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Remarks on some quasilinear equations with gradient terms and measure data

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Laurent VERON‡

Abstract

Let $\Omega \subset \mathbb{R}^N$ be a smooth bounded domain, H a Caratheodory function defined in $\Omega \times \mathbb{R} \times \mathbb{R}^N$, and μ a bounded Radon measure in Ω . We study the problem

$$-\Delta_p u + H(x, u, \nabla u) = \mu \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega,$$

where Δ_p is the p -Laplacian ($p > 1$), and we emphasize the case $H(x, u, \nabla u) = \pm |\nabla u|^q$ ($q > 0$). We obtain an existence result under subcritical growth assumptions on H , we give necessary conditions of existence in terms of capacity properties, and we prove removability results of eventual singularities. In the supercritical case, when $\mu \geq 0$ and H is an absorption term, i.e. $H \geq 0$, we give two sufficient conditions for existence of a nonnegative solution.

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1 Introduction

Let Ω be a smooth bounded domain in \mathbb{R}^N ($N \geq 2$). In this article we consider problems of the form

$$-\Delta_p u + H(x, u, \nabla u) = \mu \quad \text{in } \Omega, \quad (1.1)$$

where $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2} \nabla u)$ is the p -Laplace operator, with $1 < p \leq N$, H is a Caratheodory function defined in $\Omega \times \mathbb{R} \times \mathbb{R}^N$, and μ is a possibly signed Radon measure on Ω . We study the existence of solutions for the Dirichlet problem in Ω

$$-\Delta_p u + H(x, u, \nabla u) = \mu \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega, \quad (1.2)$$

and some questions of removability of the singularities. Our main motivation is the case where μ is nonnegative, H involves only ∇u , and either H is nonnegative, hence H is an absorption term, or H is nonpositive, hence H is a source one. The model cases are

$$-\Delta_p u + |\nabla u|^q = \mu \quad \text{in } \Omega, \quad (1.3)$$

where $q > 0$, for the absorption case and

$$-\Delta_p u = |\nabla u|^q + \mu \quad \text{in } \Omega. \quad (1.4)$$

for the source case.

The equations without gradient terms,

$$-\Delta_p u + H(x, u) = \mu \quad \text{in } \Omega, \quad (1.5)$$

such as the quasilinear Emden-Fowler equations

$$-\Delta_p u \pm |u|^{Q-1} u = \mu \quad \text{in } \Omega,$$

where $Q > 0$, have been the object of a huge literature when $p = 2$. In the general case $p > 1$, among many works we refer to [5], [6], [7] and the references therein, and to [8] for new recent results in the case of absorption.

We set

$$Q_c = \frac{N(p-1)}{N-p}, \quad q_c = \frac{N(p-1)}{N-1}, \quad (Q_c = \infty \text{ if } p = N), \quad \tilde{q} = p - 1 + \frac{p}{N} \quad (1.6)$$

(hence $q_c, \tilde{q} < p < N$ or $q_c = \tilde{q} = p = N$), and

$$q_* = \frac{q}{q+1-p}, \quad (1.7)$$

(thus $q_* = q'$ in case $p = 2$).

In Section 2 we recall the main notions of solutions of the problem $-\Delta_p u = \mu$, such as weak solutions, renormalized or locally renormalized solutions, and convergence results. In Section 3 we prove a general existence result for problem (1.2) in the subcritical case, see Theorem 3.1. Then in Section 4 we give necessary conditions for existence and removability results for the local solutions of problem (1.1), extending former results of [19] and [41], see Theorem 4.5. In Section 5 we study the problem (1.2) in the supercritical case, where many questions are still open. We give two partial results of existence in Theorems 5.5 and 5.8. Finally in Section 5 we make some remarks of regularity for the problem

$$-\Delta_p u + H(x, u, \nabla u) = 0 \quad \text{in } \Omega.$$

2 Notions of solutions

Let ω be any domain of \mathbb{R}^N . For any $r > 1$, the capacity $cap_{1,r}$ associated to $W_0^{1,r}(\omega)$ is defined by

$$cap_{1,r}(K, \omega) = \inf \left\{ \|\psi\|_{W_0^{1,r}(\omega)}^r : \psi \in \mathcal{D}(\omega), \chi_K \leq \psi \leq 1 \right\},$$

for any compact set $K \subset \omega$, and then the notion is extended to any Borel set in ω . In \mathbb{R}^N we denote by G_1 the Bessel kernel of order 1 (defined by $\widehat{G_1}(y) = (1 + |y|^2)^{-1/2}$), and we consider the Bessel capacity defined for any compact $K \subset \mathbb{R}^N$ by

$$Cap_{1,r}(K, \mathbb{R}^N) = \inf \left\{ \|f\|_{L^r(\mathbb{R}^N)}^r : f \geq 0, G_1 * f \geq \chi_K \right\}.$$

On \mathbb{R}^N the two capacities are equivalent, see [2].

We denote by $\mathcal{M}(\omega)$ the set of Radon measures in ω , and $\mathcal{M}_b(\omega)$ the subset of bounded measures, and define $\mathcal{M}^+(\omega)$, $\mathcal{M}_b^+(\omega)$ the corresponding cones of nonnegative measures. Any measure $\mu \in \mathcal{M}(\omega)$ admits a positive and a negative parts, denoted by μ^+ and μ^- . For any Borel set E , $\mu \llcorner E$ is the restriction of μ to E ; we say that μ is concentrated on E if $\mu = \mu \llcorner E$.

For any $r > 1$, we call $\mathcal{M}^r(\omega)$ the set of measures $\mu \in \mathcal{M}(\omega)$ which do not charge the sets of null capacity, that means $\mu(E) = 0$ for every Borel set $E \subset \omega$ with $cap_{1,r}(E, \omega) = 0$. Any measure concentrated on a set E with $cap_{1,r}(E, \omega) = 0$ is called r -singular. Similarly we define the subsets $\mathcal{M}_b^r(\omega)$ and $\mathcal{M}_b^{r+}(\omega)$.

For fixed $r > 1$, any measure $\mu \in \mathcal{M}(\omega)$ admits a unique decomposition of the form $\mu = \mu_0 + \mu_s$, where $\mu_0 \in \mathcal{M}^r(\omega)$, and $\mu_s = \mu_s^+ - \mu_s^-$ is r -singular. If $\mu \geq 0$, then $\mu_0 \geq 0$ and $\mu_s \geq 0$.

Remark 2.1 *Any measure $\mu \in \mathcal{M}_b(\omega)$ belongs to $\mathcal{M}^r(\omega)$ if and only if there exist $f \in L^1(\omega)$ and $g \in (L^{r'}(\omega))^N$ such that $\mu = f + \operatorname{div} g$, see [11, Theorem 2.1]. However this decomposition is not unique; if μ is nonnegative there exists a decomposition such that f is nonnegative, but one cannot ensure that $\operatorname{div} g$ is nonnegative.*

For any $k > 0$ and $s \in \mathbb{R}$, we define the truncation $T_k(s) = \max(-k, \min(k, s))$. If u is measurable and finite a.e. in ω , and $T_k(u)$ belongs to $W_0^{1,p}(\omega)$ for every $k > 0$, one can define the gradient ∇u a.e. in ω by $\nabla T_k(u) = \nabla u \cdot \chi_{\{|u| \leq k\}}$ for any $k > 0$.

For any $f \in \mathcal{M}^+(\mathbb{R}^N)$, we denote the Bessel potential of f by $J_1(f) = G_1 * f$.

2.1 Renormalized solutions

Let $\mu \in \mathcal{M}_b(\Omega)$. Let us recall some known results for the problem

$$-\Delta_p u = \mu \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega, \quad (2.1)$$

Under the assumption $p > 2 - 1/N$, from [9], problem (2.1) admits a solution $u \in W_0^{1,r}(\Omega)$ for every $r \in [1, q_c)$, satisfying the equation in $\mathcal{D}'(\Omega)$. When $p < 2 - 1/N$, then $q_c < 1$; this leads to introduce the concept of renormalized solutions developed in [16], see also [33], [45]. Here we recall one of their definitions, among four equivalent ones given in [16].

Definition 2.2 Let $\mu = \mu^0 + \mu_s \in \mathcal{M}_b(\Omega)$, where $\mu^0 \in \mathcal{M}^p(\Omega)$ and $\mu_s = \mu_s^+ - \mu_s^-$ is p -singular. A function u is a **renormalized solution**, called **R-solution** of problem (2.1), if u is measurable and finite a.e. in Ω , such that $T_k(u)$ belongs to $W_0^{1,p}(\Omega)$ for any $k > 0$, and $|\nabla u|^{p-1} \in L^\tau(\Omega)$, for any $\tau \in [1, N/(N-1))$; and for any $h \in W^{1,\infty}(\mathbb{R})$ such that h' has a compact support, and any $\varphi \in W^{1,s}(\Omega)$ for some $s > N$, such that $h(u)\varphi \in W_0^{1,p}(\Omega)$,

$$\int_{\Omega} |\nabla u|^{p-2} \nabla u \cdot \nabla (h(u)\varphi) dx = \int_{\Omega} h(u)\varphi d\mu_0 + h(\infty) \int_{\Omega} \varphi d\mu_s^+ - h(-\infty) \int_{\Omega} \varphi d\mu_s^-. \quad (2.2)$$

As a consequence, any R-solution u of problem (2.1) satisfies $|u|^{p-1} \in L^\sigma(\Omega)$, $\forall \sigma \in [1, N/(N-p)$. More precisely, u and $|\nabla u|$ belong to some Marcinkiewicz spaces

$$L^{s,\infty}(\Omega) = \left\{ u \text{ measurable in } \Omega : \sup_{k>0} k^s |\{x \in \Omega : |u(x)| > k\}| < \infty \right\},$$

see [9], [5], [16], [27], and one gets useful convergence properties, see [16, Theorem 4.1 and §5] for the proof:

Lemma 2.3 (i) Let $\mu \in \mathcal{M}_b(\Omega)$ and u be any R-solution of problem (2.1). Then for any $k > 0$,

$$\frac{1}{k} \int_{\{m \leq u \leq m+k\}} |\nabla u|^p dx \leq |\mu|(\Omega), \forall m \geq 0.$$

If $p < N$, then $u \in L^{Q_c, \infty}(\Omega)$ and $|\nabla u| \in L^{q_c, \infty}(\Omega)$,

$$|\{|u| \geq k\}| \leq C(N, p) k^{-Q_c} (|\mu|(\Omega))^{\frac{N}{N-p}}, \quad |\{|\nabla u| \geq k\}| \leq C(N, p) k^{-q_c} (|\mu|(\Omega))^{\frac{N}{N-1}}. \quad (2.3)$$

If $p = N$ (where u is unique), then for any $r > 1$ and $s \in (1, N)$,

$$|\{|u| \geq k\}| \leq C(N, p, r) k^{-r} (|\mu|(\Omega))^{\frac{r}{p-1}}, \quad |\{|\nabla u| \geq k\}| \leq C(N, p, s) k^{-N} (|\mu|(\Omega))^{\frac{s}{N-1}}. \quad (2.4)$$

(ii) Let (μ_n) be a sequence of measures $\mu_n \in \mathcal{M}_b(\Omega)$, uniformly bounded in $\mathcal{M}_b(\Omega)$, and u_n be any R-solution of

$$-\Delta_p u_n = \mu_n \quad \text{in } \Omega, \quad u_n = 0 \quad \text{on } \partial\Omega.$$

Then there exists a subsequence (μ_ν) such that (u_ν) converges a.e. in Ω to a function u , such that $T_k(u) \in W_0^{1,p}(\Omega)$, and $(T_k(u_\nu))$ converges weakly in $W_0^{1,p}(\Omega)$ to $T_k(u)$, and (∇u_ν) converges a.e. in Ω to ∇u .

