, x \rangle + \gamma$. Here ∂ coincides with derivation. Then, for all $x \in X$, the system $g'(u) = h'(v)$, $u + v = x$ is equivalent to

$$\begin{aligned} Bu - Cv &= b - c \\ u + v &= x \end{aligned}$$

and its unique solution is given by

$$\begin{aligned} u &= (B + C)^{-1}(Cx + b - c), \\ v &= (B + C)^{-1}(Bx - b + c). \end{aligned}$$

Thus (g,h) is ∂-convolable. Note that $g \Box_\partial h$ is also a quadratic polynomial: for $A := (B+C)^{-1}$, since $BAC = B - BAB = CAB$, one has

$$(g \Box_\partial h)(x) = \frac{1}{2}\langle BACx, x \rangle - \langle CAb + BAc, x \rangle - \frac{1}{2}\langle b - c, A(b - c) \rangle + \beta + \gamma.$$

Moreover, for all $x \in X$ one has $D(g\Box_\partial h)(x) = BACx - CAb - BAc = Dg(u) = Dh(v)$ for u, v as above. Then, setting $y := D(g\Box_\partial h)(x)$, one has

$$(g\Box_\partial h)^E(y) = \langle x, y\rangle - (g\Box_\partial h)(x) = \langle u+v, y\rangle - (g(u)+h(v)) = g^E(y) + h^E(y).$$

(d) If g and h are differentiable and if for all $x \in X$ the equation $g'(u) - h'(x-u) = 0$ has a unique solution u_x, then (g,h) is convolable and $(g\Box_\partial h)(x) = g(u_x) + h(x-u_x)$. This fact occurs when the map $(u,v) \mapsto (g'(u) - h'(v), u+v)$ is injective, as in the preceding example or when the minimization of $u \mapsto g(u) + h(x-u)$ has a unique solution.

(e) If g, h are saddle functions on $X' \times X''$ then (g,h) are ∂-convolable for the subdifferential ∂ defined in Example 6 provided that when $y_1 \in \partial g(u_1) \cap \partial h(v_1)$ and $y_2 \in \partial g(u_2) \cap \partial h(v_2)$ with $u_1 + v_1 = u_2 + v_2 = x$ one has $\langle x, y_1\rangle = \langle x, y_2\rangle$. In fact, if $x := u_i + v_i$ with $u_i := (u_i', u_i'')$, $v_i := (v_i', v_i'')$ and if $y_i := (y_i', y_i'') \in \partial g(u_i', u_i'') \cap \partial h(v_i', v_i'')$ for $i = 1, 2$ one has

$$\begin{aligned} g(u_1', u_1'') - \langle y_1', u_1'\rangle - \langle y_1'', u_1''\rangle &= g(u_2', u_2'') - \langle y_2', u_2'\rangle - \langle y_2'', u_2''\rangle, \\ h(v_1', v_1'') - \langle y_1', v_1'\rangle - \langle y_1'', v_1''\rangle &= h(v_2', v_2'') - \langle y_2', v_2'\rangle - \langle y_2'', v_2''\rangle \end{aligned}$$

hence, by addition

$$g(u_1) + h(v_1) - \langle y_1, x\rangle = g(u_2) + h(v_2) - \langle y_2, x\rangle$$

and $g(u_1) + h(v_1) = g(u_2) + h(v_2)$.

(f) Let $(S, \mathcal{S}, \mu)$ be a σ-finite complete measure space, let E be a seaparable Banach space and let $G, H : S \times E \to \overline{\mathbb{R}}$ be normal integrands such that for a.e. $s \in S$ the functions $G_s := G(s, \cdot)$ and $H_s := H(s, \cdot)$ are ∂-convolable for $\partial = \partial_F$ or $\partial = \partial_L$. Then the integral functionals $g : x \mapsto I_G(x) = \int_S G(s, x(s))d\mu(s)$ and $h : x \mapsto I_H(x) = \int_S H(s, x(s))d\mu(s)$ on $L_1(S, E)$ are ∂-convolable and $g\Box_\partial h$ is the integral functional associated with the integrand F given by $F_s = G_s \Box_\partial H_s$.

(g) Two indicator functions ι_C, ι_D are ∂-convolable since $\partial\iota_C(u) \cap \partial\iota_D(v) \neq \varnothing$ implies that $u \in C$, $v \in D$, hence $\iota_C(u) + \iota_D(v) = 0$. Then $\iota_C \Box_\partial \iota_D$ is an indicator function. $\Box$

A comparison of the ∂-convolution with the infimal convolution is in order. We first generalize Example 11 (a).

Proposition 4.11 *If ∂ is any elementary subdifferential, if g and h are such that for some $x \in C(g,h)$ the function $k_x : u \mapsto g(u) + h(x-u)$ is convex, then (g,h) is ∂-convolable at x and $(g\Box_\partial h)(x) = (g\Box h)(x)$.*

Proof. Let $\overline{u} \in X$ be such that $\partial g(\overline{u}) \cap \partial h(x - \overline{u}) \neq \varnothing$. Then $0 \in \partial g(\overline{u}) - \partial h(x - \overline{u}) \subset \partial k_x(\overline{u})$ since ∂ is an elementary subdifferential. Since k_x is convex, $\overline{u}$ is a minimizer of k_x. Hence $k_x(\overline{u})$ is independent of the choice of $\overline{u}$: (g,h) is ∂-convolable at x and $(g\Box_\partial h)(x) = g(\overline{u}) + h(x - \overline{u}) = (g\Box h)(x)$. $\Box$

Proposition 4.12 *Let ∂ be an arbitrary subdifferential.*

(a) If the infimal convolution $f := g\Box h$ is exact at $x \in \mathrm{dom}\partial f$, then

$$u, v \in X,\ u + v = x,\ g(u) + h(v) = f(x),\ y \in \partial f(x) \Longrightarrow y \in \partial g(u) \cap \partial h(v)$$

and if (g, h) is ∂-convolable at x one has $(g\Box_\partial h)(x) = (g\Box h)(x)$.

