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# Remarks on some quasilinear equations with gradient terms and measure data

Marie-Françoise BIDAUT-VERON\*      Marta GARCIA-HUIDOBRO†

Laurent VERON‡

## Abstract

Let  $\Omega \subset \mathbb{R}^N$  be a smooth bounded domain,  $H$  a Caratheodory function defined in  $\Omega \times \mathbb{R} \times \mathbb{R}^N$ , and  $\mu$  a bounded Radon measure in  $\Omega$ . We study the problem

$$-\Delta_p u + H(x, u, \nabla u) = \mu \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega,$$

where  $\Delta_p$  is the  $p$ -Laplacian ( $p > 1$ ), and we emphasize the case  $H(x, u, \nabla u) = \pm |\nabla u|^q$  ( $q > 0$ ). We obtain an existence result under subcritical growth assumptions on  $H$ , we give necessary conditions of existence in terms of capacity properties, and we prove removability results of eventual singularities. In the supercritical case, when  $\mu \geq 0$  and  $H$  is an absorption term, i.e.  $H \geq 0$ , we give two sufficient conditions for existence of a nonnegative solution.

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\*Laboratoire de Mathématiques et Physique Théorique, CNRS UMR 7350, Faculté des Sciences, 37200 Tours France. E-mail: veronmf@univ-tours.fr

†Departamento de Matematicas, Pontifica Universidad Catolica de Chile, Casilla 307, Correo 2, Santiago de Chile. E-mail: mgarcia@mat.puc.cl

‡Laboratoire de Mathématiques et Physique Théorique, CNRS UMR 7350, Faculté des Sciences, 37200 Tours France. E-mail: veronl@univ-tours.fr

# 1 Introduction

Let  $\Omega$  be a smooth bounded domain in  $\mathbb{R}^N$  ( $N \geq 2$ ). In this article we consider problems of the form

$$-\Delta_p u + H(x, u, \nabla u) = \mu \quad \text{in } \Omega, \quad (1.1)$$

where  $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2} \nabla u)$  is the  $p$ -Laplace operator, with  $1 < p \leq N$ ,  $H$  is a Caratheodory function defined in  $\Omega \times \mathbb{R} \times \mathbb{R}^N$ , and  $\mu$  is a possibly signed Radon measure on  $\Omega$ . We study the existence of solutions for the Dirichlet problem in  $\Omega$

$$-\Delta_p u + H(x, u, \nabla u) = \mu \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega, \quad (1.2)$$

and some questions of removability of the singularities. Our main motivation is the case where  $\mu$  is nonnegative,  $H$  involves only  $\nabla u$ , and either  $H$  is nonnegative, hence  $H$  is an absorption term, or  $H$  is nonpositive, hence  $H$  is a source one. The model cases are

$$-\Delta_p u + |\nabla u|^q = \mu \quad \text{in } \Omega, \quad (1.3)$$

where  $q > 0$ , for the absorption case and

$$-\Delta_p u = |\nabla u|^q + \mu \quad \text{in } \Omega. \quad (1.4)$$

for the source case.

The equations without gradient terms,

$$-\Delta_p u + H(x, u) = \mu \quad \text{in } \Omega, \quad (1.5)$$

such as the quasilinear Emden-Fowler equations

$$-\Delta_p u \pm |u|^{Q-1} u = \mu \quad \text{in } \Omega,$$

where  $Q > 0$ , have been the object of a huge literature when  $p = 2$ . In the general case  $p > 1$ , among many works we refer to [5], [6], [7] and the references therein, and to [8] for new recent results in the case of absorption.

We set

$$Q_c = \frac{N(p-1)}{N-p}, \quad q_c = \frac{N(p-1)}{N-1}, \quad (Q_c = \infty \text{ if } p = N), \quad \tilde{q} = p - 1 + \frac{p}{N} \quad (1.6)$$

(hence  $q_c, \tilde{q} < p < N$  or  $q_c = \tilde{q} = p = N$ ), and

$$q_* = \frac{q}{q+1-p}, \quad (1.7)$$

(thus  $q_* = q'$  in case  $p = 2$ ).

In Section 2 we recall the main notions of solutions of the problem  $-\Delta_p u = \mu$ , such as weak solutions, renormalized or locally renormalized solutions, and convergence results. In Section 3 we prove a general existence result for problem (1.2) in the subcritical case, see Theorem 3.1. Then in Section 4 we give necessary conditions for existence and removability results for the local solutions of problem (1.1), extending former results of [19] and [41], see Theorem 4.5. In Section 5 we study the problem (1.2) in the supercritical case, where many questions are still open. We give two partial results of existence in Theorems 5.5 and 5.8. Finally in Section 5 we make some remarks of regularity for the problem

$$-\Delta_p u + H(x, u, \nabla u) = 0 \quad \text{in } \Omega.$$

## 2 Notions of solutions

Let  $\omega$  be any domain of  $\mathbb{R}^N$ . For any  $r > 1$ , the capacity  $cap_{1,r}$  associated to  $W_0^{1,r}(\omega)$  is defined by

$$cap_{1,r}(K, \omega) = \inf \left\{ \|\psi\|_{W_0^{1,r}(\omega)}^r : \psi \in \mathcal{D}(\omega), \chi_K \leq \psi \leq 1 \right\},$$

for any compact set  $K \subset \omega$ , and then the notion is extended to any Borel set in  $\omega$ . In  $\mathbb{R}^N$  we denote by  $G_1$  the Bessel kernel of order 1 (defined by  $\widehat{G_1}(y) = (1 + |y|^2)^{-1/2}$ ), and we consider the Bessel capacity defined for any compact  $K \subset \mathbb{R}^N$  by

$$Cap_{1,r}(K, \mathbb{R}^N) = \inf \left\{ \|f\|_{L^r(\mathbb{R}^N)}^r : f \geq 0, G_1 * f \geq \chi_K \right\}.$$

On  $\mathbb{R}^N$  the two capacities are equivalent, see [2].

We denote by  $\mathcal{M}(\omega)$  the set of Radon measures in  $\omega$ , and  $\mathcal{M}_b(\omega)$  the subset of bounded measures, and define  $\mathcal{M}^+(\omega)$ ,  $\mathcal{M}_b^+(\omega)$  the corresponding cones of nonnegative measures. Any measure  $\mu \in \mathcal{M}(\omega)$  admits a positive and a negative parts, denoted by  $\mu^+$  and  $\mu^-$ . For any Borel set  $E$ ,  $\mu \llcorner E$  is the restriction of  $\mu$  to  $E$ ; we say that  $\mu$  is concentrated on  $E$  if  $\mu = \mu \llcorner E$ .

For any  $r > 1$ , we call  $\mathcal{M}^r(\omega)$  the set of measures  $\mu \in \mathcal{M}(\omega)$  which do not charge the sets of null capacity, that means  $\mu(E) = 0$  for every Borel set  $E \subset \omega$  with  $cap_{1,r}(E, \omega) = 0$ . Any measure concentrated on a set  $E$  with  $cap_{1,r}(E, \omega) = 0$  is called  $r$ -singular. Similarly we define the subsets  $\mathcal{M}_b^r(\omega)$  and  $\mathcal{M}_b^{r+}(\omega)$ .

For fixed  $r > 1$ , any measure  $\mu \in \mathcal{M}(\omega)$  admits a unique decomposition of the form  $\mu = \mu_0 + \mu_s$ , where  $\mu_0 \in \mathcal{M}^r(\omega)$ , and  $\mu_s = \mu_s^+ - \mu_s^-$  is  $r$ -singular. If  $\mu \geq 0$ , then  $\mu_0 \geq 0$  and  $\mu_s \geq 0$ .

**Remark 2.1** *Any measure  $\mu \in \mathcal{M}_b(\omega)$  belongs to  $\mathcal{M}^r(\omega)$  if and only if there exist  $f \in L^1(\omega)$  and  $g \in (L^{r'}(\omega))^N$  such that  $\mu = f + \operatorname{div} g$ , see [11, Theorem 2.1]. However this decomposition is not unique; if  $\mu$  is nonnegative there exists a decomposition such that  $f$  is nonnegative, but one cannot ensure that  $\operatorname{div} g$  is nonnegative.*

For any  $k > 0$  and  $s \in \mathbb{R}$ , we define the truncation  $T_k(s) = \max(-k, \min(k, s))$ . If  $u$  is measurable and finite a.e. in  $\omega$ , and  $T_k(u)$  belongs to  $W_0^{1,p}(\omega)$  for every  $k > 0$ , one can define the gradient  $\nabla u$  a.e. in  $\omega$  by  $\nabla T_k(u) = \nabla u \cdot \chi_{\{|u| \leq k\}}$  for any  $k > 0$ .

For any  $f \in \mathcal{M}^+(\mathbb{R}^N)$ , we denote the Bessel potential of  $f$  by  $J_1(f) = G_1 * f$ .

### 2.1 Renormalized solutions

Let  $\mu \in \mathcal{M}_b(\Omega)$ . Let us recall some known results for the problem

$$-\Delta_p u = \mu \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega, \quad (2.1)$$

Under the assumption  $p > 2 - 1/N$ , from [9], problem (2.1) admits a solution  $u \in W_0^{1,r}(\Omega)$  for every  $r \in [1, q_c)$ , satisfying the equation in  $\mathcal{D}'(\Omega)$ . When  $p < 2 - 1/N$ , then  $q_c < 1$ ; this leads to introduce the concept of renormalized solutions developed in [16], see also [33], [45]. Here we recall one of their definitions, among four equivalent ones given in [16].

**Definition 2.2** Let  $\mu = \mu^0 + \mu_s \in \mathcal{M}_b(\Omega)$ , where  $\mu^0 \in \mathcal{M}^p(\Omega)$  and  $\mu_s = \mu_s^+ - \mu_s^-$  is  $p$ -singular. A function  $u$  is a **renormalized solution**, called **R-solution** of problem (2.1), if  $u$  is measurable and finite a.e. in  $\Omega$ , such that  $T_k(u)$  belongs to  $W_0^{1,p}(\Omega)$  for any  $k > 0$ , and  $|\nabla u|^{p-1} \in L^\tau(\Omega)$ , for any  $\tau \in [1, N/(N-1))$ ; and for any  $h \in W^{1,\infty}(\mathbb{R})$  such that  $h'$  has a compact support, and any  $\varphi \in W^{1,s}(\Omega)$  for some  $s > N$ , such that  $h(u)\varphi \in W_0^{1,p}(\Omega)$ ,

$$\int_{\Omega} |\nabla u|^{p-2} \nabla u \cdot \nabla (h(u)\varphi) dx = \int_{\Omega} h(u)\varphi d\mu_0 + h(\infty) \int_{\Omega} \varphi d\mu_s^+ - h(-\infty) \int_{\Omega} \varphi d\mu_s^-. \quad (2.2)$$

As a consequence, any R-solution  $u$  of problem (2.1) satisfies  $|u|^{p-1} \in L^\sigma(\Omega)$ ,  $\forall \sigma \in [1, N/(N-p))$ . More precisely,  $u$  and  $|\nabla u|$  belong to some Marcinkiewicz spaces

$$L^{s,\infty}(\Omega) = \left\{ u \text{ measurable in } \Omega : \sup_{k>0} k^s |\{x \in \Omega : |u(x)| > k\}| < \infty \right\},$$

see [9], [5], [16], [27], and one gets useful convergence properties, see [16, Theorem 4.1 and §5] for the proof:

**Lemma 2.3** (i) Let  $\mu \in \mathcal{M}_b(\Omega)$  and  $u$  be any R-solution of problem (2.1). Then for any  $k > 0$ ,

$$\frac{1}{k} \int_{\{m \leq u \leq m+k\}} |\nabla u|^p dx \leq |\mu|(\Omega), \forall m \geq 0.$$

If  $p < N$ , then  $u \in L^{Q_c, \infty}(\Omega)$  and  $|\nabla u| \in L^{q_c, \infty}(\Omega)$ ,

$$|\{|u| \geq k\}| \leq C(N, p) k^{-Q_c} (|\mu|(\Omega))^{\frac{N}{N-p}}, \quad |\{|\nabla u| \geq k\}| \leq C(N, p) k^{-q_c} (|\mu|(\Omega))^{\frac{N}{N-1}}. \quad (2.3)$$

If  $p = N$  (where  $u$  is unique), then for any  $r > 1$  and  $s \in (1, N)$ ,

$$|\{|u| \geq k\}| \leq C(N, p, r) k^{-r} (|\mu|(\Omega))^{\frac{r}{p-1}}, \quad |\{|\nabla u| \geq k\}| \leq C(N, p, s) k^{-N} (|\mu|(\Omega))^{\frac{s}{N-1}}. \quad (2.4)$$

(ii) Let  $(\mu_n)$  be a sequence of measures  $\mu_n \in \mathcal{M}_b(\Omega)$ , uniformly bounded in  $\mathcal{M}_b(\Omega)$ , and  $u_n$  be any R-solution of

$$-\Delta_p u_n = \mu_n \quad \text{in } \Omega, \quad u_n = 0 \quad \text{on } \partial\Omega.$$

Then there exists a subsequence  $(\mu_\nu)$  such that  $(u_\nu)$  converges a.e. in  $\Omega$  to a function  $u$ , such that  $T_k(u) \in W_0^{1,p}(\Omega)$ , and  $(T_k(u_\nu))$  converges weakly in  $W_0^{1,p}(\Omega)$  to  $T_k(u)$ , and  $(\nabla u_\nu)$  converges a.e. in  $\Omega$  to  $\nabla u$ .

