

HYPERDETERMINANTS AND SYMMETRIC FUNCTIONS

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in collaboration with Christophe Carré

24 novembre 2012

HYPERDETERMINANTS

A LITTLE HISTORY

Simplest generalization of the determinant to higher tensor (arrays

$$\mathbf{M} = (M_{i_1, \dots, i_k})_{1 \leq i_1, \dots, i_k}$$

$$\text{Det}(\mathbf{M}) = \frac{1}{n!} \sum_{\sigma_1, \dots, \sigma_k \in \mathfrak{S}_n} \epsilon(\sigma_1) \dots \epsilon(\sigma_k) M_{\sigma_1(1), \dots, \sigma_k(1)} \dots M_{\sigma_1(n), \dots, \sigma_k(n)},$$

$\epsilon(\sigma)$ is the sign of the permutation σ .

HYPERDETERMINANTS

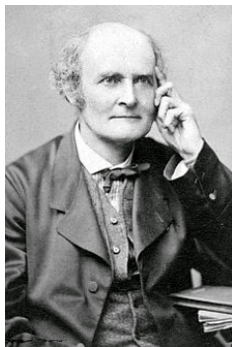
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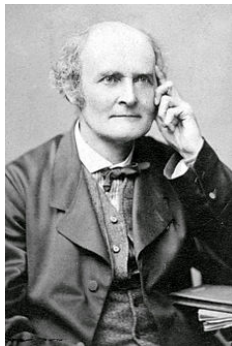
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For $k = 2$ we recover the classical determinant.

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Nineteenth century, very few other contributors for instance :

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Gegenbauer

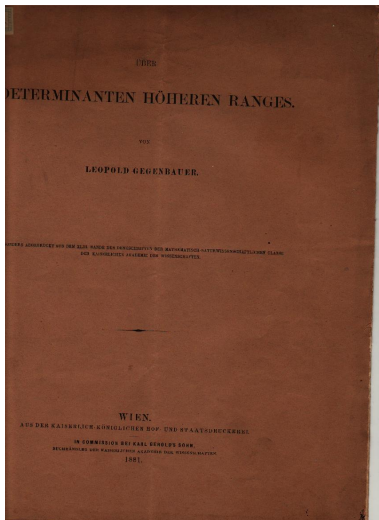
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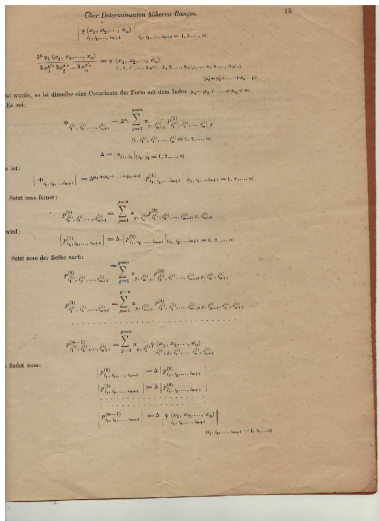
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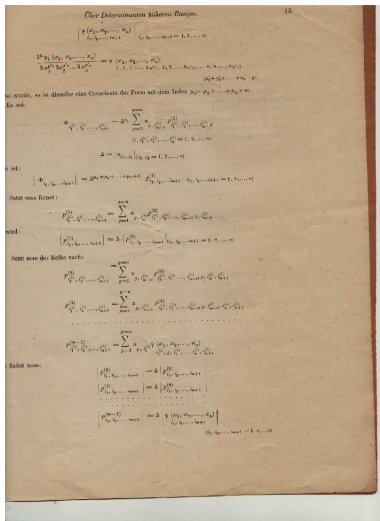
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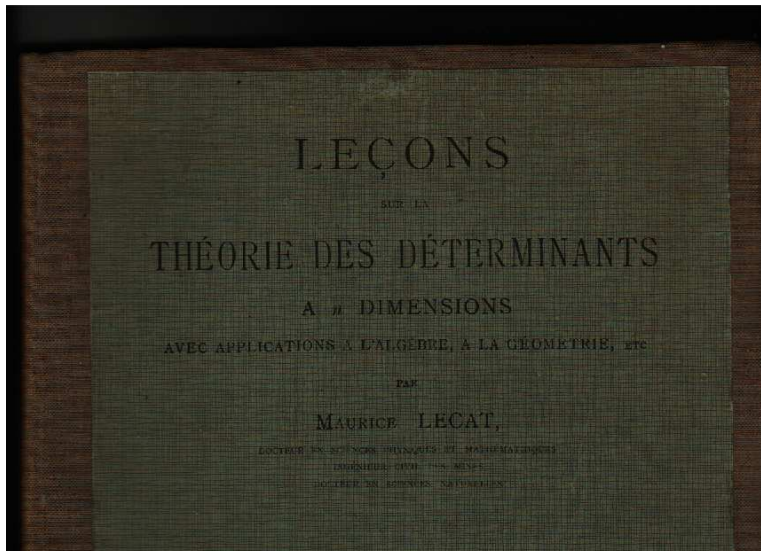


