



HAL
open science

Rigidity, internality and analysability

Daniel Palacin, Frank Olaf Wagner

► **To cite this version:**

| Daniel Palacin, Frank Olaf Wagner. Rigidity, internality and analysability. 2012. hal-00758656v1

HAL Id: hal-00758656

<https://hal.science/hal-00758656v1>

Preprint submitted on 29 Nov 2012 (v1), last revised 4 Sep 2013 (v2)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

RIGIDITY, INTERNALITY AND ANALYSABILITY

DANIEL PALACÍN AND FRANK O. WAGNER

ABSTRACT. We prove a version of Hrushovski's *socle lemma* for rigid groups in an arbitrary simple theory.

1. INTRODUCTION

One of the main consequences of the *canonical base property* (in short, CBP) for a simple theory of finite SU -rank is a certain generalization of the so-called *socle lemma* due to Hrushovski ([8, Proposition 4.3] and [9, Proposition 3.6.2]). Namely, in a group of finite SU -rank with the CBP, every type with finite stabilizer is almost internal to the family of types of rank 1. This was noted in [15] and elaborated in [10].

The formulation of the canonical base property in model-theoretic terms was influenced by the results of Campana [5] and Fujiki [7] in compact complex spaces and an analogous results due to Pillay and Ziegler [15] on differential (and difference) algebraic varieties in characteristic 0. In particular, the group-like version in the case of differentially closed fields of characteristic 0 yields an account of the Mordell-Lang Conjecture for function fields in characteristic 0 without using Zariski geometries, see [15]. The CBP, named by Moosa and Pillay [13], states that for any tuple a of finite SU -rank and any b , the type of the canonical base of $\text{tp}(a/b)$ over a is almost internal to the family of types of rank 1. Clearly, this is a property of the finite-rank context; nevertheless, not all theories of finite rank satisfy the CBP [10].

A better understanding of the CBP was provided in [6], where Chatzidakis studied simple theories with the CBP in general. Extending the work of Pillay and Ziegler, she moreover showed that the theory of

2000 *Mathematics Subject Classification.* 03C45.

Key words and phrases. stable; simple; internal; analysable; canonical base property.

Partially supported by ANR-09-BLAN-0047 Modig.

Both authors thank Anand Pillay for a discussion around the definition of the Canonical Base Property in the general setting.

existentially closed difference fields of any characteristic has the CBP. On the other hand, Chatzidakis proved that replacing internality by analysability in the statement of the CBP, and considering the family of non one-based types of rank 1, one obtains a weak version of the CBP which is satisfied by all types of finite rank in any simple theory.

Recall that a partial type π over a set A in a simple theory is *one-based* if for any tuple \bar{a} of realizations of π and any $B \supseteq A$ the canonical base of $\text{tp}(\bar{a}/B)$ over $A\bar{a}$ is bounded. One-basedness implies that the forking geometry is particularly well-behaved; for instance one-based groups are *rigid*. That is, every subgroup is commensurable with one hyperdefinable over $\text{bdd}(\emptyset)$ — see Definition 3.6. It turns out that the CBP and the weak version of the CBP correspond to a relative version of one-basedness with respect to the family of types of rank 1. This connection was noticed by Kowalski and Pillay in [12], and used to describe the structure of type-definable groups in stable theories satisfying the CBP. The relation between one-basedness and the weak CBP was then used by the authors [14] to generalize and study the weak CBP outside the finite SU -rank setting by replacing the family of types of rank 1 by an arbitrary family of partial types. These ideas also appear in [4] where Blossier, Martín-Pizarro and the second author study a generalization of the CBP in a different direction.

In [10] Hrushovski, Pillay and the first author proved the CBP for non-multidimensional theories of finite Morley rank, assuming that all Galois groups are rigid. The aim of this paper is to obtain directly the socle lemma for rigid groups in arbitrary simple theories, without passing through the CBP (and hence without any assumption of finiteness of SU -rank). In addition, we remark that the non-multidimensionality assumption of [10] is not required.

Our notation is standard and follows [16]. Throughout the paper, the ambient theory will be simple, and we shall be working in \mathfrak{M}^{heq} , where \mathfrak{M} is a sufficiently saturated model of the ambient theory. Thus tuples are tuples of hyperimaginaries, and $\text{dcl} = \text{dcl}^{heq}$.

2. THE CANONICAL BASE PROPERTY

For the rest of the paper Σ will be an \emptyset -invariant family of partial types. Recall first the definitions of internality, analysability and orthogonality.

Definition 2.1. Let π be a partial type over A . Then π is

- (almost) Σ -internal if for every realization a of π there is $B \downarrow_A a$ and a tuple \bar{b} of realizations of types in Σ based on B , such that $a \in \text{dcl}(B\bar{b})$ (or $a \in \text{bdd}(B\bar{b})$, respectively).
- Σ -analysable if for any realization a of π there are $(a_i : i < \alpha) \in \text{dcl}(Aa)$ such that $\text{tp}(a_i/A, a_j : j < i)$ is Σ -internal for all $i < \alpha$, and $a \in \text{bdd}(A, a_i : i < \alpha)$.