**Remark 2.4** These properties do not require any regularity of  $\Omega$ . If  $\mathbb{R}^N \setminus \Omega$  is geometrically dense, i.e. there exists  $c > 0$  such that  $|B(x, r) \setminus \Omega| \geq cr^N$  for any  $x \in \mathbb{R}^N \setminus \Omega$  and  $r > 0$ , then (2.4) holds with  $s = N$ , and  $C$  depends also on the geometry of  $\Omega$ . Then  $|\nabla u| \in L^{N, \infty}(\Omega)$ , hence  $u \in BMO(\Omega)$ , see [17], [27].

Next we recall the fundamental stability result of [16, Theorem 3.1]:

**Definition 2.5** For any measure  $\mu = \mu^0 + \mu_s^+ - \mu_s^- \in \mathcal{M}_b(\Omega)$ , where  $\mu^0 = f - \operatorname{div}g \in \mathcal{M}^p(\Omega)$ , and  $\mu_s^+, \mu_s^-$  are  $p$ -singular we say that a sequence  $(\mu_n)$  is a **good approximation** of  $\mu$  in  $\mathcal{M}_b(\Omega)$  if it can be decomposed as

$$\mu_n = \mu_n^0 + \lambda_n - \eta_n, \quad \text{with} \quad \mu_n^0 = f_n - \operatorname{div}g_n, \quad f_n \in L^1(\Omega), \quad g_n \in (L^{p'}(\Omega))^N, \quad \lambda_n, \eta_n \in \mathcal{M}_b^+(\Omega), \quad (2.5)$$

such that  $(f_n)$  converges to  $f$  weakly in  $L^1(\Omega)$ ,  $(g_n)$  converges to  $g$  strongly in  $(L^{p'}(\Omega))^N$  and  $(\operatorname{div}g_n)$  is bounded in  $\mathcal{M}_b(\Omega)$ , and  $(\rho_n)$  converges to  $\mu_s^+$  and  $(\eta_n)$  converges to  $\mu_s^-$  in the narrow topology.

**Theorem 2.6** ([16]) Let  $\mu \in \mathcal{M}_b(\Omega)$ , and let  $(\mu_n)$  be a good approximation of  $\mu$ . Let  $u_n$  be a  $R$ -solution of

$$-\Delta_p u_n = \mu_n \quad \text{in } \Omega, \quad u_n = 0 \quad \text{on } \partial\Omega.$$

Then there exists a subsequence  $(u_\nu)$  converging a.e. in  $\Omega$  to a  $R$ -solution  $u$  of problem (2.1). And  $(T_k(u_\nu))$  converges to  $T_k(u)$  strongly in  $W_0^{1,p}(\Omega)$ .

**Remark 2.7** As a consequence, for any measure  $\mu \in \mathcal{M}_b(\Omega)$ , there exists at least a solution of problem (2.1). Indeed, it is pointed out in [16] that any measure  $\mu \in \mathcal{M}_b(\Omega)$  can be approximated by such a sequence: extending  $\mu$  by 0 to  $\mathbb{R}^N$ , one can take  $g_n = g$ ,  $f_n = \rho_n * f$ ,  $\lambda_n = \rho_n * \mu_s^+$ ,  $\eta_n = \rho_n * \mu_s^-$ , where  $(\rho_n)$  is a regularizing sequence; then  $f_n, \lambda_n, \eta_n \in C_b^\infty(\Omega)$ . Notice that this approximation does not respect the sign:  $\mu \in \mathcal{M}_b^+(\Omega)$  does not imply that  $\mu_n \in \mathcal{M}_b^+(\Omega)$ .

In the sequel we precise the approximation property, still partially used in [20, Theorem 2.18] for problem (1.5).

**Lemma 2.8** Let  $\mu \in \mathcal{M}_b(\Omega)$ . Then

(i) there exists a sequence  $(\mu_n)$  of good approximations of  $\mu$ , such  $\mu_n \in W^{-1,p'}(\Omega)$ , and  $\mu_n^0$  has a compact support in  $\Omega$ ,  $\lambda_n, \eta_n \in C_b^\infty(\Omega)$ ,  $(f_n)$  converges to  $f$  strongly in  $L^1(\Omega)$ , and

$$|\mu_n|(\Omega) \leq 4|\mu|(\Omega), \quad \forall n \in \mathbb{N} \quad (2.6)$$

Moreover, if  $\mu \in \mathcal{M}_b^+(\Omega)$ , then one can find the approximation such that  $\mu_n \in \mathcal{M}_b^+(\Omega)$  and  $(\mu_n)$  is nondecreasing.

(ii) there exists another sequence  $(\mu_n)$  of good approximations of  $\mu$ , with  $f_n, g_n \in \mathcal{D}(\Omega)$ ,  $\lambda_n, \eta_n \in C_b^\infty(\Omega)$ , such that  $(f_n)$  converges to  $f$  strongly in  $L^1(\Omega)$ , satisfying (2.6); if  $\mu \in \mathcal{M}_b^+(\Omega)$ , one can take  $\mu_n^0 \in \mathcal{D}^+(\Omega)$ .

**Proof.** (i) Let  $\mu = \mu^0 + \mu_s^+ - \mu_s^-$ , where  $\mu^0 \in \mathcal{M}^p(\Omega)$ ,  $\mu_s^+, \mu_s^-$  are  $p$ -singular and  $\mu_1 = (\mu^0)^+, \mu_2 = (\mu^0)^-$ ; thus  $\mu_1(\Omega) + \mu_2(\Omega) + \mu_s^+(\Omega) + \mu_s^-(\Omega) \leq 2|\mu(\Omega)|$ . Following [11], for  $i = 1, 2$ , one has

$$\mu_i = \varphi_i \gamma_i, \quad \text{with } \gamma_i \in \mathcal{M}_b^+(\Omega) \cap W^{-1,p'}(\Omega) \text{ and } \varphi_i \in L^1(\Omega, \gamma_i).$$

Let  $(K_n)_{n \geq 1}$  be an increasing sequence of compacts of union  $\Omega$ ; set

$$\nu_{1,i} = T_1(\varphi_i \chi_{K_1}) \gamma_i, \quad \nu_{n,i} = T_n(\varphi_i \chi_{K_n}) \gamma_i - T_{n-1}(\varphi_i \chi_{K_{n-1}}) \gamma_i, \quad \mu_{n,i}^0 = \sum_1^n \nu_{n,i} = T_n(\varphi_i \chi_{K_n}) \gamma_i.$$

Thus  $\mu_{n,i}^0 \in \mathcal{M}_b^+(\Omega) \cap W^{-1,p'}(\Omega)$ . Regularizing by  $(\rho_n)$ , there exists  $\phi_{n,i} \in \mathcal{D}^+(\Omega)$  such that  $\|\phi_{n,i} - \nu_{n,i}\|_{W^{-1,p'}(\Omega)} \leq 2^{-n} \mu_i(\Omega)$ . Then  $\xi_{n,i} = \sum_1^n \phi_{k,i} \in \mathcal{D}^+(\Omega)$ ;  $(\eta_{n,i})$  converges strongly in  $L^1(\Omega)$  to a function  $\xi_i$  and  $\|\xi_{n,i}\|_{L^1(\Omega)} \leq \mu_i(\Omega)$ . Also setting

$$G_{n,i} = \mu_{n,i}^0 - \xi_{n,i} = \sum_1^n (\nu_{n,i} - \phi_{k,i}) \in W^{-1,p'}(\Omega) \cap \mathcal{M}_b(\Omega),$$

then  $(G_{n,i})$  converges strongly in  $W^{-1,p'}(\Omega)$  to some  $G_i$ , and  $\mu_i = \xi_i + G_i$ , and  $\|G_{n,i}\|_{\mathcal{M}_b(\Omega)} \leq 2\mu_i(\Omega)$ . Otherwise  $\lambda_n = \rho_n * \mu_s^+$  and  $\eta_n = \rho_n * \mu_s^- \in C_b^\infty(\Omega)$  converge respectively to  $\mu_s^+, \mu_s^-$  in the narrow topology, with  $\|\lambda_n\|_{L^1(\Omega)} \leq \mu_s^+(\Omega)$ ,  $\|\eta_n\|_{L^1(\Omega)} \leq \mu_s^-(\Omega)$ . Then we set

$$\mu_n = \mu_n^0 + \rho_n - \eta_n \quad \text{with } \mu_n^0 = \xi_n + G_n, \quad \xi_n = \xi_{n,1} - \xi_{n,2} \in \mathcal{D}(\Omega), \quad G_n = G_{n,1} - G_{n,2} \in W^{-1,p'}(\Omega)$$

thus  $\mu_n^0$  has a compact support. Moreover  $\mu_0 = \xi + G$  with  $\xi = \xi_1 - \xi_2 \in \mathcal{D}(\Omega)$ , and  $G = G_1 - G_2 = \varphi + \text{div}g$  for some  $\varphi \in L^{p'}(\Omega)$  and  $g \in (L^{p'}(\Omega))^N$ , and  $(G_n)$  converges to  $G$  in  $W^{-1,p'}(\Omega)$ . Then we can find  $\psi_n \in L^{p'}(\Omega)$ ,  $\phi_n \in (L^{p'}(\Omega))^N$ , such that  $G_n - G = \psi_n + \text{div}\phi_n$  and  $\|G_n - G\|_{W^{-1,p'}(\Omega)} = \max(\|\psi_n\|_{L^{p'}(\Omega)}, \|\phi_n\|_{(L^{p'}(\Omega))^N})$ ; then  $\mu_0 = f + \text{div}g$  with  $f = \xi + \varphi$  and  $\mu_n^0 = f_n + \text{div}g_n$ , with  $f_n = \xi_n + \varphi + \psi_n$ ,  $g_n = g + \phi_n$ . Thus  $(\mu_n)$  is a good approximation of  $\mu$ , and satisfies (2.6). If  $\mu$  is nonnegative, then  $\mu_n$  is nonnegative.

(ii) We replace  $\mu_n^0$  by  $\rho_m * \mu_n^0 = \rho_m * f_n + \text{div}(\rho_m * g_n)$ ,  $m \in \mathbb{N}$ , and observe that  $|\rho_m * \mu_n^0|(\Omega) \leq |\mu_n^0|(\Omega)$ ; then we can construct another sequence satisfying the conditions.  $\blacksquare$

## 2.2 Locally renormalized solutions

Let  $\mu \in \mathcal{M}(\Omega)$ . Following the notion introduced in [6], we say that  $u$  is a **locally renormalized solution**, called **LR-solution**, of problem

$$-\Delta_p u = \mu, \quad \text{in } \Omega, \quad (2.7)$$

if  $u$  is measurable and finite a.e. in  $\Omega$ ,  $T_k(u) \in W_{loc}^{1,p}(\Omega)$  for any  $k > 0$ , and

$$|u|^{p-1} \in L_{loc}^\sigma(\Omega), \forall \sigma \in [1, N/(N-p)]; \quad |\nabla u|^{p-1} \in L_{loc}^\tau(\Omega), \forall \tau \in [1, N/(N-1)], \quad (2.8)$$

and for any  $h \in W^{1,\infty}(\mathbb{R})$  such that  $h'$  has a compact support, and  $\varphi \in W^{1,m}(\Omega)$  for some  $m > N$ , with compact support, such that  $h(u)\varphi \in W^{1,p}(\Omega)$ , there holds

$$\int_\Omega |\nabla u|^{p-2} \nabla u \cdot \nabla (h(u)\varphi) dx = \int_\Omega h(u)\varphi d\mu_0 + h(+\infty) \int_\Omega \varphi d\mu_s^+ - h(-\infty) \int_\Omega \varphi d\mu_s^-. \quad (2.9)$$

**Remark 2.9** Hence the LR-solutions are solutions in  $\mathcal{D}'(\Omega)$ . From a recent result of [28], if  $\mu \in \mathcal{M}^+(\Omega)$ , any  $p$ -superharmonic function is a LR-solution, and reciprocally any LR-solution admits a  $p$ -superharmonic representant.

