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# On the product formula on non-compact Grassmannians

P. Graczyk and P. Sawyer \*

## Abstract

We study the absolute continuity of the convolution  $\delta_{e^X}^{\natural} \star \delta_{e^Y}^{\natural}$  of two orbital measures on the symmetric space  $\mathbf{SO}_0(p, q)/\mathbf{SO}(p) \times \mathbf{SO}(q)$ ,  $q > p$ . We prove sharp conditions on  $X, Y \in \mathfrak{a}$  for the existence of the density of the convolution measure. This measure intervenes in the product formula for the spherical functions. We show that the sharp criterion developed for  $\mathbf{SO}_0(p, q)/\mathbf{SO}(p) \times \mathbf{SO}(q)$  will also serve for the spaces  $\mathbf{SU}(p, q)/\mathbf{S}(\mathbf{U}(p) \times \mathbf{U}(q))$  and  $\mathbf{Sp}(p, q)/\mathbf{Sp}(p) \times \mathbf{Sp}(q)$ ,  $q > p$ . We also apply our results to the study of absolute continuity of convolution powers of an orbital measure  $\delta_{e^X}^{\natural}$ .

## 1 Introduction

The spaces  $\mathbf{SO}_0(p, q)/\mathbf{SO}(p) \times \mathbf{SO}(q)$  where  $q > p$  (which we will assume throughout the paper), are the noncompact duals of real Grassmannians. They are Riemannian symmetric spaces of noncompact type corresponding to root systems of type  $B_p$ . The harmonic analysis on these spaces has been intensely developed in recent years ([1, 13, 14, 15]).

We use throughout the paper the usual notations of the harmonic analysis on Riemannian symmetric spaces. The books [9, 10] constitute a standard reference on these topics.

Let  $X, Y \in \mathfrak{a}$  and let  $m_K$  denote the Haar measure of the group  $K$ . We define  $\delta_{e^X}^{\natural} = m_K \star \delta_{e^X} \star m_K$ . The question of the absolute continuity of the convolution  $\delta_{e^X}^{\natural} \star \delta_{e^Y}^{\natural}$  of two  $K$ -invariant orbital measures that we address in our paper has important applications in harmonic analysis itself (the product formula for the spherical functions) and in probability theory (random walks,  $I_0$  characterization of Gaussian measures).

The spherical Fourier transform of the measure  $\delta_{e^X}^{\natural}$  is equal to the spherical function  $\phi_{\lambda}(e^X)$ , where  $\lambda$  is a complex-valued linear form on  $\mathfrak{a}$ . Thus the product  $\phi_{\lambda}(e^X)\phi_{\lambda}(e^Y)$  is the spherical Fourier transform of the convolution  $m_{X,Y} = \delta_{e^X}^{\natural} \star \delta_{e^Y}^{\natural}$ . If we denote by  $\mu_{X,Y}$  the projection of the measure  $m_{X,Y}$  on  $\mathfrak{a}$  via the Cartan decomposition  $G = KAK$ , then

$$\phi_{\lambda}(e^X)\phi_{\lambda}(e^Y) = \int_{\mathfrak{a}} \phi_{\lambda}(e^H) d\mu_{X,Y}(H).$$

Let  $\delta$  be the density of the invariant measure on  $\mathfrak{a}$  in polar coordinates. The existence of a kernel in the last product formula

$$\phi_{\lambda}(e^X)\phi_{\lambda}(e^Y) = \int_{\mathfrak{a}^+} \phi_{\lambda}(e^H) k(H, X, Y) \delta(H) dH \quad (1)$$

is equivalent to the absolute continuity of the measure  $\mu_{X,Y}$  with respect to the Lebesgue measure on  $\mathfrak{a}$  and to the existence of the density of  $m_{X,Y}$  on  $G$ , with respect to the invariant measure  $dg$ . When the formula (1) holds, we say that we have a product formula for  $X$  and  $Y \in \mathfrak{a}$ . Provided that  $X, Y \in \mathfrak{a}^+$ , the product formula (1) has been shown previously (see [2] in the rank one case,

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[3] in the complex case and [4] in the general case). In [4] we were able to relax these conditions and show that  $\mu_{X,Y}$  is absolutely continuous provided one of  $X$  or  $Y$  is in  $\mathfrak{a}^+$  as long as the other is nonzero. The density can however exist in some cases when both  $X$  and  $Y$  are singular. It is a challenging problem to characterize all such pairs  $X$  and  $Y$ .

This problem was solved in [7] for symmetric spaces with root system of type  $A_n$ . We solve it in this paper for the space  $\mathbf{SO}_0(p, q)/\mathbf{SO}(p) \times \mathbf{SO}(q)$ : we give a definition of an eligible pair  $(X, Y)$  (Definition 3) and next we prove the necessity (Proposition 5) and the sufficiency (Proposition 5 and Theorem 13) of this property for the absolute continuity of  $m_{X,Y}$ .

=====>By [3, 4], the density  $k(H, X, Y)$  exists if and only if  $\mathcal{S}_{X,Y} = a(e^X K e^Y)$ , the support of the measure  $\mu_{X,Y}|_{\mathfrak{a}^+}$ , has nonempty interior. Similarly, the density of the measure  $m_{X,Y}$  exists if and only if its support  $K e^X K e^Y K$  has nonempty interior as seen in [7]. These facts are crucial in the proofs of the results of this paper.

We show in Corollary 15 that the result for the space  $\mathbf{SO}_0(p, q)/\mathbf{SO}(p) \times \mathbf{SO}(q)$  also implies the result for the spaces  $\mathbf{SU}(p, q)/\mathbf{S}(\mathbf{U}(p) \times \mathbf{U}(q))$  and  $\mathbf{Sp}(p, q)/\mathbf{Sp}(p) \times \mathbf{Sp}(q)$ . We conclude the paper with two further applications of our main result. One of them is a characterization of an optimal convolution power  $l$  of the measure  $\delta_{e^X}^\natural$ , which is absolutely continuous for any  $X \neq 0$ ,  $X \in \mathfrak{a}$ . Theorem 17 solves on non-compact Grassmannians a problem raised by Ragozin in [12].  
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## 2 Basic properties

We start by reviewing some useful information on the Lie group  $\mathbf{SO}_0(p, q)$ , its Lie algebra  $\mathfrak{so}(p, q)$  and the corresponding root system. Most of this material was given in [15]. For the convenience of the reader, we gather below the properties we will need in the sequel.

In this paper,  $E_{ij}$  is a rectangular matrix with 0's everywhere except at the position  $(i, j)$  where it is 1.

Recall that  $\mathbf{SO}(p, q)$  is the group of matrices  $g \in \mathbf{SL}(p+q, \mathbf{R})$  such that  $g^T I_{p,q} g = I_{p,q}$  where  $I_{p,q} = \begin{bmatrix} -I_p & 0_{p \times q} \\ 0_{q \times p} & I_q \end{bmatrix}$ . Unless otherwise specified, all  $2 \times 2$  block decompositions in this paper follow the same pattern.

The group  $\mathbf{SO}_0(p, q)$  is the connected component of  $\mathbf{SO}(p, q)$  containing the identity. The Lie algebra  $\mathfrak{so}(p, q)$  of  $\mathbf{SO}_0(p, q)$  consists of the matrices

$$\begin{bmatrix} A & B \\ B^T & D \end{bmatrix}$$

where  $A$  and  $D$  are skew-symmetric.

A very important element in our investigations is the Cartan decomposition of  $\mathfrak{so}(p, q)$  and  $\mathbf{SO}(p, q)$ . The maximal compact subgroup  $K$  is the subgroup of  $\mathbf{SO}(p, q)$  consisting of the matrices

$$\begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix}$$

of size  $(p+q) \times (p+q)$  such that  $A \in \mathbf{SO}(p)$  and  $D \in \mathbf{SO}(q)$  (hence  $K \simeq \mathbf{SO}(p) \times \mathbf{SO}(q)$ ). If  $\mathfrak{k}$

is the Lie algebra of  $K$  and  $\mathfrak{p}$  is the set of matrices

$$\begin{bmatrix} 0 & B \\ B^T & 0 \end{bmatrix}$$

then the Cartan decomposition is given by  $\mathfrak{so}(p, q) = \mathfrak{k} \oplus \mathfrak{p}$  with corresponding Cartan involution  $\theta(X) = -X^T$ .

The Cartan space  $\mathfrak{a} \subset \mathfrak{p}$  is the set of matrices

$$H = \begin{bmatrix} 0_{p \times p} & \mathcal{D}_H & 0_{p \times (q-p)} \\ \mathcal{D}_H & 0_{p \times p} & 0_{p \times (q-p)} \\ 0_{(q-p) \times p} & 0_{(q-p) \times p} & 0_{(q-p) \times (q-p)} \end{bmatrix}$$

where  $\mathcal{D}_H = \text{diag}[H_1, \dots, H_p]$ . Its canonical basis is given by the matrices

$$A_i := E_{i, p+i} + E_{p+i, i}, \quad 1 \leq i \leq p.$$

The restricted roots and associated root vectors for the Lie algebra  $\mathfrak{so}(p, q)$  with respect to  $\mathfrak{a}$  are given in Table 1.

root $\alpha$	multiplicity	root vectors $X_\alpha$
$\alpha(H) = \pm H_i$ $1 \leq i \leq p$	$q - p$	$X_{ir}^\pm = E_{i, 2p+r} + E_{2p+r, i} \pm (E_{p+i, 2p+r} - E_{2p+r, p+i})$ $r = 1, \dots, q - p$
$\alpha(H) = \pm(H_i - H_j)$ $1 \leq i, j \leq p, i < j$	1	$Y_{ij}^\pm = \pm(E_{ij} - E_{ji} + E_{p+i, p+j} - E_{p+j, p+i}) + E_{i, p+j} + E_{p+j, i}$ $+ E_{j, p+i} + E_{p+i, j}$
$\alpha(H) = \pm(H_i + H_j)$ $1 \leq i, j \leq p, i < j$	1	$Z_{ij}^\pm = \pm(E_{ij} - E_{ji} - E_{p+i, p+j} + E_{p+j, p+i}) - (E_{i, p+j} + E_{p+j, i})$ $+ E_{j, p+i} + E_{p+i, j}$

Table 1: Restricted roots and associated root vectors

The positive roots can be chosen as  $\alpha(H) = H_i \pm H_j$ ,  $1 \leq i < j \leq p$  and  $\alpha(H) = H_i$ ,  $i = 1, \dots, p$ . We therefore have the positive Weyl chamber

$$\mathfrak{a}^+ = \{H \in \mathfrak{a}: H_1 > H_2 > \dots > H_p > 0\}.$$

The simple roots are given by  $\alpha_i(H) = H_i - H_{i+1}$ ,  $i = 1, \dots, p - 1$  and  $\alpha_p(H) = H_p$ .