Finally, $p \in S(A)$ is *orthogonal* to $q \in S(B)$ if for all $C \supseteq AB$, $a \models p$, and $b \models q$ with $a \downarrow_A C$ and $b \downarrow_B C$ we have $a \downarrow_C b$. The type p is *orthogonal to B* if it is orthogonal to all types over B .

We shall say that a is (almost) Σ -internal or Σ -analysable over b if $\text{tp}(a/b)$ is.

Now we introduce a general version of the canonical base property.

Definition 2.2. A simple theory has the *canonical base property* with respect to Σ , if (possibly after naming some constants) whenever $\text{tp}(\text{Cb}(a/b))$ is Σ -analysable then $\text{tp}(\text{Cb}(a/b)/a)$ is almost Σ -internal.

When Σ is the family of types of SU -rank 1, this corresponds to the usual canonical base property. In [4] a similar property, named 1-tight, is defined, but without the condition on $\text{tp}(\text{Cb}(a/b))$ to be Σ -analysable. Instead, 1-tightness is defined for a family of partial types. Our canonical base property with respect to Σ corresponds to 1-tightness with respect to Σ of the family of all Σ -analysable types.

Recall that if Σ consists of partial types over \emptyset in a stable theory, if $p(x) \in S(\emptyset)$ is Σ -internal, there is an \emptyset -type-definable group G and a faithfully \emptyset -definable action of G on the set of realizations of p which is isomorphic (as a group action) to the group of permutations induced by the automorphisms of \mathfrak{M} fixing Σ pointwise. We call such a group the *Galois group* of p with respect to Σ .

In a stable theory an A -type-definable group is said to be *rigid* if all its type-definable connected subgroups are type-definable over $\text{acl}(A)$. In [10, Theorem 2.5], Hrushovski, Pillay and the first author obtain a strong version of the CBP for certain families of types when all Galois groups are rigid.

Fact 2.3. *Let Σ be a family of partial types over \emptyset in a stable theory, and assume all Galois groups (with respect to Σ) are rigid. If $\text{tp}(\text{Cb}(a/b))$ is Σ -analysable then $\text{Cb}(a/b) \subseteq \text{acl}(a, \Sigma)$.*

Lemma 2.4. *Let X be an \emptyset -invariant set. Then for any a we have*

$$a \downarrow_{\text{bdd}(a) \cap \text{dcl}(X)} X.$$

Proof: Consider a small subset X_0 of X such that $a \downarrow_{X_0} X$ and put $a_0 = \text{Cb}(X_0/a)$. Then a_0 is definable over a Morley sequence I in $\text{Lstp}(X_0/a)$, and $I \subset X$ by invariance of X . Therefore $a_0 \in \text{dcl}(X)$, whence $a_0 \in \text{bdd}(a) \cap \text{dcl}(X)$. As $a \downarrow_{a_0} X$ by transitivity, the conclusion follows. \square

Proposition 2.5. *Let Σ be a family of partial types over \emptyset in a simple theory. The following are equivalent:*

- (1) *Whenever $\text{tp}(b/a)$ is Σ -analysable, $\text{Cb}(a/b) \in \text{bdd}(a, \Sigma)$.*
- (2) *Whenever $\text{tp}(b/a)$ is Σ -analysable, if $c \in \text{bdd}(a, \text{Cb}(a/b))$ is Σ -internal over a , then either $c \in \text{bdd}(a)$ or $c \not\downarrow_a \Sigma$.*

If the theory is stable, both conditions are equivalent to:

- (3) *Whenever $\text{tp}(b/a)$ is Σ -analysable, if $c \in \text{acl}(a, \text{Cb}(a/b))$ is Σ -internal over a , then the connected component of the Galois group of $\text{stp}(c/a)$ with respect to Σ acts trivially.*

Proof: (1) \Rightarrow (2) is immediate. For the other direction, assume (2) and consider some Σ -analysable b over a . Put

$$a_0 = \text{bdd}(a, \text{Cb}(a/b)) \cap \text{bdd}(a, \Sigma).$$

Then $\text{Cb}(a/b) \downarrow_{a_0} \Sigma$ by Lemma 2.4. Assume $\text{Cb}(a/b) \notin \text{bdd}(a_0)$. As b is Σ -analysable over a_0 , so is $\text{Cb}(a/b)$. Hence there is some Σ -internal $c \in \text{bdd}(a_0, \text{Cb}(a/b)) \setminus \text{bdd}(a_0)$ over a_0 ; note that $c \in \text{bdd}(a_0, \text{Cb}(a_0/b))$ since $a \subseteq a_0$. As $c \notin \text{bdd}(a_0)$, we have $c \not\downarrow_{a_0} \Sigma$ by hypothesis, whence $\text{Cb}(a/b) \not\downarrow_{a_0} \Sigma$, a contradiction.