(b) If the infimal convolution $f := g\Box h$ is exact at all $x \in \mathrm{dom}\partial f$, and if g and h are Ekeland functions, then f is an Ekeland function and $(g\Box h)^E = g^E + h^E$ on $\mathrm{dom}\,(g\Box h)^E$.

Proof. (a) Let $y \in \partial f(x)$, the infimal convolution being exact at x, so that there exist $u, v \in X$ such that $u + v = x$, $g(u) + h(v) = f(x)$. Let us check that $y \in \partial g(u) \cap \partial h(v)$. Introducing the function $j : X \times X \to \overline{\mathbb{R}}$ given by $j(w, v) := g(w - v) + h(v)$ and noting that $f(w) := \inf_{v\in X} j(w, v)$, from $y \in \partial f(x)$ we get $(y, 0) \in \partial j(w, v)$ and $y \in \partial g(x - v) \cap \partial h(v)$ by [33, Prop. 2.6] or an easy computation. Thus, if (g, h) is ∂-convolable at x one has $(g\Box_\partial h)(x) = g(u) + h(v) = (g\Box h)(x)$.

(b) If the infimal convolution $f := g\Box h$ is exact at all $x \in \mathrm{dom}\partial f$, and if g and h are Ekeland functions, for every $y \in \partial f(X)$ one gets that $\langle y, x\rangle - f(x)$ does not depend on the choice of $x \in (\partial f)^{-1}(y)$: given u, $v \in X$ satisfying $u + v = x$, $g(u) + h(v) = f(x)$, one has $y \in \partial g(u) \cap \partial h(v)$ hence

$$\langle y, x\rangle - f(x) = \langle y, u\rangle - g(u) + \langle y, v\rangle - h(v) = g^E(y) + h^E(y). \tag{9}$$

□

Now let us turn to the ∂-convolution.

Proposition 4.13 *(a) If g and h are ∂-convolable Ekeland functions and if the following condition holds, then $f := g\Box_\partial h$ is an Ekeland function and one has $(g\Box_\partial h)^E = g^E + h^E$:*

$$y \in \partial f(x),\ x = u + v,\ \partial g(u) \cap \partial h(v) \neq \varnothing \Longrightarrow y \in \partial g(u) \cap \partial h(v).$$

(b) This condition is satisfied if g and h are both convex or if g and h are quadratic polynomials given by $g(x) := (1/2)\langle Bx, x\rangle - \langle b, x\rangle + \beta$, $h(x) := (1/2)\langle Cx, x\rangle - \langle c, x\rangle + \gamma$ with $B + C$ invertible.

Proof. (a) Under our assumption, given x_1, $x_2 \in X$ and $y \in \partial f(x_1) \cap \partial f(x_2)$, taking u_i, $v_i \in X$ such that $x_i = u_i + v_i$, $f(x_i) = g(u_i) + h(v_i)$ for $i = 1,\ 2$, using our assumption, one has $y \in \partial g(u_i) \cap \partial h(v_i)$ for $i = 1, 2$ hence

$$\langle y, x_i\rangle - f(x_i) = \langle y, u_i\rangle - g(u_i) + \langle y, v_i\rangle - h(v_i) = g^E(y) + h^E(y)$$

which is independent of $i \in \{1, 2\}$. That ensures that f is an Ekeland function and that $f^E(y) = g^E(y) + h^E(y)$.

(b) If g and h are convex we have seen that $g\Box_\partial h = g\Box h$ and that the condition is satisfied. If g and h are the quadratic forms described above, with $B + C$ invertible, we have seen that

$y \in \partial f(x)$ means that $y = BAC(x) - BAc - CAb$ for $A := (B+C)^{-1}$ and the relations $x := u + v$, $\partial g(u) \cap \partial h(v) \neq \varnothing$ imply that $y = g'(u)$, $y = h'(v)$. $\square$

Now let us show that the introduction of the ∂-convolution is justified by the transforms of sums. In the next result ∂ is an arbitrary subdifferential but the assumption requires a sum rule satisfied if g and h are Lipschitzian and if $\partial = \partial_C$ or if $\partial = \partial_L$, X being an Asplund space.

Proposition 4.14 *Let $f := g + h$, where g, $h : X \to \mathbb{R}_\infty$ are Legendre functions such that (g^L, h^L) is ∂-convolable at $y \in \partial f(X)$. If for all $x \in (\partial f)^{-1}(y)$ one has $\partial f(x) \subset \partial g(x) + \partial h(x)$ then $y \in C(g^L, h^L)$ and $(g+h)^E(y) = (g^L \Box_\partial h^L)(y)$.*

If (g^L, h^L) is ∂-convolable and if $\partial f(x) \subset \partial g(x) + \partial h(x)$ for all $x \in \mathrm{dom}\partial f$, then f is an Ekeland function and $(g+h)^E = (g^L \Box_\partial h^L)$ on $\mathrm{dom} f^E$.