### 3 Existence in the subcritical case

We first give a general existence result, where  $H$  satisfies some subcritical growth assumptions on  $u$  and  $\nabla u$ , without any assumption on the sign of  $H$  or  $\mu$ : we consider the problem

$$-\Delta_p u + H(x, u, \nabla u) = \mu \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega, \quad (3.1)$$

where  $\mu \in \mathcal{M}_b(\Omega)$ . We say that  $u$  is a R-solution of problem (1.2)  $T_k(u) \in W_0^{1,p}(\Omega)$  for any  $k > 0$ , and  $H(x, u, \nabla u) \in L^1(\Omega)$  and  $u$  is a R-solution of

$$-\Delta_p u = \mu - H(x, u, \nabla u), \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega.$$

**Theorem 3.1** *Let  $\mu \in \mathcal{M}_b(\Omega)$ , and assume that*

$$|H(x, u, \xi)| \leq f(x) |u|^Q + g(x) |\xi|^q + \ell(x) \quad (3.2)$$

with  $Q, q > 0$  and  $f \in L^r(\Omega)$  with  $Qr' < Q_c$ ,  $g \in L^s(\Omega)$  with  $qs' < q_c$ , and  $\ell \in L^1(\Omega)$ .

Then there exists a R-solution of (3.1) if, either  $\max(Q, q) > p - 1$  and  $|\mu|(\Omega)$  and  $\|\ell\|_{L^1(\Omega)}$  are small enough, or  $q = p - 1 > Q$  and  $\|f\|_{L^r(\Omega)}$  is small enough, or  $Q = p - 1 > q$  and  $\|g\|_{L^s(\Omega)}$  is small enough, or  $q, Q < p - 1$ .

**Proof. (i) Construction of a sequence of approximations.** We consider a sequence  $(\mu_n)_{n \geq 1}$  of good approximations of  $\mu$ , given in Lemma 2.8 (i). For any fixed  $n \in \mathbb{N}^*$ , and any  $v \in W_0^{1,p}(\Omega)$  we define

$$M(v) = |\Omega|^{\frac{N-p}{N} - \frac{p-1}{Qr'}} \left( \int_{\Omega} |v|^{Qr'} dx \right)^{\frac{p-1}{Qr'}} + |\Omega|^{\frac{N-1}{N} - \frac{p-1}{qs'}} \left( \int_{\Omega} |\nabla v|^{qs'} dx \right)^{\frac{p-1}{qs'}},$$

$$\Phi_n(v)(x) = - \frac{H(x, v(x), \nabla v(x))}{1 + \frac{1}{n} (f(x) |v(x)|^Q + g(x) |\nabla v(x)|^q + \ell(x))}$$

so that  $|\Phi_n(v)(x)| \leq n$  a.e. in  $\Omega$ . Let  $\lambda > 0$  be a parameter. Starting from  $u_1 \in W_0^{1,p}(\Omega)$  such that  $M(u_1) \leq \lambda$ , we define  $u_2 \in W_0^{1,p}(\Omega)$  as the solution of the problem

$$-\Delta_p u_2 = \Phi_1(u_1) + \mu_1 \quad \text{in } \Omega, \quad u_2 = 0 \quad \text{on } \partial\Omega,$$

and by induction we define  $u_n \in W_0^{1,p}(\Omega)$  as the solution of

$$-\Delta_p u_n = \Phi_{n-1}(u_{n-1}) + \mu_n \quad \text{in } \Omega, \quad u_n = 0 \quad \text{on } \partial\Omega.$$

From (2.3), for any  $\sigma \in (0, N/(N-p))$  and  $\tau \in (0, N/(N-1))$ ,

$$|\Omega|^{\frac{N-p}{N} - \frac{1}{\sigma}} \left( \int_{\Omega} |u_n|^{(p-1)\sigma} dx \right)^{\frac{1}{\sigma}} + |\Omega|^{\frac{N-1}{N} - \frac{1}{\tau}} \left( \int_{\Omega} |\nabla u_n|^{(p-1)\tau} dx \right)^{\frac{1}{\tau}} \leq C \left( \int_{\Omega} |\Phi_{n-1}(u_{n-1})| dx + 4 |\mu|(\Omega) \right),$$

with  $C = C(N, p, \sigma, \tau)$ . We take  $\sigma = Qr'/(p-1)$  and  $\tau = qs'/(p-1)$ ; since

$$\int_{\Omega} |H(x, u_{n-1}, \nabla u_{n-1})| dx \leq \|f\|_{L^r(\Omega)} \int_{\Omega} |u_{n-1}|^{Qr'} dx^{\frac{1}{r'}} + \|g\|_{L^s(\Omega)} \left( \int_{\Omega} |\nabla u_{n-1}|^{qs'} dx \right)^{\frac{1}{s'}} + \|\ell\|_{L^1(\Omega)} \quad (3.3)$$



we obtain

$$M(u_n) \leq C \left( \int_{\Omega} |H(x, u_{n-1}, \nabla u_{n-1})| dx + 4 |\mu|(\Omega) \right) \leq b_1 M(u_{n-1})^{Q/(p-1)} + b_2 M(u_{n-1})^{q/(p-1)} + \eta + a$$

with  $C = C(N, p, q, Q)$ , and  $b_1 = C \|f\|_{L^r(\Omega)} |\Omega|^{\frac{1}{r} - \frac{Q}{Q_c}}$ ,  $b_2 = C \|g\|_{L^s(\Omega)} |\Omega|^{1/s' - q/q_c}$ ,  $\eta = C \|\ell\|_{L^1(\Omega)}$ ,  $a = 4C |\mu|(\Omega)$ . Then by induction,  $M(u_n) \leq \lambda$  for any  $n \geq 1$  if

$$b_1 \lambda^{Q/(p-1)} + b_2 \lambda^{q/(p-1)} + \eta + a \leq \lambda. \quad (3.4)$$

When  $Q < p - 1$  and  $q < p - 1$ , (3.4) holds for  $\lambda$  large enough. In the other cases, we note that it holds as soon as

$$b_1 \lambda^{Q/(p-1)-1} + b_2 \lambda^{q/(p-1)-1} \leq 1/2, \quad \text{and } \eta \leq \lambda/4, \quad a \leq \lambda/4. \quad (3.5)$$

First suppose that  $Q > p - 1$  or  $q > p - 1$ . We take  $\lambda \leq 1$ , small enough so that  $(b_1^{Q/(p-1)} + b_2^{q/(p-1)}) \lambda^{\max(Q, q)/(p-1)-1} \leq 1/2$ , and then  $\eta, a \leq \lambda/4$ . Next suppose for example that  $Q = p - 1 > q$ ,  $a$  is arbitrary. If  $b_1$  small enough, and  $\eta, a$  are arbitrary, then we obtain (3.5) for  $\lambda$  large enough.

**(ii) Convergence:** Since  $M(u_n) \leq \lambda$ , in turn from (3.3),  $(H(x, u_n, \nabla u_n))$  is bounded in  $L^1(\Omega)$ , and then also  $\Phi_n(u_n)$ . Thus

$$\int_{\Omega} |\Phi_{n-1}(u_{n-1})| dx + |\mu_n|(\Omega) \leq C_{\lambda} := b_1 \lambda^{Q/(p-1)} + b_2 \lambda M^{q/(p-1)} + \eta + 4 |\mu|(\Omega).$$

From Lemma 2.3, up to a subsequence,  $(u_n)$  converges *a.e.* to a function  $u$ ,  $(\nabla u_n)$  converges *a.e.* to  $\nabla u$ , and  $(u_n^{p-1})$  converges strongly in  $L^{\sigma}(\Omega)$ , for any  $\sigma \in [1, N/(N - p))$ , and finally  $(|\nabla u_n|^{p-1})$  converges strongly in  $L^{\tau}(\Omega)$ , for any  $\tau \in [1, N/(N - 1))$ . Therefore  $(u_n^{Qr'})$  and  $(|\nabla u_n|^{qr'})$  converge strongly in  $L^1(\Omega)$ , in turn  $(\Phi_n(x, u_n, \nabla u_n))$  converges strongly to  $H(x, u, \nabla u)$  in  $L^1(\Omega)$ . Then  $(\Phi_n(x, u_n, \nabla u_n) + \mu_n)$  is a sequence of good approximations of  $H(x, u, \nabla u) + \mu$ . From Theorem 2.6,  $u$  is a R-solution of problem (3.1).  $\blacksquare$

**Remark 3.2** *Our proof is not based on the Schauder fixed point theorem, so we do not need that  $1 \leq Qr'$  or  $1 \leq qs'$ . Hence we improve the former result of [20] for problem (1.5) where  $H$  only depends on  $u$ , proved for  $1 \leq Qr'$ , implying  $1 < Q_c$ . Here we have no restriction on  $Q_c$  and  $q_c$ .*

Next we consider the case where  $H$  and  $\mu$  are nonnegative; then we do not need that the data are small:

**Theorem 3.3** *Consider the problem (3.1)*

$$-\Delta_p u + H(x, u, \nabla u) = \mu \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega, \quad (3.6)$$

where  $\mu \in \mathcal{M}_b^+(\Omega)$ , and

$$0 \leq H(x, u, \xi) \leq C(|u|^Q + |\xi|^q) + \ell(x) \quad (3.7)$$

with  $0 < Q < Q_c, 0 < q < q_c, C > 0, \ell \in L^1(\Omega)$ . Then there exists a nonnegative R-solution of problem (3.6).

**Proof.** We use the good approximation of  $\mu$  by a sequence of measures  $\mu_n = \mu_n^0 + \lambda_n$ , with  $\mu_n^0 \in \mathcal{D}^+(\Omega)$ ,  $\lambda_n \in C_b^+(\Omega)$ , given at Lemma 2.8 (ii). Then there exists a weak nonnegative solution  $u_n \in W_0^{1,p}(\Omega)$  of the problem

$$-\Delta_p u_n + H(x, u_n, \nabla u_n) = \mu_n \quad \text{in } \Omega, \quad u_n = 0 \quad \text{on } \partial\Omega.$$

Indeed 0 is a subsolution, and the solution  $\psi_n \in W_0^{1,p}(\Omega)$  of  $-\Delta_p \psi_n = \mu_n$  in  $\Omega$ , is a supersolution. Since  $\mu_n \in L^\infty(\Omega)$ , there holds  $\psi \in C^{1,\alpha}(\overline{\Omega})$  for some  $\alpha \in (0, 1)$ , thus  $\psi \in W^{1,\infty}(\Omega)$ . From [12, Theorem 2.1], since  $Q_c \leq p$ , there exists a weak solution  $u_n \in W_0^{1,p}(\Omega)$ , such that  $0 \leq u_n \leq \psi_n$ , hence  $u_n \in L^\infty(\Omega)$ , and  $u_n \in W_{loc}^{1,r}(\Omega)$  for some  $r > p$ . Taking  $\varphi = k^{-1}T_k(u_n - m)$  with  $m \geq 0$ ,  $k > 0$ , as a test function, we get from (2.6)

$$\frac{1}{k} \int_{\{m \leq u \leq m+k\}} |\nabla u_n|^p dx \leq \mu_n(\Omega) \leq 4\mu(\Omega), \quad (3.8)$$

then from Lemma 2.3, up to a subsequence,  $(u_n)$  converges *a.e.* to a function  $u$ ,  $(T_k(u_n))$  converges weakly in  $W_0^{1,p}(\Omega)$  and  $(\nabla u_n)$  converges *a.e.* to  $\nabla u$ , and  $(|\nabla u_n|^p)$ ,  $(u_n^{p-1})$  converges strongly in  $L^\sigma(\Omega)$  for any  $\sigma \in [1, N/(N-p))$ ,  $(|\nabla u_n|^{p-1})$  converges strongly in  $L^\tau(\Omega)$ , for any  $\tau \in [1, N/(N-1))$ . Then  $(u_n^{Qr'})$  and  $(|\nabla u_n|^{qr'})$  converge strongly in  $L^1(\Omega)$ , in turn  $(H(x, u_n, \nabla u_n))$  converges strongly to  $H(x, u, \nabla u)$  in  $L^1(\Omega)$ . Applying Theorem 2.6 to  $\mu_n - H(x, u_n, \nabla u_n)$  as above, we still obtain that  $u$  is a R-solution of (3.6).  $\blacksquare$

## 4 Necessary conditions for existence and removability results

Let  $\mu \in \mathcal{M}(\Omega)$ . We consider the local solutions of

$$-\Delta_p u + H(x, u, \nabla u) = \mu \quad \text{in } \Omega, \quad (4.1)$$

We say that  $u$  is a **weak solution** of (4.1) if  $u$  is measurable and finite a.e. in  $\Omega$ ,  $T_k(u) \in W_{loc}^{1,p}(\Omega)$  for any  $k > 0$ ,  $H(x, u, \nabla u) \in L_{loc}^1(\Omega)$  and (4.1) holds in  $\mathcal{D}'(\Omega)$ . We say that  $u$  is a **LR-solution** of (4.1) if  $T_k(u) \in W_{loc}^{1,p}(\Omega)$  for any  $k > 0$ , and  $|\nabla u|^q \in L_{loc}^1(\Omega)$  and  $u$  is a LR-solution of

$$-\Delta_p u = \mu - H(x, u, \nabla u), \quad \text{in } \Omega.$$

**Remark 4.1** If  $q \geq 1$  and  $u$  is a weak solution, then  $u$  satisfies (2.8), see for example [31, Lemma 2.2 and 2.3], thus  $u \in W_{loc}^{1,q}(\Omega)$ .