Finally, if T is stable and $\text{tp}(c/a)$ is Σ -internal, then the Galois group of $\text{stp}(c/a)$ with respect to Σ acts transitively. Thus its connected component acts trivially if and only if $c \in \text{acl}(a, \Sigma)$. Therefore (1) \Rightarrow (3) \Rightarrow (2). \square

Recall that a stable (or simple) theory is *non-multidimensional* if every type is non-orthogonal to \emptyset . Hence there is a set A of parameters such that every type of finite SU -rank is non-orthogonal to the family Σ of types over A of SU -rank 1. Thus Fact 2.3 yields immediately the CBP for non-multidimensional stable theories with rigid Galois groups, in the strong form that $\text{Cb}(a/b) \subseteq \text{acl}(a, A, \Sigma)$ whenever $SU(\text{Cb}(a/b)/A)$ is finite.

Using the following result, which corresponds to [6, Theorem 1.16], we can drop the non-multidimensionality hypothesis.

Fact 2.6. *In a simple theory, if $\text{tp}(\text{Cb}(a/b))$ has finite rank, then $\text{tp}(\text{Cb}(a/b)/\text{bdd}(a) \cap \text{bdd}(b))$ is analysable in the family of non one-based types of SU -rank 1 non-orthogonal to $\text{bdd}(a) \cap \text{bdd}(b)$.*

Now if T is stable and all Galois groups with respect to families of types of SU -rank 1 are rigid, then given a canonical base $\text{Cb}(a/b)$ of finite SU -rank, we can choose a set A of parameters independent of ab over $\text{bdd}(a) \cap \text{bdd}(b)$ such that every type of SU -rank 1 non-orthogonal to $\text{bdd}(a) \cap \text{bdd}(b)$ is non-orthogonal to the family Σ of types of SU -rank 1 over A . So $\text{Cb}(a/b) \subseteq \text{acl}(a, A, \Sigma)$ by Fact 2.3. In particular $\text{tp}(\text{Cb}(a/b)/a)$ is almost Σ -internal, i.e., T has the CBP.

Question. Is the result true for simple theories?

The problem with a generalization to simple theories is that the Galois group given in [2] is only *almost* hyperdefinable.

3. STABILIZERS AND RIGIDITY

From now on, G will be an \emptyset -hyperdefinable group in a simple theory T .

Let H be an A -hyperdefinable subgroup of G and $g \in G$. The *canonical parameter* \bar{g} of the coset gH over A is the class of g modulo the A -hyperdefinable equivalence relation given by $x^{-1}y \in H$. Similarly, we define the canonical parameter for the right coset. Even though the canonical parameter \bar{g} of gH is an A -hyperimaginary, there is a hyperimaginary which is interdefinable with \bar{g} over A , see [16, Remark 3.1.5]. Working over A we may thus identify the canonical parameter of gH with an ordinary hyperimaginary.

However, in general it need not be true that a hyperdefinable subgroup has a canonical parameter, since equality of hyperdefinable subgroups need not be a type-definable equivalence relation on their parameters. For canonical parameters to exist we have to assume *local connectivity*.

Recall that a subgroup H of G is *locally connected* if for any model-theoretic or group-theoretic conjugate H^* of H , either $H = H^*$ or $H \cap H^*$ has unbounded index in H . For every hyperdefinable subgroup H of G there exists a unique minimal hyperdefinable locally connected subgroup commensurable with H , its *locally connected component* H^{lc} ,

see [16, Corollary 4.5.16]. An inspection of the proof yields that such a subgroup is hyperdefinable over the parameters needed for H . Moreover, a locally connected hyperdefinable subgroup, or a coset thereof, has a canonical parameter.

Definition 3.1. For an A -hyperdefinable subgroup H of G and $g \in G$, put

$$S(gH/A) = \{h \in G : \exists x [xH \downarrow_A h \wedge xH \equiv_A^{Lstp} hxH \equiv_A^{Lstp} gH]\},$$

and $\text{stab}(gH/A) = S(gH/A) \cdot S(gH/A)$, the (left) stabilizer of gH in G .

Proposition 3.2. $S(gH/A)$ is hyperdefinable over $\text{bdd}(A)$, and $\text{stab}(gH/A)$ is a hyperdefinable subgroup of G whose generic types are contained in $S(gH/A)$.

Proof: Clearly $S(gH/A)$ is hyperdefinable over $\text{bdd}(A)$; we claim it is closed under inverse. So suppose $h \in S(gH/A)$ as witnessed by x . Then for every left stratified local rank D

$D(hxH/A) \geq D(hxH/A, h) = D(xH/A, h) = D(xH/A) = D(hxH/A)$, whence equality holds and $hxH \downarrow_A h$. In particular hx witnesses that $h^{-1} \in S(gH/A)$.

Now if x and x' witness that h and h' are in $S(gH/A)$ where $h \downarrow_A h'$, by the Independence Theorem we may assume

$$xH = h'x'H \quad \text{and} \quad h'x'H \downarrow_A h, h'.$$

So $x'H \downarrow_{A, h'} h$; as $x'H \downarrow_A h'$ we get $x'H \downarrow_A hh'$. Thus x' witnesses $hh' \in S(gH/A)$.