Proof. Let $y \in C(g^L, h^L) \cap \partial f(X)$ and let $x_i \in (\partial f)^{-1}(y)$ for $i = 1, 2$. By assumption, there exist $w_i \in \partial g(x_i)$, $z_i \in \partial h(x_i)$ such that $y = w_i + z_i$ for $i = 1, 2$. Then $x_i \in \partial g^L(w_i) \cap \partial h^L(z_i)$ so that $y \in C(g, h)$ and since (g^L, h^L) is ∂-convolable at y, one has $g^L(w_1) + h^L(z_1) = g^L(w_2) + h^L(z_2)$ and, for $i = 1, 2$ one gets

$$\langle x_i, y\rangle - f(x_i) = \langle x_i, w_i + z_i\rangle - g(x_i) - h(x_i) = g^L(w_i) + h^L(z_i)$$

which does not depend on the choice of i. Thus f is an Ekeland function and $(g+h)^E(y) = (g^L \Box_\partial h^L)(y)$. $\square$

One can give a converse in the case ∂ is an elementary subdifferential.

Proposition 4.15 *Let g, $h : X \to \mathbb{R}_\infty$ be Legendre functions such that $f := g + h$ is an Ekeland function for an elementary subdifferential ∂. Then g^L and h^L are ∂-convolable and $f^E = g^L \Box_\partial h^L$ on $C(g^L, h^L)$.*

Proof. Let $y \in C(g^L, h^L)$: there exist some $w, z \in Y$ such that $y = w + z$ and $\partial g^L(w) \cap \partial h^L(z) \neq \varnothing$. Let $x \in \partial g^L(w) \cap \partial h^L(z)$. Since g and h are Legendre functions and since ∂ is an elementary subdifferential, one has $w \in \partial g(x)$, $z \in \partial h(x)$ and $y = w + z \in \partial f(x)$. Then, by definition of g^L, h^L, and f^E,

$$g^L(w) + h^L(z) = \langle w, x\rangle - g(x) + \langle z, x\rangle - h(x) = \langle y, x\rangle - f(x) = f^E(y)$$

so that this sum is independent of the choices of x, w, z: g^L and h^L are ∂-convolable and one has $(g^L \Box_\partial h^L)(y) = g^L(w) + h^L(z) = f^E(y)$. $\square$

Let us turn to composition properties; they encompass Proposition 4.5.

Proposition 4.16 *Let W_0, X_0 be open subsets of Banach spaces W, X respectively, let $G : X_0 \to W_0$ be a mapping of class C^1 whose derivative at every point of X_0 is surjective, G being the restriction of a positively homogeneous mapping from X to W. Let $g : W_0 \to \mathbb{R}_\infty$ be a function such that for any $x_1, x_2 \in X_0$ and $z_1, z_2 \in W^*$ satisfying $z_1 \circ G'(x_1) = z_2 \circ G'(x_2)$ and $z_i \in \partial g(G(x_i))$ for $i = 1, 2$, one has $z_1 = z_2$.*

If g is an Ekeland function, then $f := g \circ G$ is an Ekeland function. Moreover, for any $y \in \partial f(X)$, $x \in (\partial f)^{-1}(y)$, $z \in \partial g(G(x))$ with $y = z \circ G'(x)$, one has $f^E(y) = g^E(z)$.

In particular, this conclusion holds under each of the following assumptions on g, G being as above:

(a) g is a continuous linear form;

(b) $g : W \to \mathbb{R}$ is an Ekeland function and $G = A$ is a surjective linear continuous map;

(c) $g : W \to \mathbb{R}_\infty$ is a closed proper convex function such that

$$\forall x \in X_0 \qquad G'(x)(X) - \mathbb{R}_+(\operatorname{dom} g - G(x)) = W.$$

Proof. Using the fact that G is open at a linear rate at each point of X_0, one can easily check that for any $x \in X_0$ one has $\partial f(x) = \partial g(G(x)) \circ G'(x)$. Moreover, by the Euler relation, for all $x \in X_0$ one has $G'(x).x = G(x)$. Then if $y \in \partial f(x_i)$ for $i = 1, 2$, one can find $z_i \in \partial g(G(x_i))$ such that $y = z_i \circ G'(x_i)$. Our assumption ensures that $z_1 = z_2$, and the Euler relation entails that $G'(x_i).x_i = G(x_i)$, so that

$$\langle y, x_i\rangle - f(x_i) = \langle z_i \circ G'(x_i), x_i\rangle - g(G(x_i)) = \langle z_i, G(x_i)\rangle - g(G(x_i)) = g^E(z_i)$$

is independent of i : f is an Ekeland function. The relation $f^E(y) = g^E(z)$ for $y \in \partial f(x)$, $z \in \partial g(G(x))$ with $y = z \circ G'(x)$ stems from the preceding string of equalities.

(a) When g is a continuous linear form, the relations $z_i \in \partial g(G(x_i))$ imply that $z_i = g$.

(b) When G is a surjective continuous linear map, the relation $z_1 \circ G'(x_1) = z_2 \circ G'(x_2)$ amounts to $z_1 \circ G = z_2 \circ G$, hence $z_1 = z_2$ since G is surjective.