Remark 2.4 These properties do not require any regularity of Ω . If $\mathbb{R}^N \setminus \Omega$ is geometrically dense, i.e. there exists $c > 0$ such that $|B(x, r) \setminus \Omega| \geq cr^N$ for any $x \in \mathbb{R}^N \setminus \Omega$ and $r > 0$, then (2.4) holds with $s = N$, and C depends also on the geometry of Ω . Then $|\nabla u| \in L^{N, \infty}(\Omega)$, hence $u \in BMO(\Omega)$, see [17], [27].

Next we recall the fundamental stability result of [16, Theorem 3.1]:

Definition 2.5 For any measure $\mu = \mu^0 + \mu_s^+ - \mu_s^- \in \mathcal{M}_b(\Omega)$, where $\mu^0 = f - \operatorname{div}g \in \mathcal{M}^p(\Omega)$, and μ_s^+, μ_s^- are p -singular we say that a sequence (μ_n) is a **good approximation** of μ in $\mathcal{M}_b(\Omega)$ if it can be decomposed as

$$\mu_n = \mu_n^0 + \lambda_n - \eta_n, \quad \text{with} \quad \mu_n^0 = f_n - \operatorname{div}g_n, \quad f_n \in L^1(\Omega), \quad g_n \in (L^{p'}(\Omega))^N, \quad \lambda_n, \eta_n \in \mathcal{M}_b^+(\Omega), \quad (2.5)$$

such that (f_n) converges to f weakly in $L^1(\Omega)$, (g_n) converges to g strongly in $(L^{p'}(\Omega))^N$ and $(\operatorname{div}g_n)$ is bounded in $\mathcal{M}_b(\Omega)$, and (ρ_n) converges to μ_s^+ and (η_n) converges to μ_s^- in the narrow topology.

Theorem 2.6 ([16]) Let $\mu \in \mathcal{M}_b(\Omega)$, and let (μ_n) be a good approximation of μ . Let u_n be a R -solution of

$$-\Delta_p u_n = \mu_n \quad \text{in } \Omega, \quad u_n = 0 \quad \text{on } \partial\Omega.$$

Then there exists a subsequence (u_ν) converging a.e. in Ω to a R -solution u of problem (2.1). And $(T_k(u_\nu))$ converges to $T_k(u)$ strongly in $W_0^{1,p}(\Omega)$.

Remark 2.7 As a consequence, for any measure $\mu \in \mathcal{M}_b(\Omega)$, there exists at least a solution of problem (2.1). Indeed, it is pointed out in [16] that any measure $\mu \in \mathcal{M}_b(\Omega)$ can be approximated by such a sequence: extending μ by 0 to \mathbb{R}^N , one can take $g_n = g$, $f_n = \rho_n * f$, $\lambda_n = \rho_n * \mu_s^+$, $\eta_n = \rho_n * \mu_s^-$, where (ρ_n) is a regularizing sequence; then $f_n, \lambda_n, \eta_n \in C_b^\infty(\Omega)$. Notice that this approximation does not respect the sign: $\mu \in \mathcal{M}_b^+(\Omega)$ does not imply that $\mu_n \in \mathcal{M}_b^+(\Omega)$.

In the sequel we precise the approximation property, still partially used in [20, Theorem 2.18] for problem (1.5).

Lemma 2.8 Let $\mu \in \mathcal{M}_b(\Omega)$. Then

(i) there exists a sequence (μ_n) of good approximations of μ , such $\mu_n \in W^{-1,p'}(\Omega)$, and μ_n^0 has a compact support in Ω , $\lambda_n, \eta_n \in C_b^\infty(\Omega)$, (f_n) converges to f strongly in $L^1(\Omega)$, and

$$|\mu_n|(\Omega) \leq 4|\mu|(\Omega), \quad \forall n \in \mathbb{N} \quad (2.6)$$

Moreover, if $\mu \in \mathcal{M}_b^+(\Omega)$, then one can find the approximation such that $\mu_n \in \mathcal{M}_b^+(\Omega)$ and (μ_n) is nondecreasing.

(ii) there exists another sequence (μ_n) of good approximations of μ , with $f_n, g_n \in \mathcal{D}(\Omega)$, $\lambda_n, \eta_n \in C_b^\infty(\Omega)$, such that (f_n) converges to f strongly in $L^1(\Omega)$, satisfying (2.6); if $\mu \in \mathcal{M}_b^+(\Omega)$, one can take $\mu_n^0 \in \mathcal{D}^+(\Omega)$.

Proof. (i) Let $\mu = \mu^0 + \mu_s^+ - \mu_s^-$, where $\mu^0 \in \mathcal{M}^p(\Omega)$, μ_s^+, μ_s^- are p -singular and $\mu_1 = (\mu^0)^+, \mu_2 = (\mu^0)^-$; thus $\mu_1(\Omega) + \mu_2(\Omega) + \mu_s^+(\Omega) + \mu_s^-(\Omega) \leq 2|\mu(\Omega)|$. Following [11], for $i = 1, 2$, one has

$$\mu_i = \varphi_i \gamma_i, \quad \text{with } \gamma_i \in \mathcal{M}_b^+(\Omega) \cap W^{-1,p'}(\Omega) \text{ and } \varphi_i \in L^1(\Omega, \gamma_i).$$

Let $(K_n)_{n \geq 1}$ be an increasing sequence of compacts of union Ω ; set

$$\nu_{1,i} = T_1(\varphi_i \chi_{K_1}) \gamma_i, \quad \nu_{n,i} = T_n(\varphi_i \chi_{K_n}) \gamma_i - T_{n-1}(\varphi_i \chi_{K_{n-1}}) \gamma_i, \quad \mu_{n,i}^0 = \sum_1^n \nu_{n,i} = T_n(\varphi_i \chi_{K_n}) \gamma_i.$$

Thus $\mu_{n,i}^0 \in \mathcal{M}_b^+(\Omega) \cap W^{-1,p'}(\Omega)$. Regularizing by (ρ_n) , there exists $\phi_{n,i} \in \mathcal{D}^+(\Omega)$ such that $\|\phi_{n,i} - \nu_{n,i}\|_{W^{-1,p'}(\Omega)} \leq 2^{-n} \mu_i(\Omega)$. Then $\xi_{n,i} = \sum_1^n \phi_{k,i} \in \mathcal{D}^+(\Omega)$; $(\eta_{n,i})$ converges strongly in $L^1(\Omega)$ to a function ξ_i and $\|\xi_{n,i}\|_{L^1(\Omega)} \leq \mu_i(\Omega)$. Also setting

$$G_{n,i} = \mu_{n,i}^0 - \xi_{n,i} = \sum_1^n (\nu_{n,i} - \phi_{k,i}) \in W^{-1,p'}(\Omega) \cap \mathcal{M}_b(\Omega),$$

then $(G_{n,i})$ converges strongly in $W^{-1,p'}(\Omega)$ to some G_i , and $\mu_i = \xi_i + G_i$, and $\|G_{n,i}\|_{\mathcal{M}_b(\Omega)} \leq 2\mu_i(\Omega)$. Otherwise $\lambda_n = \rho_n * \mu_s^+$ and $\eta_n = \rho_n * \mu_s^- \in C_b^\infty(\Omega)$ converge respectively to μ_s^+, μ_s^- in the narrow topology, with $\|\lambda_n\|_{L^1(\Omega)} \leq \mu_s^+(\Omega)$, $\|\eta_n\|_{L^1(\Omega)} \leq \mu_s^-(\Omega)$. Then we set

$$\mu_n = \mu_n^0 + \rho_n - \eta_n \quad \text{with } \mu_n^0 = \xi_n + G_n, \quad \xi_n = \xi_{n,1} - \xi_{n,2} \in \mathcal{D}(\Omega), \quad G_n = G_{n,1} - G_{n,2} \in W^{-1,p'}(\Omega)$$

thus μ_n^0 has a compact support. Moreover $\mu_0 = \xi + G$ with $\xi = \xi_1 - \xi_2 \in \mathcal{D}(\Omega)$, and $G = G_1 - G_2 = \varphi + \text{div}g$ for some $\varphi \in L^{p'}(\Omega)$ and $g \in (L^{p'}(\Omega))^N$, and (G_n) converges to G in $W^{-1,p'}(\Omega)$. Then we can find $\psi_n \in L^{p'}(\Omega)$, $\phi_n \in (L^{p'}(\Omega))^N$, such that $G_n - G = \psi_n + \text{div}\phi_n$ and $\|G_n - G\|_{W^{-1,p'}(\Omega)} = \max(\|\psi_n\|_{L^{p'}(\Omega)}, \|\phi_n\|_{(L^{p'}(\Omega))^N})$; then $\mu_0 = f + \text{div}g$ with $f = \xi + \varphi$ and $\mu_n^0 = f_n + \text{div}g_n$, with $f_n = \xi_n + \varphi + \psi_n$, $g_n = g + \phi_n$. Thus (μ_n) is a good approximation of μ , and satisfies (2.6). If μ is nonnegative, then μ_n is nonnegative.

(ii) We replace μ_n^0 by $\rho_m * \mu_n^0 = \rho_m * f_n + \text{div}(\rho_m * g_n)$, $m \in \mathbb{N}$, and observe that $|\rho_m * \mu_n^0|(\Omega) \leq |\mu_n^0|(\Omega)$; then we can construct another sequence satisfying the conditions. \blacksquare

2.2 Locally renormalized solutions

Let $\mu \in \mathcal{M}(\Omega)$. Following the notion introduced in [6], we say that u is a **locally renormalized solution**, called **LR-solution**, of problem

$$-\Delta_p u = \mu, \quad \text{in } \Omega, \quad (2.7)$$

if u is measurable and finite a.e. in Ω , $T_k(u) \in W_{loc}^{1,p}(\Omega)$ for any $k > 0$, and

$$|u|^{p-1} \in L_{loc}^\sigma(\Omega), \forall \sigma \in [1, N/(N-p)]; \quad |\nabla u|^{p-1} \in L_{loc}^\tau(\Omega), \forall \tau \in [1, N/(N-1)], \quad (2.8)$$

and for any $h \in W^{1,\infty}(\mathbb{R})$ such that h' has a compact support, and $\varphi \in W^{1,m}(\Omega)$ for some $m > N$, with compact support, such that $h(u)\varphi \in W^{1,p}(\Omega)$, there holds

$$\int_\Omega |\nabla u|^{p-2} \nabla u \cdot \nabla (h(u)\varphi) dx = \int_\Omega h(u)\varphi d\mu_0 + h(+\infty) \int_\Omega \varphi d\mu_s^+ - h(-\infty) \int_\Omega \varphi d\mu_s^-. \quad (2.9)$$

Remark 2.9 Hence the LR-solutions are solutions in $\mathcal{D}'(\Omega)$. From a recent result of [28], if $\mu \in \mathcal{M}^+(\Omega)$, any p -superharmonic function is a LR-solution, and reciprocally any LR-solution admits a p -superharmonic representant.