It now follows from [16, Lemma 4.4.8] that $\text{stab}(gH/A)$ is a hyperdefinable subgroup of G whose generic types are contained in $S(gH/A)$. \square

Remark 3.3. We also have

$$S(gH/A) = \{h \in G : h\pi \cup \pi \text{ does not fork over } A\},$$

where $\pi(x)$ is given by

$$\pi(x) = \text{Lstp}(gH/A) = \exists y [y^{-1}x \in H \wedge y \models \text{Lstp}(g/A)].$$

Lemma 3.4. Let $(H_i : i < \alpha)$ be a continuous descending sequence of A -hyperdefinable subgroups of G and $g \in G$. Then

- $\text{stab}(gH_i/A) \geq \text{stab}(gH_j/A)$ for $i \leq j$.

- $\text{stab}(gH_\lambda/A) = \bigcap_{i < \lambda} \text{stab}(gH_i/A)$ for λ limit.

If

$$X_i = \{n \in G : \exists x [x \downarrow_A n \wedge x \equiv_A^{Lstp} g \wedge n \in H_i^x]\}$$

and $N_i = X_i \cdot X_i$, then N_i is a hyperdefinable subgroup contained in $\text{stab}(gH_i/A)$.

Proof: Suppose $h \downarrow_A gH_j$ with $hgH_j \equiv_A^{Lstp} gH_j$. Since $H_i \geq H_j$ we have $h \downarrow_A gH_i$ and $hgH_i \equiv_A^{Lstp} gH_i$. Hence $\text{stab}(gH_j/A) \leq \text{stab}(gH_i/A)$.

From the first statement we obtain $\text{stab}(gH_\lambda/A) \leq \bigcap_{i < \lambda} \text{stab}(gH_i/A)$. Now suppose $h \in \bigcap_{i < \lambda} \text{stab}(gH_i/A)$ is generic. Then every formula $\phi(x) \in \text{tp}(h/A)$ is generic in $\text{stab}(gH_i/A)$ for $i < \lambda$ sufficiently big. By the first paragraph, there is $h_i \models \phi$ such that $h_i \downarrow_A gH_j$ and $h_i gH_j \equiv_A^{Lstp} gH_j$ for all $j \leq i$. By compactness, there is $h' \models \text{tp}(h/A)$ such that $h' \downarrow_A gH_i$ and $h' gH_i \equiv_A^{Lstp} gH_i$ for all $i < \lambda$. But this implies $h' \downarrow_A gH_\lambda$ and $h' gH_\lambda \equiv_A^{Lstp} gH_\lambda$, whence $h' \in \text{stab}(gH_\lambda/A)$.

Finally, X_i is clearly hyperdefinable over $\text{bdd}(A)$ and closed under inverse; by the Independence Theorem it is closed under products of independent elements. So N_i is a hyperdefinable subgroup of G whose generic types are contained in X_i by [16, Lemma 4.4.8]. But if $n \in N_i$ is generic, as witnessed by x , then $n \in H_i^x$, whence $nxH_i = xH_i$. Thus x also witnesses $n \in S(gH_i/A)$. \square

Recall that for any set A , the A -connected component G_A^0 of G is the smallest A -hyperdefinable subgroup of bounded index; note that it is normal. Whereas in a stable theory the A -connected component of a group does not depend on A and is locally connected, this need not be true in a simple theory.

Remark 3.5. If in the Proposition above H_i is G_\emptyset^0 -invariant, then $X_i = N_i = H_i^{g^{-1}}$.

Proof: Let $n \in H_i^{g^{-1}}$ and take a principal generic g' over A , g, n with $gg' \equiv_A^{Lstp} g$. Then $n \in H_i^{g^{-1}} = H_i^{(gg')^{-1}}$ and $gg' \downarrow_A n$, whence $n \in X_i$.

Conversely, if $n \in X_i$ as witnessed by x , then $x^{-1}G_\emptyset^0 = g^{-1}G_\emptyset^0$, as this coset is $\text{bdd}(\emptyset)$ -hyperdefinable. Hence $n \in H_i^{x^{-1}} = H_i^{g^{-1}}$; the assertion follows. \square

Recall that two subgroups H_1 and H_2 of G are *commensurable* if their intersection has bounded index both in H_1 and in H_2 .

Definition 3.6. An \emptyset -hyperdefinable group is *rigid* if every hyperdefinable subgroup is commensurable with one hyperdefinable over $\text{bdd}(\emptyset)$.

Remark 3.7. An \emptyset -hyperdefinable group is rigid if and only if every locally connected subgroup is hyperdefinable over $\text{bdd}(\emptyset)$.

In the pure theory of an algebraically closed field, semi-abelian varieties are rigid. On the other hand, one-based groups are rigid [11].

Lemma 3.8. *If G is rigid, then every hyperdefinable locally connected subgroup is normalized by G_\emptyset^0 .*

Proof: Let H be a hyperdefinable locally connected subgroup of G . Then every G -conjugate of H is also locally connected, and hence hyperdefinable over $\text{bdd}(\emptyset)$. So there are only boundedly many G -conjugates, and G_\emptyset^0 must normalize them all. \square

Proposition 3.9. *A type-definable superstable rigid group G is nilpotent-by-finite.*

Proof: Clearly, we may assume that G is connected. If $U(G) = \omega^\alpha \cdot n + \beta$ with $\beta < \omega^\alpha$, then by [3] there is a type-definable connected abelian subgroup A with $U(A) \geq \omega^\alpha$. So A is normal by Lemma 3.8, and $U(G/A) < U(G)$. By inductive hypothesis G/A is nilpotent.