$$\left| \Phi_{i_1, i_2, \dots, i_{m+1}} \right| = \Delta^{\mu_1 + \mu_2 + \dots + \mu_m + 1} \left| p_{i_1, i_2, \dots, i_{m+1}}^{(1)} \right|$$

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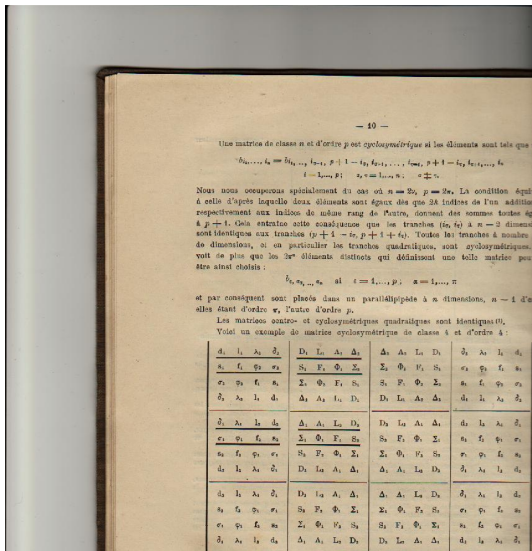
Early twentieth century, an important contributor : Maurice Lecat



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A LITTLE HISTORY

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HYPERDETERMINANTS

A LITTLE HISTORY

The reference book : Sokolov, N.P., *Introduction à la théorie des matrices multidimensionnelles*, Kiev : Nukova Dumka, En Russe, 1972.

... называются идеальтерминантами, называются *перманентами*. Детерминанта, род которых больше нуля и меньше p , называются *смешанными*.

Детерминанты (включая перманент), порожденные одной и той же многомерной матрицей, объединяются под общим названием *кодетерминантов*. Два кодетерминанта четного числа измерений будут *союзными*, если альтернативные индексы одного являются идеальтерминантами индексами другого, и наоборот.

Так, например, у четырехмерной матрицы второго порядка $\|A_{ij}\|$, $i, j, i_1, i_2 = 1, 2$) детерминантом высшего рода 4 будет гипердетерминант

$$\left| \begin{array}{c} A_{11} A_{22} \\ A_{12} A_{21} \end{array} \right|_2 = A_{111} A_{222} - A_{112} A_{221} - A_{121} A_{212} + A_{122} A_{211} - A_{121} A_{212} + A_{122} A_{211} + A_{121} A_{212} - A_{122} A_{211}.$$

Союзным кодетерминантом является перманент

$$\left| \begin{array}{c} A_{11} A_{22} \\ A_{12} A_{21} \end{array} \right|_2 = A_{111} A_{222} + A_{112} A_{221} + A_{121} A_{212} + A_{122} A_{211} + A_{121} A_{212} + A_{122} A_{211} + A_{121} A_{212} + A_{122} A_{211}.$$

Кроме того, имеются три пары союзных кодетерминантов рода 2, из которых отнимем пару смешанных детерминантов

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Детерминанты двух матриц, обладающие одной и той же сигнатурой, называются *коиндетерминантами*.