**Lemma 4.2** Let  $\mu \in \mathcal{M}(\Omega)$ . Assume that (4.1) admits a weak solution  $u$ .

(i) If

$$|H(x, u, \xi)| \leq C_1 |\xi|^q + \ell(x) \quad (4.2)$$

with  $C_1 > 0$  and  $\ell \in L^1(\Omega)$ , then setting  $C_2 = C_1 + q_* - 1$ , for any  $\zeta \in \mathcal{D}^+(\Omega)$ ,

$$\left| \int_{\Omega} \zeta^{q_*} d\mu \right| \leq C_2 \int_{\Omega} |\nabla u|^q \zeta^{q_*} dx + \int_{\Omega} |\nabla \zeta|^{q_*} dx + \int_{\Omega} \ell \zeta^{q_*} dx. \quad (4.3)$$

(ii) If  $H$  has a constant sign, and

$$C_0 |\xi|^q - \ell(x) \leq |H(x, u, \xi)|, \quad (4.4)$$

then for some  $C = C(C_0, p, q)$ ,

$$\int_{\Omega} |\nabla u|^q \zeta^{q_*} dx \leq C \left( \left| \int_{\Omega} \zeta^{q_*} d\mu \right| + \int_{\Omega} |\nabla \zeta|^{q_*} dx + \int_{\Omega} \ell \zeta^{q_*} dx \right) \quad (4.5)$$

**Proof.** By density, we can take  $\zeta^{q_*}$  as a test function, and get

$$\int_{\Omega} \zeta^{q_*} d\mu = - \int_{\Omega} H(x, u, \nabla u) \zeta^{q_*} dx + q_* \int_{\Omega} |\nabla u|^{p-2} \nabla u \cdot \zeta^{q_*-1} \nabla \zeta dx$$

and from the Hölder inequality, for any  $\varepsilon > 0$ ,

$$q_* \int_{\Omega} |\nabla u|^{p-1} \zeta^{q_*-1} |\nabla \zeta| dx \leq (q_* - 1) \varepsilon \int_{\Omega} |\nabla u|^q \zeta^{q_*} dx + \varepsilon^{1-q_*} \int_{\Omega} |\nabla \zeta|^{q_*} dx \quad (4.6)$$

which implies (4.3). If  $H$  has a constant sign, then

$$\begin{aligned} C_0 \int_{\Omega} |\nabla u|^q \zeta^{q_*} dx - \int_{\Omega} \ell dx &\leq \int_{\Omega} |H(x, u, \nabla u)| \zeta^{q_*} dx = \left| \int_{\Omega} H(x, u, \nabla u) \zeta^{q_*} dx \right| \\ &\leq \left| \int_{\Omega} \zeta^{q_*} d\mu \right| + q_* \int_{\Omega} |\nabla u|^{p-1} \zeta^{q_*-1} |\nabla \zeta| dx, \end{aligned}$$

thus (4.5) follows after taking  $\varepsilon$  small enough. ■

**Proposition 4.3** *Let  $\mu \in \mathcal{M}(\Omega)$ , and assume that (4.1) admits a weak solution  $u$ .*

(i) *If (4.2) holds, then  $\mu \in \mathcal{M}^{q_*}(\Omega)$ .*

(ii) *If  $H(x, u, \xi) \leq -C_0 |\xi|^q$  and  $\mu$  and  $u$  are nonnegative, then in addition there exists  $C = C(C_0, p, q) > 0$  such that for any compact  $K \subset \Omega$ ,*

$$\mu(K) \leq C \text{cap}_{1, q_*}(K, \Omega). \quad (4.7)$$

**Proof.** (i) Let  $E$  be a Borel set such that  $\text{cap}_{1, q_*}(E, \Omega) = 0$ . There exist two measurable disjoint sets  $A, B$  such that  $\Omega = A \cup B$  and  $\mu^+(B) = \mu^-(A) = 0$ . Let us show that  $\mu^+(A \cap E) = 0$ . Let  $K$  be any fixed compact set in  $A \cap E$ . Since  $\mu^-(K) = 0$ , for any  $\delta > 0$  there exists a regular domain  $\omega \subset\subset \Omega$  containing  $K$ , such that  $\mu^-(\omega) < \delta$ . Then there exists  $\zeta_n \in \mathcal{D}(\omega)$  such that  $0 \leq \zeta_n \leq 1$ , and  $\zeta_n = 1$  on a neighborhood of  $K$  contained in  $\omega$ , and  $(\zeta_n)$  converges to in  $W^{1, q_*}(\mathbb{R}^N)$  and *a.e.* in  $\Omega$ , see [2]. There holds

$$\mu^+(K) \leq \int_{\omega} \zeta_n^{q_*} d\mu^+ = \int_{\omega} \zeta_n^{q_*} d\mu + \int_{\omega} \zeta_n^{q_*} d\mu^- \leq \int_{\omega} \zeta_n^{q_*} d\mu + \delta$$

and from (4.3),

$$\left| \int_{\Omega} \zeta_n^{q_*} d\mu \right| \leq C_2 \int_{\Omega} |\nabla u|^q \zeta_n^{q_*} dx + \int_{\Omega} |\nabla \zeta_n|^{q_*} dx + \int_{\Omega} \ell \zeta_n^{q_*} dx$$

And  $\lim_{n \rightarrow \infty} \int_{\Omega} |\nabla u|^q \zeta_n^{q^*} dx = 0$ , from the dominated convergence theorem, thus  $|\int_{\Omega} \zeta_n^{q^*} d\mu| \leq \delta$  for large  $n$ ; then  $\mu^+(K) \leq 2\delta$  for any  $\delta > 0$ , thus  $\mu^+(K) = 0$ , hence  $\mu^+(A \cap E) = 0$ ; similarly we get  $\mu^-(B \cap E) = 0$ , hence  $\mu(E) = 0$ .

(ii) Here we find

$$\int_{\Omega} \zeta^{q^*} d\mu + C_0 \int_{\Omega} |\nabla u|^q \zeta^{q^*} dx \leq q_* \int_{\Omega} |\nabla u|^{p-2} \nabla u \cdot \zeta^{q^*-1} \nabla \zeta dx$$

and hence from (4.6) with  $\varepsilon > 0$  small enough, for some  $C = C(C_0, p, q)$ ,

$$\int_{\Omega} \zeta^{q^*} d\mu \leq C \int_{\Omega} |\nabla \zeta|^{q^*} dx$$

and (4.7) follows, see [34]. ■

**Remark 4.4** Property (ii) extends the results of [19] and [41, Theorem 3.1] for equation (1.4).

Next we show a removability result:

**Theorem 4.5** Assume that  $H$  has a constant sign and satisfies (4.2) and (4.4). Let  $F$  be any relatively closed subset of  $\Omega$ , such that  $\text{cap}_{1, q^*}(F, \mathbb{R}^N) = 0$ , and  $\mu \in \mathcal{M}^{q^*}(\Omega)$ .

(i) Let  $1 < q \leq p$ . Let  $u$  be any LR-solution of

$$-\Delta_p u + H(x, u, \nabla u) = \mu \quad \text{in } \Omega \setminus K \quad (4.8)$$

Then  $u$  is a LR-solution of

$$-\Delta_p u + H(x, u, \nabla u) = \mu \quad \text{in } \Omega. \quad (4.9)$$

(ii) Let  $q > p$  and  $u$  be a weak solution of (4.8), then  $u$  is a weak solution of (4.9).

**Proof.** (i) Let  $1 < q \leq p$ . From our assumption,  $T_k(u) \in W_{loc}^{1,p}(\Omega \setminus F)$ , for any  $k > 0$ , and  $|u|^{p-1} \in L_{loc}^{\sigma}(\Omega)$ , for any  $\sigma \in [1, N/(N-p))$ , and  $|\nabla u|^{p-1} \in L_{loc}^{\tau}(\Omega \setminus F)$ , for any  $\tau \in [1, N/(N-1))$ , and  $|\nabla u|^q \in L_{loc}^1(\Omega \setminus F)$ . For any compact  $K \subset \Omega$ , there holds  $\text{cap}_{1,p}(F \cap K, \mathbb{R}^N) = 0$ , because  $p \leq q_*$ , thus  $T_k(u) \in W_{loc}^{1,p}(\Omega)$ , see [24, Theorem 2.44]. And  $u$  is measurable on  $\Omega$  and finite a.e. in  $\Omega$ , thus we can define  $\nabla u$  a.e. in  $\Omega$  by the formula  $\nabla u(x) = \nabla T_k(u)(x)$  a.e. on the set  $\{x \in \Omega : |u(x)| \leq k\}$ .

Let us consider a fixed function  $\zeta \in \mathcal{D}^+(\Omega)$  and let  $\omega \subset\subset \Omega$  such that  $\text{supp} \zeta \subset \omega$  and set  $K_{\zeta} = F \cap \text{supp} \zeta$ . Then  $K_{\zeta}$  is a compact and  $\text{cap}_{1, q^*}(K, \mathbb{R}^N) = 0$ . Thus there exists  $\zeta_n \in \mathcal{D}(\omega)$  such that  $0 \leq \zeta_n \leq 1$ , and  $\zeta_n = 1$  on a neighborhood of  $K$  contained in  $\omega$ , and  $(\zeta_n)$  converges to 0 in  $W^{1, q^*}(\mathbb{R}^N)$ ; we can assume that the convergence holds everywhere on  $\mathbb{R}^N \setminus N$ , where  $\text{cap}_{1, q^*}(N, \mathbb{R}^N) = 0$ , see for example [4, Lemmas 2.1, 2.2]. From Lemma 4.2 applied to  $\xi_n = \zeta(1 - \zeta_n)$  in  $\Omega \setminus F$ , we have

$$\begin{aligned} \int_{\Omega} |\nabla u|^q \xi_n^{q^*} dx &\leq C \left( \int_{\Omega} \xi_n^{q^*} d|\mu| + \int_{\Omega} |\nabla \xi_n|^{q^*} dx + \int_{\Omega} \ell \xi_n^{q^*} dx \right) \\ &\leq C \left( \int_{\Omega} \zeta^{q^*} d|\mu| + \int_{\Omega} |\nabla \zeta|^{q^*} dx + \int_{\Omega} |\nabla \zeta_n|^{q^*} dx + \int_{\Omega} \ell \zeta^{q^*} dx \right). \end{aligned} \quad (4.10)$$

Then from the Fatou Lemma, we get  $|\nabla u|^q \zeta^{q^*} \in L^1(\Omega)$  and

$$\int_{\Omega} |\nabla u|^q \zeta^{q^*} dx \leq C_{\zeta} := C \left( \int_{\Omega} \zeta^{q^*} d|\mu| + \int_{\Omega} |\nabla \zeta|^{q^*} dx + \int_{\Omega} \ell \zeta^{q^*} dx \right), \quad (4.11)$$

where  $C_\zeta$  also depends on  $\zeta$ . Taking as test function  $T_k(u)\xi_n^{q^*}$ , we obtain

$$\begin{aligned} & \int_{\Omega} |\nabla(T_k(u))|^p \xi_n^{q^*} dx + \int_{\Omega} H(x, u, \nabla u) T_k(u) \xi_n^{q^*} dx \\ &= \int_{\Omega} T_k(u) \xi_n^{q^*} d\mu_0 + k \left( \int_{\Omega} \xi_n^{q^*} d\mu_s^+ + \int_{\Omega} \xi_n^{q^*} d\mu_s^- \right) + \int_{\Omega} T_k(u) |\nabla u|^{p-2} \nabla u \cdot \nabla(\xi_n^{q^*}) dx; \end{aligned}$$