3 Existence in the subcritical case

We first give a general existence result, where H satisfies some subcritical growth assumptions on u and ∇u , without any assumption on the sign of H or μ : we consider the problem

$$-\Delta_p u + H(x, u, \nabla u) = \mu \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega, \quad (3.1)$$

where $\mu \in \mathcal{M}_b(\Omega)$. We say that u is a R-solution of problem (1.2) $T_k(u) \in W_0^{1,p}(\Omega)$ for any $k > 0$, and $H(x, u, \nabla u) \in L^1(\Omega)$ and u is a R-solution of

$$-\Delta_p u = \mu - H(x, u, \nabla u), \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega.$$

Theorem 3.1 *Let $\mu \in \mathcal{M}_b(\Omega)$, and assume that*

$$|H(x, u, \xi)| \leq f(x) |u|^Q + g(x) |\xi|^q + \ell(x) \quad (3.2)$$

with $Q, q > 0$ and $f \in L^r(\Omega)$ with $Qr' < Q_c$, $g \in L^s(\Omega)$ with $qs' < q_c$, and $\ell \in L^1(\Omega)$.

Then there exists a R-solution of (3.1) if, either $\max(Q, q) > p - 1$ and $|\mu|(\Omega)$ and $\|\ell\|_{L^1(\Omega)}$ are small enough, or $q = p - 1 > Q$ and $\|f\|_{L^r(\Omega)}$ is small enough, or $Q = p - 1 > q$ and $\|g\|_{L^s(\Omega)}$ is small enough, or $q, Q < p - 1$.

Proof. (i) Construction of a sequence of approximations. We consider a sequence $(\mu_n)_{n \geq 1}$ of good approximations of μ , given in Lemma 2.8 (i). For any fixed $n \in \mathbb{N}^*$, and any $v \in W_0^{1,p}(\Omega)$ we define

$$M(v) = |\Omega|^{\frac{N-p}{N} - \frac{p-1}{Qr'}} \left(\int_{\Omega} |v|^{Qr'} dx \right)^{\frac{p-1}{Qr'}} + |\Omega|^{\frac{N-1}{N} - \frac{p-1}{qs'}} \left(\int_{\Omega} |\nabla v|^{qs'} dx \right)^{\frac{p-1}{qs'}},$$

$$\Phi_n(v)(x) = - \frac{H(x, v(x), \nabla v(x))}{1 + \frac{1}{n}(f(x) |v(x)|^Q + g(x) |\nabla v(x)|^q + \ell(x))}$$

so that $|\Phi_n(v)(x)| \leq n$ a.e. in Ω . Let $\lambda > 0$ be a parameter. Starting from $u_1 \in W_0^{1,p}(\Omega)$ such that $M(u_1) \leq \lambda$, we define $u_2 \in W_0^{1,p}(\Omega)$ as the solution of the problem

$$-\Delta_p u_2 = \Phi_1(u_1) + \mu_1 \quad \text{in } \Omega, \quad u_2 = 0 \quad \text{on } \partial\Omega,$$

and by induction we define $u_n \in W_0^{1,p}(\Omega)$ as the solution of

$$-\Delta_p u_n = \Phi_{n-1}(u_{n-1}) + \mu_n \quad \text{in } \Omega, \quad u_n = 0 \quad \text{on } \partial\Omega.$$

From (2.3), for any $\sigma \in (0, N/(N-p))$ and $\tau \in (0, N/(N-1))$,

$$|\Omega|^{\frac{N-p}{N} - \frac{1}{\sigma}} \left(\int_{\Omega} |u_n|^{(p-1)\sigma} dx \right)^{\frac{1}{\sigma}} + |\Omega|^{\frac{N-1}{N} - \frac{1}{\tau}} \left(\int_{\Omega} |\nabla u_n|^{(p-1)\tau} dx \right)^{\frac{1}{\tau}} \leq C \left(\int_{\Omega} |\Phi_{n-1}(u_{n-1})| dx + 4 |\mu|(\Omega) \right),$$

with $C = C(N, p, \sigma, \tau)$. We take $\sigma = Qr'/(p-1)$ and $\tau = qs'/(p-1)$; since

$$\int_{\Omega} |H(x, u_{n-1}, \nabla u_{n-1})| dx \leq \|f\|_{L^r(\Omega)} \int_{\Omega} |u_{n-1}|^{Qr'} dx^{\frac{1}{r'}} + \|g\|_{L^s(\Omega)} \left(\int_{\Omega} |\nabla u_{n-1}|^{qs'} dx \right)^{\frac{1}{s'}} + \|\ell\|_{L^1(\Omega)} \quad (3.3)$$

we obtain

$$M(u_n) \leq C \left(\int_{\Omega} |H(x, u_{n-1}, \nabla u_{n-1})| dx + 4 |\mu|(\Omega) \right) \leq b_1 M(u_{n-1})^{Q/(p-1)} + b_2 M(u_{n-1})^{q/(p-1)} + \eta + a$$

with $C = C(N, p, q, Q)$, and $b_1 = C \|f\|_{L^r(\Omega)} |\Omega|^{\frac{1}{r} - \frac{Q}{Q_c}}$, $b_2 = C \|g\|_{L^s(\Omega)} |\Omega|^{1/s' - q/q_c}$, $\eta = C \|\ell\|_{L^1(\Omega)}$, $a = 4C |\mu|(\Omega)$. Then by induction, $M(u_n) \leq \lambda$ for any $n \geq 1$ if

$$b_1 \lambda^{Q/(p-1)} + b_2 \lambda^{q/(p-1)} + \eta + a \leq \lambda. \quad (3.4)$$

When $Q < p - 1$ and $q < p - 1$, (3.4) holds for λ large enough. In the other cases, we note that it holds as soon as

$$b_1 \lambda^{Q/(p-1)-1} + b_2 \lambda^{q/(p-1)-1} \leq 1/2, \quad \text{and } \eta \leq \lambda/4, \quad a \leq \lambda/4. \quad (3.5)$$

First suppose that $Q > p - 1$ or $q > p - 1$. We take $\lambda \leq 1$, small enough so that $(b_1^{Q/(p-1)} + b_2^{q/(p-1)}) \lambda^{\max(Q, q)/(p-1)-1} \leq 1/2$, and then $\eta, a \leq \lambda/4$. Next suppose for example that $Q = p - 1 > q$, a is arbitrary. If b_1 small enough, and η, a are arbitrary, then we obtain (3.5) for λ large enough.

(ii) Convergence: Since $M(u_n) \leq \lambda$, in turn from (3.3), $(H(x, u_n, \nabla u_n))$ is bounded in $L^1(\Omega)$, and then also $\Phi_n(u_n)$. Thus

$$\int_{\Omega} |\Phi_{n-1}(u_{n-1})| dx + |\mu_n|(\Omega) \leq C_{\lambda} := b_1 \lambda^{Q/(p-1)} + b_2 \lambda M^{q/(p-1)} + \eta + 4 |\mu|(\Omega).$$

From Lemma 2.3, up to a subsequence, (u_n) converges *a.e.* to a function u , (∇u_n) converges *a.e.* to ∇u , and (u_n^{p-1}) converges strongly in $L^{\sigma}(\Omega)$, for any $\sigma \in [1, N/(N - p))$, and finally $(|\nabla u_n|^{p-1})$ converges strongly in $L^{\tau}(\Omega)$, for any $\tau \in [1, N/(N - 1))$. Therefore $(u_n^{Qr'})$ and $(|\nabla u_n|^{qr'})$ converge strongly in $L^1(\Omega)$, in turn $(\Phi_n(x, u_n, \nabla u_n))$ converges strongly to $H(x, u, \nabla u)$ in $L^1(\Omega)$. Then $(\Phi_n(x, u_n, \nabla u_n) + \mu_n)$ is a sequence of good approximations of $H(x, u, \nabla u) + \mu$. From Theorem 2.6, u is a R-solution of problem (3.1). \blacksquare

Remark 3.2 *Our proof is not based on the Schauder fixed point theorem, so we do not need that $1 \leq Qr'$ or $1 \leq qs'$. Hence we improve the former result of [20] for problem (1.5) where H only depends on u , proved for $1 \leq Qr'$, implying $1 < Q_c$. Here we have no restriction on Q_c and q_c .*

Next we consider the case where H and μ are nonnegative; then we do not need that the data are small:

Theorem 3.3 *Consider the problem (3.1)*

$$-\Delta_p u + H(x, u, \nabla u) = \mu \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega, \quad (3.6)$$

where $\mu \in \mathcal{M}_b^+(\Omega)$, and

$$0 \leq H(x, u, \xi) \leq C(|u|^Q + |\xi|^q) + \ell(x) \quad (3.7)$$

with $0 < Q < Q_c, 0 < q < q_c, C > 0, \ell \in L^1(\Omega)$. Then there exists a nonnegative R-solution of problem (3.6).

