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Partitioning the Cartesian product of an arbitrarily partitionable graph and a complete graph

Olivier Baudon and Julien Bensmail

Univ. Bordeaux, LaBRI, UMR 5800, F-33400 Talence, France
CNRS, LaBRI, UMR 5800, F-33400 Talence, France
{baudon, jbensmai}@labri.fr

Abstract

A graph G is *arbitrarily partitionable* (AP for short) if for every sequence (n_1, \dots, n_p) of positive integers summing up to $|V(G)|$ there exists a partition (V_1, \dots, V_p) of $V(G)$ such that $G[V_i]$ is a connected graph on n_i vertices for every $i \in \{1, \dots, p\}$. We show that the Cartesian product $G \square K_l$ is AP whenever G is AP and K_l is a complete graph on $l \geq 1$ vertices.

1 Result

Please refer to [1] to understand our terminology and notation. We start with the following lemma.

Proposition 1:

Let $l \geq 1$ be a positive integer, and $\tau = (n_1, \dots, n_p)$ be a sequence of positive integers such that $\|\tau\| \equiv 0 \pmod{l}$. If $p > l$, then τ can be divided into two non empty subsequences τ_1 and τ_2 such that $\|\tau_1\| \equiv 0 \pmod{l}$ and $\|\tau_2\| \equiv 0 \pmod{l}$.

Proof:

If τ contains an element n_i such that $n_i \equiv 0 \pmod{l}$, then it suffices to consider $\tau_1 = (n_i)$ and $\tau_2 = \tau - (n_i)$. Let us then consider that for every $i \in \{1, \dots, p\}$, we have $n_i \not\equiv 0 \pmod{l}$. For every $x \in \{1, \dots, p\}$, let $s_x = \sum_{i=1}^x n_i$ be the sum of the x first elements of τ . Because τ has more than l elements, there exist two values x_1 and x_2 such that $s_{x_1} \equiv s_{x_2} \pmod{l}$. Thus, $\tau_1 = \bigcup_{i=x_1+1}^{x_2} (n_i)$ and $\tau_2 = \tau - \tau_1$ satisfy our conditions ■

We will also be needing the following two connectivity theorems.

Theorem 1 ([4]):

If G and H are connected graphs, then the Cartesian product $G \square H$ is $(k_1 + k_2)$ -connected, where k_1 and k_2 are the connectivity of G and H , respectively.

Theorem 2 ([2, 3]):

If G is a k -connected graph, then G can be partitioned following every sequence of length at most k .

We now prove our main result.

Theorem 3:

The Cartesian product $G \square K_l$ is AP whenever G is AP and $l \geq 1$.

Proof:

If $l = 1$, then $G \square K_l$ is isomorphic to G and is AP by assumption. Let us thus consider that $l \geq 2$, and let consider any sequence $\tau = (n_1, \dots, n_p)$ of positive integers summing up to $|V(G \square K_l)|$. Since $G \square K_l$ is l -connected by Theorem 1, we may also suppose that $|\tau| \geq l + 1$ since otherwise an obvious realization of τ in $G \square K_l$ could be deduced thanks to Theorem 2.

By repeatedly applying Proposition 1, our sequence τ can be divided into $q \geq 2$ subsequences τ_1, \dots, τ_q such that $|\tau_i| \leq l$ and $\|\tau_i\| \equiv 0 \pmod{l}$ for every $i \in \{1, \dots, q\}$. Let us put $\lambda_i = \frac{\|\tau_i\|}{l}$ for every $i \in \{1, \dots, q\}$. These α_i 's are integers, we have $\alpha_1 + \dots + \alpha_q = |V(G)|$ and, because G is AP, there exists a realization (V_1, \dots, V_q) of $(\alpha_1, \dots, \alpha_q)$ in G . Consider now U_i the extension of V_i in $G \square K_l$ for every $i \in \{1, \dots, q\}$ ($U_i = V_i^1 \cup \dots \cup V_i^l$). Clearly, we have $|U_i| = \|\tau_i\|$ and, because $|\tau_i| \leq l$ and $(G \square K_l)[U_i]$ is l -connected, there exists a realization R_i of τ_i in $(G \square K_l)[U_i]$ for every $i \in \{1, \dots, q\}$ according to Theorem 2. It follows that $\bigcup_{i=1}^q R_i$ is a realization of τ in $G \square K_l$. ■

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