From the Hölder inequality, we deduce

$$\begin{aligned} & \frac{1}{k} \left| \int_{\Omega} T_k(u) |\nabla u|^{p-2} \nabla u \cdot \nabla(\xi_n^{q^*}) dx \right| \\ & \leq q_* \left( \int_{\Omega} \zeta^{q_*-1} |\nabla u|^{p-1} |\nabla \zeta| dx + \int_{\Omega} \zeta^{q_*} |\nabla u|^{p-1} |\nabla \zeta_n| dx \right) \\ & \leq (q_* - 1) \int_{\Omega} |\nabla u|^q \zeta^{q_*} dx + \int_{\Omega} |\nabla \zeta|^{q_*} dx + q_* \left( \int_{\Omega} |\nabla u|^q \zeta^{q_*} dx + \int_{\Omega} \zeta^{q_*} |\nabla \zeta_n|^{q_*} dx \right) \\ & \leq 2q_* C_\zeta + \int_{\Omega} |\nabla \zeta|^{q_*} dx + o(n). \end{aligned}$$

Thus from (4.2), with a new constant  $C_\zeta$ ,

$$\int_{\Omega} |\nabla(T_k(u))|^p \xi_n^{q^*} dx \leq (k+1)C_\zeta + o(n);$$

hence from the Fatou Lemma,

$$\int_{\Omega} |\nabla(T_k(u))|^p \zeta^{q^*} dx \leq (k+1)C_\zeta.$$

Therefore  $|u|^{p-1} \in L_{loc}^\sigma(\Omega)$ ,  $\forall \sigma \in [1, N/(N-p))$  and  $|\nabla u|^{p-1} \in L_{loc}^\tau(\Omega)$ ,  $\forall \tau \in [1, N/(N-1))$ , from a variant of the estimates of [5] and [10], see [39, Lemma 3.1].

Finally we show that  $u$  is a LR-solution in  $\Omega$ : let  $h \in W^{1,\infty}(\mathbb{R})$  such that  $h'$  has a compact support, and  $\varphi \in W^{1,m}(\Omega)$  for some  $m > N$ , with compact support in  $\Omega$ , such that  $h(u)\varphi \in W^{1,p}(\Omega)$ ; let  $\omega \subset\subset \Omega$  such that  $\text{supp} \zeta \subset \omega$  and set  $K = F \cap \text{supp} \zeta$ , and consider  $\zeta_n \in \mathcal{D}(\mathbb{R}^N)$  as above; then  $(1 - \zeta_n)\varphi \in W^{1,m}(\Omega \setminus F)$  and  $h(u)(1 - \zeta_n)\varphi \in W^{1,p}(\Omega \setminus F)$  and has a compact support in  $\Omega \setminus F$ , then we can write

$$I_1 + I_2 + I_3 + I_4 = \int_{\Omega} h(u)\varphi(1 - \zeta_n) d\mu_0 + h(+\infty) \int_{\Omega} \varphi(1 - \zeta_n) d\mu_s^+ - h(-\infty) \int_{\Omega} \varphi(1 - \zeta_n) d\mu_s^-,$$

with

$$\begin{aligned} I_1 &= \int_{\Omega} |\nabla u|^{p-2} \nabla u \cdot h'(u)\varphi(1 - \zeta_n) dx, & I_2 &= - \int_{\Omega} |\nabla u|^{p-2} \nabla u \cdot h(u)\varphi \nabla \zeta_n dx \\ I_3 &= \int_{\Omega} |\nabla u|^{p-2} \nabla u \cdot h(u)(1 - \zeta_n) \nabla \varphi dx, & I_4 &= \int_{\Omega} H(x, u, \nabla u) h(u)\varphi(1 - \zeta_n) dx \end{aligned}$$

We can go to the limit in  $I_1$  as  $n \rightarrow \infty$ , from the dominated convergence theorem, since there exists  $a > 0$  such that

$$\int_{\Omega} |\nabla u|^{p-2} \nabla u \cdot h'(u)\varphi(1 - \zeta_n) dx = \int_{\Omega} |\nabla T_a(u)|^{p-2} \nabla T_a(u) \cdot h'(T_a(u))\varphi(1 - \zeta_n) dx.$$

Furthermore  $I_2 = o(n)$ , because

$$\left| \int_{\Omega} |\nabla u|^{p-2} \nabla u \cdot h(u) \varphi \nabla \zeta_n dx \right| \leq \|h\|_{L^\infty(\mathbb{R})} \left( \int_{\Omega} |\nabla u|^q \varphi dx \right)^{1/q} \|\nabla \zeta_n\|_{L^{q^*}(\mathbb{R}^N)};$$

we also go to the limit in  $I_3$  because  $|\nabla \varphi| \in L^m(\Omega)$  and  $|\nabla u|^{p-1} \in L_{loc}^\tau(\Omega), \forall \tau \in [1, N/(N-1))$ ; in  $I_4$  from (4.11) and (4.2), and in the right hand side because  $h(u)\varphi \in L^1(\Omega, d\mu_0)$ , see [16, Remark 2.26] and  $\zeta_n \rightarrow 0$  everywhere in  $\mathbb{R}^N \setminus N$  and  $\mu(N) = 0$ . Then we can conclude:

$$\begin{aligned} & \int_{\Omega} |\nabla u|^{p-2} \nabla u \cdot \nabla (h(u)\varphi) dx + \int_{\Omega} H(x, u, \nabla u) h(u) \varphi dx \\ &= \int_{\Omega} h(u) \varphi d\mu_0 + h(+\infty) \int_{\Omega} \varphi d\mu_s^+ - h(-\infty) \int_{\Omega} \varphi d\mu_s^-. \end{aligned}$$

(ii) Assume that  $q > p > 1$  (hence  $1 < q_* < p$ ) and  $u$  is a weak solution in  $\Omega \setminus F$ . Then  $u \in W_{loc}^{1,q}(\Omega \setminus F)$  implies  $u \in W_{loc}^{1,q_*}(\Omega \setminus F) = W_{loc}^{1,q_*}(\Omega)$ , hence  $|\nabla u|$  is well defined in  $L_{loc}^1(\Omega)$ . As in part (i) we obtain that  $|\nabla u|^q \zeta^{q_*} \in L^1(\Omega)$ , hence  $|\nabla u|^q \in L_{loc}^1(\Omega)$ . For any  $\varphi \in \mathcal{D}(\Omega)$ , and  $\omega$  containing  $\text{supp} \varphi$ , we have  $\varphi(1 - \zeta_n) \in \mathcal{D}(\Omega \setminus F)$ , then we can write  $J_1 + J_2 + J_3 = \int_{\Omega} \varphi(1 - \zeta_n) d\mu$ , with

$$J_1 = \int_{\Omega} (1 - \zeta_n) |\nabla u|^{p-2} \nabla u \cdot \nabla \varphi dx, \quad J_2 = - \int_{\Omega} \varphi |\nabla u|^{p-2} \nabla u \cdot \nabla \zeta_n dx, \quad J_3 = \int_{\Omega} H(x, u, \nabla u) \varphi (1 - \zeta_n) dx.$$

Now we can go to the limit in  $J_1$  and  $J_3$  from the dominated convergence theorem, because  $|\nabla u|^q \in L_{loc}^1(\Omega)$  and  $q > p - 1$ ; and  $(\int_{\Omega} \varphi(1 - \zeta_n) d\mu)$  converges to  $\int_{\Omega} \varphi d\mu$  as above. And  $J_2$  converges to 0, because  $|\nabla u|^{p-1} \in L_{loc}^{q/(p-1)}(\Omega)$  and  $|\nabla \zeta_n|$  tends to 0 in  $L^{q_*}(\Omega)$ . Then  $u$  is a weak solution in  $\Omega$ . ■

## 5 Existence in the supercritical case

Here the problem is delicate and many problems are still unsolved.

### 5.1 Case of a source term

Here we consider problem

$$-\Delta_p u = |\nabla u|^q + \mu \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega. \quad (5.1)$$

The main question is the following:

*If  $\mu \in M_b^{q_*}(\Omega)$  satisfies condition (4.7) with a constant  $C > 0$  small enough, does (5.1) admit a solution?*

In the case  $p = 2 < q$ , the problem has been solved in [19]. In that case one can define the solutions in a very weak sense. According to [14], setting  $\rho(x) = \text{dist}(x, \partial\Omega)$ , a function  $u$  is called a *very weak solution* of (5.1) if  $u \in W_{loc}^{1,q}(\Omega) \cap L^1(\Omega)$ ,  $|\nabla u|^q \in L^1(\Omega, \rho dx)$  and for any  $\varphi \in C^2(\overline{\Omega})$  such that  $\varphi = 0$  on  $\partial\Omega$ ,

$$- \int_{\Omega} u \Delta \varphi dx = \int_{\Omega} |\nabla u|^q \varphi dx + \int_{\Omega} \varphi d\mu.$$

**Theorem 5.1** ([19]) *Let  $\mu \in \mathcal{M}^+(\Omega)$ . If  $1 < q$  and  $p = 2$  and (5.1) has a very weak solution, then*

$$\mu(K) \leq C \text{Cap}_{1,q'}(K, \Omega) \quad (5.2)$$

*holds for any compact  $K \subset \Omega$ , and some  $C < C_1(N, q)$ . Conversely, if  $2 < q$  and (5.2) holds for some  $C < C_2(N, q, \Omega)$  then (5.1) has a very weak nonnegative solution.*

In the general case  $p > 1$ , such a notion of solution does not exist. The problem (5.1) with  $p < q$  was studied by [41] for signed measures  $\mu \in \mathcal{M}_b(\Omega)$  such that

$$[\mu]_{1,q^*,\Omega} = \sup \left\{ \frac{|\mu(K \cap \Omega)|}{\text{Cap}_{1,q^*}(K, \mathbb{R}^N)} : K \text{ compact of } \mathbb{R}^N, \text{Cap}_{1,q^*}(K, \mathbb{R}^N) > 0 \right\} < \infty.$$

**Theorem 5.2** ([41]) *Let  $1 < p < q$ . Let  $\mu \in \mathcal{M}_b(\Omega)$ . There exists  $C_1 = C_1(N, p, q, \Omega)$  such that if*

$$|\mu(K \cap \Omega)| \leq C \text{Cap}_{1,q^*}(K, \mathbb{R}^N) \quad (5.3)$$

*for any compact  $K \subset \mathbb{R}^N$ , and some  $C < C_1$ , then (5.1) has a weak solution  $u \in W_0^{1,q}(\Omega)$ , such that  $[\|\nabla u\|^q]_{1,q^*,\Omega}$  is finite. In particular this holds for any  $\mu \in L^{N/q^*,\infty}(\Omega)$ .*

Very recently the case  $p = q$ , has been studied in [23] for signed measures satisfying a trace inequality: setting  $p^\# = (p-1)^{2-p}$  if  $p \geq 2$ ,  $p^\# = 1$  if  $p < 2$ , they show in particular the following:

**Theorem 5.3** ([23]) *Let  $1 < p = q$ . Let  $\mu \in \mathcal{M}_b(\Omega)$  such that*

$$-C_1 \int_{\Omega} |\nabla \zeta|^p dx \leq \int_{\Omega} |\zeta|^p d\mu \leq C_2 \int_{\Omega} |\nabla \zeta|^p dx, \quad \forall \zeta \in \mathcal{D}(\Omega), \quad (5.4)$$

*with  $C_1 > 0$  and  $C_2 \in (0, p^\#)$ . Then (5.1) has a weak solution  $u \in W_{loc}^{1,p}(\Omega)$ .*

The existence for problem (5.1) is still open in the case  $q < p$  for  $p \neq 2$

## 5.2 Case of an absorption term

Here we consider problem (1.2) in case of absorption, where  $\mu \in M_b^+(\Omega)$  and we look for a nonnegative solution. In the model case

$$-\Delta_p u + |\nabla u|^q = \mu \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega, \quad (5.5)$$

the main question is the following: *If  $\mu \in M_b^{q^*+}(\Omega)$ , hence  $\mu = f + \text{div}g$ , with  $f \in L^1(\Omega)$  and  $g \in (L^{q/(p-1)}(\Omega))^N$ , does (5.5) admits a nonnegative solution?*

**Remark 5.4** *Up to changing  $u$  into  $-u$ , the results of Theorem 5.2 and 5.3 are also available for the problem (5.5) but we have **no information on the sign of  $u$** .*

In the sequel we give two partial results of existence.