**The action of the Weyl group.** The elements of the Weyl group  $W$  act as permutations of the diagonal entries of  $\mathcal{D}_X$  with eventual sign changes of any number of these entries.

The Lie algebra  $\mathfrak{k}$  is generated by the vectors  $X_\alpha + \theta X_\alpha$ . We will use the notation

$$k_{X_\alpha}^t = e^{t(X_\alpha + \theta X_\alpha)}.$$

The linear space  $\mathfrak{p}$  has a basis formed by  $A_i \in \mathfrak{a}$ ,  $1 \leq i \leq p$  and by the symmetric matrices  $X_\alpha^s := \frac{1}{2}(X_\alpha - \theta X_\alpha)$  which have the following form

$$\begin{aligned} X_{ir} &:= E_{i, 2p+r} + E_{2p+r, i}, & 1 \leq i \leq p, & \quad 1 \leq r \leq q - p; \\ Y_{ij} &:= E_{i, p+j} + E_{j, p+i} + E_{p+j, i} + E_{p+i, j}, & 1 \leq i < j \leq p; \\ Z_{ij} &:= E_{i, p+j} - E_{j, p+i} + E_{p+j, i} - E_{p+i, j}, & 1 \leq i < j \leq p. \end{aligned}$$

If we followed the notation of [7], we should write  $(X_{ir}^+)^s$ , etc. but we simplify the notation to  $X_{ir}$ ,  $Y_{ij}$  and  $Z_{ij}$ . If we write a matrix from the space  $\mathfrak{p}$  in the form

$$\begin{bmatrix} 0 & B \\ B^T & 0 \end{bmatrix} = \begin{bmatrix} 0 & B_1 & B_2 \\ B_1^T & 0 & 0 \\ B_2^T & 0 & 0 \end{bmatrix}$$

where  $B_1$  is a square  $p \times p$  matrix and  $B_2$  is a  $p \times (q - p)$  matrix, then the matrices

$$\begin{bmatrix} 0 & B_1 & 0 \\ B_1^T & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

are generated by the vectors  $A_i$  (for the diagonal entries of  $B_1$  and  $B_1^T$ ),  $Y_{ij}$  and  $Z_{ij}$  (for the non-diagonal entries), whereas the matrices

$$\begin{bmatrix} 0 & 0 & B_2 \\ 0 & 0 & 0 \\ B_2^T & 0 & 0 \end{bmatrix}$$

are spanned by the vectors  $X_{ir}$ .

We now recall the useful matrix  $S \in \mathbf{SO}(p + q)$  which allows us to diagonalize simultaneously all the elements of  $\mathfrak{a}$ . Let

$$S = \begin{bmatrix} \frac{\sqrt{2}}{2} I_p & 0_{p \times (q-p)} & \frac{\sqrt{2}}{2} J_p \\ \frac{\sqrt{2}}{2} I_p & 0_{p \times (q-p)} & -\frac{\sqrt{2}}{2} J_p \\ 0_{(q-p) \times p} & I_{q-p} & 0_{(q-p) \times p} \end{bmatrix}$$

where  $J_p = (\delta_{i,p+1-i})$  is a matrix of size  $p \times p$ . If  $H = \begin{bmatrix} 0 & \mathcal{D}_H & 0 \\ \mathcal{D}_H & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$  with  $\mathcal{D}_H = \text{diag}[H_1, \dots, H_p]$

then  $S^T H S = \text{diag}[H_1, \dots, H_p, \overbrace{0, \dots, 0}^{q-p}, -H_p, \dots, -H_1]$ .

The ‘‘group’’ version of this result is as follows:

$$S^T e^H S = \text{diag}[e^{H_1}, \dots, e^{H_p}, \overbrace{1, \dots, 1}^{q-p}, e^{-H_p}, \dots, e^{-H_1}].$$

**Remark 1** *The Cartan projection  $a(g)$  on the group  $\mathbf{SO}_0(p, q)$ , defined as usual by*

$$g = k_1 e^{a(g)} k_2, \quad a(g) \in \overline{\mathfrak{a}^+}, k_1, k_2 \in K$$

*is related to the singular values of  $g \in \mathbf{SO}(p, q)$  in the following way. Recall that the singular values of  $g$  are defined as the non-negative square roots of the eigenvalues of  $g^T g$ . Let us write  $H = a(g)$ . We have*

$$g^T g = k_2^T e^{2H} k_2 = (k_2^T S) (S^T e^{2H} S) (S^T k_2)$$

where  $S^T e^{2H} S$  is a diagonal matrix with nonzero entries  $e^{2H_1}, \dots, e^{2H_p}, \overbrace{1, \dots, 1}^{q-p}, e^{-2H_p}, \dots, e^{-2H_1}$ . Let us write  $a_j = e^{H_j}$ . Thus the set of  $p+q$  singular values of  $g$  contains the value 1 repeated  $q-p$  times and the  $2p$  values  $a_1, \dots, a_p, a_1^{-1}, \dots, a_p^{-1}$ .

Hence, in order to determine  $a(g)$ , we can compute the  $p+q$  singular values of  $g^T g$  and omit  $q-p$  values 1 always appearing among them. The  $2p$  remaining singular values may be ordered  $a_1 \geq \dots \geq a_p \geq a_p^{-1} \geq \dots \geq a_1^{-1}$  with  $a_1 \geq \dots \geq a_p \geq 1$ . Then

$$a(g) = \begin{bmatrix} 0 & \mathcal{D}_{a(g)} & 0 \\ \mathcal{D}_{a(g)} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{with } \mathcal{D}_{a(g)} = \text{diag}[\log a_1, \dots, \log a_p]$$

Summarizing, if for  $g \in \mathbf{SO}(p, q) \subset \mathbf{SL}(p+q)$  the  $\mathbf{SL}(p+q)$ -Cartan decomposition writes  $g = k_1 e^{\tilde{a}(g)} k_2$ ,  $k_1, k_2 \in \mathbf{SO}(p+q)$ , then  $\mathcal{D}_{a(g)} = \pi_p(\tilde{a}(g))$ , where  $\pi_p$  denotes the projection  $\pi_p(\text{diag}[h_1, \dots, h_{p+q}]) = \text{diag}[h_1, \dots, h_p]$ .

**Singular elements of  $\mathfrak{a}$ .** In what follows, we will consider singular elements  $X, Y \in \partial\mathfrak{a}^+$ . As in [7], we need to control the irregularity of  $X$  and  $Y$ , i.e. consider the simple positive roots annihilating  $X$  and  $Y$ . A special attention must be paid to the last simple root  $\alpha_p$ , different from the roots  $\alpha_i$ ,  $i = 1, \dots, p-1$ , that generate a root subsystem of type  $A_{p-1}$ . We introduce the following definition of the configuration of  $X \in \overline{\mathfrak{a}^+}$ .

**Definition 2** Let  $X \in \overline{\mathfrak{a}^+}$ . There exist nonnegative integers  $s_1 \geq 1, \dots, s_r \geq 1, u \geq 0$  such that

$$\mathcal{D}_X = \text{diag}[\overbrace{x_1, \dots, x_1}^{s_1}, \overbrace{x_2, \dots, x_2}^{s_2}, \dots, \overbrace{x_r, \dots, x_r}^{s_r}, \overbrace{0, \dots, 0}^u]$$

with  $x_1 > x_2 > \dots > x_r > 0$  and  $\sum s_i + u = p$ . We say that  $[s_1, \dots, s_r; u]$  is the **configuration** of  $X$ . Writing  $\mathbf{s} = (s_1, \dots, s_r)$ , we will shorten the notation of the configuration of  $X$  to  $[\mathbf{s}; u]$ . We will also write  $X = X[\mathbf{s}; u]$ .

Note that  $X = 0$  is equivalent to  $u = p$  and has configuration  $[0; p]$ . A regular  $X \in \mathfrak{a}^+$  has the configuration  $[\mathbf{1}; 0] = [1^p; 0]$ . We extend naturally the definition of configuration to any  $X \in \mathfrak{a}$ , whose configuration is defined as that of the projection  $\pi(X)$  of  $X$  on  $\overline{\mathfrak{a}^+}$ .