(c) It is known that the qualification condition we assume implies that $\partial f(x) = \partial g(G(x)) \circ G'(x)$ both for the firm and the directional subdifferentials (see for instance [28], Prop. 4.1). The rest of the proof is similar to the preceding case. □

Proposition 4.17 *Let $A : X \to W$ be a surjective continuous linear map between Banach spaces and let g, $h : W \to \mathbb{R}_\infty$ be ∂-convolable Ekeland functions. Then $g \circ A$, $h \circ A$ are ∂-convolable and $(g \circ A)\Box_\partial(h \circ A) = (g\Box_\partial h) \circ A$.*

Proof. Let $x \in C(g \circ A, h \circ A)$ and for $i = 1, 2$ let u_i, $v_i \in X$, $y_i \in \partial(g \circ A)(u_i) \cap \partial(h \circ A)(v_i)$ be such that $u_i + v_i = x$. Then there exists $z_i \in \partial g(A(u_i)) \cap \partial h(A(v_i))$ satisfying $y_i = z_i \circ A$. Since g, h are ∂-convolable one has $g(A(u_1)) + h(A(v_1)) = g(A(u_2)) + h(A(v_2))$. Thus $g \circ A$, $h \circ A$ are ∂-convolable. Moreover, since $A(u_i) + A(v_i) = A(x)$ and $z_i \in \partial g(A(u_i)) \cap \partial h(A(v_i))$, one has

$$((g \circ A)\Box_\partial(h \circ A))(x) = g(A(u_i)) + h(A(v_i)) = (g\Box_\partial h)(A(x)).$$

5 Ekeland sets

The first part of the following definition has been introduced in [31]. Here we use the notion of normal cone to a set S at $x \in S$ associated with a subdifferential ∂ by $N(S, x) := \partial\iota_S(x)$, where ι_S is the indicator function of S.

Definition 5.1 *A subset S of a Banach space will be called an Ekeland set if its indicator function ι_S is an Ekeland function, i.e. if for any $x_1, x_2 \in S$ and any $y \in N(S, x_1) \cap N(S, x_2)$ one has $\langle x_1, y\rangle = \langle x_2, y\rangle$.*

A subset S of a Banach space will be called an Legendre set if its indicator function ι_S is a Legendre function.

Any closed convex subset S is a Legendre set and ι_S^E is the support function of S.

Any cone S is an Ekeland set when ∂ is contained in the directional subdifferential ∂_D; moreover, in such a case, ι_S^E is an indicator function. In particular, for $S := \mathbb{R}_+ \times \{0\} \cup \{0\} \times \mathbb{R}_+$, one has $\iota_S^E = \iota_S$. If S is a closed convex cone, then ι_S^E is the indicator function of the polar cone of S.

Let us note that if S is an Ekeland set, then ι_S^E is a positively homogeneous function.

In the following statement, we use the Fréchet subdifferential ∂_F.

Proposition 5.2 *If the distance function d_S to a closed subset S of a n.v.s. is an Ekeland function then S is an Ekeland set. Moreover, for all $y \in \partial_F \iota_F(S)$ one has*

$$\iota_S^E(y) = \|y\|\, d_S^E(y).$$

Conversely, if a closed subset S of a Hilbert space (or a reflexive Banach space with the Kadec-Klee property and a Gâteaux smooth norm off 0) X is an Ekeland set, then its associated distance function d_S is an Ekeland function.

Proof. Suppose d_S is an Ekeland function. Let $y \in X^*$ and let $x_1, x_2 \in S$ be such that $y \in \partial_F \iota_S(x_i)$ for $i = 1, 2$. Let u be an element of the unit sphere S_{X^*} of X^* be such that $y = \|y\|\, u$. Then, since by a well known result (see [34, Lemma 4.21] for instance), one has

$$\partial_F d_S(x) = N_F(S, x) \cap B_{X^*}$$

for all $x \in S$, where B_{X^*} is the unit ball of X^*, and since the Fréchet normal cone $N_F(S, x)$ is $\partial_F \iota_S(x)$, one has $u \in \partial_F d_S(x_i)$ for $i = 1, 2$. Since d_S is an Ekeland function, one has

$$\langle x_1, u\rangle = \langle x_1, u\rangle - d_S(x_1) = \langle x_2, u\rangle - d_S(x_2) = \langle x_2, u\rangle$$

It follows that $\langle x_1, y\rangle = \|y\|\, \langle x_1, u\rangle = \|y\|\, \langle x_2, u\rangle = \langle x_2, y\rangle$ and S is an Ekeland subset of X. Moreover, for $i = 1, 2$ one has $\iota_S^E(y) = \langle x_i, y\rangle = \|y\|\, \langle x_i, u\rangle - \|y\|\, d_S(x_i) = \|y\|\, d_S^E(y)$.

Conversely, let S be a closed Ekeland subset of X. Let $y \in X^*$ and let $x \in X$ be such that $y \in \partial_F d_S(x)$. If $y = 0$, then $x \in S$ and $\langle x, y\rangle - d_S(x) = 0$, independently of the choice of $x \in (\partial d_S)^{-1}(y)$. Suppose $y \neq 0$. Then, under our assumptions, by [6, Lemma 6], there exists some $w \in S$ such that $d_S(x) = \|x - w\|$ and $y \in S(x - w)$ where

$$S(x - w) := \{y \in S_{X^*} : \langle x - w, y\rangle = \|x - w\|\}.$$

Moreover $y \in N_F(S, w)$. Then,

$$\langle x, y\rangle - d_S(x) = \langle x, y\rangle - \|x - w\| = \langle x, y\rangle - \langle x - w, y\rangle = \langle w, y\rangle = \iota_S^S(y)$$

is independent of the choice of $x \in (\partial d_S)^{-1}(y)$. Thus d_S is an Ekeland function. $\square$

Example 12. Let S be the epigraph of the function $x \mapsto -x^2$ from $\mathbb{R}$ to $\mathbb{R}$. Then Example 4 and the following proposition show that S is an Ekeland set. It uses the next elementary lemma which is valid for any usual subdifferential ∂ contained in the directional subdifferential ∂_D (or in the limiting subdifferential associated with ∂_D if f is continuous).