Обозначение детерминантов, порожденных многомерной матрицей, можем сделать более подробным, выписывая полностью элементы матрицы с помощью ее двумерных сечений и заменяя выписанные элементы...

$$\left| \begin{array}{c} A_{11} A_{12} \\ A_{21} A_{22} \end{array} \right|_2 = \left| \begin{array}{cc} A_{11} & A_{12} \\ A_{21} & A_{22} \end{array} \right|_2 \xrightarrow{+0} = A_{111} A_{222} - A_{112} A_{221} + A_{121} A_{212} - A_{122} A_{211}.$$

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Детерминант и перманент обычной матрицы n -го порядка $\|A_{ij}\|$ ($i, j = 1, 2, \dots, n$) также могут быть обозначены символами $\left| \begin{array}{c} A_{11} A_{12} \\ A_{21} A_{22} \end{array} \right|_n$ и $\left| \begin{array}{c} A_{11} A_{12} \\ A_{21} A_{22} \end{array} \right|_n$ как двумерные детерминанты с сигнатурами $(\bar{1} \bar{1})$ и $(\bar{1} \bar{1})$. Таким образом, имеем

$$\left| \begin{array}{c} A_{11} A_{12} \\ A_{21} A_{22} \end{array} \right|_2 = \left| \begin{array}{cc} A_{11} & A_{12} \\ A_{21} & A_{22} \end{array} \right|_2 = A_{11} A_{22} - A_{12} A_{21}.$$

$$\left| \begin{array}{c} A_{11} A_{12} \\ A_{21} A_{22} \end{array} \right|_2 = \left| \begin{array}{cc} A_{11} & A_{12} \\ A_{21} & A_{22} \end{array} \right|_2 = A_{11} A_{22} + A_{12} A_{21}.$$

Замечание 1. В дальнейшем будем говорить об элементах, сечений и строках детерминантов многомерной матрицы...

WHY STUDY HYPERDETERMINANTS ?

A NATURAL GENERALIZATION OF THE DETERMINANT

These properties are similar to those of the case of the determinant

- 1 Invariance properties
- 2 $\text{Det}(M \circ_i N) = \text{Det}(M)\text{Det}(N)$
- 3 $\text{Det}(M + N) = \dots$ (minor summation formula)
- 4 Laplace expansion formula
- 5 ...

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These properties can not be *easily* generalized :

- 1 The rank notion
- 2 Geometric interpretation of the variety $\text{Det} = 0$
- 3 How to efficiently compute Det ?
- 4 Eigenvalues, eigenfunctions ...

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MANY APPLICATIONS AND CONNEXIONS WITH OTHER DISCIPLINES

- 1 Statistic Physic and random matrices (multiple integrals)
- 2 Fractional Quantum Hall effect (expansion of the Laughlin wavefunction)
- 3 Algebra : Det is the smallest invariants of hypermatrices.
- 4 Algebraic combinatorics : rectangular Jack polynomials.
- 5 Orthogonal multivariate polynomials.
- 6 Combinatorics. For instance : the Alon-Tarsi conjecture (sum of the signs of latin squares).
- 7 ...

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- 3 Computer science

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Tensor (Algebraic geometry), Multiindexed arrays (computer science), hypermatrices (Algebraic combinatorics)

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We study the special case :