In what follows, we will write  $\max \mathbf{s} = \max_i s_i$  and  $\max(\mathbf{s}, u) = \max(\max \mathbf{s}, u)$ . We will show that in the case of the symmetric spaces  $\mathbf{SO}_0(p, q)/\mathbf{SO}(p) \times \mathbf{SO}(q)$ ,  $q > p$ , the criterion for the existence of the density of the convolution  $\delta_{e^X}^{\mathfrak{h}} \star \delta_{e^Y}^{\mathfrak{h}}$  is given by the following definition of an eligible pair  $X$  and  $Y$ :

**Definition 3** Let  $X = X[\mathbf{s}; u]$  and  $Y = Y[\mathbf{t}; v]$  be two elements of  $\mathfrak{a}$ . We say that  $X$  and  $Y$  are eligible if

$$\max(\mathbf{s}, 2u) + \max(\mathbf{t}, 2v) \leq 2p.$$

Observe that if  $X$  and  $Y$  are eligible, then  $X \neq 0$  and  $Y \neq 0$ .

### 3 Necessity of the eligibility condition

In the proof of the necessity of the eligibility condition we will use the result stated in [8, Step 1, page 1767]:

**Lemma 4** Let  $U = \text{diag}([\overbrace{u_0, \dots, u_0}^r, u_1, \dots, u_{N-r}])$  and  $V = \text{diag}([\overbrace{v_0, \dots, v_0}^{N-s}, v_1, \dots, v_s])$ . where  $s+1 \leq r < N$ ,  $s \geq 1$ , and the  $u_i$ 's and  $v_j$ 's are arbitrary. Then each element of  $\tilde{a}(e^U \mathbf{SU}(N, \mathbf{F}) e^V)$  has at least  $r - s$  entries equal to  $u_0 + v_0$ .

We will use Lemma 4 with  $N = p + q$  in the proofs of Proposition 5 and Theorem 17.

**Proposition 5** If  $X[\mathbf{s}; u]$  and  $Y[\mathbf{t}, v]$  are not eligible then the measure  $\mu_{X,Y}$  is not absolutely continuous with respect to the Lebesgue measure on  $\mathfrak{a}$ .

**Proof:** Suppose  $\max(\mathbf{s}, 2u) + \max(\mathbf{t}, 2v) > 2p$  and consider the matrices  $a(e^X k e^Y)$ ,  $k \in \mathbf{SO}(p) \times \mathbf{SO}(q)$ . Applying Remark 1, the diagonal  $p \times p$  matrix  $\mathcal{D}_{a(e^X k e^Y)}$  contains the  $p$  biggest diagonal terms of the matrix

$$\tilde{a}(e^X k e^Y) = \tilde{a}(\overbrace{(S^T e^X S)}^{e^{s^T X s}} \overbrace{(S^T k S)}^{\in \mathbf{SO}(p+q)} \overbrace{(S^T e^Y S)}^{e^{s^T Y s}})$$

If  $u + v > p$  then there are  $r - s = r + (N - s) - N = (2u + q - p) + (2v + q - p) - (p + q) = 2(u + v - p) + (q - p)$  repetitions of  $0 + 0 = 0$  in coefficients of  $\tilde{a}(e^X k e^Y)$ . Therefore 0 occurs at least  $u + v - p > 0$  times as a diagonal entry of  $\mathcal{D}_H$  for every  $H \in a(e^X K e^Y)$  which implies that  $a(e^X K e^Y)$  has empty interior.

If  $2u + \max(\mathbf{t}) > 2p$  denote  $t = \max(\mathbf{t})$ . Let  $Y_i \neq 0$  be repeated  $t$  times in  $\mathcal{D}_Y$ . Then there are  $r - s = r + (N - s) - N = (2u + q - p) + t - (p + q) = 2u + t - 2p$  repetitions of  $Y_i + 0$  in coefficients of  $\tilde{a}(e^X k e^Y)$ . Therefore  $Y_i$  occurs at least  $2u + t - 2p > 0$  times as a diagonal entry of  $\mathcal{D}_H$  for every  $H \in a(e^X K e^Y)$  which implies that  $a(e^X K e^Y)$  has empty interior. ■

### 4 Sufficiency of the eligibility condition

We use basic ideas and some results and notations of [7, Section 3].

**Proposition 6** (i) The density of the measure  $m_{X,Y}$  exists if and only if its support  $Ke^X Ke^Y K$  has nonempty interior.

(ii) Consider the analytic map  $T: K \times K \times K \rightarrow \mathbf{SO}_0(p, q)$  defined by

$$T(k_1, k_2, k_3) = k_1 e^X k_2 e^Y k_3.$$

If the derivative of  $T$  is surjective for some choice of  $\mathbf{k} = (k_1, k_2, k_3)$ , then the set  $T(K \times K \times K) = Ke^X Ke^Y K$  contains an open set.

**Proof:** Part (i) follows from arguments explained in [4] in the case of the support of the measure  $\mu_{X,Y}$ , equal to  $a(e^X K e^Y)$ . Part (ii) is justified for example in [10, p. 479]. ■

**Proposition 7** Let  $U_Z = \mathfrak{k} + \text{Ad}(Z)\mathfrak{k}$ . If there exists  $k \in K$  such that

$$U_{-X} + \text{Ad}(k)U_Y = \mathfrak{g} \quad (2)$$

then the measure  $m_{X,Y}$  is absolutely continuous.

**Proof:** We want to show that the derivative of  $T$  is surjective for some choice of  $\mathbf{k} = (k_1, k_2, k_3)$ .

Let  $A, B, C \in \mathfrak{k}$ . The derivative of  $T$  at  $\mathbf{k}$  in the direction of  $(A, B, C)$  equals

$$\begin{aligned} dT_{\mathbf{k}}(A, B, C) &= \left. \frac{d}{dt} \right|_{t=0} e^{tA} k_1 e^{tX} e^{tB} k_2 e^{tY} e^{tC} k_3 \\ &= A k_1 e^X k_2 e^Y k_3 + k_1 e^X B k_2 e^Y k_3 + k_1 e^X k_2 e^Y C k_3 \end{aligned} \quad (3)$$

We now transform the space of all matrices of the form (3) without modifying its dimension:

$$\begin{aligned} &\dim\{A k_1 e^X k_2 e^Y k_3 + k_1 e^X B k_2 e^Y k_3 + k_1 e^X k_2 e^Y C k_3 : A, B, C \in \mathfrak{k}\} \\ &= \dim\{k_1^{-1} A k_1 e^X k_2 e^Y + e^X B k_2 e^Y + e^X k_2 e^Y C : A, B, C \in \mathfrak{k}\} \\ &= \dim\{A e^X k_2 e^Y + e^X B k_2 e^Y + e^X k_2 e^Y C : A, B, C \in \mathfrak{k}\} \\ &= \dim\{e^{-X} A e^X + B + k_2 e^Y C e^{-Y} k_2^{-1} : A, B, C \in \mathfrak{k}\} \end{aligned}$$

The space in the last line equals  $\mathfrak{k} + \text{Ad}(e^{-X})(\mathfrak{k}) + \text{Ad}(k_2)(\text{Ad}(e^Y)(\mathfrak{k})) = U_{-X} + \text{Ad}(k_2)U_Y$ . ■

In order to apply the condition (2), we will consider convenient root vectors and their symmetrizations. For  $Z \in \mathfrak{a}$ , we define the space

$$V_Z = \text{span}\{X_\alpha^s \mid \alpha(Z) \neq 0\},$$

where  $X_\alpha^s = X_\alpha - \theta X_\alpha$ . Note that this space would be called  $V_Z^S$  in the notation of [7].

**Lemma 8** Let  $Z \in \mathfrak{a}$ . The vector space  $U_Z = \mathfrak{k} + \text{Ad}(e^Z)(\mathfrak{k})$  contains the root vectors  $X_\alpha$  for which  $\alpha(Z) \neq 0$ .  $\implies$  Consequently,  $V_Z = V_{-Z} \subset U_{\pm Z}$ .  $\Leftarrow$

**Proof:**  $\implies$  Suppose  $\alpha$  is a root such that  $\alpha(Z) \neq 0$ .  $\Leftarrow$  Note that  $[Z, X_\alpha] = \alpha(Z) X_\alpha$  and  $[Z, \theta(X_\alpha)] = -\alpha(Z) \theta(X_\alpha)$ . Let  $U = X_\alpha + \theta(X_\alpha) \in \mathfrak{k}$ . Now,

$$\begin{aligned} \text{Ad}(e^Z)U &= e^{\text{ad } Z}(X_\alpha + \theta(X_\alpha)) = \sum_{k=0}^{\infty} \frac{(\text{ad } Z)^k}{k!} (X_\alpha + \theta(X_\alpha)) \\ &= \sum_{k=0}^{\infty} \frac{(\text{ad } Z)^k}{k!} X_\alpha + \sum_{k=0}^{\infty} \frac{(\text{ad } Z)^k}{k!} \theta(X_\alpha) \\ &= \sum_{k=0}^{\infty} \frac{(\alpha(Z))^k}{k!} X_\alpha + \sum_{k=0}^{\infty} \frac{(-1)^k (\alpha(Z))^k}{k!} \theta(X_\alpha) \\ &= e^{\alpha(Z)} X_\alpha + e^{-\alpha(Z)} \theta(X_\alpha). \end{aligned}$$

Therefore  $X_\alpha = (e^{\alpha(Z)} - e^{-\alpha(Z)})^{-1} (-e^{-\alpha(Z)} U + \text{Ad}(e^Z)U) \in \mathfrak{k} + \text{Ad}(e^Z)(\mathfrak{k}) = U_Z$ . The vector  $\theta X_\alpha$  is a root vector for the root  $-\alpha$ , so we also have  $\theta X_\alpha \in U_Z$ . ■

$\implies$



**Proposition 9** *If there exists  $k \in K$  such that*

$$V_X + \text{Ad}(k) V_Y = \mathfrak{p} \quad (4)$$

*then the measure  $m_{X,Y}$  is absolutely continuous.*

**Proof:** We want to prove formula (2). By Lemma 8, we know that  $V_X = V_{-X} \subset U_{-X}$  and  $V_Y \subset U_Y$ . As  $\mathfrak{k} \subset U_X$ , we see that the formula (4) implies (2). ■

←=====

Later in this section, (Theorem 13), we will show that the hypotheses of the last Proposition are always satisfied for  $X$  and  $Y$  eligible. For technical reasons, in order to make an induction proof work, we will show more, i.e. that a “better” matrix  $k \in K$  exists such that the formula (4) holds. The meaning of a “better”  $k$  will be similar to the notion of a total matrix given in Definition 10. Here is a definition and a lemma about total matrices in  $K$ . The reasoning of the proof of this lemma will be used in a more general setting in Steps 2 and 3 of the proof of Theorem 13.