If $g \in G$ is generic, let C be the centralizer-connected component of $C_G(g)$. Then C is locally connected and of finite index n in $C_G(g)$; since it is $\text{bdd}(\emptyset)$ -definable, it does not depend on g . Therefore C is centralized by all generic elements, and thus by the whole of G . Put $N = A \cdot Z(G)$. Then N is normal and abelian, and G/N is nilpotent of exponent at most n . So G is nilpotent by a theorem of Baudisch and Wilson [1]. \square

Rigidity implies a monotonicity property for stabilizers.

Lemma 3.10. *Suppose G is rigid, $g \in G$ and $A \subseteq B$. Then $\text{stab}(g/B)_B^0$ is contained in $\text{stab}(g/A)$.*

Proof: By rigidity $\text{stab}(g/B)_B^0$ is commensurate with a group H hyperdefinable over $\text{bdd}(\emptyset)$; by B -connectivity it has bounded index in H . Consider a generic $h \in \text{stab}(g/B)_B^0$. Then h is generic in H and so $h \perp_B B$. Also, since h is generic in $\text{stab}(g/B)$, there is $g' \models \text{Lstp}(g/B)$ with $g' \perp_B h$ and $hg' \models \text{Lstp}(g/B)$. In particular $g', hg' \models \text{Lstp}(g/A)$. Since $h \perp_B B$ we have $h \perp_A B$, whence $h \perp_A g'$ by transitivity and hence $h \in \text{stab}(g/A)$. \square

Recall that $\ell_1^\Sigma(g/A)$ is the set of all elements in $\text{bdd}(gA)$ whose type over A is almost Σ -internal.

Lemma 3.11. *If $a \perp_A b$ and $\text{tp}(c/Ab)$ is almost Σ -internal, then $a \perp_B bc$ where $B = \ell_1^\Sigma(a/A)$.*

Proof: [14, Corollary 2.4] yields that $\text{tp}(\text{Cb}(bc/Aa)/A)$ is almost Σ -internal. Thus $\text{Cb}(bc/Aa) \in B$. \square

In order to obtain equality in Lemma 3.4, we have to assume rigidity, Σ -analysability and work with $\ell_1^\Sigma(g/A)$ -connected components.

Proposition 3.12. *Let G be Σ -analysable and rigid. If H is a locally connected hyperdefinable subgroup, $g \in G$ and $B = \ell_1^\Sigma(g/A)$, then $\text{stab}(gH/B)_B^0 = \text{stab}(g/B)_B^0 (H_B^0)^{g^{-1}}$.*

Proof: Note first that $H^{g^{-1}}$ is again locally connected and therefore hyperdefinable over $\text{bdd}(\emptyset)$. In particular, $(H^{g^{-1}})_B^0 = (H_B^0)^{g^{-1}}$ is hyperdefinable over B and B -connected. Moreover, by Remark 3.5,

$$(H_B^0)^{g^{-1}} \leq \text{stab}(gH_B^0/B) \leq \text{stab}(gH/B).$$

One inclusion now follows from Lemma 3.4.

For the other direction, since G is Σ -analysable, there is a continuous descending sequence $(G_i : i \leq \alpha)$ of \emptyset -hyperdefinable normal subgroups of G with $G_0 = G$ and $G_\alpha = \{1\}$, such that G_i/G_{i+1} is almost Σ -internal for all $i < \alpha$. Put $H_i = H \cap G_i$ and note that H_j is normal in H_i for $j > i$.

Claim. $\text{stab}(gH_i/B)_B^0 = \text{stab}(gH_j/B)_B^0 (H_i^{g^{-1}})_B^0$ for all $j \geq i$.

Proof of Claim: Consider a generic element $h \in \text{stab}(gH_i/B)_B^0$. We may assume that $h \perp_B g$ and $hgH_i \equiv_B gH_i$. By rigidity, the B -connected component of the stabilizer has bounded index in a group hyperdefinable over $\text{bdd}(\emptyset)$, and h must be generic in that group as well. Therefore $h \perp_B$, and $h \perp_A g$ by transitivity.