Proof. We use the good approximation of μ by a sequence of measures $\mu_n = \mu_n^0 + \lambda_n$, with $\mu_n^0 \in \mathcal{D}^+(\Omega)$, $\lambda_n \in C_b^+(\Omega)$, given at Lemma 2.8 (ii). Then there exists a weak nonnegative solution $u_n \in W_0^{1,p}(\Omega)$ of the problem

$$-\Delta_p u_n + H(x, u_n, \nabla u_n) = \mu_n \quad \text{in } \Omega, \quad u_n = 0 \quad \text{on } \partial\Omega.$$

Indeed 0 is a subsolution, and the solution $\psi_n \in W_0^{1,p}(\Omega)$ of $-\Delta_p \psi_n = \mu_n$ in Ω , is a supersolution. Since $\mu_n \in L^\infty(\Omega)$, there holds $\psi \in C^{1,\alpha}(\overline{\Omega})$ for some $\alpha \in (0, 1)$, thus $\psi \in W^{1,\infty}(\Omega)$. From [12, Theorem 2.1], since $Q_c \leq p$, there exists a weak solution $u_n \in W_0^{1,p}(\Omega)$, such that $0 \leq u_n \leq \psi_n$, hence $u_n \in L^\infty(\Omega)$, and $u_n \in W_{loc}^{1,r}(\Omega)$ for some $r > p$. Taking $\varphi = k^{-1}T_k(u_n - m)$ with $m \geq 0$, $k > 0$, as a test function, we get from (2.6)

$$\frac{1}{k} \int_{\{m \leq u \leq m+k\}} |\nabla u_n|^p dx \leq \mu_n(\Omega) \leq 4\mu(\Omega), \quad (3.8)$$

then from Lemma 2.3, up to a subsequence, (u_n) converges *a.e.* to a function u , $(T_k(u_n))$ converges weakly in $W_0^{1,p}(\Omega)$ and (∇u_n) converges *a.e.* to ∇u , and $(|\nabla u_n|^p)$, (u_n^{p-1}) converges strongly in $L^\sigma(\Omega)$ for any $\sigma \in [1, N/(N-p))$, $(|\nabla u_n|^{p-1})$ converges strongly in $L^\tau(\Omega)$, for any $\tau \in [1, N/(N-1))$. Then $(u_n^{Qr'})$ and $(|\nabla u_n|^{qr'})$ converge strongly in $L^1(\Omega)$, in turn $(H(x, u_n, \nabla u_n))$ converges strongly to $H(x, u, \nabla u)$ in $L^1(\Omega)$. Applying Theorem 2.6 to $\mu_n - H(x, u_n, \nabla u_n)$ as above, we still obtain that u is a R-solution of (3.6). \blacksquare

4 Necessary conditions for existence and removability results

Let $\mu \in \mathcal{M}(\Omega)$. We consider the local solutions of

$$-\Delta_p u + H(x, u, \nabla u) = \mu \quad \text{in } \Omega, \quad (4.1)$$

We say that u is a **weak solution** of (4.1) if u is measurable and finite a.e. in Ω , $T_k(u) \in W_{loc}^{1,p}(\Omega)$ for any $k > 0$, $H(x, u, \nabla u) \in L_{loc}^1(\Omega)$ and (4.1) holds in $\mathcal{D}'(\Omega)$. We say that u is a **LR-solution** of (4.1) if $T_k(u) \in W_{loc}^{1,p}(\Omega)$ for any $k > 0$, and $|\nabla u|^q \in L_{loc}^1(\Omega)$ and u is a LR-solution of

$$-\Delta_p u = \mu - H(x, u, \nabla u), \quad \text{in } \Omega.$$

Remark 4.1 If $q \geq 1$ and u is a weak solution, then u satisfies (2.8), see for example [31, Lemma 2.2 and 2.3], thus $u \in W_{loc}^{1,q}(\Omega)$.

Lemma 4.2 Let $\mu \in \mathcal{M}(\Omega)$. Assume that (4.1) admits a weak solution u .

(i) If

$$|H(x, u, \xi)| \leq C_1 |\xi|^q + \ell(x) \quad (4.2)$$

with $C_1 > 0$ and $\ell \in L^1(\Omega)$, then setting $C_2 = C_1 + q_* - 1$, for any $\zeta \in \mathcal{D}^+(\Omega)$,

$$\left| \int_{\Omega} \zeta^{q_*} d\mu \right| \leq C_2 \int_{\Omega} |\nabla u|^q \zeta^{q_*} dx + \int_{\Omega} |\nabla \zeta|^{q_*} dx + \int_{\Omega} \ell \zeta^{q_*} dx. \quad (4.3)$$

(ii) If H has a constant sign, and

$$C_0 |\xi|^q - \ell(x) \leq |H(x, u, \xi)|, \quad (4.4)$$

then for some $C = C(C_0, p, q)$,

$$\int_{\Omega} |\nabla u|^q \zeta^{q_*} dx \leq C \left(\left| \int_{\Omega} \zeta^{q_*} d\mu \right| + \int_{\Omega} |\nabla \zeta|^{q_*} dx + \int_{\Omega} \ell \zeta^{q_*} dx \right) \quad (4.5)$$

Proof. By density, we can take ζ^{q_*} as a test function, and get

$$\int_{\Omega} \zeta^{q_*} d\mu = - \int_{\Omega} H(x, u, \nabla u) \zeta^{q_*} dx + q_* \int_{\Omega} |\nabla u|^{p-2} \nabla u \cdot \zeta^{q_*-1} \nabla \zeta dx$$

and from the Hölder inequality, for any $\varepsilon > 0$,

$$q_* \int_{\Omega} |\nabla u|^{p-1} \zeta^{q_*-1} |\nabla \zeta| dx \leq (q_* - 1) \varepsilon \int_{\Omega} |\nabla u|^q \zeta^{q_*} dx + \varepsilon^{1-q_*} \int_{\Omega} |\nabla \zeta|^{q_*} dx \quad (4.6)$$

which implies (4.3). If H has a constant sign, then

$$\begin{aligned} C_0 \int_{\Omega} |\nabla u|^q \zeta^{q_*} dx - \int_{\Omega} \ell dx &\leq \int_{\Omega} |H(x, u, \nabla u)| \zeta^{q_*} dx = \left| \int_{\Omega} H(x, u, \nabla u) \zeta^{q_*} dx \right| \\ &\leq \left| \int_{\Omega} \zeta^{q_*} d\mu \right| + q_* \int_{\Omega} |\nabla u|^{p-1} \zeta^{q_*-1} |\nabla \zeta| dx, \end{aligned}$$

thus (4.5) follows after taking ε small enough. ■

Proposition 4.3 *Let $\mu \in \mathcal{M}(\Omega)$, and assume that (4.1) admits a weak solution u .*

(i) *If (4.2) holds, then $\mu \in \mathcal{M}^{q_*}(\Omega)$.*

(ii) *If $H(x, u, \xi) \leq -C_0 |\xi|^q$ and μ and u are nonnegative, then in addition there exists $C = C(C_0, p, q) > 0$ such that for any compact $K \subset \Omega$,*

$$\mu(K) \leq C \text{cap}_{1, q_*}(K, \Omega). \quad (4.7)$$

Proof. (i) Let E be a Borel set such that $\text{cap}_{1, q_*}(E, \Omega) = 0$. There exist two measurable disjoint sets A, B such that $\Omega = A \cup B$ and $\mu^+(B) = \mu^-(A) = 0$. Let us show that $\mu^+(A \cap E) = 0$. Let K be any fixed compact set in $A \cap E$. Since $\mu^-(K) = 0$, for any $\delta > 0$ there exists a regular domain $\omega \subset\subset \Omega$ containing K , such that $\mu^-(\omega) < \delta$. Then there exists $\zeta_n \in \mathcal{D}(\omega)$ such that $0 \leq \zeta_n \leq 1$, and $\zeta_n = 1$ on a neighborhood of K contained in ω , and (ζ_n) converges to in $W^{1, q_*}(\mathbb{R}^N)$ and *a.e.* in Ω , see [2]. There holds

$$\mu^+(K) \leq \int_{\omega} \zeta_n^{q_*} d\mu^+ = \int_{\omega} \zeta_n^{q_*} d\mu + \int_{\omega} \zeta_n^{q_*} d\mu^- \leq \int_{\omega} \zeta_n^{q_*} d\mu + \delta$$

and from (4.3),

$$\left| \int_{\Omega} \zeta_n^{q_*} d\mu \right| \leq C_2 \int_{\Omega} |\nabla u|^q \zeta_n^{q_*} dx + \int_{\Omega} |\nabla \zeta_n|^{q_*} dx + \int_{\Omega} \ell \zeta_n^{q_*} dx$$

And $\lim_{n \rightarrow \infty} \int_{\Omega} |\nabla u|^q \zeta_n^{q^*} dx = 0$, from the dominated convergence theorem, thus $|\int_{\Omega} \zeta_n^{q^*} d\mu| \leq \delta$ for large n ; then $\mu^+(K) \leq 2\delta$ for any $\delta > 0$, thus $\mu^+(K) = 0$, hence $\mu^+(A \cap E) = 0$; similarly we get $\mu^-(B \cap E) = 0$, hence $\mu(E) = 0$.

(ii) Here we find

$$\int_{\Omega} \zeta^{q^*} d\mu + C_0 \int_{\Omega} |\nabla u|^q \zeta^{q^*} dx \leq q_* \int_{\Omega} |\nabla u|^{p-2} \nabla u \cdot \zeta^{q^*-1} \nabla \zeta dx$$

and hence from (4.6) with $\varepsilon > 0$ small enough, for some $C = C(C_0, p, q)$,

$$\int_{\Omega} \zeta^{q^*} d\mu \leq C \int_{\Omega} |\nabla \zeta|^{q^*} dx$$

and (4.7) follows, see [34]. ■

Remark 4.4 Property (ii) extends the results of [19] and [41, Theorem 3.1] for equation (1.4).

Next we show a removability result:

Theorem 4.5 Assume that H has a constant sign and satisfies (4.2) and (4.4). Let F be any relatively closed subset of Ω , such that $\text{cap}_{1, q^*}(F, \mathbb{R}^N) = 0$, and $\mu \in \mathcal{M}^{q^*}(\Omega)$.

(i) Let $1 < q \leq p$. Let u be any LR-solution of

$$-\Delta_p u + H(x, u, \nabla u) = \mu \quad \text{in } \Omega \setminus K \quad (4.8)$$

Then u is a LR-solution of

$$-\Delta_p u + H(x, u, \nabla u) = \mu \quad \text{in } \Omega. \quad (4.9)$$

(ii) Let $q > p$ and u be a weak solution of (4.8), then u is a weak solution of (4.9).

Proof. (i) Let $1 < q \leq p$. From our assumption, $T_k(u) \in W_{loc}^{1,p}(\Omega \setminus F)$, for any $k > 0$, and $|u|^{p-1} \in L_{loc}^{\sigma}(\Omega)$, for any $\sigma \in [1, N/(N-p))$, and $|\nabla u|^{p-1} \in L_{loc}^{\tau}(\Omega \setminus F)$, for any $\tau \in [1, N/(N-1))$, and $|\nabla u|^q \in L_{loc}^1(\Omega \setminus F)$. For any compact $K \subset \Omega$, there holds $\text{cap}_{1,p}(F \cap K, \mathbb{R}^N) = 0$, because $p \leq q_*$, thus $T_k(u) \in W_{loc}^{1,p}(\Omega)$, see [24, Theorem 2.44]. And u is measurable on Ω and finite a.e. in Ω , thus we can define ∇u a.e. in Ω by the formula $\nabla u(x) = \nabla T_k(u)(x)$ a.e. on the set $\{x \in \Omega : |u(x)| \leq k\}$.