### 5.2.1 Case $q \leq p$ and $\mu \in \mathcal{M}_b^{p+}(\Omega)$

Here we assume that  $\mu \in \mathcal{M}_b^{p+}(\Omega)$ , subspace of  $\mathcal{M}_b^{q*+}(\Omega)$ . Our proof is directly inspired from the results of [11] for the problem (3.6), where  $q = p$  and  $H(x, u, \xi)u \geq 0$ .

**Theorem 5.5** *Let  $p - 1 < q \leq p$ . Let  $\mu \in \mathcal{M}_b^{p+}(\Omega)$ , and*

$$0 \leq H(x, u, \xi) \leq C_1 |\xi|^p + \ell(x), \quad (5.6)$$

$$H(x, u, \xi) \geq C_0 |\xi|^q \text{ for } u \geq L, \quad (5.7)$$

with  $\ell(x) \in L^1(\Omega)$ ,  $C_k, C_0, L \geq 0$ . Then there exists a nonnegative R-solution of problem (1.2).

**Remark 5.6** *The result was known in the case where  $H(x, u, \nabla u) = |\nabla u|^q$ ,  $p = 2$ , and  $\mu \in L^1(\Omega)$  (see for example [1], where the existence for any  $\mu \in \mathcal{M}_b^{2+}(\Omega)$  is also claimed, without proof). For  $p \neq 2$ , the case  $q < p$ ,  $\mu \in L^1(\Omega)$  is partially treated in [40].*

**Proof.** Let  $\mu = f - \text{div}g$  with  $f \in L^1(\Omega)$  and  $g = (g_i) \in (L^{p'}(\Omega))^N$ . Here again we use the good approximation of  $\mu$  by a sequence of measures  $\mu_n \in \mathcal{M}_b^+(\Omega)$  given at Lemma 2.8 (ii),  $\lambda_n = 0$ , thus  $\mu_n = \mu_n^0 = f_n - \text{div}g_n$ , with  $f_n \in \mathcal{D}^+(\Omega)$  and  $g_n = (g_{n,i}) \in (\mathcal{D}(\Omega))^N$ . Hence there exists a weak nonnegative solution  $u_n \in W_0^{1,p}(\Omega)$  of the problem

$$-\Delta_p u_n + H(x, u_n, \nabla u_n) = \mu_n \quad \text{in } \Omega, \quad u_n = 0 \quad \text{on } \partial\Omega.$$

Since  $H(x, u, \xi) \geq 0$ , taking  $\varphi = k^{-1}T_k(u_n - m)$  with  $m \geq 0$ ,  $k > 0$ , as a test function, we still obtain (3.8). From Lemma 2.3, up to a subsequence,  $(u_n)$  converges *a.e.* to a function  $u$ ,  $(T_k(u_n))$  converges weakly in  $W_0^{1,p}(\Omega)$ ,  $(\nabla u_n)$  converges *a.e.* to  $\nabla u$ , and  $(u_n^{p-1})$  converges strongly in  $L^\sigma(\Omega)$ , for any  $\sigma \in [1, N/(N-p))$ . Thus  $\lim_{k \rightarrow \infty} \sup_{n \in \mathbb{N}} |\{u_n > k\}| = 0$ , and  $(|\nabla u_n|^{p-1})$  converges strongly in  $L^\tau(\Omega)$ , for any  $\tau \in [1, N/(N-1))$ . Moreover the choice of  $\varphi$  with  $m + k > L$  gives

$$\frac{1}{k} \int_{\{m \leq u \leq m+k\}} |\nabla u_n|^p dx + C_0 \int_{\{u_n \geq m+k\}} |\nabla u_n|^q dx \leq \mu_n(\Omega) \leq 4\mu(\Omega).$$

Taking  $m = 0$  we obtain

$$\int_{\Omega} |\nabla u_n|^q dx \leq \int_{\{u_n \geq k\}} |\nabla u_n|^q dx + \int_{\Omega} |\nabla T_k(u_n)|^q dx \leq 4C_0^{-1}\mu(\Omega) + \int_{\Omega} |\nabla T_k(u_n)|^p dx + |\Omega|$$

since  $q \leq p$ ; thus from the Fatou Lemma,  $|\nabla u|^q \in L^1(\Omega)$ . Moreover using  $\varphi = T_1(u_n - k)$ ,

$$\int_{\{k-1 \leq u_n \leq k\}} |\nabla u_n|^p dx + \int_{\{u_n \geq k\}} H(x, u_n, \nabla u_n) dx \leq \int_{\{u_n \geq k-1\}} f_n dx + \int_{\{k-1 \leq u_n \leq k\}} |g_n \cdot \nabla u_n| dx.$$

Therefore, from the Hölder inequality,

$$\int_{\{k-1 \leq u_n \leq k\}} |\nabla u_n|^p dx + p' \int_{\{u_n \geq k\}} H(x, u_n, \nabla u_n) dx \leq \int_{\{u_n \geq k-1\}} f_n dx + \left( \sum_{i=1}^N \int_{\{k-1 \leq u_n \leq k\}} |g_{n,i}|^{p'} dx \right).$$



From Lemma 2.8, there holds

$$\limsup_{k \rightarrow \infty} \sup_{n \in \mathbb{N}} \left( \int_{\{k-1 \leq u_n \leq k\}} |\nabla u_n|^p dx + \int_{\{u_n \geq k\}} H(x, u_n, \nabla u_n) dx \right) = 0. \quad (5.8)$$

Next we prove the strong convergence of the truncates in  $W_0^{1,p}(\Omega)$  as in [11]: we take as test function

$$\varphi_n = \Phi(T_k(u_n) - T_k(u)), \text{ where } \Phi(s) = se^{\theta^2 s^2/4},$$

where  $\theta > 0$  will be chosen after, thus  $\Phi'(s) \geq \theta |\Phi(s)| + 1/2$ . Then  $\varphi_n \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ , and we have  $|\varphi_n| \leq \Phi(k)$ ; setting  $\psi_n = \Phi'(T_k(u_n) - T_k(u))$ , we have  $0 \leq \psi_n \leq \Phi'(k)$ . Then  $\varphi_n \rightarrow 0$ ,  $\psi_n \rightarrow 1$  in  $L^\infty(\Omega)$  weak \* and *a.e.* in  $\Omega$ . We set  $a(\xi) = |\xi|^{p-2}\xi$ , and

$$X = \int_{\Omega} (a(\nabla(T_k(u_n))) - a(\nabla(T_k(u)))) \cdot \nabla(T_k(u_n) - T_k(u)) \psi_n dx,$$

and get  $X + I_1 = I_2 + I_3 + I_4$  with

$$I_1 = \int_{\Omega} H(x, u_n, \nabla u_n) \varphi_n dx, \quad I_2 = \int_{\Omega} a(\nabla(T_k(u))) \cdot \nabla(T_k(u) - T_k(u_n)) \psi_n dx,$$

$$I_3 = \int_{\Omega} f_n \varphi_n dx + \int_{\Omega} \operatorname{div}(g_n - g) \varphi_n dx + \int_{\Omega} g \cdot \nabla(T_k(u_n) - T_k(u)) \psi_n dx,$$

$$I_4 = - \int_{\Omega} a(\nabla(u_n - T_k(u_n))) \cdot \nabla(T_k(u_n) - T_k(u)) \psi_n dx = \int_{\{u_n \geq k\}} a(\nabla(u_n - T_k(u_n))) \cdot \nabla(T_k(u)) \psi_n dx.$$

From this, one can easily see that  $|I_2| + |I_3| + |I_4| = o(n)$ . Since  $H(x, u_n, \nabla u_n) \geq 0$  for  $u_n \geq k$ , then  $X \leq I_5 + o(n)$ , where

$$I_5 = \left| \int_{\{u_n < k\}} H(x, u_n, \nabla u_n) \varphi_n dx \right| \leq C_1 \int_{\Omega} |\nabla(T_k u_n)|^p |\varphi_n| dx + \int_{\Omega} l |\varphi_n| dx \leq C_1(Y + I_7) + o(n),$$

with

$$Y = \int_{\Omega} (a(\nabla(T_k(u_n))) - a(\nabla(T_k(u)))) \cdot \nabla(T_k(u_n) - T_k(u)) |\varphi_n| dx,$$

$$I_7 = \int_{\Omega} a(\nabla(T_k(u))) \cdot \nabla(T_k(u_n) - T_k(u)) |\varphi_n| dx + \int_{\Omega} (a(\nabla(T_k(u_n))) \cdot \nabla(T_k(u)) |\varphi_n| dx$$

and then  $I_7 = o(n)$ . We get finally  $X \leq C_1 Y + o(n)$ ; choosing  $\theta = 2C_1$ , we deduce that

$$\int_{\Omega} (a(\nabla(T_k(u_n))) - a(\nabla(T_k(u)))) \cdot \nabla(T_k(u_n) - T_k(u)) dx = o(n).$$

Hence  $(T_k(u_n))$  converges strongly to  $T_k(u)$  in  $W_0^{1,p}(\Omega)$ . Therefore  $H(x, u_n, \nabla u_n)$  is equi-integrable, from (5.6) and (5.8), since for any measurable set  $E \subset \Omega$ ,

$$\int_E H(x, u_n, \nabla u_n) dx \leq C_1 \int_E |\nabla(T_k u_n)|^p dx + \int_E \ell dx + \int_{\{u_n \geq k\}} H(x, u_n, \nabla u_n) dx.$$

Then  $(H(x, u_n, \nabla u_n))$  converges to  $H(x, u, \nabla u)$  strongly in  $L^1(\Omega)$ ; thus  $(\mu_n - H(x, u_n, \nabla u_n))$  is a good approximation of  $\mu - H(x, u, \nabla u)$ , and  $u$  is a R-solution of problem (3.1) from Theorem 2.6.  $\blacksquare$

**Remark 5.7** In the case  $p - 1 < q < p$ , and if (5.6) is replaced by

$$0 \leq H(x, u, \xi) \leq C_1 |\xi|^q + \ell(x), \quad (5.9)$$

the proof is much shorter: in order to prove the equi-integrability of  $(H(x, u_n, \nabla u_n))$  we do not need to prove the strong convergence of the truncates: indeed for any measurable set  $E \subset \Omega$ ,

$$\int_E |\nabla u_n|^q dx \leq \int_E |\nabla(T_k u_n)|^q dx + \int_{\{u_n \geq k\}} |\nabla u_n|^q dx$$

and  $(\nabla T_k(u_n))$  converges strongly to  $\nabla T_k(u)$  in  $L^q(\Omega)$  and (5.8) holds. Then  $(H(x, u_n, \nabla u_n))$  converges to  $H(x, u, \nabla u)$  strongly in  $L^1(\Omega)$ .

### 5.2.2 Case where $\mu$ satisfies (4.7)

Here we assume that  $\mu \in \mathcal{M}_b^+(\Omega)$  satisfies a capacity condition of type (4.7). For simplicity we assume that  $\mu$  has a compact support in  $\Omega$ . In the sequel we prove the following:

**Theorem 5.8** Let  $1 < q \leq p$  or  $p = 2$ . Assume that  $\mu \in \mathcal{M}_b^+(\Omega)$ , has a compact support and satisfies

$$\mu(K) \leq C_1 \text{cap}_{1, q^*}(K, \Omega), \quad \text{for any compact } K \subset \Omega, \quad (5.10)$$

for some  $C_1 = C(N, q, \Omega) > 0$  (non necessarily small). Then there exists a nonnegative  $R$ -solution  $u$  of problem (5.5), such that  $[|\nabla u|^q]_{1, q^*, \Omega}$  is finite.

First recall some equivalent properties of measures, see [35, Theorem 1.2], [19, Lemma 3.3], see also [41]:

**Remark 5.9** 1) Let  $\mu \in \mathcal{M}_b^+(\Omega)$ , extended by 0 to  $\mathbb{R}^N$ . Then (5.10) holds if and only if there exists  $C_2 > 0$  such that

$$\int_{\Omega} \zeta^{q^*} d\mu \leq C_2 \int_{\Omega} |\nabla \zeta|^{q^*} dx, \quad \forall \zeta \in \mathcal{D}^+(\Omega); \quad (5.11)$$

the constants of equivalence between  $C_1, C_2$  only depend on  $N, q_*, \Omega$ .

If moreover  $\mu$  has a compact support  $K_0 \subset \Omega$ , then (5.10) holds if and only if there exists  $C_3 > 0$  such that

$$\mu(K) \leq C_3 \text{Cap}_{1, q^*}(K, \mathbb{R}^N) \quad \text{for any compact } K \subset \mathbb{R}^N; \quad (5.12)$$

the constants of equivalence between  $C_1, C_3$  only depend on  $N, q_*, K_0$ .