**Definition 10** *We say that a square  $n \times n$  matrix  $k$  is total if by removing any  $r < n$  rows and  $r$  columns of  $k$  we always obtain a nonsingular matrix.*

Note that this definition of totality is more restrictive than in [7, Definition 3.7].

**Lemma 11** *The set of matrices in  $\mathbf{SO}(n)$  which are total is dense and open in  $\mathbf{SO}(n)$ .*

**Proof:** Consider first the set  $M_{I,J} = M_{\{i_1, \dots, i_j\}, \{j_1, \dots, j_r\}} \subset \mathbf{SO}(n)$  of orthogonal matrices which remain nonsingular once the rows of indices  $i_1, \dots, i_j$  and the columns of indices  $j_1, \dots, j_r$  are removed. To see that such matrices exist, take the identity matrix (whose determinant is 1 if we remove, say, the first  $r$  rows and columns). By taking convenient permutations of the rows and columns of the identity matrix, we obtain an element of  $M_{I,J}$ . Given that  $\mathbf{SO}(n) \setminus M_{I,J}$  corresponds to the set of zeros of a certain determinant function, it must be closed and nowhere dense in  $\mathbf{SO}(n)$ .

To conclude, it suffices to notice that the set of total matrices in  $\mathbf{SO}(n)$  is the finite intersection of all the sets  $M_{I,J}$ . ■

In the proof of the main Theorem 13 we will need the following technical lemma.

**Lemma 12** (i) *For the root vectors  $X_{ir}^+$ ,  $Z_{ij}^+$ ,  $Y_{ij}^+$ , we have*

$$\begin{aligned} \text{Ad}(e^{t(X_{ir}^+ + \theta X_{ir}^+)}) (X_{ir}) &= \cos(2t) X_{ir} + 2 \sin(2t) A_i, \\ \text{Ad}(e^{t(Y_{ij}^+ + \theta Y_{ij}^+)}) (Y_{ij}) &= \cos(4t) Y_{ij} + 2 \sin(4t) (A_i - A_j), \\ \text{Ad}(e^{t(Z_{ij}^+ + \theta Z_{ij}^+)}) (Z_{ij}) &= \cos(4t) Z_{ij} + 2 \sin(4t) (A_i + A_j). \end{aligned}$$

(ii) *The functions  $\text{Ad}(e^{t(X_{ir}^+ + \theta X_{ir}^+)})$ ,  $\text{Ad}(e^{t(Y_{ij}^+ + \theta Y_{ij}^+)})$  and  $\text{Ad}(e^{t(Z_{ij}^+ + \theta Z_{ij}^+)})$  applied to the other symmetrized root vectors do not produce any components in  $\mathfrak{a}$ .*

**Proof:** It is just a matter of carefully evaluating

$$\text{Ad}(e^{t(Z+\theta Z)})(W) = e^{t \text{ad}(Z+\theta Z)}(W) = \sum_{k=0}^{\infty} (\text{ad}(Z + \theta Z))^k(W) \frac{t^k}{k!}.$$

For (ii), use the well known properties of the root system:  $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \in \mathfrak{g}_{\alpha+\beta}$  and  $[X_\alpha, \theta X_\alpha] \in \mathfrak{a}$ . ■

By Proposition 9, in order to justify the sufficiency of the eligibility condition, it is enough to prove the following theorem. This is the main result of this section.

**Theorem 13** *Let  $G = \mathbf{SO}_0(p, q)$  and let  $X, Y \in \mathfrak{a}$ . If  $X$  and  $Y$  are eligible then there exists a matrix  $k \in K$  such that*

$$V_X + \text{Ad}(k) V_Y = \mathfrak{p}. \quad (5)$$

**Proof:** We will assume that  $X = X[\mathfrak{s}; u]$  and  $Y = Y[\mathfrak{t}; v]$ . Observe that the spaces  $V_X$  and  $V_Y$  depend on the Weyl chambers where  $X$  and  $Y$  belong. However, see [7, Lemma 3.3 and Reduction 1, p. 759], the property (5) is equivalent to  $V_{w_1 X} + \text{Ad}(k') V_{w_2 Y} = \mathfrak{p}$  for any  $w_1, w_2 \in W$  and a convenient  $k' \in K$ . Throughout the proof we will assume that the diagonal entries of  $\mathcal{D}_X$  and  $\mathcal{D}_Y$  are non-negative and we will arrange (permute) them conveniently.

To lighten the notation, for a matrix  $c$  of size  $p \times q$ , we will consider the  $(p+q) \times (p+q)$  symmetric matrix

$$c^s = \begin{bmatrix} 0 & c \\ c^T & 0 \end{bmatrix} \in \mathfrak{p}.$$

The proof will be organized in the following way:

1. Proof for  $q = p + 1$  using induction on  $p$ 
  - (a) Proof for  $p = 2$  and  $q = 3$
  - (b) Proof of the induction step
    - i. Proof in the case  $u > 0$  or  $v > 0$
    - ii. Proof in the case  $X[p; 0], Y[p; 0]$
2. Proof that the case  $(p, q)$  implies the case  $(p, q + 1)$ .

## 1. Proof for $q = p + 1$ using induction on $p$

### (a) Proof for $p = 2$ and $q = 3$

This corresponds to the space  $\mathbf{SO}_0(2, 3)$ . Only two configurations  $[2; 0]$  and  $[1; 1]$  may be realized by singular non-zero  $X$  and  $Y$ . When  $Z \in \bar{\mathfrak{a}}^+$ , we have  $\mathcal{D}_{Z[1;1]} = \text{diag}[z, 0]$ , and  $\mathcal{D}_{Z[2;0]} = \text{diag}[z, z]$ ,  $z \neq 0$ . It is easy to check that in all 3 possible cases:

$$(i) X[2; 0], Y[2; 0] \quad (ii) X[2; 0], Y[1; 1] \text{ or } X[1; 1], Y[2; 0] \quad (iii) X[1; 1], Y[1; 1],$$

$X$  and  $Y$  are eligible. Note that

$$\begin{aligned} \mathfrak{p} &= \left\{ \begin{bmatrix} h_1 & a & b \\ c & h_2 & d \end{bmatrix}^s : h_1, h_2, a, b, c, d \in \mathbf{R} \right\}, \\ V_{Z[2;0]} &= \left\{ \begin{bmatrix} 0 & a & b \\ -a & 0 & c \end{bmatrix}^s : a, b, c \in \mathbf{R} \right\}, \\ V_{Z[1;1]} &= \left\{ \begin{bmatrix} 0 & a & c \\ b & 0 & 0 \end{bmatrix}^s : a, b, c \in \mathbf{R} \right\}. \end{aligned}$$

$$\text{If } k_1 = \begin{bmatrix} \sqrt{2}/2 & -\sqrt{2}/2 & 0 & 0 & 0 \\ \sqrt{2}/2 & \sqrt{2}/2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{2}/2 & -\sqrt{2}/2 \\ 0 & 0 & 0 & \sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix} \text{ then}$$

$$\text{Ad}(k_1) V_{Z[2;0]} = \left\{ \begin{bmatrix} \sqrt{2}a/2 & (a-b+c)/2 & (a+b-c)/2 \\ -\sqrt{2}a/2 & (a-b-c)/2 & (a+b+c)/2 \end{bmatrix}^s : a, b, c \in \mathbf{R} \right\},$$

and

$$\text{Ad}(k_1) V_{Z[1;1]} = \left\{ \begin{bmatrix} -\sqrt{2}b/2 & (a-c)/2 & (a+c)/2 \\ \sqrt{2}b/2 & (a-c)/2 & (a+c)/2 \end{bmatrix}^s : a, b, c \in \mathbf{R} \right\}$$

$$\text{If } k_2 = \begin{bmatrix} \sqrt{2}/2 & -\sqrt{2}/2 & 0 & 0 & 0 \\ \sqrt{2}/2 & \sqrt{2}/2 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{2}/2 & 0 & -\sqrt{2}/2 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & \sqrt{2}/2 & 0 & \sqrt{2}/2 \end{bmatrix} \text{ then}$$

$$\text{Ad}(k_2) V_{Z[1;1]} = \left\{ \begin{bmatrix} -(b+c)/2 & \sqrt{2}a/2 & (-b+c)/2 \\ (b-c)/2 & \sqrt{2}a/2 & (b+c)/2 \end{bmatrix}^s : a, b, c \in \mathbf{R} \right\}.$$

We verify easily that in the cases (i) and (iii) we have  $V_X + \text{Ad}(k_1) V_Y = \mathfrak{p}$ . For  $X[2;0]$  and  $Y[1;1]$ , we can see that  $V_X + \text{Ad}(k_2) V_Y = \mathfrak{p}$ .