$$\mathbf{M} := (M_{i_1, \dots, i_k})_{1 \leq i_1, \dots, i_k \leq n}$$

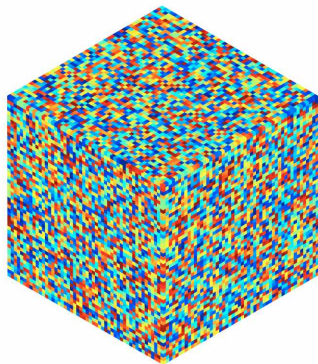
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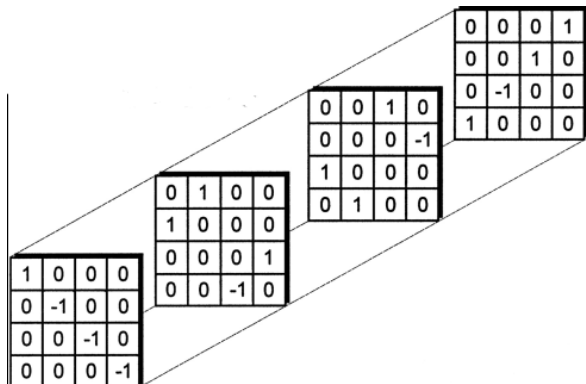
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$$\mathbf{M}^n = n! \text{Det}(\mathbf{M}) \eta_1 \dots \eta_n \otimes \dots \otimes \eta_1 \dots \eta_n.$$

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$$\begin{aligned} \mathbf{M} &= M_{11}\eta_1 \otimes \eta_1 + M_{12}\eta_1 \otimes \eta_2 + M_{21}\eta_2 \otimes \eta_1 + M_{22}\eta_2 \otimes \eta_2 \\ \mathbf{M}^2 &= (M_{11}\eta_1 \otimes \eta_1)^2 + (M_{12}\eta_1 \otimes \eta_2)^2 + (M_{21}\eta_2 \otimes \eta_1)^2 + (M_{22}\eta_2 \otimes \eta_2)^2 \\ &\quad + (M_{11}M_{12}\eta_1^2 \otimes \eta_1\eta_2) + \dots \\ &\quad + M_{11}M_{22}\eta_1\eta_2 \otimes \eta_1\eta_2 + M_{12}M_{21}\eta_1\eta_2 \otimes \eta_2\eta_1 \\ &\quad + M_{22}M_{11}\eta_2\eta_1 \otimes \eta_2\eta_1 + M_{21}M_{12}\eta_2\eta_1 \otimes \eta_1\eta_2 \end{aligned}$$

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$$\mathbf{M}^2 = 0 + 0 + 0 + 0$$

$$+ 0 + \dots$$

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WHAT IS AN HYPERDETERMINANT ?

GRASSMANIAN VARIABLES

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BASIC PROPERTIES

SUMMATION FORMULA

Let $\mathbf{M} = \sum_{1 \leq i_1, \dots, i_k \leq n} M_{i_1, \dots, i_k} \eta_{i_1} \otimes \cdots \otimes \eta_{i_k}$ and
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\mathbf{M} and \mathbf{N} are anticommutative variables $\mathbf{MN} = -\mathbf{NM}$. Hence,
 $\mathbf{M}^2 = \mathbf{N}^2 = 0$ and $\text{Det}(\mathbf{M}) = \text{Det}(\mathbf{N}) = \text{Det}(\mathbf{M} + \mathbf{N}) = 0$.

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Note that :

$$\mathbf{M}^i = i! \sum_{J_1, \dots, J_{2k}} \text{Det} \left(\mathbf{M} \begin{bmatrix} J_1 \\ \vdots \\ J_{2k} \end{bmatrix} \right) \eta_{J_1} \otimes \dots \otimes \eta_{J_{2k}}$$

where $\mathbf{M} \begin{bmatrix} J_1 \\ \vdots \\ J_{2k} \end{bmatrix}$ denotes a hyperminor and the sum is over the

$2k$ -tuple of sets $J_1, \dots, J_{2k} \subset \{1, \dots, n\}$ of size i .

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$$\text{Det}(\mathbf{M} + \mathbf{N}) = \sum_{\substack{i_1, \dots, i_{2k}, \\ j_1, \dots, j_{2k}}} \pm \text{Det} \left(\mathbf{M} \begin{bmatrix} i_1 \\ \vdots \\ i_{2k} \end{bmatrix} \right) \text{Det} \left(\mathbf{N} \begin{bmatrix} j_1 \\ \vdots \\ j_{2k} \end{bmatrix} \right)$$

BASIC PROPERTIES

PRODUCT (COMPOSITION ?) OF TENSOR

Composition :