Choose some $h' \in H_i^{g^{-1}}$ with $hh'g \equiv_B g$. Since $H_i^{g^{-1}}/H_{i+1}^{g^{-1}}$ is $\text{bdd}(\emptyset)$ -hyperdefinable and almost Σ -internal, $\text{tp}(h'H_{i+1}^{g^{-1}})$ is almost Σ -internal. Hence $h, h'H_{i+1}^{g^{-1}} \perp_B g$ by Lemma 3.11. Therefore there is some $h'' \in hh'H_{i+1}^{g^{-1}}$ with $h'' \perp_B g$. But then

$$h''gH_{i+1} = h''H_{i+1}^{g^{-1}}g = hh'H_{i+1}^{g^{-1}}g = hh'gH_{i+1} \equiv_B gH_{i+1}.$$

Thus $h'' \in \text{stab}(gH_{i+1}/B)$; as $h \in h''H_i^{g^{-1}}$ we obtain

$$h \in \text{stab}(gH_{i+1}/B) H_i^{g^{-1}}.$$

Taking B -connected components yields

$$\text{stab}(gH_i/B)_B^0 = \text{stab}(gH_{i+1}/B)_B^0 (H_i^{g^{-1}})_B^0;$$

the claim now follows by transfinite induction from Lemma 3.4. \square

Taking $i = 0$ and $j = \alpha$, we obtain the proposition. \square

4. THE SOCLE LEMMA

As pointed out in the introduction, one of the main consequences of the canonical base property is a group-version, the so-called *socle lemma*. For the sake of completeness we sketch a proof for arbitrary simple theories. We follow [6, Proposition 2.7].

Fact 4.1. *Suppose T has the CBP with respect to Σ and G is Σ -analysable. If $g \in G$ and \bar{g} is the canonical parameter of $\text{stab}(g/A)g$, then $\text{tp}(\bar{g}/A)$ is almost Σ -internal.*

Proof: We may assume $A = \emptyset$. Let $c \in G$ be generic over g . Then one can check as in [6, Proposition 2.7] that $\text{stab}(g)g$ is hyperdefinable over $\text{bdd}(\text{Cb}(c/gc), c)$, as is \bar{g} . Since $\text{tp}(c)$ is Σ -analysable, so is $\text{tp}(\text{Cb}(c/gc))$; the CBP with respect to Σ yields that $\text{tp}(\text{Cb}(c/gc)/c)$ is almost Σ -internal. As $\bar{g} \in \text{bdd}(\text{Cb}(c/gc), c)$, we obtain that $\text{tp}(\bar{g}/c)$ is almost Σ -internal, and so is $\text{tp}(\bar{g})$ since $\bar{g} \in \text{dcl}(g)$ and $g \perp c$ by genericity of c . \square

When $\text{stab}(g/A)$ is bounded the statement above coincides with the usual version of the *socle lemma*, see for example [10, Fact 1.3]. Now we generalize [6, Corollary 2.8].

Remark 4.2. Suppose T has the CBP with respect to Σ and G is Σ -analysable. If $g \in G$ then $\text{stab}(g/\ell_1^\Sigma(g/A)) \leq \text{stab}(g/A)$.

Proof: Let $B = \ell_1^\Sigma(g/A)$ and let $h \in \text{stab}(g/B)$ be generic over B, g with $hg \models \text{Lstp}(g/B)$. Then by Fact 4.1 the canonical parameter of $\text{stab}(g/A)g$ belongs to B . So $\text{stab}(g/A)g$ is hyperdefinable over B , whence $hg \in \text{stab}(g/A)g$. Thus $h \in \text{stab}(g/A)$. \square

An inspection of the proof of Fact 4.1 gives the following variation without assuming the CBP.

Proposition 4.3. *Let $g \in G$ and suppose that there is some $h \in G$ such that $gh \downarrow_A g$ and $\text{tp}(h/A)$ is almost Σ -internal (e.g. if $g \in \text{stab}(h/A)$ is generic). If \bar{g} is the canonical parameter of $\text{stab}(g/A)$, then $\text{tp}(\bar{g}/A)$ is almost Σ -internal.*

Proof: Let g, h, A be as in the statement and set $B = \ell_1^\Sigma(gh/A)$. As $\text{tp}(h/A)$ is almost Σ -internal we have $h \downarrow_B gh$ by Lemma 3.11.

Let h_1, h_2 be two independent realizations of $\text{Lstp}(gh/B)$, and note that $B \subseteq \text{bdd}(h_1, A)$. By the Independence Theorem there is some $h' \downarrow_B h_1, h_2$ with $h', h_i \models \text{Lstp}(h, gh/B)$ for $i = 1, 2$. Put $g_i = h_i h'^{-1}$ for $i = 1, 2$ and note that $g_i \models \text{Lstp}(g/B)$. Since $h' \downarrow_{B, h_1} h_2 h_1^{-1}$ we get $g_1 \downarrow_{B, h_1} h_2 h_1^{-1}$. Moreover $g_1, h_1 \models \text{Lstp}(g, gh/A)$, whence $g_1 \downarrow_A h_1$ and $g_1 \downarrow_A B, h_1$. Thus $g_1 \downarrow_A h_2 h_1^{-1}$ by transitivity. But

$$h_2 h_1^{-1} g_1 = h_2 h_1^{-1} h_1 h'^{-1} = h_2 h'^{-1} = g_2.$$

Thus $h_2 h_1^{-1} \in \text{stab}(g/A)$.

It follows that for any two realizations h_1, h_2 of $\text{Lstp}(gh/B)$ the product $h_2 h_1^{-1} \in \text{stab}(g/A)$. Then $\text{stab}(g/A)gh$ is hyperdefinable over B and $\text{stab}(g/A)g$ is hyperdefinable over B, h . Thus $\text{tp}(\bar{g}/A)$ is almost Σ -internal over A . \square

Finally, we prove a general version of the socle lemma.