Lemma 5.3 *If E is the epigraph of a lower semicontinuous function $f : X \to \mathbb{R}_\infty$, if $(x, t) \in E$ and if $(x^*, -t^*) \in N(E, (x, t))$ with $t^* > 0$, then $t = f(x)$ and $x^*/t^* \in \partial f(x)$.*

Proof. Assume $t > f(x)$. Then $(0, -1) \in T(E, (x, t))$, as easily seen. Then, the relation

$$t^* = \langle (x^*, -t^*), (0, -1)\rangle \leq 0$$

leads to a contradiction to the assumption $t^* > 0$. Thus $t = f(x)$. The inclusion $x^*/t^* \in \partial f(x)$ is then a consequence of the well known fact that $u^* \in \partial f(x)$ if, and only if $(u^*, -1) \in N(E, (x, t))$. $\square$

In the next statement, we say that a function $f : X \to \mathbb{R}_\infty$ is *quiet* if f is calm at x for all $x \in \mathrm{dom} f$, i.e. if there exists $c > 0$ such that $f(w) - f(x) \leq c\,\|w - x\|$ for all w in a neighborhood of x.

Proposition 5.4 *Let E be the epigraph E of a lower semicontinuous function f and let S be the hypograph of $-f$: $E := \{(x, r) : r \geq f(x)\}$, $S := \{(x, s) : s \leq -f(x)\}$. If E is an Ekeland set, then f is an Ekeland function and $\iota_S^E(y, s) = sf^E(s^{-1}y) = (sf)^E(y)$ for every $(y, s) \in \partial f(X) \times (0, +\infty)$; in particular $f^E(y) = \iota_S^E(y, 1)$ for all $y \in \partial f(X)$.*

Conversely, if f is a quiet lower semicontinuous Ekeland function, then its epigraph E is an Ekeland set.

Proof. Suppose E is an Ekeland set. Let $x_1, x_2 \in X$, $y \in X^*$ be such that $y \in \partial f(x_1) \cap \partial f(x_2)$. Then, for $i = 1, 2$, one has $(y, -1) \in N(E, (x_i, f(x_i)))$, hence

$$\langle x_1, y\rangle - f(x_1) = \langle (x_1, f(x_1)), (y, -1)\rangle = \langle (x_2, f(x_2)), (y, -1)\rangle = \langle x_2, y\rangle - f(x_2),$$

so that f is an Ekeland function. Moreover, by the preceding relations, for every $(y, s) \in \partial f(X) \times (0, +\infty)$ one has $f^E(y) = \iota_E^E(y, -1) = \iota_S^E(y, 1)$ for all $y \in X^*$. Thus, since ι_S^E is positively homogeneous, for $(y, s) \in \partial f(X) \times (0, +\infty)$ one has $(sf)^E(y) = sf^E(s^{-1}y) = s\iota_S^E(s^{-1}y, 1) = \iota_S^E(y, s)$.

Conversely, let f be a quiet Ekeland function and let (x_1, r_1), $(x_2, r_2) \in E$, $(y, s) \in X^* \times \mathbb{R}$ be such that $(y, -s) \in N(E, (x_i, r_i))$ for $i = 1, 2$. When $(y, -s) = (0, 0)$, one has $\langle (y, -s), (x_i, r_i)\rangle = 0$ for $i = 1, 2$. Because f is continuous, that happens whenever $r_i > f(x_i)$ for some $i \in \{1, 2\}$.

Thus, one may suppose $r_i = f(x_i)$. Since $(0,r) \in T(E,(x_i,r_i))$ for every $r > 0$, one has $s \geq 0$. Let $c > 0$ be such that $f(x_i + v) - f(x_i) \leq c\,\|v\|$ for $\|v\|$ small enough. Then, for every $u \in X$ one has

$$df(x_i,u) := \liminf_{(t,v)\to(0_+,u)} (1/t)(f(x_i+tv) - f(x_i)) \leq c\,\|u\|\,,$$

and since $(u,r) \in T(E,(x_i,f(x_i)))$ for $r \geq df(x_i,u)$, one gets $\langle u,y\rangle - cs\,\|u\| \leq 0$ and $\|y\| \leq cs$. Thus, when $s = 0$ one has $(y,-s) = (0,0)$. Therefore we may suppose $s > 0$, so that $s^{-1}y \in \partial f(x_i)$ and

$$\langle (x_i,r_i),(y,-s)\rangle = s\left(\langle x_i, s^{-1}y\rangle - f(x_i)\right) = sf^E(s^{-1}y).$$

Therefore E is an Ekeland set. □

Proposition 5.5 *Let A and B be Ekeland sets. Then $S := A + B := \{a + b : a \in A,\ b \in B\}$ is an Ekeland set.*

Proof. Since $\iota_S = \iota_A \Box \iota_B$ and the infimal convolution is exact, that follows from Proposition 4.12. □

Proposition 5.6 *Let $A : X \to Y$ be a surjective continuous linear map and let G be an Ekeland set in Y. Then $F := A^{-1}(G)$ is an Ekeland set in X.*

Proof. For $x_1,\ x_2 \in F$ such that there exists $x^* \in N(F,x_1) \cap N(F,x_2)$, we can find $y^* \in N(G,A(x_1)) \cap N(G,A(x_2))$ such that $x^* = A^T(y^*)$ (see [34, Thm 2.111]). Then one has $\langle x^*,x_1\rangle = \langle y^*,A(x_1)\rangle = \langle y^*,A(x_2)\rangle = \langle x^*,x_2\rangle$. Thus F is an Ekeland set. □

A notion of Ekeland relation can easily be introduced and used jointly with a notion of coderivative (see [31]), but we shall not use it here.