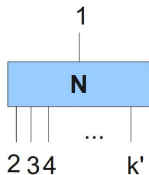
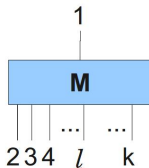
$$(\mathbf{M} \circ_{\ell} \mathbf{N})_{i_1, \dots, i_{k+k'-2}} = \sum_j M_{i_1, \dots, i_{\ell-1}, j, i_{\ell}, \dots, i_k} N_{i_{\ell}, i_{k+1}, \dots, i_{k+k'-2}}$$

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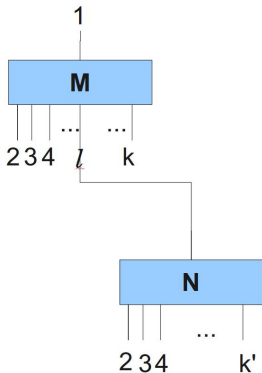


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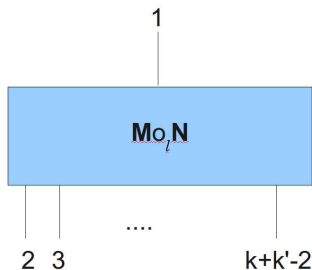


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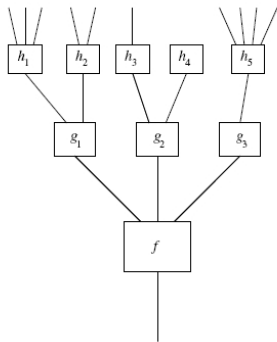


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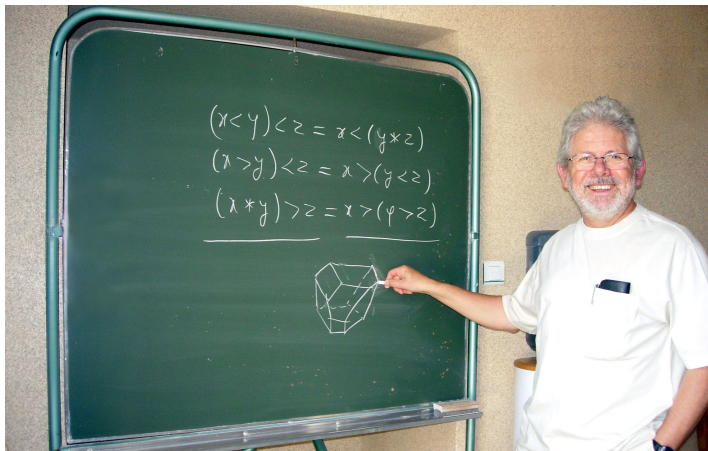
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$$\text{Det}(\mathbf{M} \circ_{\ell} \mathbf{N}) = \text{Det}(\mathbf{M})\text{Det}(\mathbf{N})$$

$$N \in SL_n \Rightarrow \text{Det}(\mathbf{M} \circ_{\ell} \mathbf{N}) = \text{Det}(\mathbf{M})$$

Det, polynomial invariant for the action of SL_n^{2k} .

SYMMETRIC FUNCTIONS

DEFINITION

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Stable by linear combination + product $\Rightarrow \text{Sym}(\mathbb{X})$ is an algebra.

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- ③ Jack $J_\lambda^{(\alpha)}$, Hall-Littlewood $Q_\lambda(\mathbb{X}; q)$, Macdonald polynomials $M_\lambda(\mathbb{X}; q, t)$.

SYMMETRIC FUNCTIONS

VIRTUAL ALPHABET

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Interest : relate some sequences of numbers.

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VIRTUAL ALPHABET

$$\begin{aligned}\text{Sym}(\mathbb{X}) &= \mathbb{C}[p_1(\mathbb{X}), p_2(\mathbb{X}), \dots,] \\ &= \mathbb{C}[h_1(\mathbb{X}), h_2(\mathbb{X}), \dots,] \\ &= \mathbb{C}[e_1(\mathbb{X}), e_2(\mathbb{X}), \dots,]\end{aligned}$$

Consequences : $A = (a_i)_{i \in \mathbb{N}}$ be any number sequences. $h_i(\mathbb{X}) \rightarrow a_i$, morphism of algebra.