Let us consider a fixed function $\zeta \in \mathcal{D}^+(\Omega)$ and let $\omega \subset\subset \Omega$ such that $\text{supp} \zeta \subset \omega$ and set $K_{\zeta} = F \cap \text{supp} \zeta$. Then K_{ζ} is a compact and $\text{cap}_{1, q^*}(K, \mathbb{R}^N) = 0$. Thus there exists $\zeta_n \in \mathcal{D}(\omega)$ such that $0 \leq \zeta_n \leq 1$, and $\zeta_n = 1$ on a neighborhood of K contained in ω , and (ζ_n) converges to 0 in $W^{1, q^*}(\mathbb{R}^N)$; we can assume that the convergence holds everywhere on $\mathbb{R}^N \setminus N$, where $\text{cap}_{1, q^*}(N, \mathbb{R}^N) = 0$, see for example [4, Lemmas 2.1, 2.2]. From Lemma 4.2 applied to $\xi_n = \zeta(1 - \zeta_n)$ in $\Omega \setminus F$, we have

$$\begin{aligned} \int_{\Omega} |\nabla u|^q \xi_n^{q^*} dx &\leq C \left(\int_{\Omega} \xi_n^{q^*} d|\mu| + \int_{\Omega} |\nabla \xi_n|^{q^*} dx + \int_{\Omega} \ell \xi_n^{q^*} dx \right) \\ &\leq C \left(\int_{\Omega} \zeta^{q^*} d|\mu| + \int_{\Omega} |\nabla \zeta|^{q^*} dx + \int_{\Omega} |\nabla \zeta_n|^{q^*} dx + \int_{\Omega} \ell \zeta^{q^*} dx \right). \end{aligned} \quad (4.10)$$

Then from the Fatou Lemma, we get $|\nabla u|^q \zeta^{q^*} \in L^1(\Omega)$ and

$$\int_{\Omega} |\nabla u|^q \zeta^{q^*} dx \leq C_{\zeta} := C \left(\int_{\Omega} \zeta^{q^*} d|\mu| + \int_{\Omega} |\nabla \zeta|^{q^*} dx + \int_{\Omega} \ell \zeta^{q^*} dx \right), \quad (4.11)$$

where C_ζ also depends on ζ . Taking as test function $T_k(u)\xi_n^{q^*}$, we obtain

$$\begin{aligned} & \int_{\Omega} |\nabla(T_k(u))|^p \xi_n^{q^*} dx + \int_{\Omega} H(x, u, \nabla u) T_k(u) \xi_n^{q^*} dx \\ &= \int_{\Omega} T_k(u) \xi_n^{q^*} d\mu_0 + k \left(\int_{\Omega} \xi_n^{q^*} d\mu_s^+ + \int_{\Omega} \xi_n^{q^*} d\mu_s^- \right) + \int_{\Omega} T_k(u) |\nabla u|^{p-2} \nabla u \cdot \nabla(\xi_n^{q^*}) dx; \end{aligned}$$

From the Hölder inequality, we deduce

$$\begin{aligned} & \frac{1}{k} \left| \int_{\Omega} T_k(u) |\nabla u|^{p-2} \nabla u \cdot \nabla(\xi_n^{q^*}) dx \right| \\ & \leq q_* \left(\int_{\Omega} \zeta^{q^*-1} |\nabla u|^{p-1} |\nabla \zeta| dx + \int_{\Omega} \zeta^{q^*} |\nabla u|^{p-1} |\nabla \zeta_n| dx \right) \\ & \leq (q_* - 1) \int_{\Omega} |\nabla u|^q \zeta^{q^*} dx + \int_{\Omega} |\nabla \zeta|^{q^*} dx + q_* \left(\int_{\Omega} |\nabla u|^q \zeta^{q^*} dx + \int_{\Omega} \zeta^{q^*} |\nabla \zeta_n|^{q^*} dx \right) \\ & \leq 2q_* C_\zeta + \int_{\Omega} |\nabla \zeta|^{q^*} dx + o(n). \end{aligned}$$

Thus from (4.2), with a new constant C_ζ ,

$$\int_{\Omega} |\nabla(T_k(u))|^p \xi_n^{q^*} dx \leq (k+1)C_\zeta + o(n);$$

hence from the Fatou Lemma,

$$\int_{\Omega} |\nabla(T_k(u))|^p \zeta^{q^*} dx \leq (k+1)C_\zeta.$$

Therefore $|u|^{p-1} \in L_{loc}^\sigma(\Omega)$, $\forall \sigma \in [1, N/(N-p))$ and $|\nabla u|^{p-1} \in L_{loc}^\tau(\Omega)$, $\forall \tau \in [1, N/(N-1))$, from a variant of the estimates of [5] and [10], see [39, Lemma 3.1].

Finally we show that u is a LR-solution in Ω : let $h \in W^{1,\infty}(\mathbb{R})$ such that h' has a compact support, and $\varphi \in W^{1,m}(\Omega)$ for some $m > N$, with compact support in Ω , such that $h(u)\varphi \in W^{1,p}(\Omega)$; let $\omega \subset\subset \Omega$ such that $\text{supp} \zeta \subset \omega$ and set $K = F \cap \text{supp} \zeta$, and consider $\zeta_n \in \mathcal{D}(\mathbb{R}^N)$ as above; then $(1 - \zeta_n)\varphi \in W^{1,m}(\Omega \setminus F)$ and $h(u)(1 - \zeta_n)\varphi \in W^{1,p}(\Omega \setminus F)$ and has a compact support in $\Omega \setminus F$, then we can write

$$I_1 + I_2 + I_3 + I_4 = \int_{\Omega} h(u)\varphi(1 - \zeta_n) d\mu_0 + h(+\infty) \int_{\Omega} \varphi(1 - \zeta_n) d\mu_s^+ - h(-\infty) \int_{\Omega} \varphi(1 - \zeta_n) d\mu_s^-,$$

with

$$\begin{aligned} I_1 &= \int_{\Omega} |\nabla u|^{p-2} \nabla u \cdot h'(u)\varphi(1 - \zeta_n) dx, & I_2 &= - \int_{\Omega} |\nabla u|^{p-2} \nabla u \cdot h(u)\varphi \nabla \zeta_n dx \\ I_3 &= \int_{\Omega} |\nabla u|^{p-2} \nabla u \cdot h(u)(1 - \zeta_n) \nabla \varphi dx, & I_4 &= \int_{\Omega} H(x, u, \nabla u) h(u)\varphi(1 - \zeta_n) dx \end{aligned}$$

We can go to the limit in I_1 as $n \rightarrow \infty$, from the dominated convergence theorem, since there exists $a > 0$ such that

$$\int_{\Omega} |\nabla u|^{p-2} \nabla u \cdot h'(u)\varphi(1 - \zeta_n) dx = \int_{\Omega} |\nabla T_a(u)|^{p-2} \nabla T_a(u) \cdot h'(T_a(u))\varphi(1 - \zeta_n) dx.$$

Furthermore $I_2 = o(n)$, because

$$\left| \int_{\Omega} |\nabla u|^{p-2} \nabla u \cdot h(u) \varphi \nabla \zeta_n dx \right| \leq \|h\|_{L^\infty(\mathbb{R})} \left(\int_{\Omega} |\nabla u|^q \varphi dx \right)^{1/q} \|\nabla \zeta_n\|_{L^{q^*}(\mathbb{R}^N)};$$

we also go to the limit in I_3 because $|\nabla \varphi| \in L^m(\Omega)$ and $|\nabla u|^{p-1} \in L_{loc}^\tau(\Omega), \forall \tau \in [1, N/(N-1))$; in I_4 from (4.11) and (4.2), and in the right hand side because $h(u)\varphi \in L^1(\Omega, d\mu_0)$, see [16, Remark 2.26] and $\zeta_n \rightarrow 0$ everywhere in $\mathbb{R}^N \setminus N$ and $\mu(N) = 0$. Then we can conclude:

$$\begin{aligned} & \int_{\Omega} |\nabla u|^{p-2} \nabla u \cdot \nabla (h(u)\varphi) dx + \int_{\Omega} H(x, u, \nabla u) h(u) \varphi dx \\ &= \int_{\Omega} h(u) \varphi d\mu_0 + h(+\infty) \int_{\Omega} \varphi d\mu_s^+ - h(-\infty) \int_{\Omega} \varphi d\mu_s^-. \end{aligned}$$

(ii) Assume that $q > p > 1$ (hence $1 < q_* < p$) and u is a weak solution in $\Omega \setminus F$. Then $u \in W_{loc}^{1,q}(\Omega \setminus F)$ implies $u \in W_{loc}^{1,q_*}(\Omega \setminus F) = W_{loc}^{1,q_*}(\Omega)$, hence $|\nabla u|$ is well defined in $L_{loc}^1(\Omega)$. As in part (i) we obtain that $|\nabla u|^q \zeta^{q_*} \in L^1(\Omega)$, hence $|\nabla u|^q \in L_{loc}^1(\Omega)$. For any $\varphi \in \mathcal{D}(\Omega)$, and ω containing $\text{supp} \varphi$, we have $\varphi(1 - \zeta_n) \in \mathcal{D}(\Omega \setminus F)$, then we can write $J_1 + J_2 + J_3 = \int_{\Omega} \varphi(1 - \zeta_n) d\mu$, with

$$J_1 = \int_{\Omega} (1 - \zeta_n) |\nabla u|^{p-2} \nabla u \cdot \nabla \varphi dx, \quad J_2 = - \int_{\Omega} \varphi |\nabla u|^{p-2} \nabla u \cdot \nabla \zeta_n dx, \quad J_3 = \int_{\Omega} H(x, u, \nabla u) \varphi (1 - \zeta_n) dx.$$

Now we can go to the limit in J_1 and J_3 from the dominated convergence theorem, because $|\nabla u|^q \in L_{loc}^1(\Omega)$ and $q > p - 1$; and $(\int_{\Omega} \varphi(1 - \zeta_n) d\mu)$ converges to $\int_{\Omega} \varphi d\mu$ as above. And J_2 converges to 0, because $|\nabla u|^{p-1} \in L_{loc}^{q/(p-1)}(\Omega)$ and $|\nabla \zeta_n|$ tends to 0 in $L^{q_*}(\Omega)$. Then u is a weak solution in Ω . ■

5 Existence in the supercritical case

Here the problem is delicate and many problems are still unsolved.

5.1 Case of a source term

Here we consider problem

$$-\Delta_p u = |\nabla u|^q + \mu \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega. \quad (5.1)$$

The main question is the following:

If $\mu \in M_b^{q_}(\Omega)$ satisfies condition (4.7) with a constant $C > 0$ small enough, does (5.1) admit a solution?*

In the case $p = 2 < q$, the problem has been solved in [19]. In that case one can define the solutions in a very weak sense. According to [14], setting $\rho(x) = \text{dist}(x, \partial\Omega)$, a function u is called a *very weak solution* of (5.1) if $u \in W_{loc}^{1,q}(\Omega) \cap L^1(\Omega)$, $|\nabla u|^q \in L^1(\Omega, \rho dx)$ and for any $\varphi \in C^2(\overline{\Omega})$ such that $\varphi = 0$ on $\partial\Omega$,

$$- \int_{\Omega} u \Delta \varphi dx = \int_{\Omega} |\nabla u|^q \varphi dx + \int_{\Omega} \varphi d\mu.$$

Theorem 5.1 ([19]) *Let $\mu \in \mathcal{M}^+(\Omega)$. If $1 < q$ and $p = 2$ and (5.1) has a very weak solution, then*

$$\mu(K) \leq C \text{Cap}_{1,q'}(K, \Omega) \quad (5.2)$$

holds for any compact $K \subset \Omega$, and some $C < C_1(N, q)$. Conversely, if $2 < q$ and (5.2) holds for some $C < C_2(N, q, \Omega)$ then (5.1) has a very weak nonnegative solution.