6 The Lax formula and characteristics

Let us return to the Hamilton-Jacobi equation in the case the Hamiltonian H depends only on its variable y in $Y := X^*$:

$$D_t u(x,t) + H(D_x u(x,t)) = 0 \qquad (x,t) \in X \times \mathbb{P}, \tag{10}$$

$$u(x,0) = g(x) \qquad x \in X. \tag{11}$$

Here g and H are given functions on X and X^* respectively with values in $\mathbb{R}_\infty$. We assume that the epigraph E of H (or the hypograph S of $-H$) is a Legendre set, so that H is a Legendre function. We consider the function u defined by analogy with the Lax-Oleinik-Hopf formula by

$$u(x,t) := \left(g \Box (tH)^L\right)(x). \tag{12}$$

In particular, when H is a closed proper convex function, the preceding formula coincides with the Lax-Oleinik-Hopf formula given by

$$u(x,t) := \inf_{w\in X} \sup_{p\in Y} \left(g(w) + \langle p, x-w\rangle - tH(p)\right) = \left(g\Box(tH)^*\right)(x),$$

where $(tH)^*$ is the Fenchel conjugate of tH. Following [17], [35], [36], let us introduce the functions F, G given by

$$\begin{aligned} F(y,r) &:= \iota_E(y,-r) := \iota_S(y,r) \qquad\qquad (p,r) \in Y\times\mathbb{R},\\ G(x,t) &:= g(x) + \iota_{\{0\}}(t) \qquad\quad (x,t)\in X\times\mathbb{R}. \end{aligned}$$

Since the indicator function ι_E of E is a Legendre function, its composition F with the isomorphism $A : (x,r) \mapsto (x,-r)$ also is a Legendre function.

Lemma 6.1 *The function u given by (12) coincides with the function $F^L\Box G$ on $X\times(0,+\infty)$. Moreover, for $(x,t)\in X\times(0,+\infty)$ the infimal convolution $\left(F^L\Box G\right)(x,t)$ is exact if, and only if, the infimal convolution $\left(g\Box(tH)^L\right)(x)$ is exact.*

Proof. Proposition 5.4 ensures that $F^L(x,t) = (tH)^L(x)$ for all $(x,t)\in X\times\mathbb{P}$. Moreover

$$\begin{aligned} \left(F^L\Box G\right)(x,t) &= \inf\{g(w) + \iota_{\{0\}}(t-s) + (sH)^L(x-w) : (w,s)\in X\times\mathbb{R}\}\\ &= \inf\{g(w) + (tH)^L(x-w) : w\in X\} = \left(g\Box(tH)^L\right)(x). \end{aligned}$$

The assertion about exactness (i.e. attainment) ensues.

When H is a closed proper convex function, since F is closed proper convex, one has $F^L = F^*$. Hence, for $(x,t)\in X\times\mathbb{R}_+$, one has

$$\begin{aligned} F^L(x,t) &= F^*(x,t) = \sup\{\langle x,p\rangle - rt : (p,r)\in E\}\\ &= \sup\{\langle x,p\rangle - rt : p\in \operatorname{dom} H,\ r\geq H(p)\} = (tH)^*(x). \end{aligned}$$

□

In the next two results we take for ∂ a subdifferential contained in ∂_D (or the limiting subdifferential associated with ∂_D if H is continuous).

Proposition 6.2 *Let H be a function whose epigraph is a Legendre set and let g be an arbitrary lower semicontinuous function such that for some $\omega > 0$ and all $t\in(0,\omega)$ the infimal convolution $u(\cdot,t) := g\Box(tH)^L$ is exact. Then u is an unilateral solution of equation (10) on $X\times(0,\omega)$ in the sense that for all $(x,t)\in X\times(0,\omega)$ and all $(p,q)\in\partial u(x,t)$ one has*

$$q + H(p) = 0.$$

Proof. Let $(x,t) \in X \times (0,\omega)$ and let $(p,q) \in \partial u(x,t)$. Since the infimal convolution $\big(F^L \Box G\big)$ is exact at (x,t) there exists $w \in X$ such that $u(x,t) = F^L(x-w,t) + G(w,0)$. Then we have $(p,q) \in \partial F^L(x-w,t)$. Thus, we have $(x-w,t) \in \partial F(p,q)$ or $(x-w,-t) \in \partial \iota_E(p,-q)$. As t is positive, this inclusion means that $t^{-1}(x-w) \in \partial H(p)$ and the relation $-q = H(p)$ holds in view of Lemma 5.3. $\Box$

The condition that the infimal convolution is exact is satisfied under lower semicontinuity and coercivity assumptions in the finite dimensional case or in the reflexive case.