Specialization or virtual alphabet $h_i(\mathbb{A}) = a_i$.

Interest : relate some sequences of numbers.

Example :

$h_i(\mathbb{F}) = F_i$ (Fibonacci numbers : $F_1 = 1$, $F_2 = 2$, $F_{n+2} = F_n + F_{n+1}$)

$e_1(\mathbb{F}) = -e_2(\mathbb{F}) = 1$,

$p_i = L_i$ (Lucas number $L_1 = 1$, $L_2 = 3$, $L_{n+2} = L_n + L_{n+1}$)

GENERALIZED HEINE IDENTITY

Let $h_n(\mu) = \int z^n d\mu(z)$ be a moment.

$$V(z_1, \dots, z_n) = \prod_{1 \leq i < j \leq n} (z_i - z_j)$$

$$\begin{aligned} I(\mu; n, k) &:= \frac{1}{n!} \int \dots \int V(z_1, \dots, z_n)^{2k} d\mu(z_1) \dots d\mu(z_n) \\ &= \text{Det}(h_{i_1 + \dots + i_{2k}}(\mu))_{0 \leq i_1, \dots, i_{2k} \leq n-1}. \end{aligned}$$

SELBERG INTEGRAL



$$d\mu_{a,b}^\beta(c) = \mathbf{1}_{[0,1]} x^{a-1} (1-x)^{b-1} dx$$

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Atle Selberg, *Bemerkinger om et multiplet integral*, Norsk Matematisk Tidsskrift **26** (1944), 71-78.

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$$\text{Det} \left(h_{i_1 + \dots + i_{2k}}(\mu_{a,b}^\beta) \right) = \prod_{i=0}^{n-1} \frac{\Gamma(a + ik)\Gamma(b + ik)((i + 1)k)!}{\Gamma(a + b + (n + j - 1)k)k!}$$

$$h_n(\mu_{a,b}^\beta) = \int_0^1 z^{n+a-1} (1-z)^{b-1} dz = \beta(n+a, b) = \frac{\Gamma(n+a)\Gamma(b)}{\Gamma(a+b+n)}.$$

$$\begin{aligned}
 I(\mu_{a,b}^\beta; \lambda, k) &:= \frac{1}{n!} \int \cdots \int J_\lambda^{(\frac{1}{k})}(z_1, \dots, z_n) V(z_1, \dots, z_n)^{2k} \\
 &\quad d\mu_{a-k(n-1), b-k(n-1)}(z_1) \cdots d\mu_{a-k(n-1), b-k(n-1)}(z_n) \\
 &= J_\lambda^{(\frac{1}{k})}(1, \dots, 1) I(\mu_{a,b}^\beta; n, k).
 \end{aligned}$$

$J_\lambda^{(\alpha)}$ is a Jack polynomials.

K.W. J. Kadell , The Selberg-Jack polynomials Adv. in Math.

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More generally : for any measure μ ,

$$I(\mu; \lambda, k) = (*)_\lambda J_{\lambda+n^{k(n-1)}}^{(k)}(\tilde{\mu})$$

$(*)_\lambda$ is an explicit factor and $h_n(\tilde{\mu}) = e_n(\mu)$.

If $Z = \{z_1, \dots, z_n\}$ then $J_{m^n}^{(k)}(Z) = (*) (z_1 \dots z_n)^m$.

$$I(\mu; \lambda, k) = (*) \text{Det}(h_{i_1+\dots+i_k+m}(\mu)) = (*) J_{n^{m+k(n-1)}}^{(k)}(\tilde{\mu})$$

More generally, true for any specialization :

$$\text{Det}(h_{i_1+\dots+i_k+m}(\mathbb{A})) = (*) J_{n^{m+k(n-1)}}^{(k)}(-\mathbb{A}),$$

$$h_n(-\mathbb{A}) = e_n(\mathbb{A}).$$

(Matsumoto 2006, Belbachir, Boussicault, Luque 2008)

$$\text{Det}(e_{i_1+\dots+i_k+m}(\mathbb{A})) = (*) J_{n^{m+k(n-1)}}^{(k)}(\mathbb{A}),$$

EVEN POWER OF THE VANDERMONDE

Context : Fractional quantum Hall effect (Laughlin wave function)



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Expansion of $V(x_1, \dots, x_n)^{2k}$ in terms of Schur functions.