## (b) Proof of the induction step

### (i) Proof in the case $u > 0$ or $v > 0$

We consider the space  $\mathbf{SO}_0(p, p+1)/\mathbf{SO}(p) \times \mathbf{SO}(p+1)$  with  $p > 2$  and the case when  $X[\mathbf{s}; u]$  and  $Y[\mathbf{t}; v]$  are such that  $u > 0$  or  $v > 0$ . We assume  $u \geq v$ . We choose the predecessors in  $\mathbf{SO}_0(p-1, p)/\mathbf{SO}(p-1) \times \mathbf{SO}(p)$  in the following way:

$$X' = X'[\mathbf{s}; u-1], \quad Y' = Y'[\mathbf{t}'; v]$$

where  $\mathbf{t}'$  means that we suppress one term from the longest block of size  $\max \mathbf{t}$ . Note that if  $p > 2$  then  $\mathbf{t}'$  is not the zero partition (otherwise,  $\mathbf{t}$  would have been the partition  $[1]$  meaning that  $u \geq v = p-1$  which would make  $X$  and  $Y$  ineligible).

We arrange  $X, X', Y, Y'$  in the following way.

1. The first diagonal entry of  $\mathcal{D}_X$  is zero and all the other zeros are at the end. The diagonal entries of  $\mathcal{D}_{X'}$  are those of  $\mathcal{D}_X$  without the first zero:

$$\mathcal{D}_X = \text{diag}[0, \overbrace{x_1, \dots, x_{p-u}}^{\neq 0}, \overbrace{0, \dots, 0}^{u-1}], \quad \mathcal{D}_{X'} = \text{diag}[\overbrace{x_1, \dots, x_{p-u}}^{\neq 0}, \overbrace{0, \dots, 0}^{u-1}]$$

2. We put a longest block of size  $t$  of equal diagonal entries  $y_1$  of  $\mathcal{D}_Y$  in the beginning of  $Y$ . The diagonal entries of  $\mathcal{D}_{Y'}$  are those of  $\mathcal{D}_Y$  with the first entry omitted:

$$\mathcal{D}_Y = \text{diag}[\overbrace{y_1, \dots, y_1}^t, y_2, \dots, y_s], \quad \mathcal{D}_{Y'} = \text{diag}[\overbrace{y_1, \dots, y_1}^{t-1}, y_2, \dots, y_s].$$

It is easy to check that if  $X, Y$  are eligible in  $\mathbf{SO}_0(p, p+1)$  then  $X', Y'$  are eligible in  $\mathbf{SO}_0(p-1, p)$ .

**Step 1.** By the induction hypothesis, there exists a matrix  $k_0 \in \mathbf{SO}(p-1) \times \mathbf{SO}(p)$  such that

$$V_{X'} + \text{Ad}(k_0)V_{Y'} = \mathfrak{p}'. \quad (6)$$

We embed  $K' = \mathbf{SO}(p-1) \times \mathbf{SO}(p)$  in  $\mathbf{SO}(p) \times \mathbf{SO}(p+1)$  in the following way

$$K' = \begin{bmatrix} 1 & & & \\ & \mathbf{SO}(p-1) & & \\ & & 1 & \\ & & & \mathbf{SO}(p) \end{bmatrix} \subset \begin{bmatrix} \mathbf{SO}(p) & \\ & \mathbf{SO}(p+1) \end{bmatrix}.$$

Hence, we have (taking the natural embedding of  $\mathfrak{p}'$  into  $\mathfrak{p}$ )

$$V_1 := V_{X'} + \text{Ad}(k_0)V_{Y'} = \mathfrak{p}' = \begin{bmatrix} 0 & B' \\ B'^T & 0 \end{bmatrix} \quad (7)$$

where  $B' = \left[ \begin{array}{c|c} 0_{1 \times 1} & 0_{1 \times p} \\ \hline 0_{p \times 1} & B''_{(p-1) \times p} \end{array} \right]$  and  $B''$  is arbitrary (note that  $\mathfrak{p}'$  is of dimension  $(p-1)p$ ). We must show that for some  $k \in K$ , the space  $V_X + \text{Ad}(k)V_Y = \mathfrak{p}$ , i.e. that

- (i)  $V_X + \text{Ad}(k)V_Y$  contains  $\mathfrak{p}'$  embedded into  $\mathfrak{p}$  as in (7).
- (ii)  $V_X + \text{Ad}(k)V_Y$  contains all the matrices of the form

$$C = \left[ \begin{array}{c|c} * & * \dots * \\ * & \\ \vdots & \\ * & 0_{(p-1) \times p} \end{array} \right]^s.$$

*New vectors in  $V_X$  and  $V_Y$ .* In order to prove the induction conclusion, we must now use the elements of  $V_X$  and  $V_Y$  which do not come from  $V_{X'}$  or  $V_{Y'}$ . They appear by the interaction of, respectively, the first diagonal entry of  $\mathcal{D}_X$  with the others of  $\mathcal{D}_X$  and the interaction of the first

entry of  $\mathcal{D}_Y$  with the others of  $\mathcal{D}_Y$ . We see that the new independent root vectors in  $V_X$  and  $V_Y$  are respectively

$$N_X = \{Y_{1j}, Z_{1j}, j = 2, \dots, p+1-u\}, \quad N_Y = \{X_1, Y_{1i}, i = t+1, \dots, p, Z_{1j}, j = 2, \dots, p\}$$

where  $t = \max \mathbf{t} > 1$  and we wrote  $X_1$  for  $X_{11}$ . Note that  $N_X$  has  $2p - 2u$  elements while  $N_Y$  has  $2p - t$ .

**Step 2.** We show that there exists  $k'_0 \in \mathbf{SO}(p-1) \times \mathbf{SO}(p)$  for which (6) holds, and with the following property:

The space  $V_2 := \text{Ad}(k'_0)\text{span}(N_Y)$  is of dimension  $2p - t$  and its elements can be written in the form

$$\left[ \begin{array}{c|cccccc} 0 & \sigma_1 & \dots & \sigma_r & a_1 & \dots & a_{p-r} \\ \hline \tau_1 & & & & & & \\ \vdots & & & & & & \\ \tau_s & & & & 0 & & \\ a_{p-r+1} & & & & & & \\ \vdots & & & & & & \\ a_{2p-t} & & & & & & \end{array} \right]^s \quad (8)$$

with  $r = \lceil (t-1)/2 \rceil$ ,  $s = t-1-r$ ,  $a_i \in \mathbf{R}$  arbitrary,  $\sigma_i = \sigma_i(a_1, \dots, a_{2p-t})$  and  $\tau_j = \tau_j(a_1, \dots, a_{2p-t})$ ,  $i \leq r, j \leq s$ .

We will not need to write explicitly the functions  $\sigma_i$  and  $\tau_j$ . Note that  $s = r$  if  $t$  is odd and  $s = r + 1$  if  $t$  is even.

To justify Step 2, we write

$$k_0 = \begin{bmatrix} 1 & & & \\ & k_{01} & & \\ & & 1 & \\ & & & k_{02} \end{bmatrix}$$

where  $k_{01} \in \mathbf{SO}(p-1)$  and  $k_{02} \in \mathbf{SO}(p)$ . Let  $\alpha_1, \dots, \alpha_{p-1}$  be the columns of the matrix  $k_{01}$  and  $\beta_1, \dots, \beta_p$  the columns of the matrix  $k_{02}$ . A simple block multiplication to compute the action of  $\text{Ad}(k_0)$  on the elements of  $N_Y$  gives the linearly independent matrices

$$\begin{aligned} \text{Ad}(k_0)X_1 &= \left[ \begin{array}{c|c} 0 & \beta_p^T \\ \hline 0 & 0 \end{array} \right]^s, \quad \text{Ad}(k_0)Y_{1i} = \left[ \begin{array}{c|c} 0 & \beta_{i-1}^T \\ \hline \alpha_{i-1} & 0 \end{array} \right]^s, \quad i = t+1, \dots, p, \\ \text{Ad}(k_0)Z_{1i} &= \left[ \begin{array}{c|c} 0 & \beta_{i-1}^T \\ \hline -\alpha_{i-1} & 0 \end{array} \right]^s, \quad i = 2, \dots, p. \end{aligned} \quad (9)$$

Let us write  $\beta'_i$  for a column  $\beta_i$  from which we have removed the first  $r$  entries and  $\alpha'_i$  for a column  $\alpha_i$  with the first  $s$  entries omitted. In order to prove the statement of Step 2, we must show that the matrices obtained by replacing  $\beta_i$  by  $\beta'_i$  and  $\alpha_i$  by  $\alpha'_i$  in (9) are still linearly independent. This is equivalent to the linear independence of the matrices

$$\left[ \begin{array}{c|c} 0 & \beta_i'^T \\ \hline -\alpha'_i & 0 \end{array} \right]^s, \quad i = 1, \dots, t-1, \quad \left[ \begin{array}{c|c} 0 & \beta_i'^T \\ \hline 0 & 0 \end{array} \right]^s, \quad i = t, \dots, p, \quad \left[ \begin{array}{c|c} 0 & 0 \\ \hline \alpha'_i & 0 \end{array} \right]^s, \quad i = t, \dots, p-1. \quad (10)$$

We will reason in the same way as in Lemma 11.

It is enough to show that there exists at least a choice of matrices  $k_{01}$  and  $k_{02}$  such that the matrices in (10) are linearly independent. Then, as in Lemma 11, it will follow that such matrices form a dense open subset in  $\mathbf{SO}(p-1) \times \mathbf{SO}(p)$ . By choosing  $k'_0$  with the matrices in (10) linearly independent and close enough to  $k_0$ , the property (6) will be preserved for  $k'_0$ .