In the general case $p > 1$, such a notion of solution does not exist. The problem (5.1) with $p < q$ was studied by [41] for signed measures $\mu \in \mathcal{M}_b(\Omega)$ such that

$$[\mu]_{1,q^*,\Omega} = \sup \left\{ \frac{|\mu(K \cap \Omega)|}{\text{Cap}_{1,q^*}(K, \mathbb{R}^N)} : K \text{ compact of } \mathbb{R}^N, \text{Cap}_{1,q^*}(K, \mathbb{R}^N) > 0 \right\} < \infty.$$

Theorem 5.2 ([41]) *Let $1 < p < q$. Let $\mu \in \mathcal{M}_b(\Omega)$. There exists $C_1 = C_1(N, p, q, \Omega)$ such that if*

$$|\mu(K \cap \Omega)| \leq C \text{Cap}_{1,q^*}(K, \mathbb{R}^N) \quad (5.3)$$

for any compact $K \subset \mathbb{R}^N$, and some $C < C_1$, then (5.1) has a weak solution $u \in W_0^{1,q}(\Omega)$, such that $[\|\nabla u\|^q]_{1,q^,\Omega}$ is finite. In particular this holds for any $\mu \in L^{N/q^*,\infty}(\Omega)$.*

Very recently the case $p = q$, has been studied in [23] for signed measures satisfying a trace inequality: setting $p^\# = (p-1)^{2-p}$ if $p \geq 2$, $p^\# = 1$ if $p < 2$, they show in particular the following:

Theorem 5.3 ([23]) *Let $1 < p = q$. Let $\mu \in \mathcal{M}_b(\Omega)$ such that*

$$-C_1 \int_{\Omega} |\nabla \zeta|^p dx \leq \int_{\Omega} |\zeta|^p d\mu \leq C_2 \int_{\Omega} |\nabla \zeta|^p dx, \quad \forall \zeta \in \mathcal{D}(\Omega), \quad (5.4)$$

with $C_1 > 0$ and $C_2 \in (0, p^\#)$. Then (5.1) has a weak solution $u \in W_{loc}^{1,p}(\Omega)$.

The existence for problem (5.1) is still open in the case $q < p$ for $p \neq 2$

5.2 Case of an absorption term

Here we consider problem (1.2) in case of absorption, where $\mu \in M_b^+(\Omega)$ and we look for a nonnegative solution. In the model case

$$-\Delta_p u + |\nabla u|^q = \mu \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega, \quad (5.5)$$

the main question is the following: *If $\mu \in M_b^{q^*+}(\Omega)$, hence $\mu = f + \text{div}g$, with $f \in L^1(\Omega)$ and $g \in (L^{q/(p-1)}(\Omega))^N$, does (5.5) admits a nonnegative solution?*

Remark 5.4 *Up to changing u into $-u$, the results of Theorem 5.2 and 5.3 are also available for the problem (5.5) but we have **no information on the sign of u** .*

In the sequel we give two partial results of existence.

5.2.1 Case $q \leq p$ and $\mu \in \mathcal{M}_b^{p+}(\Omega)$

Here we assume that $\mu \in \mathcal{M}_b^{p+}(\Omega)$, subspace of $\mathcal{M}_b^{q*+}(\Omega)$. Our proof is directly inspired from the results of [11] for the problem (3.6), where $q = p$ and $H(x, u, \xi)u \geq 0$.

Theorem 5.5 *Let $p - 1 < q \leq p$. Let $\mu \in \mathcal{M}_b^{p+}(\Omega)$, and*

$$0 \leq H(x, u, \xi) \leq C_1 |\xi|^p + \ell(x), \quad (5.6)$$

$$H(x, u, \xi) \geq C_0 |\xi|^q \text{ for } u \geq L, \quad (5.7)$$

with $\ell(x) \in L^1(\Omega)$, $C_k, C_0, L \geq 0$. Then there exists a nonnegative R-solution of problem (1.2).

Remark 5.6 *The result was known in the case where $H(x, u, \nabla u) = |\nabla u|^q$, $p = 2$, and $\mu \in L^1(\Omega)$ (see for example [1], where the existence for any $\mu \in \mathcal{M}_b^{2+}(\Omega)$ is also claimed, without proof). For $p \neq 2$, the case $q < p$, $\mu \in L^1(\Omega)$ is partially treated in [40].*

Proof. Let $\mu = f - \text{div}g$ with $f \in L^1(\Omega)$ and $g = (g_i) \in (L^{p'}(\Omega))^N$. Here again we use the good approximation of μ by a sequence of measures $\mu_n \in \mathcal{M}_b^+(\Omega)$ given at Lemma 2.8 (ii), $\lambda_n = 0$, thus $\mu_n = \mu_n^0 = f_n - \text{div}g_n$, with $f_n \in \mathcal{D}^+(\Omega)$ and $g_n = (g_{n,i}) \in (\mathcal{D}(\Omega))^N$. Hence there exists a weak nonnegative solution $u_n \in W_0^{1,p}(\Omega)$ of the problem

$$-\Delta_p u_n + H(x, u_n, \nabla u_n) = \mu_n \quad \text{in } \Omega, \quad u_n = 0 \quad \text{on } \partial\Omega.$$

Since $H(x, u, \xi) \geq 0$, taking $\varphi = k^{-1}T_k(u_n - m)$ with $m \geq 0$, $k > 0$, as a test function, we still obtain (3.8). From Lemma 2.3, up to a subsequence, (u_n) converges *a.e.* to a function u , $(T_k(u_n))$ converges weakly in $W_0^{1,p}(\Omega)$, (∇u_n) converges *a.e.* to ∇u , and (u_n^{p-1}) converges strongly in $L^\sigma(\Omega)$, for any $\sigma \in [1, N/(N-p))$. Thus $\lim_{k \rightarrow \infty} \sup_{n \in \mathbb{N}} |\{u_n > k\}| = 0$, and $(|\nabla u_n|^{p-1})$ converges strongly in $L^\tau(\Omega)$, for any $\tau \in [1, N/(N-1))$. Moreover the choice of φ with $m + k > L$ gives

$$\frac{1}{k} \int_{\{m \leq u \leq m+k\}} |\nabla u_n|^p dx + C_0 \int_{\{u_n \geq m+k\}} |\nabla u_n|^q dx \leq \mu_n(\Omega) \leq 4\mu(\Omega).$$

Taking $m = 0$ we obtain

$$\int_{\Omega} |\nabla u_n|^q dx \leq \int_{\{u_n \geq k\}} |\nabla u_n|^q dx + \int_{\Omega} |\nabla T_k(u_n)|^q dx \leq 4C_0^{-1}\mu(\Omega) + \int_{\Omega} |\nabla T_k(u_n)|^p dx + |\Omega|$$

since $q \leq p$; thus from the Fatou Lemma, $|\nabla u|^q \in L^1(\Omega)$. Moreover using $\varphi = T_1(u_n - k)$,

$$\int_{\{k-1 \leq u_n \leq k\}} |\nabla u_n|^p dx + \int_{\{u_n \geq k\}} H(x, u_n, \nabla u_n) dx \leq \int_{\{u_n \geq k-1\}} f_n dx + \int_{\{k-1 \leq u_n \leq k\}} |g_n \cdot \nabla u_n| dx.$$

Therefore, from the Hölder inequality,

$$\int_{\{k-1 \leq u_n \leq k\}} |\nabla u_n|^p dx + p' \int_{\{u_n \geq k\}} H(x, u_n, \nabla u_n) dx \leq \int_{\{u_n \geq k-1\}} f_n dx + \left(\sum_{i=1}^N \int_{\{k-1 \leq u_n \leq k\}} |g_{n,i}|^{p'} dx \right).$$

From Lemma 2.8, there holds

$$\limsup_{k \rightarrow \infty} \sup_{n \in \mathbb{N}} \left(\int_{\{k-1 \leq u_n \leq k\}} |\nabla u_n|^p dx + \int_{\{u_n \geq k\}} H(x, u_n, \nabla u_n) dx \right) = 0. \quad (5.8)$$

Next we prove the strong convergence of the truncates in $W_0^{1,p}(\Omega)$ as in [11]: we take as test function

$$\varphi_n = \Phi(T_k(u_n) - T_k(u)), \text{ where } \Phi(s) = se^{\theta^2 s^2/4},$$

where $\theta > 0$ will be chosen after, thus $\Phi'(s) \geq \theta |\Phi(s)| + 1/2$. Then $\varphi_n \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$, and we have $|\varphi_n| \leq \Phi(k)$; setting $\psi_n = \Phi'(T_k(u_n) - T_k(u))$, we have $0 \leq \psi_n \leq \Phi'(k)$. Then $\varphi_n \rightarrow 0$, $\psi_n \rightarrow 1$ in $L^\infty(\Omega)$ weak * and *a.e.* in Ω . We set $a(\xi) = |\xi|^{p-2}\xi$, and

$$X = \int_{\Omega} (a(\nabla(T_k(u_n))) - a(\nabla(T_k(u)))) \cdot \nabla(T_k(u_n) - T_k(u)) \psi_n dx,$$

and get $X + I_1 = I_2 + I_3 + I_4$ with

$$I_1 = \int_{\Omega} H(x, u_n, \nabla u_n) \varphi_n dx, \quad I_2 = \int_{\Omega} a(\nabla(T_k(u))) \cdot \nabla(T_k(u) - T_k(u_n)) \psi_n dx,$$

$$I_3 = \int_{\Omega} f_n \varphi_n dx + \int_{\Omega} \operatorname{div}(g_n - g) \varphi_n dx + \int_{\Omega} g \cdot \nabla(T_k(u_n) - T_k(u)) \psi_n dx,$$

$$I_4 = - \int_{\Omega} a(\nabla(u_n - T_k(u_n))) \cdot \nabla(T_k(u_n) - T_k(u)) \psi_n dx = \int_{\{u_n \geq k\}} a(\nabla(u_n - T_k(u_n))) \cdot \nabla(T_k(u)) \psi_n dx.$$