2) Let  $\nu \in \mathcal{M}_b^+(\mathbb{R}^N)$ . Then (5.12) holds if and only if there exists  $C_4 > 0$  such that  $J_1(\nu)$  is finite a.e. and

$$J_1((J_1(\nu))^{q^*}) \leq C_4 J_1(\nu) \quad \text{a.e. in } \mathbb{R}^N; \quad (5.13)$$

the constants of equivalence between  $C_3, C_4$  do not depend on  $\nu$ .

Following the ideas of [41, Theorem 3.4] we prove a convergence Lemma:

**Lemma 5.10** Let  $(z_n)$  be a sequence of nonnegative functions, converging a.e. in  $L^1(\Omega)$ . Extending  $z_n$  by 0 in  $\mathbb{R}^N \setminus \Omega$ , assume that for some  $C > 0$ ,

$$\int_{\Omega} z_n^{\frac{q}{p-1}} \xi^{q^*} dx \leq C \int_{\Omega} |\nabla \xi|^{q^*} dx \quad \forall n \in \mathbb{N}, \forall \xi \in \mathcal{D}^+(\mathbb{R}^N).$$

Then  $(z_n)$  converges strongly in  $L^{q/(p-1)}(\Omega)$ .

**Proof.** From our assumption,  $(z_n)$  is bounded in  $L^{q/(p-1)}(\Omega)$ , then up to a subsequence, it converges to some  $z$  weakly in  $L^{q/(p-1)}(\Omega)$  and *a.e.* in  $\Omega$ . Consider a ball  $B \supset \Omega$  of radius  $2\text{diam}\Omega$ , and denote by  $G$  the Green function associated to  $-\Delta$  in  $B$ . Set  $w_n = z_n^{q/(p-1)}$ , and extend  $w_n$  by 0 to  $\mathbb{R}^N \setminus \Omega$ . Then for any compact  $K \subset \mathbb{R}^N$ ,

$$\int_{K \cap \Omega} w_n dx = \int_{K \cap B} w_n dx \leq CCap_{1,q^*}(K, \mathbb{R}^N),$$

which means that  $[w_n]_{1,q^*,B}$  is bounded, and

$$|\nabla G(w_n)(x)| \leq \int_B |\nabla_x G(x, y)| w_n(y) dy \leq CG_1 * w_n(x),$$

with  $C = C(N, \text{diam}\Omega)$ . In turn from [41, Corollary 2.5], we get

$$\left[ |\nabla G(w_n)|^{q/(p-1)} \right]_{1,q^*,B} \leq C \left[ |G_1 * w_n|^{q/(p-1)} \right]_{1,q^*,B} \leq C [w_n]_{1,q^*,B}^{q/(p-1)},$$

hence it is bounded. Therefore  $|\nabla G(w_n)|$  is bounded in  $L^{q/(p-1)}(B)$ , thus  $|\nabla G(w_n - w)|$  is bounded in  $L^{q/(p-1)}(B)$ . Let  $\varphi \in \mathcal{D}(B)$  and  $\varepsilon > 0$  be fixed. Since  $(z_n)$  converges *a.e.* to  $z$ , from the Egoroff theorem, there exists a measurable set  $\omega_\varepsilon \subset B$  such that  $(w_n)$  converges to  $w = z^{q/(p-1)}$  uniformly on  $\omega_\varepsilon$ , and  $\|\nabla \varphi\|_{L^{q^*}(B \setminus \omega_\varepsilon)} \leq \varepsilon$ . Then it follows that

$$\begin{aligned} \left| \int_\Omega (w_n - w) \varphi dx \right| &= \left| \int_B (w_n - w) \varphi dx \right| = \\ &= \left| - \int_B (\Delta(G(w_n - w)) \varphi dx \right| = \left| - \int_B \nabla(G(w_n - w)) \cdot \nabla \varphi dx \right| \end{aligned}$$

Considering the two integrals on  $B \setminus \omega_\varepsilon$  and  $\omega_\varepsilon$  we find  $\lim \int_\Omega (w_n - w) \varphi dx = 0$ . Taking  $\varphi = 1$  on  $\Omega$ , it follows that  $\lim \int_\Omega z_n^{q/(p-1)} dx = \int_\Omega z^{q/(p-1)} dx$  and the proof is done.  $\blacksquare$

**Proof of Theorem 5.8.** From our assumption,  $\mu \in \mathcal{M}^{q^*}(\Omega)$ . We consider the problem associated to  $\mu_n = \mu * \rho_n$

$$-\Delta_p u_n + |\nabla u_n|^q = \mu_n \quad \text{in } \Omega, \quad u_n = 0 \quad \text{on } \partial\Omega. \quad (5.14)$$

For  $q \leq p$ , from [12, Theorem 2.1], as in the proof of Theorem 3.3, (5.14) admits a nonnegative solution  $u_n \in W_0^{1,p}(\Omega) \cap C^{1,\alpha}(\overline{\Omega})$ . Moreover we can approximate  $u_n$  in  $C^{1,\alpha}(\overline{\Omega})$  by the solution  $u_{n,\varepsilon}$  ( $\varepsilon > 0$ ) of the problem

$$-\text{div}((\varepsilon^2 + |\nabla u_{n,\varepsilon}|^2)^{\frac{p-2}{2}} \nabla u_{n,\varepsilon}) + (\varepsilon^2 + |\nabla u_{n,\varepsilon}|^2)^{\frac{q}{2}} = \mu_n \quad \text{in } \Omega, \quad u_{n,\varepsilon} = 0 \quad \text{on } \partial\Omega.$$

Multiplying this equation by  $\xi^{q^*}$  with  $\xi \in \mathcal{D}^+(\mathbb{R}^N)$ , we obtain

$$\begin{aligned} & q_* \int_\Omega (\varepsilon^2 + |\nabla u_{n,\varepsilon}|^2)^{\frac{p-2}{2}} \nabla u_{n,\varepsilon} \cdot \xi^{q^*-1} \nabla \xi dx + \int_\Omega (\varepsilon^2 + |\nabla u_{n,\varepsilon}|^2)^{\frac{q}{2}} \xi^{q^*} dx \\ &= \int_\Omega \xi^{q^*} \mu_n dx + q_* \int_{\partial\Omega} \xi^{q^*} (\varepsilon^2 + |\nabla u_{n,\varepsilon}|^2)^{\frac{p-2}{2}} \nabla u_{n,\varepsilon} \cdot \nu ds. \end{aligned}$$

The boundary term is nonpositive, hence going to the limit as  $\varepsilon \rightarrow 0$ , we get

$$\int_{\Omega} |\nabla u_n|^q \xi^{q_*} dx \leq \int_{\Omega} \xi^{q_*} \mu_n dx + q_* \int_{\Omega} |\nabla u_n|^{p-2} \nabla u_n \cdot \xi^{q_*-1} \nabla \xi dx \quad (5.15)$$

When  $p = 2$ , existence also holds for  $q > 2$ , from [32]; and then  $u_n \in C^2(\overline{\Omega})$ , thus (5.15) is still true. As in Lemma 4.2, it follows that for any  $\xi \in \mathcal{D}^+(\mathbb{R}^N)$

$$\int_{\Omega} |\nabla u_n|^q \xi^{q_*} dx \leq C \left( \int_{\Omega} \xi^{q_*} d\mu_n + \int_{\Omega} |\nabla \xi|^{q_*} dx \right). \quad (5.16)$$

Otherwise, since  $\mu_n(\Omega) \leq \mu(\Omega)$ , from Lemma 2.3, up to a subsequence  $(u_n)$  converges *a.e.* to a function  $u$ ,  $(T_k(u_n))$  converges weakly in  $W_0^{1,p}(\Omega)$  and  $(\nabla u_n)$  converges *a.e.* to  $\nabla u$  in  $\Omega$ . Note also that  $(\mu_n)$  is a sequence of good approximations of  $\mu$ , since  $\mu$  has a compact support (see [8]). From (4.5), for any  $\xi \in \mathcal{D}^+(\mathbb{R}^N)$ , we have  $\lim \int_{\Omega} \xi^{q_*} d\mu_n = \int_{\Omega} \xi^{q_*} d\mu$ , since  $\xi^{q_*} \in C_c(\mathbb{R}^N)$ . Then  $\int_{\Omega} \xi^{q_*} d\mu \leq C \int_{\Omega} |\nabla \xi|^{q_*} dx$ . From the Fatou Lemma, we obtain

$$\int_{\Omega} |\nabla u|^q \xi^{q_*} dx \leq C \left( \int_{\Omega} \xi^{q_*} d\mu + \int_{\Omega} |\nabla \xi|^{q_*} dx \right) \leq C \int_{\Omega} |\nabla \xi|^{q_*} dx, \quad (5.17)$$

hence  $|\nabla u|^q \in L^1(\Omega)$ . And then for any compact  $K \subset \mathbb{R}^N$ , taking  $\xi = 1$  on  $K$ ,

$$\int_{K \cap \Omega} |\nabla u|^q dx \leq C C a p_{1,q_*}(K, \mathbb{R}^N),$$

thus  $[|\nabla u|^q]_{1,q_*,\Omega}$  is finite. Moreover, extending  $\mu$  by 0 to  $\mathbb{R}^N \setminus \Omega$ , we see from Remark 5.9 that  $\mu$  satisfies condition (5.11), which is equivalent to (5.13). By convexity,  $\mu_n$  also satisfies (5.13) and hence (5.11), with the same constants, *i.e.* for any  $n \in \mathbb{N}$  and any  $\xi \in \mathcal{D}^+(\mathbb{R}^N)$ ,

$$\int_{\Omega} \xi^{q_*} d\mu_n \leq C_2 \int_{\Omega} |\nabla \xi|^{q_*} dx \quad (5.18)$$

Then from (5.16) with another  $C > 0$ ,

$$\int_{\Omega} |\nabla u_n|^q \xi^{q_*} dx \leq C \int_{\Omega} |\nabla \xi|^{q_*} dx \quad (5.19)$$

Next we can apply Lemma 5.10 to  $z_n = |\nabla u_n|^{p-1}$ , since  $(\nabla u_n)$  converges *a.e.* to  $\nabla u$  in  $\Omega$ . Then  $(|\nabla u_n|^q)$  converges strongly in  $L^1(\Omega)$  to  $|\nabla u|^q$ . Thus  $(\mu_n - |\nabla u_n|^q)$  is a good approximation of  $(\mu - |\nabla u|^q)$ . From Theorem 2.6,  $u$  is a R-solution of the problem.

From [23, Theorem 1.4], condition (5.17) (for  $N \geq 2$ ) implies that  $q_* < N$ , that means  $q > q_c$ , or  $|\nabla u|^q = 0$  in  $\Omega$ , thus  $\mu = 0$ . If  $\mu = \operatorname{div} g$  with  $g \in (L^{N(q+1-p)/(p-1),\infty}(\Omega))^N$  with compact support, then  $|g|^{\frac{q}{p-1}} \in L^{N/q_*,\infty}(\Omega)$ , thus

$$\int_{\Omega} \zeta^{q_*} |g|^{\frac{q}{p-1}} dx \leq C_2 \int_{\Omega} |\nabla \zeta|^{q_*} dx, \quad \forall \zeta \in \mathcal{D}^+(\Omega).$$

Hence  $\mu$  satisfies (5.11) from the Hölder inequality. Note that  $\mu \in \mathcal{M}^{q_*}(\Omega)$ , since  $q > q_c$  implies  $|g| \in L^{q/(p-1)}(\Omega)^N$ .  $\blacksquare$

**Remark 5.11** Let  $q \leq p$  and  $\mu = \operatorname{div}g$ , where  $g$  has a compact support in  $\Omega$ . From Theorems 5.5 and 5.8, we have existence when  $g \in (L^{p'}(\Omega))^N$ , or when  $g \in (L^{N(q+1-p)/(p-1),\infty}(\Omega))^N$ . Observe that  $L^{p'}(\Omega) \supset L^{N(q+1-p)/(p-1)}(\Omega)$  if and only if  $\tilde{q} \leq q$ , where  $\tilde{q}$  is defined at (1.6). Hence Theorem 5.5 brings better results than Theorem 5.8 when  $\tilde{q} \leq q \leq p$ .

**Remark 5.12** The extension of this result to the case  $p < q, p \neq 2$  will be studied in a further article.

## 6 Some regularity results

In this section we give some regularity properties for the problem:

$$-\Delta_p u + H(x, u, \nabla u) = 0 \quad \text{in } \Omega. \quad (6.1)$$

We first recall some local estimates of the gradient for renormalized solutions, see [20], following the first results of [9], and many others, see among them [3], [26].