Pick  $k_{01} = I_{p-1}$  which implies that  $\alpha'_i = \mathbf{0}$  for  $i = 1, \dots, s$  and  $\alpha'_i = \mathbf{e}_{i-s}$  for  $i > s$ . With this choice, (10) becomes

$$\left[ \begin{array}{c|c} 0 & \beta_i'^T \\ \hline 0 & 0 \end{array} \right]^s, \quad i = 1, \dots, s, \quad \left[ \begin{array}{c|c} 0 & \beta_i'^T \\ \hline -\mathbf{e}_{i-s} & 0 \end{array} \right]^s, \quad i = s+1, \dots, t-1, \quad \left[ \begin{array}{c|c} 0 & \beta_i'^T \\ \hline 0 & 0 \end{array} \right]^s, \quad i = t, \dots, p, \\ \left[ \begin{array}{c|c} 0 & 0 \\ \hline \mathbf{e}_{i-s} & 0 \end{array} \right]^s \quad i = t, \dots, p-1$$

which are linearly independent provided that

$$\left[ \begin{array}{c|c} 0 & \beta_i'^T \\ \hline 0 & 0 \end{array} \right]^s, \quad i = 1, \dots, s, \quad \left[ \begin{array}{c|c} 0 & \beta_i'^T \\ \hline 0 & 0 \end{array} \right]^s, \quad i = t, \dots, p$$

are linearly independent. This is the case for a total matrix  $k_{02} \in \mathbf{SO}(p)$  or by taking convenient permutations of the rows and columns of the identity matrix  $I_p$ .

**Step 3.** We show that there exists a proper subset  $N'_X$  of  $N_X$  such that, if

$$V_3 := \text{span}(N'_X) + V_{X'} + \text{Ad}(k'_0)V_Y = \text{span}(N'_X) + V_1 + V_2$$

then  $\dim V_3 = pq - 1 = \dim \mathfrak{p} - 1$  and  $V_3$  is given by

$$V_3 = \left\{ \left[ \begin{array}{c|cccc} 0 & a_1 & a_2 & \dots & a_p \\ \hline a_{p+1} & & & & \\ \vdots & & & & \\ a_{2p-1} & & & & \end{array} \right]^s, \quad a_1, \dots, a_{2p-1} \in \mathbf{R} \right\}. \quad (11)$$

Note that in matrices from the space  $V_2$ , there are  $r = \lfloor (t-1)/2 \rfloor$  pairs  $(\sigma_i, \tau_i)$  plus possibly an extra  $\tau_s$  if  $t$  is even and therefore  $s = r + 1$ . Note also that  $t + 2u \leq 2p$  implies that  $p - u \geq s \geq r$ . For  $j \leq r \leq p - u$ , each pair  $Y_{1j} = \left[ \begin{array}{c|c} 0 & \mathbf{e}_{j-1}^T \\ \hline \mathbf{e}_{j-1} & 0 \end{array} \right]^s$ ,  $Z_{1j} = \left[ \begin{array}{c|c} 0 & \mathbf{e}_{j-1}^T \\ \hline -\mathbf{e}_{j-1} & 0 \end{array} \right]^s$  of elements of  $N_X$  allows us to replace  $\sigma_j$  and  $\tau_j$  by independent variables. If  $t$  is odd, all the  $\sigma_j$ 's and  $\tau_j$ 's will be taken care off and at least 2 elements of  $N_X$  will remain off  $N'_X$ . If  $t$  is even, all the  $\sigma_j$  and  $\tau_j$ 's,  $1 \leq j \leq r$  will be replaced by independent variables and only  $\tau_s$  will remain. Now, letting the coefficient  $a_1$  "vis-à-vis" the remaining  $\tau_s$  be equal to 1 and all the other variables  $a_i$  equal to 0, either  $\tau_s = 1$  or  $-1$  or  $\tau_s \neq \pm 1$ . If  $\tau_s = 1$  then  $Z_{1s}$  allows us to introduce the missing independent variable, if  $\tau_s = -1$  then adding  $Y_{1s}$  to  $N'_X$  will do the trick. In the case  $\tau_s \neq \pm 1$  we choose indifferently between  $Y_{1s}$  and  $Z_{1s}$ . In all cases the set  $N_X \setminus N'_X$  has at least one element.

**Step 4.** Let  $v_1$  be the positive root vector corresponding to an element of  $N_X \setminus N'_X$ . We denote  $k_1^t = k_{v_1}^t$ . There exists  $\epsilon > 0$  such that for  $t \in (0, \epsilon)$

$$V_4^t := \text{Ad}(k_1^t)(\text{span}(N'_X) + V_{X'}) + \text{Ad}(k'_0)V_Y = V_3.$$

Observe that  $v_1$  is equal to  $Z_{1j}^+$  or  $Y_{1j}^+$  for one of the remaining  $Z_{1j}$  or  $Y_{1j}$  that was not used in the preceding step. The space  $V_4^t \subset V_4^0$  for all  $t$  according to Lemma 12 (ii).

Recall the definition of  $k_{X_\alpha}^t = e^{t(X_\alpha + \theta X_\alpha)}$ ,  $t > 0$ . Let  $d(t) = \dim V_4^t$ ; for  $t = 0$  we have  $k_1^0 = Id$ , and  $\text{Ad}(k_1^0)(\text{span}(N'_X) + V_{X'}) + \text{Ad}(k'_0)V_Y = V_3$  is of dimension  $pq - 1$ , so  $d(0) = pq - 1$ . The equality  $d(t) = d(0)$  is equivalent to non-nullity of an appropriate determinant continuous in  $t$ . Thus  $d(t) = pq - 1$  holds for  $t \in (0, \epsilon)$  for some  $\epsilon > 0$ . As  $V_4^t \subset V_4^0$ , the statement of Step 4 follows.

**Step 5.** *Generation of  $A_1$ .* By Lemma 12, we have  $\text{Ad}(k_1^t)v_1^s = a_tv_1^s + b_tA_1 + c_tA_j$  with  $j \neq 1$  and  $b_t \neq 0$  for  $t \in (0, \epsilon)$  with  $\epsilon$  small enough. Consequently

$$\text{Ad}(k_1^t)\text{span}(v_1^s) + V_4^t = \mathfrak{p}.$$

**Conclusion.** We have  $\mathfrak{p} = \text{Ad}(k_1^t)(\mathbf{R}v_1^s + \text{span}N'_X + V_{X'}) + \text{Ad}(k'_0)V_Y \subset \text{Ad}(k_1^t)V_X + \text{Ad}(k'_0)V_Y$ , so  $\text{Ad}(k_1^t)V_X + \text{Ad}(k'_0)V_Y = \mathfrak{p}$ . It follows that

$$V_X + \text{Ad}((k_1^t)^{-1}k'_0)V_Y = \mathfrak{p}.$$

**(ii) Proof in the case  $X[p; 0], Y[p; 0]$**

This case must be treated separately because the predecessors  $X', Y'$  and consequently the sets  $N_X$  and  $N_Y$  are different than in case (i). The structure of the induction proof is identical as in (i), with the Steps 2 and 3 executed together.

We choose both predecessors  $X'[p-1; 0], Y'[p-1; 0]$  and arrange  $X, X', Y, Y'$  in the same way as we did in the first part of the proof with  $Y[\mathbf{t}; v]$  and  $Y'[\mathbf{t}'; v]$ . In that case,  $N_X = \{X_1, Z_{12}, \dots, Z_{1p}\} = N_Y$  and the space  $\text{Ad}(k'_0)(N_Y)$  is generated by

$$\left[ \begin{array}{c|c} 0 & \beta_i^T \\ \hline -\alpha_i & 0 \end{array} \right]^s, \quad i = 1, \dots, p-1 \quad \text{and} \quad \left[ \begin{array}{c|c} 0 & \beta_p^T \\ \hline 0 & 0 \end{array} \right]^s. \quad (12)$$

Recall that

$$Z_{1j} = \left[ \begin{array}{c|c} 0 & \mathbf{e}_{j-1}^T \\ \hline -\mathbf{e}_{j-1} & 0 \end{array} \right]^s, \quad j = 2, \dots, p. \quad (13)$$

We want to show that the matrices in (12) together with those of (13) are linearly independent for a  $k'_0 \in \mathbf{SO}(p-1) \times \mathbf{SO}(p)$  for which the equality (6) holds. Note that if  $k'_0 = \begin{bmatrix} -I_{p-1} & 0 \\ 0 & I_p \end{bmatrix}$

( $p$  odd) or  $k'_0 = \begin{bmatrix} I_{p-1} & 0 \\ 0 & -I_p \end{bmatrix}$  ( $p$  even) then the matrices (12) and (13) are linearly independent.

Using once more the reasoning in Lemma 11, this implies that the set of matrices  $k'_0$  for which this is true, is open and dense in  $\mathbf{SO}(p-1) \times \mathbf{SO}(p)$ .

We conclude that if  $N'_X = N_X \setminus \{X_1\}$  then  $\text{span}(N'_X + V_{X'}) + \text{Ad}(k'_0)V_Y$  has the form given in (11).

We reproduce the previous Step 4 and Step 5 using  $v_1 = X_1^+$ . The rest follows.

**2. Proof that the case  $(p, q)$  implies the case  $(p, q+1)$**

=====>We will show by induction that for any  $q > p$ , there exists a matrix  $k \in K$  such that (5) holds.<=====<We know by the first part of the proof that this is true for  $\mathbf{SO}_0(p, p+1)$ .

Assume that  $X$  and  $Y$  are eligible in  $\mathbf{SO}_0(p, q+1)$ . Their configurations are eligible in  $\mathbf{SO}_0(p, q)$ . We write  $X', Y'$  when we work in  $\mathbf{SO}_0(p, q)$ .