From this, one can easily see that $|I_2| + |I_3| + |I_4| = o(n)$. Since $H(x, u_n, \nabla u_n) \geq 0$ for $u_n \geq k$, then $X \leq I_5 + o(n)$, where

$$I_5 = \left| \int_{\{u_n < k\}} H(x, u_n, \nabla u_n) \varphi_n dx \right| \leq C_1 \int_{\Omega} |\nabla(T_k u_n)|^p |\varphi_n| dx + \int_{\Omega} l |\varphi_n| dx \leq C_1(Y + I_7) + o(n),$$

with

$$Y = \int_{\Omega} (a(\nabla(T_k(u_n))) - a(\nabla(T_k(u)))) \cdot \nabla(T_k(u_n) - T_k(u)) |\varphi_n| dx,$$

$$I_7 = \int_{\Omega} a(\nabla(T_k(u))) \cdot \nabla(T_k(u_n) - T_k(u)) |\varphi_n| dx + \int_{\Omega} (a(\nabla(T_k(u_n))) \cdot \nabla(T_k(u)) |\varphi_n| dx$$

and then $I_7 = o(n)$. We get finally $X \leq C_1 Y + o(n)$; choosing $\theta = 2C_1$, we deduce that

$$\int_{\Omega} (a(\nabla(T_k(u_n))) - a(\nabla(T_k(u)))) \cdot \nabla(T_k(u_n) - T_k(u)) dx = o(n).$$

Hence $(T_k(u_n))$ converges strongly to $T_k(u)$ in $W_0^{1,p}(\Omega)$. Therefore $H(x, u_n, \nabla u_n)$ is equi-integrable, from (5.6) and (5.8), since for any measurable set $E \subset \Omega$,

$$\int_E H(x, u_n, \nabla u_n) dx \leq C_1 \int_E |\nabla(T_k u_n)|^p dx + \int_E \ell dx + \int_{\{u_n \geq k\}} H(x, u_n, \nabla u_n) dx.$$

Then $(H(x, u_n, \nabla u_n))$ converges to $H(x, u, \nabla u)$ strongly in $L^1(\Omega)$; thus $(\mu_n - H(x, u_n, \nabla u_n))$ is a good approximation of $\mu - H(x, u, \nabla u)$, and u is a R-solution of problem (3.1) from Theorem 2.6. \blacksquare

Remark 5.7 In the case $p - 1 < q < p$, and if (5.6) is replaced by

$$0 \leq H(x, u, \xi) \leq C_1 |\xi|^q + \ell(x), \quad (5.9)$$

the proof is much shorter: in order to prove the equi-integrability of $(H(x, u_n, \nabla u_n))$ we do not need to prove the strong convergence of the truncates: indeed for any measurable set $E \subset \Omega$,

$$\int_E |\nabla u_n|^q dx \leq \int_E |\nabla(T_k u_n)|^q dx + \int_{\{u_n \geq k\}} |\nabla u_n|^q dx$$

and $(\nabla T_k(u_n))$ converges strongly to $\nabla T_k(u)$ in $L^q(\Omega)$ and (5.8) holds. Then $(H(x, u_n, \nabla u_n))$ converges to $H(x, u, \nabla u)$ strongly in $L^1(\Omega)$.

5.2.2 Case where μ satisfies (4.7)

Here we assume that $\mu \in \mathcal{M}_b^+(\Omega)$ satisfies a capacity condition of type (4.7). For simplicity we assume that μ has a compact support in Ω . In the sequel we prove the following:

Theorem 5.8 Let $1 < q \leq p$ or $p = 2$. Assume that $\mu \in \mathcal{M}_b^+(\Omega)$, has a compact support and satisfies

$$\mu(K) \leq C_1 \text{cap}_{1, q^*}(K, \Omega), \quad \text{for any compact } K \subset \Omega, \quad (5.10)$$

for some $C_1 = C(N, q, \Omega) > 0$ (non necessarily small). Then there exists a nonnegative R -solution u of problem (5.5), such that $[|\nabla u|^q]_{1, q^*, \Omega}$ is finite.

First recall some equivalent properties of measures, see [35, Theorem 1.2], [19, Lemma 3.3], see also [41]:

Remark 5.9 1) Let $\mu \in \mathcal{M}_b^+(\Omega)$, extended by 0 to \mathbb{R}^N . Then (5.10) holds if and only if there exists $C_2 > 0$ such that

$$\int_{\Omega} \zeta^{q^*} d\mu \leq C_2 \int_{\Omega} |\nabla \zeta|^{q^*} dx, \quad \forall \zeta \in \mathcal{D}^+(\Omega); \quad (5.11)$$

the constants of equivalence between C_1, C_2 only depend on N, q_*, Ω .

If moreover μ has a compact support $K_0 \subset \Omega$, then (5.10) holds if and only if there exists $C_3 > 0$ such that

$$\mu(K) \leq C_3 \text{Cap}_{1, q^*}(K, \mathbb{R}^N) \quad \text{for any compact } K \subset \mathbb{R}^N; \quad (5.12)$$

the constants of equivalence between C_1, C_3 only depend on N, q_*, K_0 .

2) Let $\nu \in \mathcal{M}_b^+(\mathbb{R}^N)$. Then (5.12) holds if and only if there exists $C_4 > 0$ such that $J_1(\nu)$ is finite a.e. and

$$J_1((J_1(\nu))^{q^*}) \leq C_4 J_1(\nu) \quad \text{a.e. in } \mathbb{R}^N; \quad (5.13)$$

the constants of equivalence between C_3, C_4 do not depend on ν .

Following the ideas of [41, Theorem 3.4] we prove a convergence Lemma:

Lemma 5.10 Let (z_n) be a sequence of nonnegative functions, converging a.e. in $L^1(\Omega)$. Extending z_n by 0 in $\mathbb{R}^N \setminus \Omega$, assume that for some $C > 0$,

$$\int_{\Omega} z_n^{\frac{q}{p-1}} \xi^{q^*} dx \leq C \int_{\Omega} |\nabla \xi|^{q^*} dx \quad \forall n \in \mathbb{N}, \forall \xi \in \mathcal{D}^+(\mathbb{R}^N).$$

Then (z_n) converges strongly in $L^{q/(p-1)}(\Omega)$.

Proof. From our assumption, (z_n) is bounded in $L^{q/(p-1)}(\Omega)$, then up to a subsequence, it converges to some z weakly in $L^{q/(p-1)}(\Omega)$ and *a.e.* in Ω . Consider a ball $B \supset \Omega$ of radius $2\text{diam}\Omega$, and denote by G the Green function associated to $-\Delta$ in B . Set $w_n = z_n^{q/(p-1)}$, and extend w_n by 0 to $\mathbb{R}^N \setminus \Omega$. Then for any compact $K \subset \mathbb{R}^N$,

$$\int_{K \cap \Omega} w_n dx = \int_{K \cap B} w_n dx \leq CCap_{1,q^*}(K, \mathbb{R}^N),$$

which means that $[w_n]_{1,q^*,B}$ is bounded, and

$$|\nabla G(w_n)(x)| \leq \int_B |\nabla_x G(x, y)| w_n(y) dy \leq CG_1 * w_n(x),$$

with $C = C(N, \text{diam}\Omega)$. In turn from [41, Corollary 2.5], we get

$$\left[|\nabla G(w_n)|^{q/(p-1)} \right]_{1,q^*,B} \leq C \left[|G_1 * w_n|^{q/(p-1)} \right]_{1,q^*,B} \leq C [w_n]_{1,q^*,B}^{q/(p-1)},$$

hence it is bounded. Therefore $|\nabla G(w_n)|$ is bounded in $L^{q/(p-1)}(B)$, thus $|\nabla G(w_n - w)|$ is bounded in $L^{q/(p-1)}(B)$. Let $\varphi \in \mathcal{D}(B)$ and $\varepsilon > 0$ be fixed. Since (z_n) converges *a.e.* to z , from the Egoroff theorem, there exists a measurable set $\omega_\varepsilon \subset B$ such that (w_n) converges to $w = z^{q/(p-1)}$ uniformly on ω_ε , and $\|\nabla \varphi\|_{L^{q^*}(B \setminus \omega_\varepsilon)} \leq \varepsilon$. Then it follows that

$$\begin{aligned} \left| \int_\Omega (w_n - w) \varphi dx \right| &= \left| \int_B (w_n - w) \varphi dx \right| = \\ &= \left| - \int_B (\Delta(G(w_n - w)) \varphi dx \right| = \left| - \int_B \nabla(G(w_n - w)) \cdot \nabla \varphi dx \right| \end{aligned}$$

Considering the two integrals on $B \setminus \omega_\varepsilon$ and ω_ε we find $\lim \int_\Omega (w_n - w) \varphi dx = 0$. Taking $\varphi = 1$ on Ω , it follows that $\lim \int_\Omega z_n^{q/(p-1)} dx = \int_\Omega z^{q/(p-1)} dx$ and the proof is done. \blacksquare

Proof of Theorem 5.8. From our assumption, $\mu \in \mathcal{M}^{q^*}(\Omega)$. We consider the problem associated to $\mu_n = \mu * \rho_n$

$$-\Delta_p u_n + |\nabla u_n|^q = \mu_n \quad \text{in } \Omega, \quad u_n = 0 \quad \text{on } \partial\Omega. \quad (5.14)$$

For $q \leq p$, from [12, Theorem 2.1], as in the proof of Theorem 3.3, (5.14) admits a nonnegative solution $u_n \in W_0^{1,p}(\Omega) \cap C^{1,\alpha}(\overline{\Omega})$. Moreover we can approximate u_n in $C^{1,\alpha}(\overline{\Omega})$ by the solution $u_{n,\varepsilon}$ ($\varepsilon > 0$) of the problem

$$-\text{div}((\varepsilon^2 + |\nabla u_{n,\varepsilon}|^2)^{\frac{p-2}{2}} \nabla u_{n,\varepsilon}) + (\varepsilon^2 + |\nabla u_{n,\varepsilon}|^2)^{\frac{q}{2}} = \mu_n \quad \text{in } \Omega, \quad u_{n,\varepsilon} = 0 \quad \text{on } \partial\Omega.$$

Multiplying this equation by ξ^{q^*} with $\xi \in \mathcal{D}^+(\mathbb{R}^N)$, we obtain

$$\begin{aligned} & q_* \int_\Omega (\varepsilon^2 + |\nabla u_{n,\varepsilon}|^2)^{\frac{p-2}{2}} \nabla u_{n,\varepsilon} \cdot \xi^{q_*-1} \nabla \xi dx + \int_\Omega (\varepsilon^2 + |\nabla u_{n,\varepsilon}|^2)^{\frac{q}{2}} \xi^{q_*} dx \\ &= \int_\Omega \xi^{q_*} \mu_n dx + q_* \int_{\partial\Omega} \xi^{q_*} (\varepsilon^2 + |\nabla u_{n,\varepsilon}|^2)^{\frac{p-2}{2}} \nabla u_{n,\varepsilon} \cdot \nu ds. \end{aligned}$$