Theorem 4.4. *Suppose G is Σ -analysable and rigid. If $g \in G$ and $B = \ell_1^\Sigma(g/A)$, then $\text{tp}(g/B)$ is generic in a B -hyperdefinable coset of its stabilizer.*

Proof: Since G is Σ -analysable, there is a continuous descending sequence $(G_i : i \leq \alpha)$ of normal \emptyset -hyperdefinable G -invariant subgroups of G with $G_0 = G$ and $G_\alpha = \{1\}$ such that G_i/G_{i+1} is almost Σ -internal for all $i < \alpha$. Note that trivially all G_i are locally connected. Hence Lemma 3.12 with $H = G_i$ yields

$$\text{stab}(gG_i/B)_B^0 = \text{stab}(g/B)_B^0(G_i)_B^0$$

for all $i < \alpha$.

We shall show inductively that $\text{tp}(gG_i/B)$ is generic in $\text{stab}(gG_i/B)g$ for all $i \leq \alpha$, and that the coset $\text{stab}(gG_i/B)g$ is hyperdefinable over B . For $i = \alpha$ this yields the result.

The assertion is clear for $i = 0$; for limit ordinals it follows from continuity of the sequence $(G_i : i \leq \alpha)$ and Lemma 3.4. So assume

that $\text{tp}(gG_i/B)$ is generic in $\text{stab}(gG_i/B)g$, and that this coset is hyperdefinable over B . Then also $\text{stab}(gG_i/B)_B^0 g$ is hyperdefinable over B . If $g' \models \text{tp}(g/B)$ independently of g over B , then

$$g'g^{-1} \in \text{stab}(gG_i/B)_B^0 \leq \text{stab}(gG_i/B)^{lc}.$$

By rigidity both $\text{stab}(gG_i/B)^{lc}$ and $\text{stab}(gG_{i+1}/B)^{lc}$ are hyperdefinable and connected over $\text{bdd}(\emptyset)$; as $\text{stab}(gG_{i+1}/B) \leq \text{stab}(gG_i/B)$, we obtain $\text{stab}(gG_{i+1}/B)^{lc} \leq \text{stab}(gG_i/B)^{lc}$. Since $\text{stab}(gG_{i+1}/B)^{lc}$ is normalized by G_\emptyset^0 , it is even normal in $\text{stab}(gG_i/B)^{lc}$. But

$$\text{stab}(gG_i/B)^{lc} / \text{stab}(gG_{i+1}/B)^{lc}$$

is $\text{bdd}(\emptyset)$ -hyperdefinable and almost Σ -internal, whence by Lemma 3.11

$$g'g^{-1} \text{stab}(gG_{i+1}/B)^{lc} \downarrow_B g.$$

But $(\text{stab}(gG_{i+1}/B)^{lc})^g = (\text{stab}(gG_{i+1}/B)^{lc})^{g'}$, so

$$g'g^{-1} \text{stab}(gG_{i+1}/B)^{lc} g (\text{stab}(gG_{i+1}/B)^{lc})^g = g' (\text{stab}(gG_{i+1}/B)^{lc})^{g'}.$$

It follows that

$$g'g^{-1} \text{stab}(gG_{i+1}/B)^{lc} \leq \text{stab}(g (\text{stab}(gG_{i+1}/B)^{lc})^g / B),$$

so

$$g' (\text{stab}(gG_{i+1}/B)^{lc})^g \leq \text{stab}(g (\text{stab}(gG_{i+1}/B)^{lc})^g / B) g,$$

and this coset is hyperdefinable over B . But by Proposition 3.12

$$\begin{aligned} \text{stab}(g (\text{stab}(gG_{i+1}/B)^{lc})^g / B)_B^0 &= \text{stab}(g/B)_B^0 (\text{stab}(gG_{i+1}/B)^{lc})_B^0 \\ &= \text{stab}(g/B)_B^0 \text{stab}(gG_{i+1}/B)_B^0 \\ &= \text{stab}(gG_{i+1}/B)_B^0. \end{aligned}$$

Hence $\text{stab}(gG_{i+1}/B)g$ is also hyperdefinable over B . Since this coset contains gG_{i+1} , the type $\text{tp}(gG_{i+1}/B)$ must be generic in it. \square

Corollary 4.5. *Let G be an \emptyset -hyperdefinable group and $g \in G$. If G is Σ -analysable and rigid, then whenever $\text{stab}(g/A)$ is bounded, $\text{tp}(g/A)$ is almost Σ -internal.*

Proof: By Lemma 3.4 $\text{stab}(g/B)$ is bounded and hence g is bounded over $\ell_1^\Sigma(g/A)$ by Theorem 4.4, i.e. $\text{tp}(g/A)$ is almost Σ -internal. \square

Remark 4.6. It follows from the proof of Theorem 3.4 that we can weaken the rigidity hypothesis, either to $\text{stab}(gG_i/B)/G_i$ being bounded for all $i \geq 2$, or to $\text{stab}(g/B)$ being bounded and $\text{stab}(gG_i/B)$ being commensurable to a group hyperdefinable over A whenever $2 \leq i < \alpha$.