We embed  $K' = \mathbf{SO}(p) \times \mathbf{SO}(q)$  in  $K = \mathbf{SO}(p) \times \mathbf{SO}(q+1)$  in the following way

$$K' = \begin{bmatrix} \mathbf{SO}(p) & & \\ & \mathbf{SO}(q) & \\ & & 1 \end{bmatrix} \subset \begin{bmatrix} \mathbf{SO}(p) & & \\ & \mathbf{SO}(q+1) & \\ & & 1 \end{bmatrix}.$$

The space  $\mathfrak{p}'$  is formed by matrices

$$\begin{bmatrix} 0 & B \\ B^T & 0 \end{bmatrix},$$

where  $B$  are  $p \times q$  matrices. We embed  $\mathfrak{p}'$  in  $\mathfrak{p}$  by adding a last column of zeros to  $B$ .

**Step 1.** We suppose that there exists a matrix  $k_0 \in K'$  such that

$$V_{X'} + \text{Ad}(k_0)V_{Y'} = \mathfrak{p}'. \quad (14)$$

Then, by [7, Lemma 3.3], for any permutations  $s_1$  and  $s_2$  of the diagonal entries of  $\mathcal{D}_{X'}$  and  $\mathcal{D}_{Y'}$ , there exists  $k_0 \in K'$  such that

$$V_{s_1 X'} + \text{Ad}(k_0)V_{s_2 Y'} = \mathfrak{p}'$$

so we can permute the elements of  $X'$  and  $Y'$  in a convenient way and still have the equality (14). We will arrange them in the following way (where the stars denote nonzero entries):

$$\mathcal{D}_{X'} = \text{diag}[\overbrace{0, \dots, 0}^u, \star, \dots, \star], \quad \mathcal{D}_{Y'} = \text{diag}[\star, \dots, \star, \overbrace{0, \dots, 0}^v].$$

Let us denote  $k_{01} \in \mathbf{SO}(p)$  and  $k_{02} \in \mathbf{SO}(q)$  the matrices compositing  $k_0$  corresponding in (14) to such  $X'$  and  $Y'$ . We can suppose that the matrix  $k_{01}$  is total.

By the eligibility of  $X$  and  $Y$ ,  $u + v \leq p$ , so no two zeros in  $\mathcal{D}_{X'}$  and  $\mathcal{D}_{Y'}$  are at the same position.

Let  $N = \{X_{i,q+1}\}_{i=1}^p$ . We set  $N_X := V_X \cap N = \{X_{u+1,q+1}, \dots, X_{p,q+1}\}$ ,  $N_Y := V_Y \cap N = \{X_{1,q+1}, \dots, X_{p-v,q+1}\}$ . We have  $p - v \geq u$ .

**Step 2.** Let

$$k_1 = \begin{bmatrix} k_{01} & & \\ & k_{02} & \\ & & 1 \end{bmatrix}$$

where  $k_{01} \in \mathbf{SO}(p)$  and  $k_{02} \in \mathbf{SO}(q)$  are the blocks compositing  $k_0$ . We then have

$$V_{X'} + \text{Ad}(k_1)V_{Y'} = [\mathfrak{p}' \quad \mathbf{0}]^s. \quad (15)$$

The space  $V_X + \text{Ad}(k_1)V_Y$  contains, in addition to the matrices in (15), the linear span of  $N_X + \text{Ad}(k_1)N_Y$ .



Denote the columns of the matrix  $k_{01}$  by  $\mathbf{c}_1, \dots, \mathbf{c}_p$ . By block multiplication in  $\mathbf{SO}_0(p, q+1)$ , we obtain

$$\text{Ad}(k_1)X_{j,q+1} = \begin{bmatrix} 0_{p \times q} & \mathbf{c}_j \end{bmatrix}^s.$$

This implies that the linear span of  $N_X + \text{Ad}(k_1)N_Y$  contains the following symmetric matrices:

$$\begin{bmatrix} 0_{p \times q} & \mathbf{c}_1 \end{bmatrix}^s, \dots, \begin{bmatrix} 0_{p \times q} & \mathbf{c}_u \end{bmatrix}^s, \begin{bmatrix} 0_{p \times q} & \mathbf{e}_{u+1} \end{bmatrix}^s, \dots, \begin{bmatrix} 0_{p \times q} & \mathbf{e}_p \end{bmatrix}^s,$$

which are linearly independent by the totality of  $k_{01}$ . Thus  $V_X + \text{Ad}(k_1)V_Y = \mathfrak{p}$ . ■

=====>

We conclude this section with an example to illustrate our proof.

**Example 14** Consider  $X = X[2; 1], Y = Y[1, 1; 1]$  in  $\mathfrak{so}(3, 4)$ . We write  $X$  and  $Y$  in such a way that  $\mathcal{D}_X = \text{diag}[0, a, a]$  and  $\mathcal{D}_Y = \text{diag}[b, c, 0]$ . Their predecessors in  $\mathfrak{so}(2, 3)$  are  $X'$  and  $Y'$  such that  $\mathcal{D}_{X'} = \text{diag}[a, a]$  and  $\mathcal{D}_{Y'} = \text{diag}[c, 0]$ .

Note that  $X$  and  $Y$  form an eligible pair and so are  $X' = X[2; 0]$  and  $Y' = Y'[1; 1]$ . In Step 1,

we show that there exists a matrix  $k_0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & k_{0,1} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & k_{0,2} \end{bmatrix}$  with  $k_{01} \in \mathbf{SO}(2)$  and  $k_{0,2} \in \mathbf{SO}(3)$

such that

$$V_{X'} + \text{Ad}(k_0)V_{Y'} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & * & * & * \\ 0 & * & * & * \end{bmatrix}^s$$

where  $*$  designates an arbitrary element. We have

$$N_X = \{Z_{12}, Y_{12}, Z_{13}, Y_{13}\}, \quad N_Y = \{X_1, Z_{12}, Y_{12}, Z_{13}, Y_{13}\}.$$

In Step 2, we observe that

$$\text{Ad}(k_0)\text{span}(N_Y) = \left\{ \left[ \begin{array}{c|ccc} 0 & a_1 & a_2 & a_3 \\ a_4 & & & \\ a_5 & & & 0 \end{array} \right]^s, \quad a_1, \dots, a_6 \in \mathbf{R} \right\}$$

since the matrices

$$\left[ \begin{array}{c|c} 0 & \beta_1^T \\ -\alpha_1 & 0 \end{array} \right]^s, \left[ \begin{array}{c|c} 0 & \beta_1^T \\ \alpha_1 & 0 \end{array} \right]^s, \left[ \begin{array}{c|c} 0 & \beta_2^T \\ -\alpha_2 & 0 \end{array} \right]^s, \left[ \begin{array}{c|c} 0 & \beta_2^T \\ \alpha_2 & 0 \end{array} \right]^s, \left[ \begin{array}{c|c} 0 & \beta_3^T \\ 0 & 0 \end{array} \right]^s$$

are linearly independent. Note that in this case, there are no  $\sigma_i$  and no  $\tau_i$ .

Now,  $V_X = \text{span} \left\{ \overbrace{Z_{12}, Y_{12}, Z_{13}, Y_{13}}^{N_X} \right\} \cup V_{X'}$  while  $V_Y = \text{span} \left\{ \overbrace{X_1, Z_{12}, Z_{13}, Z_{14}, Y_{13}}^{N_Y} \right\} \cup V_{Y'}$ . We can show that

$$\begin{aligned} \text{Ad}(e^{t(Z_{1,2}^+ + \theta Z_{1,2}^+)}) (V_{X'}) + \text{Ad}(k_0) (\text{span } N_Y \cup V_{Y'}) \\ = \begin{bmatrix} 0 & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix}^s \end{aligned}$$

for  $t$  small enough (with  $t$  small enough, the dimension will not decrease). Now,

$$Ad(e^{t(Z_{1,2}^+ + \theta Z_{1,2}^+)}) \overbrace{(\text{span}\{Z_{12}\} \cup V_{X'})}^{\subset V_X} + Ad(k_0) \overbrace{(\text{span} N_Y \cup V_{Y'})}^{V_Y} = \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix}^s = \mathfrak{p}$$

if  $t$  close to 0 since  $Ad(e^{t(Z_{1,2}^+ + \theta Z_{1,2}^+)}) (Z_{12}) = \cos(4t) Z_{12} + 2 \sin(4t) (A_1 + A_2)$ . Therefore,

$$V_X + Ad(\overbrace{e^{-t(Z_{1,2}^+ + \theta Z_{1,2}^+)} k_0}^k) V_Y = Ad(e^{-t(Z_{1,2}^+ + \theta Z_{1,2}^+)}) \mathfrak{p} = \mathfrak{p}$$

which means that the density exists.

←=====

## 5 Applications

=====> We now extend our results to the complex and quaternion cases. ←=====

Recall that  $\mathbf{SU}(p, q)$  is the subgroup of  $\mathbf{SL}(p + q, \mathbf{C})$  such that  $g^* I_{p,q} g = I_{p,q}$  while  $\mathbf{Sp}(p, q)$  is the subgroup of  $\mathbf{SL}(p + q, \mathbf{H})$  such that  $g^* I_{p,q} g = I_{p,q}$ . Their respective maximal compact subgroups are  $\mathbf{S}(\mathbf{U}(p) \times \mathbf{U}(q))$  and  $\mathbf{Sp}(p) \times \mathbf{Sp}(q) \equiv \mathbf{SU}(p, \mathbf{H}) \times \mathbf{SU}(q, \mathbf{H})$ .