**Lemma 6.1** Let  $u$  be the  $R$ -solution of problem

$$-\Delta_p u = f \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega,$$

with  $f \in L^m(\Omega)$ ,  $1 < m < N$ . Set  $\bar{m} = Np/(Np - N + p) = p/\tilde{q}$ , where  $\tilde{q}$  is defined in (1.6).

(i) If  $m > N/p$ , then  $u \in L^\infty(\Omega)$ . If  $m = N/p$ , then  $u \in L^k(\Omega)$  for any  $k \geq 1$ . If  $m < N/p$ , then  $u^{p-1} \in L^k(\Omega)$  for  $k = Nm/(N - pm)$ .

(ii)  $|\nabla u|^{(p-1)} \in L^{m^*}(\Omega)$ , where  $m^* = Nm/(N - m)$ . If  $\bar{m} \leq m$ , then  $u \in W_0^{1,p}(\Omega)$ .

**Remark 6.2** The estimates on  $u$  and  $|\nabla u|$  are obtained in the case  $m < \bar{m}$  by using the classical test functions  $\phi_{\beta,\varepsilon}(T_k(u))$ , where  $\phi_{\beta,\varepsilon}(w) = \int_0^w (\varepsilon + |t|)^{-\beta} dt$ , for given real  $\beta < 1$ . Let us recall the proof in the case  $m \geq \bar{m}$ ,  $p < N$ . Then  $L^m(\Omega) \subset W^{-1,p'}(\Omega)$ , thus, from uniqueness,  $u \in W_0^{1,p}(\Omega)$  and  $u$  is a variational solution. If  $m = \bar{m}$ , then  $m^* = p'$ , and the conclusion follows. Suppose  $m > \bar{m}$ , equivalently  $m^* > p'$ . For any  $\sigma > p$ , for any  $F \in (L^\sigma(\Omega))^N$ , there exists a unique weak solution  $w$  in  $W_0^{1,\sigma}(\Omega)$  of the problem

$$-\Delta_p w = \operatorname{div}(|F|^{p-2} F) \quad \text{in } \Omega, \quad w = 0 \quad \text{on } \partial\Omega,$$

see [25], [29], [30]. Let  $v$  be the unique solution in  $W_0^{1,1}(\Omega)$  of the problem

$$-\Delta v = f \quad \text{in } \Omega, \quad v = 0 \quad \text{on } \partial\Omega. \quad (6.2)$$

Then from the classical Calderon-Zygmund theory,  $v \in W^{2,m}(\Omega)$ , then  $|\nabla v| \in L^{m^*}(\Omega)$ . Let  $F$  be defined by  $|F|^{p-2} F = \nabla v$ . Then  $F \in (L^\sigma(\Omega))^N$ , with  $\sigma = (p-1)m^* > p$ . Then  $-\Delta_p w = -\Delta v = f$ , thus  $w = u$ . Then  $u \in W_0^{1,\sigma}(\Omega)$ , thus  $|\nabla u|^{(p-1)} \in L^{m^*}(\Omega)$ .

We also obtain local estimates:

**Lemma 6.3** *Let  $u \in W_{loc}^{1,p}(\Omega)$  such that*

$$-\Delta_p u = f \quad \text{in } \Omega,$$

*with  $f \in L_{loc}^m(\Omega)$ ,  $1 < m < N$ , and  $m > \bar{m}$ . Then  $|\nabla u|^{p-1} \in L_{loc}^{m^*}(\Omega)$ . Furthermore, for any balls  $B_1 \subset\subset B_2 \subset\subset \Omega$ ,  $\left\| |\nabla u|^{p-1} \right\|_{L^{m^*}(\overline{B_1})}$  is bounded by a constant which depends only on  $N, p, B_1, B_2$  and  $\|u\|_{W^{1,p}(B_2)}$ .*

**Proof.** We consider again the function  $v$  defined in (6.2), and set  $|F|^{p-2}F = \nabla v$ . Then  $F \in (L^\sigma(\Omega))^N$  with  $\sigma = (p-1)m^*$ , and  $u \in W_{loc}^{1,p}(\Omega)$  is a solution of the problem

$$-\Delta_p u = \operatorname{div}(|F|^{p-2}F) \quad \text{in } \Omega.$$

Then, from [29],  $u \in W_{loc}^{1,\sigma}(\Omega)$  and for any balls  $B_1 \subset\subset B_2 \subset\subset \Omega$ ,  $\|u\|_{W^{1,\sigma}(B_1)}$  is controlled by the norm  $\|u\|_{W^{1,p}(B_2)}$ . ■

Next we consider problem (6.1) in the case  $q < \tilde{q}$ , where  $\tilde{q}$  is defined at (1.6).

**Theorem 6.4** *Let  $0 < q < \tilde{q}$ ,  $N \geq 2$ . Let  $H$  be a Caratheodory function on  $\Omega \times \mathbb{R}$  such that*

$$|H(x, u, \xi)| \leq g(x) + C|\xi|^q, \quad (6.3)$$

*where  $g \in L_{loc}^{N+\varepsilon}(\Omega)$ ,  $C > 0$ . Let  $u \in W_{loc}^{1,p}(\Omega)$  be any weak solution of problem (6.1). Then  $u \in C^{1,\alpha}(\Omega)$  for some  $\alpha \in (0, 1)$ . Moreover for any balls  $B_1 \subset\subset B_2 \subset\subset \Omega$ ,  $\|u\|_{C^{1,\alpha}(\overline{B_1})}$  is bounded by a constant which depends only on  $N, p, B_1, B_2$ ,  $\|g\|_{L^{N+\varepsilon}(B_2)}$ , and the norm  $\|u\|_{W^{1,p}(B_2)}$ .*

**Proof.** Since  $u \in W_{loc}^{1,p}(\Omega)$ , the function  $f = -H(x, u, \nabla u)$  satisfies  $f \in L_{loc}^{m_0}(\Omega)$  from (6.3), with  $m_0 = p/q > 1$ . Notice that  $q < \tilde{q}$  is equivalent to  $m_0 > \bar{m}$ . If  $m_0 > N$ , then from [18, Theorem 1.2],  $|\nabla u| \in L_{loc}^\infty(\Omega)$  and we get an estimate of  $\| |\nabla u| \|_{L^\infty(B_1)}$  in terms of the norm  $\|u\|_{W^{1,p}(B_2)}$  and  $\|g\|_{L^{N+\varepsilon}(B_2)}$ . Then  $u \in C(\Omega)$ ,  $f \in L_{loc}^\infty(\Omega)$ , hence  $u \in C^{1,\alpha}(\Omega)$  for some  $\alpha \in (0, 1)$ .

Next suppose that  $m_0 < N$ . Then from Lemma 6.3 we have  $|\nabla u|^{p-1} \in L^{(p-1)m_0^*}(\Omega)$ . In turn, from (6.3),  $f \in L_{loc}^{m_1}(\Omega)$  with  $m_1 = (p-1)m_0^*/q$ . Note that  $m_1/m_0 = N(p-1)/(qN-p) > 1$  since  $q < \tilde{q}$ . By induction, starting from  $m_1$ , as long as  $m_n < N$ , we can define  $m_{n+1} = (p-1)m_n^*/q$ , and we find  $m_n < m_{n+1}$ . If  $m_n < N$  for any  $n$ , then the sequence converges to  $\lambda = N(q-p+1)/q$ , which is impossible since  $p/q < \lambda$  and  $q < \tilde{q}$ . Then there exists  $n_0$  such that  $m_{n_0} \geq N$ . If  $n_0 = N$ , or if  $m_0 = N$  we modify a little  $m_0$  in order to avoid the case. Then we conclude from above. ■

**Remark 6.5** *The result, which holds without any assumption on the sign of  $H$ , is sharp. Indeed for  $\tilde{q} < q < p < N$ , the problem  $-\Delta_p u = |\nabla u|^q$  in  $B(0, 1)$  with  $u = 0$  on  $\partial B(0, 1)$  admits the solution*

$$x \mapsto u_C(x) = C(|x|^{-\frac{p-q}{q+1-p}} - 1),$$

*for suitable  $C > 0$ , and  $u_C \in W_0^{1,p}(\Omega)$  for  $\tilde{q} < q$ .*

Next we consider the absorption case, and for simplicity the model problem:

**Theorem 6.6** *Let  $0 < q < p$ . Let  $u \in W_{loc}^{1,p}(\Omega)$  be a nonnegative weak solution of*

$$-\Delta_p u + |\nabla u|^q = 0 \quad \text{in } \Omega.$$

*Then  $u \in L_{loc}^\infty(\Omega)$ , and for any balls  $B_1 \subset\subset B_2 \subset\subset \Omega$ ,  $\|u\|_{L^\infty(B_1)}$  and  $\|u\|_{W^{1,p}(B_1)}$  are controlled by the norm  $\|u\|_{L^p(B_2)}$ .*

*As a consequence, if  $q < \tilde{q}$ , then  $u \in C^{1,\alpha}(\Omega)$  for some  $\alpha \in (0, 1)$ . In particular  $\|\nabla u\|_{L^\infty(B_1)}$  is controlled by  $\|u\|_{L^p(B_2)}$ . If  $p \geq 2$ , the result is still true for any  $q < p$ .*

**Proof.** Since  $u \in W_{loc}^{1,p}(\Omega)$  and  $-\Delta_p u \leq 0$  in  $\Omega$ , from the weak Harnack inequality, for any ball  $B(x_0, 3\rho) \subset \Omega$ , and any  $\ell > p - 1$ ,

$$\sup_{B(x_0, \rho)} u \leq C \left( \int_{B(x_0, 2\rho)} u^\ell \right)^{\frac{1}{\ell}}, \quad (6.4)$$

with  $C = C(N, p, \ell)$ , in particular for  $\ell = p$ . Then  $u \in L_{loc}^\infty(\Omega)$ . For any  $\xi \in \mathcal{D}(\Omega)$ , taking  $u\xi^p$  as a test function, we get

$$\begin{aligned} \int_{\Omega} |\nabla u|^p \xi^p dx + \int_{\Omega} |\nabla u|^q u \xi^p dx &= -p \int_{\Omega} \xi^{p-1} u |\nabla u|^{p-2} \nabla u \cdot \nabla \xi dx \\ &\leq \frac{1}{2} \int_{\Omega} |\nabla u|^p \xi^p dx + C_p \int_{\Omega} u^p |\nabla \xi|^p dx. \end{aligned}$$

Then for any balls  $B_1 \subset\subset B_2 \subset\subset \Omega$ , we obtain that  $\|\nabla u\|_{L^p(B_1)}$  is bounded by a constant which depends only on  $N, p, B$  and  $\|u\|_{L^p(B_2)}$ . The conclusions hold from Theorem 6.4.

Next assume that  $p \geq 2$ , and  $q < p$ . We still have  $f = -|\nabla u|^q \in L_{loc}^{m_0}(\Omega)$ , with  $m_0 = p/q > 1$ , since  $u \in W_{loc}^{1,p}(\Omega)$ . From a local version of [36, Theorem 5], there holds  $\nabla u \in (W_{loc}^{\sigma, m_0(p-1)}(\Omega))^N$  for any  $\sigma < 1/(p-1)$ , then  $u \in W_{loc}^{k, m_0(p-1)}(\Omega)$  for any  $k < p'$ . Since we have also  $u \in L_{loc}^\infty(\Omega)$ , from Gagliardo-Nirenberg inequality,

$$\|\nabla u\|_{L^{km_0(p-1)}(B_1)} \leq \|u\|_{W^{k, m_0(p-1)}(B_1)}^{\frac{1}{k}} \|u\|_{L^\infty(B_1)}^{\frac{1}{k'}}.$$

Therefore  $|\nabla u| \in L^{m_0 s}(B_1)$  for any  $s < p$ . Then  $f \in L_{loc}^{m_1}(\Omega)$ , for any  $m_1 < m_0 p/q = m_0^2$ . By induction, for any  $r \geq 1$ ,  $f \in L_{loc}^m(\Omega)$  for any  $m < m_0^r$ . Thus  $f \in L_{loc}^m(\Omega)$  for any  $m \geq 1$ . We deduce as above that  $|\nabla u| \in L_{loc}^\infty(\Omega)$  from [18, Theorem 1.2], and then  $u \in C^{1,\alpha}(\Omega)$  for some  $\alpha \in (0, 1)$ . ■

**Remark 6.7** *Up to our knowledge, the regularity in the case  $p < 2$  is still an open problem, even if  $p > 2 - 1/N$ .*

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