In particular, Theorem 4.4 holds without any rigidity hypothesis when G is Σ -analysable in two steps:

Remark 4.7. Let G be an \emptyset -hyperdefinable group, $g \in G$ and $B = \ell_1^\Sigma(g/A)$. If G is Σ -analysable in two steps, then $\text{tp}(g/B)$ is generic in a B -hyperdefinable coset of its stabilizer.

Proof: We sketch a short proof for convenience. If G is Σ -analysable in two steps, then there is some hyperdefinable normal subgroup N of G such that G/N and N are almost Σ -internal. Thus $gN \in B$, and for any realization $g' \models \text{tp}(g/B)$ we have $g'g^{-1} \in N$. Hence $g \perp_B g'g^{-1}$ by Lemma 3.11, so $g'g^{-1} \in \text{stab}(g/B)$. Therefore $\text{stab}(g/B)g$ is hyperdefinable over B and the type $\text{tp}(g/B)$ must be generic in it. \square

Remark 4.8. In the finite rank context, the original socle lemma follows from Corollary 4.5: By the Indecomposability Theorem there is a normal almost Σ -internal \emptyset -hyperdefinable subgroup N such that any almost Σ -internal partial type π is contained in only finitely many cosets of N . Then any almost Σ -internal Lascar strong type is contained in a single coset of N ; by Corollary 4.5 this holds in particular for any Lascar strong type with bounded stabilizer. Note that we make no assumption of commutativity on the group.

5. FINAL REMARKS

As we have seen, there are two ways to deduce almost internality of types with finite stabilizer from rigidity: Either by assuming that the Galois groups are rigid and passing through the canonical base property (Fact 4.1), or by assuming directly that the ambient group is rigid (Corollary 4.5). A priori neither hypothesis implies the other one. It would thus be interesting to compare the two approaches, and to identify connections between a group G and the Galois groups of types in G .

Another question concerns nilpotent groups: In [14, Remark 6.7] it is shown that if G is Σ -analysable type-definable or supersimple, then there is a nilpotent normal subgroup N such that G/N is almost Σ -internal. Is this related to Proposition 3.9 stating that a superstable rigid group is virtually nilpotent?

REFERENCES

- [1] Andreas Baudisch and John Wilson. *Stable actions of torsion groups and stable soluble groups*, Journal of Algebra, 153:453–457, 1991.

- [2] Itai Ben Yaacov and Frank O. Wagner. *On almost orthogonality in simple theories*, Journal of Symbolic Logic, 69(2):398-408, 2004.
- [3] Chantal Berline and Daniel Lascar. *Superstable groups*, Annals of Pure and Applied Logic, 30:1-43, 1986.
- [4] Thomas Blossier, Amador Martín-Pizarro, and Frank O. Wagner. *On variants of CM-triviality*, Fundamenta Mathematicae, to appear.
- [5] F. Campana, *Algébricité et compacité dans l'espace des cycles d'un espace analytique complexe*, Math. Ann. 251:7-18, 1980.
- [6] Zoé Chatzidakis. *A note on canonical bases and modular types in supersimple theories*. Confluentes Mathematici, to appear.
- [7] A. Fujiki, *On the Douady space of a compact complex space in the category C*, Nagoya Math. J. 85 (1982), 189-211.
- [8] Ehud Hrushovski. *The Mordell-Lang conjecture for function fields*, Journal of the American Mathematical Society 9(3):667-690, 1996.
- [9] Ehud Hrushovski *The Manin-Mumford conjecture and the model theory of difference fields*, Annals of Pure and Applied Logic 112(1):43-115, 2001.
- [10] Ehud Hrushovski, Daniel Palacín, and Anand Pillay. *On the canonical base property*, preprint 2012.
- [11] Ehud Hrushovski and Anand Pillay. *Weakly normal groups*, Logic Colloquium '85, North-Holland, Amsterdam, 1987.
- [12] Piotr Kowalski and Anand Pillay. *Quantifier elimination for algebraic D-groups*, Transactions of the American Mathematical Society 358:167-181, 2006.
- [13] Rahim Moosa and Anand Pillay. *On canonical bases and internality criteria*, Illinois J. Math., 52:901-917, 2008.
- [14] Daniel Palacín and Frank O. Wagner. *Ample thoughts*, The Journal of Symbolic Logic, to appear.
- [15] Anand Pillay and Martin Ziegler. *Jet spaces of varieties over differential and difference fields*, Selecta Mathematica (N.S.) 9:579-599, 2003.
- [16] Frank O. Wagner. *Simple Theories*. Mathematics and Its Applications 503. Kluwer Academic Publishers, Dordrecht, NL, 2000.

UNIVERSITÉ DE LYON; CNRS; UNIVERSITÉ LYON 1; INSTITUT CAMILLE JORDAN UMR5208, 43 BD DU 11 NOVEMBRE 1918, 69622 VILLEURBANNE CEDEX, FRANCE

E-mail address: palacin@math.univ-lyon1.fr

E-mail address: wagner@math.univ-lyon1.fr