Their subspaces  $\mathfrak{p}$  can be described as  $\begin{bmatrix} 0 & B \\ B^* & 0 \end{bmatrix}$  where  $B$  is an arbitrary complex (respectively quaternionic) matrix of size  $p \times q$ . The Cartan subalgebra  $\mathfrak{a}$  is chosen in the same way as for  $\mathfrak{so}(p, q)$ .

**Corollary 15** Consider the symmetric spaces  $\mathbf{SO}_0(p, q)/\mathbf{SO}(p) \times \mathbf{SO}(q)$ ,  $\mathbf{SU}(p, q)/\mathbf{S}(\mathbf{U}(p) \times \mathbf{U}(q))$  and  $\mathbf{Sp}(p, q)/\mathbf{Sp}(p) \times \mathbf{Sp}(q)$ ,  $q > p$ .

Let  $X, Y \in \mathfrak{a}$ . Then the measure  $\delta_{e^X}^{\mathfrak{h}} \star \delta_{e^Y}^{\mathfrak{h}}$  is absolutely continuous if and only if  $X$  and  $Y$  are eligible, as defined in Definition 3.

**Proof:** Let  $X, Y \in \mathfrak{a}$ . If they are eligible then since

$$a(e^X (\mathbf{SO}(p) \times \mathbf{SO}(q)) e^Y) \subset a(e^X \mathbf{S}(\mathbf{U}(p) \times \mathbf{U}(q)) e^Y) \subset a(e^X (\mathbf{Sp}(p) \times \mathbf{Sp}(q)) e^Y),$$

it follows from Theorem 13 that these sets have nonempty interior. Hence the density exists in all three cases.

On the other hand, given Lemma 4, one can reproduce Proposition 5 using  $\mathbf{F} = \mathbf{C}$  and  $\mathbf{F} = \mathbf{H}$  to show that the eligibility condition is necessary in the complex and quaternionic cases. ■

We will conclude this paper with two further applications.

**Proposition 16** Let  $X$  and  $Y \in \mathfrak{a}$  be such that  $(\delta_{e^X}^{\mathfrak{h}})^{*2}$  and  $(\delta_{e^Y}^{\mathfrak{h}})^{*2}$  are absolutely continuous. Then  $\delta_{e^X}^{\mathfrak{h}} \star \delta_{e^Y}^{\mathfrak{h}}$  is absolutely continuous.

**Proof:**  $\implies$  Let  $X = X[\mathbf{s}; u]$  and  $Y = Y[\mathbf{t}; v]$ .  $\Leftarrow \implies$  We know that the couple  $(X, X)$  is eligible; therefore

$$2 \max\{\mathbf{s}, 2u\} \leq 2p.$$

In the same manner,  $\max\{\mathbf{t}, 2v\} \leq p$ . Hence,

$$\max\{\mathbf{s}, 2u\} + \max\{\mathbf{t}, 2v\} \leq p + p = 2p$$

which means that  $X$  and  $Y$  are eligible. Consequently,  $\delta_{e^X}^{\mathfrak{h}} * \delta_{e^Y}^{\mathfrak{h}}$  is absolutely continuous.  $\blacksquare$

If  $X \in \mathfrak{a}$  and  $X \neq 0$ , it is important to know for which convolution powers  $l$  the measure  $(\delta_{e^X}^{\mathfrak{h}})^l$  is absolutely continuous. This problem is equivalent to the study of the absolute continuity of convolution powers of uniform orbital measures  $\delta_g^{\mathfrak{h}} = m_K * \delta_g * m_K$  for  $g \notin K$ .

It was proved in [8, Corollary 7] that it is always the case for  $l \geq r + 1$ , where  $r$  is the rank of the symmetric space  $G/K$ . It was also conjectured ([8, Conjecture 10]) that  $r + 1$  is optimal for this property, which was effectively proved for symmetric spaces of type  $A_n$  ([8, Corollary 18]). In the following theorem, the conjecture is shown not to hold on symmetric spaces of type  $\implies B_p \Leftarrow \implies$ , where  $r = p$ . Thanks to the rich structure of the root system  $B_p$ , already all  $p$ -th powers of orbital measures are absolutely continuous and  $p$  is optimal for this property.

**Theorem 17** *On symmetric spaces  $\mathbf{SO}_0(p, q)/\mathbf{SO}(p) \times \mathbf{SO}(q)$ ,  $\mathbf{SU}(p, q)/\mathbf{S}(\mathbf{U}(p) \times \mathbf{U}(q))$  and  $\mathbf{Sp}(p, q)/\mathbf{Sp}(p) \times \mathbf{Sp}(q)$ ,  $q > p$ , for every nonzero  $X \in \mathfrak{a}$ , the measure  $(\delta_{e^X}^{\mathfrak{h}})^p$  is absolutely continuous. Moreover,  $p$  is the smallest value for which this is true: if  $X$  has a configuration  $[1; p - 1]$  then the measure  $(\delta_{e^X}^{\mathfrak{h}})^{p-1}$  is singular.*

**Proof:** We will write  $S_X^l$  for the set  $a(e^X K \dots K e^X)$  where the factor  $e^X$  appears  $l$  times. Note that  $(\delta_{e^X}^{\mathfrak{h}})^l$  is absolutely continuous if and only if  $S_X^l$  has nonempty interior.

We prove first that for  $l < p$ , the measure  $(\delta_{e^X}^{\mathfrak{h}})^l$  may not be absolutely continuous. Let  $X = X[1; p - 1]$ . Using Lemma 4 repeatedly, as in the proof of Proposition 5, we show that for  $l < p$ , there are at least  $p - l$  diagonal entries of  $\mathcal{D}_H$  which are equal to 0 for every  $H \in S_X^l$ . Consequently,  $S_X^l$  has empty interior and  $(\delta_{e^X}^{\mathfrak{h}})^l$  is not absolutely continuous when  $l \leq p - 1$ .

We will now show that  $(\delta_{e^X}^{\mathfrak{h}})^p$  has a density for every  $X \neq 0$ .

If  $X = X[\mathbf{s}; 0]$  then the measure  $(\delta_{e^X}^{\mathfrak{h}})^2$  is already absolutely continuous (the couple  $(X, X)$  is eligible). Suppose then that  $X = X[\mathbf{s}; u] \in \overline{\mathfrak{a}^+}$ ,  $u > 0$ .

Remark that if  $H \in S_X^l$  then  $a(e^X K e^H) \subset S_X^{l+1}$ . Indeed, we have  $e^X k_1 E^X \dots k_{l-1} e^X = k_a e^H k_b$  and therefore  $a(e^X K e^H) = a(e^X K k_a e^H k_b) = a(e^X K e^X k_1 \dots k_{l-1} e^X) \subset S_X^{l+1}$ .

We claim that there exists  $H \in S_X^{p-1}$  such that  $H = H[1^{p-1}; 1]$  or  $H \in \mathfrak{a}^+$ .

We prove the claim using induction on  $p$ . If  $p = 2$  then  $S_X^{p-1} = \{X\}$  and the result follows (in that case,  $u$  cannot be higher than 1 for  $X \neq 0$ ).

Suppose that the claim is true for  $p - 1 \geq 2$ . Let  $K_0 = \begin{bmatrix} \mathbf{SO}(p-1) & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \mathbf{SO}(q-1) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ .

Consider the set  $B = a(e^X K_0 e^X \dots e^X)$  with  $p - 1$  factors  $e^X$ . By the induction hypothesis, there exists  $H_0 \in B$  with  $H_0 = H_0[1^{p-2}; 2]$  or  $H_0 = H_0[1^{p-1}; 1]$ . In the second case, we are done.

If  $H_0 = H_0[1^{p-2}; 2] \in B$ , we can assume that the diagonal entries which are 0 in  $\mathcal{D}_X$  and in  $H_0$  are at the end. We note that  $X$  and  $H_0$  considered without their last entries are eligible in  $\mathbf{SO}_0(p-1, q-1)$ , their configurations being  $[\mathbf{s}; u-1]$  and  $[1^{p-2}; 1]$  respectively. Hence  $a(e^X K_0 e^{H_0})$  has nonempty interior in the subspace  $\overline{\mathfrak{a}^+} \cap \{H_p = 0\}$ . Therefore, there exists  $H \in a(e^X K_0 e^{H_0}) \subset S_X^{p-1}$  with  $H = H[1^{p-1}; 1]$  which proves the claim.

To conclude, we take  $H \in S_X^{p-1}$  with  $H \in \mathfrak{a}^+$  or  $H = H[1^{p-1}; 1]$ . In both cases,  $X$  and  $H$  are eligible, so by Corollary 15 the set  $a(e^X K e^H)$  has nonempty interior. As  $a(e^X K e^H) \subset S_X^p$ , this ends the proof. ■

## 6 Conclusion

With this paper and with [7], we have now obtained sharp criteria on singular  $X$  and  $Y$  for the existence of the density of  $\delta_{e^X}^{\mathfrak{h}} \star \delta_{e^Y}^{\mathfrak{h}}$  for the root systems of type  $A_n$  and type  $B_p$ . Thanks to [8] and Theorem 17 of the present paper, sharp criteria are now given for the  $l$ -th convolution powers  $(\delta_{e^X}^{\mathfrak{h}})^l$  to be absolutely continuous for any  $X \neq 0$ ,  $X \in \mathfrak{a}$ .

Although there is considerable similarity between the criteria for both type of spaces, a characterization of eligibility that would be applicable for all Riemannian symmetric spaces of non-compact type has yet to emerge. The solution of the second problem in Theorem 17 seems to indicate that the answer may depend on the type of the symmetric space.

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