Now let us tackle the links with the method of characteristics. In the present case, because H does not depend on x,t,z, the characteristic system can be given a simple form:

$$\begin{aligned} &\widehat{x}'(s) \in \partial H(\widehat{y}(s)) && \widehat{x}(0) = w, \\ &\widehat{y}'(s) = 0 && \widehat{y}(0) = w^*, \\ &\widehat{z}'(s) = \langle \widehat{x}'(s), \widehat{y}(s)\rangle - H(\widehat{y}(s)) && \widehat{z}(0) = g(w) \end{aligned}$$

Given $(w,w^*) \in \partial g$ and $v \in \partial H(w^*)$, a solution to this system is given by

$$\widehat{x}(s) = w + sv, \qquad \widehat{y}(s) = w^*, \qquad \widehat{z}(s) = g(w) + s(\langle v, w^*\rangle - H(w^*)).$$

The following result extends [27, Theorem 2.2] from the case of a strictly convex Hamiltonian to the case of a (generalized) Legendre Hamiltonian; see [44], [45] for related results dealing with regularity properties. It is also a partial extension of [41, Thm 2.4] since in that paper H also depends on x.

Proposition 6.3 *Let H be a Legendre function and let g be an arbitrary lower semicontinuous function such that for some $t \in (0,\omega)$ the infimal convolution $u(\cdot,t) := g\Box(tH)^L$ is exact at $x \in X$. If $\partial u(\cdot,t)(x)$ is nonempty, there exist $w \in X$ and $w^* \in \partial g(w)$ such that the characteristic curve emanating from $(w,w^*,g(w))$ satisfies $\widehat{x}(t) = x$, $\widehat{y}(t) = w^*$, $\widehat{z}(t) = u(x,t)$.*

Thus, if the infimal convolution $u(\cdot,t) := g\Box(tH)^L$ is exact at each point of $\mathrm{dom}\partial u(\cdot,t)$, the subjet of $u(\cdot,t)$ is contained in the image of the subjet of g by the flow defined by the characteristic system.

Proof. Given $(x,t) \in X \times (0,\omega)$ and $w \in X$ such that $u(x,t) = (tH)^L(x-w) + g(w)$, for all $p \in \partial u(\cdot,t)(x)$, one has $p \in \partial(tH)^L(x-w) \cap \partial g(w)$. Then, since tH is a Legendre function by Proposition 4.9 (a), one gets $t^{-1}(x-w) \in \partial H(p)$. Setting $w^* := p$, $v := t^{-1}(x-w)$ we see that x is the value at $s=t$ of the characteristic curve $\widehat{x} : s \mapsto w + sv$. Correspondingly, since $v \in \partial H(w^*)$,

$$\begin{aligned} \widehat{z}(t) &= g(w) + t\left(\langle v, w^*\rangle - H(w^*)\right) = g(w) + tH^L(v) \\ &= g(w) + tH^L(t^{-1}(x-w)) = g(w) + (tH)^L(x-w) \\ &= u(x,t). \end{aligned}$$

Thus $(x,p,u(x,t))$ is the image of $(w,w^*,g(w))$ by the flow $(\widehat{x},\widehat{y},\widehat{z})$ at time t. $\Box$

The following examples fall outside the usual convex framework of [14], [35].

Example 13. Let X be a reflexive Banach space, let B, C be symmetric linear continuous operators from X to X^*, B being invertible, and let g and H be given by $g(x) = \frac{1}{2}\langle Cx, x\rangle$, $H(p) = \frac{1}{2}\langle B^{-1}p, p\rangle$. Then H is a locally Lipschitz Legendre function and $H^L(x) = \frac{1}{2}\langle Bx, x\rangle$. Its epigraph is a Legendre set by Proposition 5.4. Then, Proposition 6.2 and a simple computation show that the function u given by

$$u(x,t) := (g\Box_\partial(tH)^L)(x) = \frac{1}{2}\langle (I + tB^{-1}C)^{-1}Cx, x\rangle$$

is a unilateral solution of equation (H-J) on $X \times (0,\omega)$ with $\omega := \|B^{-1}C\|^{-1}$ since $t^{-1}B + C$ is invertible when $t \in (0,\omega)$.

Example 14. Let $X = \mathbb{R}^2$ and let g, H be given by $g(x_1,x_2) = \frac{1}{2}x_1^2 - \frac{1}{2}x_2^2$, $H(p_1,p_2) = \frac{1}{4}p_1^4 - \frac{1}{4}p_2^4$. Then g and H are classical Legendre functions and

$$H^L(x_1,x_2) = \frac{3}{4}x_1^{4/3} - \frac{3}{4}x_2^{4/3}.$$

Let $r(s,t)$ be the unique solution of the equation $r^3 + t^{-1/3}r - s = 0$. Then u given by

$$u(x_1,x_2,t) = \frac{1}{2}\left(x_1 - r(x_1,t)^{1/3}\right)^2 + \frac{3}{4}t^{-1/3}r(x_1,t)^{4/9} - \frac{1}{2}\left(x_2 - r(x_2,t)^{1/3}\right)^2 - \frac{3}{4}t^{-1/3}r(x_2,t)^{4/9}$$

is a unilateral solution of equation (H-J) on $X \times (0,+\infty)$.

7 The Hopf formula and characteristics

Finally, let us consider the Hamilton-Jacobi equation in the case the Hamiltonian H depends on t and y only. Given $t > 0$, let us suppose that for all $r \in (0,t)$ the function $y \longmapsto H_r(y) = H(r,y)$ is convex and that for all $y \in Y$ the function $r \longmapsto H_r(y) = H(r,y)$ is continuous and let us suppose g is a Legendre function. Let h_t be the function given by

$$h_t(y) := \int_0^t H(s,y)ds \qquad y \in Y := X^*. \tag{13}$$

We use an elementary subdifferential contained in ∂_D and we assume that for some $t > 0$ $g^L + h_t$ is an Ekeland function, so that we can consider the function u_t given by

$$u_t(x) = (g^L + h_t)^E(x).$$

It can be considered as an analogue of the function $(g^* + h_t)^*$ of interest when g is convex too.