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$$\delta_{a,b} = 0 \text{ if } a \neq b, \delta_{a,a} = 1.$$

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Sparse hypermatrices, computed using Laplace expansion

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GENERALIZED LAPLACE EXPANSION

$$\text{Det}(M) = \sum_{l_2, \dots, l_{2k}} \pm \text{Det} \left(M \begin{bmatrix} l_1 \\ \vdots \\ l_{2k} \end{bmatrix} \right) \text{Det} \left(M \begin{bmatrix} \{1, \dots, n\} \setminus l_1 \\ \vdots \\ \{1, \dots, n\} \setminus l_{2k} \end{bmatrix} \right).$$

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W. Zajaczkowski, *Teoryja Wyznacznikow o p wymiarach a rzedu n^{go}* , *Pamietnik Akademii Umiejetnosci (w. Krakowie)*, Tom 6 (1881) 1-33.

Independently by Gegenbauer, *Über Determinanten höheren Ranges*, *Denkschriften der Kais. Akademie der Wissenschaften in Wien* 43, 17-32, 1882.

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Applications :

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Find recurrences and formulas for some λ .

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For instance : If $\lambda_1 + \dots + \lambda_m = km(m-1)$ for some $0 < m < n$, we have

$$g_\lambda^{n,k} = g_\mu^{n-m,k} g_\nu^{m,k}$$

with $\mu = [\lambda_1 - 2k(m-1), \dots, \lambda_m - 2k(m-1)]$ and $\nu = [\lambda_{m+1}, \dots, \lambda_n]$.

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Boussicault, Luque, Tollu (2009)

$$\mathbb{X} = \{x_1, \dots, x_n\} :$$

$$\begin{aligned} V(x_1, \dots, x_n)^{2k} &= \text{Det} (h_{i_1 + \dots + i_{2k}}(\mathbb{X}))_{0 \leq i_1, \dots, i_{2k} \leq n-1} \\ &= \pm e_n^{n-k}(\mathbb{X}) \det(p_{n+i-j}(\mathbb{X}))^k. \end{aligned}$$

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Example of Specialization : $\mathbb{X} = \left\{ \frac{1+\sqrt{5}}{2}, \frac{1-\sqrt{5}}{2} \right\}$.

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$$\text{Det} (F_{i_1+\dots+i_{2k}})_{0 \leq i_1, \dots, i_{2k} \leq 1} = \begin{vmatrix} L_2 & L_1 \\ L_3 & L_2 \end{vmatrix}^k.$$

Consider the alphabets : $X^{(i)} = \{x_1^{(i)}, \dots, x_n^{(i)}\}$ and $X = \bigcup_{i=1}^{2k-1} X^i$.

$$\frac{\text{Det} \left(h_{\lambda_{i_{2k}} + i_{2k} - 1}(x_{i_1}^{(1)}, \dots, x_{i_{2k-1}}^{(2k-1)}) \right)_{1 \leq i_1, \dots, i_{2k} \leq n}}{V(X^{(1)}) \dots V(X^{(2k-1)})}$$
 is symmetric in X

JACOBI-TRUDY TYPE EQUALITY

Consider the alphabets : $X^{(i)} = \{x_1^{(i)}, \dots, x_n^{(i)}\}$ and $X = \bigcup_{i=1}^{2k-1} X^i$.

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For $k = 1$, we recover the classical Jacobi-Trudy equality :

$$S_{\lambda}(x_1, \dots, x_n) = \det(h_{\lambda_i+i+j-n-1}(X)) = \frac{\det(x_j^{\lambda_j+j-1})}{V(X)}.$$

Thank you !