For $v \in X$, $y \in \partial g(v)$, let us consider the characteristic system

$$\begin{aligned} &\widehat{x}'(s) \in \partial H_s(\widehat{y}(s)) && \widehat{x}(0) = v \\ &\widehat{y}'(s) = 0 && \widehat{y}(0) = y \\ &\widehat{z}'(s) \in \partial H_s(\widehat{y}(s)).\widehat{y}(s) - H_s(\widehat{y}(s)) && \widehat{z}(0) = g(v). \end{aligned}$$

Any solutions $\widehat{x}$, $\widehat{y}$ of the first two equations of this system satisfy

$$\widehat{y}(s) = y, \qquad \widehat{x}(s) \in \partial h_s(y) + v.$$

Here, with the initial data, we have used the fact that $h_0 = 0$ and the inequality

$$H_r(w) - H_r(\widehat{y}(r)) \geq \langle w - \widehat{y}(r), \widehat{x}'(r)\rangle \qquad \forall (w,r) \in Y \times [0,s]$$

to get by integration, since $\widehat{y}(r) = y$ for all $r > 0$,

$$h_s(w) - h_s(\widehat{y}(s)) \geq \langle w - \widehat{y}(s), \widehat{x}(s) - v\rangle \qquad \forall w \in Y,$$

hence $\widehat{x}(s) - v \in \partial h_s(y)$. Then, taking

$$\widehat{z}(s) = \langle y, \widehat{x}(s) - v\rangle + g(v) - h_s(y)$$

we get a solution to the whole system.

Now, since g is a Legendre function and $y \in \partial g(v)$, we have $v \in \partial g^L(y)$, hence

$$\widehat{x}(t) = v + (\widehat{x}(t) - v) \in \partial g^L(y) + \partial h_t(y) \subset \partial(g^L + h_t)(y). \tag{14}$$

When $g^L + h_t$ is an Ekeland function, for $x := \widehat{x}(t)$ we get, since $g^L(y) = \langle y, v\rangle - g(v)$,

$$\begin{aligned}(g^L + h_t)^E(x) &= \langle y, x\rangle - (g^L + h_t)(y)\\ &= \langle y, x - v\rangle + g(v) - h_t(y) = \widehat{z}(t).\end{aligned}$$

If moreover $g^L + h_t$ is a Legendre function, then, by (14) $y \in \partial(g^L + h_t)^L(x) = \partial(g^L + h_t)^E(x) = \partial u_t(x)$.

Recording our findings, we get the following statement in which we write explicitely the dependence in v of $\widehat{x}(t)$ by substituting to $\widehat{x}(t)$ the notation $\widehat{x}(t,v)$. Since H is independent of x but depends on t and since g is not supposed to be convex, this statement is a partial extension of [41, Thm 6.8].

Proposition 7.1 *Let $H : (t,y) \mapsto H(t,y)$ be a function that is convex in its second variable and let g be a Legendre function such that for some $t > 0$ the function $g^L + h_t$ is a Legendre function, h_t being given by (13). Then, the flow associated with $v \mapsto \widehat{x}(t,v)$ carries the subjet $J^\partial g := \{(v,y,z) : v \in X,\ y \in \partial g(v),\ z = g(v)\}$ of g into the subjet $J^\partial u_t$ of u_t, where*

$$u_t(x) := (g^L + h_t)^L(x).$$

Moreover, if the preceding assumptions are satisfied for every t in some interval T, then the function $(t,x) \mapsto u_t(x)$ is a subsolution of the Hamilton-Jacobi equation on $X \times T$ in the sense that for all $(x,t) \in X \times T$ and all $(p,q) \in \partial u(x,t)$ one has

$$q + H(t,p) \leq 0.$$

Proof. It remains to prove the last assertion. We first observe that since $(p,q) \in \partial_D u(x,t)$ we have $p \in \partial_D u_t(x)$ (and a similar assertion holds for any bornological subdifferential such as the Fréchet subdifferential ∂_F). Moreover, taking $v \in \partial g^L(q)$, $y = p$, since $u(t, \widehat{x}(t,v)) = \widehat{z}(t,v)$, we have

$$q + \langle p, \widehat{x}'(t,v)\rangle \leq \widehat{z}'(t,v) = \langle y, \widehat{x}'(t,v)\rangle - H(t,y),$$

hence $q \leq -H(t,p)$. □

When g is closed, proper, convex it is proved in [46, Thm 9.1] under continuity and growth conditions that $(x,t) \mapsto u_t(x) = (g^* + h_t)^*(x)$ is a classical solution of the Hamilton-Jacobi equation

$$D_t u(x,t) + H(t, D_x u(x,t)) = 0, \qquad u(x,0) = g(x).$$

What precedes is not limited to the convex case as the next example shows.

Example 15. Let $X = \mathbb{R}^2$ and let g, H be functions of class C^2 such that $g(x_1,x_2) = \frac{1}{2}x_1^2 - \frac{1}{2}x_2^2$, H_t being convex with bounded second order partial derivatives. Then, g is a classical Legendre function and $g^L(y_1,y_2) = \frac{1}{2}y_1^2 - \frac{1}{2}y_2^2$ and for $t > 0$ small enough the function $g^L + h_t$ is a classical Legendre function too. Then the preceding result applies.

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