The boundary term is nonpositive, hence going to the limit as $\varepsilon \rightarrow 0$, we get

$$\int_{\Omega} |\nabla u_n|^q \xi^{q_*} dx \leq \int_{\Omega} \xi^{q_*} \mu_n dx + q_* \int_{\Omega} |\nabla u_n|^{p-2} \nabla u_n \cdot \xi^{q_*-1} \nabla \xi dx \quad (5.15)$$

When $p = 2$, existence also holds for $q > 2$, from [32]; and then $u_n \in C^2(\overline{\Omega})$, thus (5.15) is still true. As in Lemma 4.2, it follows that for any $\xi \in \mathcal{D}^+(\mathbb{R}^N)$

$$\int_{\Omega} |\nabla u_n|^q \xi^{q_*} dx \leq C \left(\int_{\Omega} \xi^{q_*} d\mu_n + \int_{\Omega} |\nabla \xi|^{q_*} dx \right). \quad (5.16)$$

Otherwise, since $\mu_n(\Omega) \leq \mu(\Omega)$, from Lemma 2.3, up to a subsequence (u_n) converges *a.e.* to a function u , $(T_k(u_n))$ converges weakly in $W_0^{1,p}(\Omega)$ and (∇u_n) converges *a.e.* to ∇u in Ω . Note also that (μ_n) is a sequence of good approximations of μ , since μ has a compact support (see [8]). From (4.5), for any $\xi \in \mathcal{D}^+(\mathbb{R}^N)$, we have $\lim \int_{\Omega} \xi^{q_*} d\mu_n = \int_{\Omega} \xi^{q_*} d\mu$, since $\xi^{q_*} \in C_c(\mathbb{R}^N)$. Then $\int_{\Omega} \xi^{q_*} d\mu \leq C \int_{\Omega} |\nabla \xi|^{q_*} dx$. From the Fatou Lemma, we obtain

$$\int_{\Omega} |\nabla u|^q \xi^{q_*} dx \leq C \left(\int_{\Omega} \xi^{q_*} d\mu + \int_{\Omega} |\nabla \xi|^{q_*} dx \right) \leq C \int_{\Omega} |\nabla \xi|^{q_*} dx, \quad (5.17)$$

hence $|\nabla u|^q \in L^1(\Omega)$. And then for any compact $K \subset \mathbb{R}^N$, taking $\xi = 1$ on K ,

$$\int_{K \cap \Omega} |\nabla u|^q dx \leq C C a p_{1,q_*}(K, \mathbb{R}^N),$$

thus $[|\nabla u|^q]_{1,q_*,\Omega}$ is finite. Moreover, extending μ by 0 to $\mathbb{R}^N \setminus \Omega$, we see from Remark 5.9 that μ satisfies condition (5.11), which is equivalent to (5.13). By convexity, μ_n also satisfies (5.13) and hence (5.11), with the same constants, *i.e.* for any $n \in \mathbb{N}$ and any $\xi \in \mathcal{D}^+(\mathbb{R}^N)$,

$$\int_{\Omega} \xi^{q_*} d\mu_n \leq C_2 \int_{\Omega} |\nabla \xi|^{q_*} dx \quad (5.18)$$

Then from (5.16) with another $C > 0$,

$$\int_{\Omega} |\nabla u_n|^q \xi^{q_*} dx \leq C \int_{\Omega} |\nabla \xi|^{q_*} dx \quad (5.19)$$

Next we can apply Lemma 5.10 to $z_n = |\nabla u_n|^{p-1}$, since (∇u_n) converges *a.e.* to ∇u in Ω . Then $(|\nabla u_n|^q)$ converges strongly in $L^1(\Omega)$ to $|\nabla u|^q$. Thus $(\mu_n - |\nabla u_n|^q)$ is a good approximation of $(\mu - |\nabla u|^q)$. From Theorem 2.6, u is a R-solution of the problem.

From [23, Theorem 1.4], condition (5.17) (for $N \geq 2$) implies that $q_* < N$, that means $q > q_c$, or $|\nabla u|^q = 0$ in Ω , thus $\mu = 0$. If $\mu = \operatorname{div} g$ with $g \in (L^{N(q+1-p)/(p-1),\infty}(\Omega))^N$ with compact support, then $|g|^{\frac{q}{p-1}} \in L^{N/q_*,\infty}(\Omega)$, thus

$$\int_{\Omega} \zeta^{q_*} |g|^{\frac{q}{p-1}} dx \leq C_2 \int_{\Omega} |\nabla \zeta|^{q_*} dx, \quad \forall \zeta \in \mathcal{D}^+(\Omega).$$

Hence μ satisfies (5.11) from the Hölder inequality. Note that $\mu \in \mathcal{M}^{q_*}(\Omega)$, since $q > q_c$ implies $|g| \in L^{q/(p-1)}(\Omega)^N$. \blacksquare

Remark 5.11 Let $q \leq p$ and $\mu = \operatorname{div} g$, where g has a compact support in Ω . From Theorems 5.5 and 5.8, we have existence when $g \in (L^{p'}(\Omega))^N$, or when $g \in (L^{N(q+1-p)/(p-1), \infty}(\Omega))^N$. Observe that $L^{p'}(\Omega) \supset L^{N(q+1-p)/(p-1)}(\Omega)$ if and only if $\tilde{q} \leq q$, where \tilde{q} is defined at (1.6). Hence Theorem 5.5 brings better results than Theorem 5.8 when $\tilde{q} \leq q \leq p$.

Remark 5.12 The extension of this result to the case $p < q, p \neq 2$ will be studied in a further article.

6 Some regularity results

In this section we give some regularity properties for the problem:

$$-\Delta_p u + H(x, u, \nabla u) = 0 \quad \text{in } \Omega. \quad (6.1)$$

We first recall some local estimates of the gradient for renormalized solutions, see [20], following the first results of [9], and many others, see among them [3], [26].

Lemma 6.1 Let u be the R -solution of problem

$$-\Delta_p u = f \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega,$$

with $f \in L^m(\Omega)$, $1 < m < N$. Set $\bar{m} = Np/(Np - N + p) = p/\tilde{q}$, where \tilde{q} is defined in (1.6).

(i) If $m > N/p$, then $u \in L^\infty(\Omega)$. If $m = N/p$, then $u \in L^k(\Omega)$ for any $k \geq 1$. If $m < N/p$, then $u^{p-1} \in L^k(\Omega)$ for $k = Nm/(N - pm)$.

(ii) $|\nabla u|^{(p-1)} \in L^{m^*}(\Omega)$, where $m^* = Nm/(N - m)$. If $\bar{m} \leq m$, then $u \in W_0^{1,p}(\Omega)$.

Remark 6.2 The estimates on u and $|\nabla u|$ are obtained in the case $m < \bar{m}$ by using the classical test functions $\phi_{\beta,\varepsilon}(T_k(u))$, where $\phi_{\beta,\varepsilon}(w) = \int_0^w (\varepsilon + |t|)^{-\beta} dt$, for given real $\beta < 1$. Let us recall the proof in the case $m \geq \bar{m}$, $p < N$. Then $L^m(\Omega) \subset W^{-1,p'}(\Omega)$, thus, from uniqueness, $u \in W_0^{1,p}(\Omega)$ and u is a variational solution. If $m = \bar{m}$, then $m^* = p'$, and the conclusion follows. Suppose $m > \bar{m}$, equivalently $m^* > p'$. For any $\sigma > p$, for any $F \in (L^\sigma(\Omega))^N$, there exists a unique weak solution w in $W_0^{1,\sigma}(\Omega)$ of the problem

$$-\Delta_p w = \operatorname{div}(|F|^{p-2} F) \quad \text{in } \Omega, \quad w = 0 \quad \text{on } \partial\Omega,$$

see [25], [29], [30]. Let v be the unique solution in $W_0^{1,1}(\Omega)$ of the problem

$$-\Delta v = f \quad \text{in } \Omega, \quad v = 0 \quad \text{on } \partial\Omega. \quad (6.2)$$

Then from the classical Calderon-Zygmund theory, $v \in W^{2,m}(\Omega)$, then $|\nabla v| \in L^{m^*}(\Omega)$. Let F be defined by $|F|^{p-2} F = \nabla v$. Then $F \in (L^\sigma(\Omega))^N$, with $\sigma = (p-1)m^* > p$. Then $-\Delta_p w = -\Delta v = f$, thus $w = u$. Then $u \in W_0^{1,\sigma}(\Omega)$, thus $|\nabla u|^{(p-1)} \in L^{m^*}(\Omega)$.

We also obtain local estimates:

Lemma 6.3 *Let $u \in W_{loc}^{1,p}(\Omega)$ such that*

$$-\Delta_p u = f \quad \text{in } \Omega,$$

with $f \in L_{loc}^m(\Omega)$, $1 < m < N$, and $m > \bar{m}$. Then $|\nabla u|^{p-1} \in L_{loc}^{m^}(\Omega)$. Furthermore, for any balls $B_1 \subset\subset B_2 \subset\subset \Omega$, $\left\| |\nabla u|^{p-1} \right\|_{L^{m^*}(\overline{B_1})}$ is bounded by a constant which depends only on N, p, B_1, B_2 and $\|u\|_{W^{1,p}(B_2)}$.*

Proof. We consider again the function v defined in (6.2), and set $|F|^{p-2}F = \nabla v$. Then $F \in (L^\sigma(\Omega))^N$ with $\sigma = (p-1)m^*$, and $u \in W_{loc}^{1,p}(\Omega)$ is a solution of the problem

$$-\Delta_p u = \operatorname{div}(|F|^{p-2}F) \quad \text{in } \Omega.$$

Then, from [29], $u \in W_{loc}^{1,\sigma}(\Omega)$ and for any balls $B_1 \subset\subset B_2 \subset\subset \Omega$, $\|u\|_{W^{1,\sigma}(B_1)}$ is controlled by the norm $\|u\|_{W^{1,p}(B_2)}$. ■

Next we consider problem (6.1) in the case $q < \tilde{q}$, where \tilde{q} is defined at (1.6).

Theorem 6.4 *Let $0 < q < \tilde{q}$, $N \geq 2$. Let H be a Caratheodory function on $\Omega \times \mathbb{R}$ such that*

$$|H(x, u, \xi)| \leq g(x) + C|\xi|^q, \quad (6.3)$$

where $g \in L_{loc}^{N+\varepsilon}(\Omega)$, $C > 0$. Let $u \in W_{loc}^{1,p}(\Omega)$ be any weak solution of problem (6.1). Then $u \in C^{1,\alpha}(\Omega)$ for some $\alpha \in (0, 1)$. Moreover for any balls $B_1 \subset\subset B_2 \subset\subset \Omega$, $\|u\|_{C^{1,\alpha}(\overline{B_1})}$ is bounded by a constant which depends only on N, p, B_1, B_2 , $\|g\|_{L^{N+\varepsilon}(B_2)}$, and the norm $\|u\|_{W^{1,p}(B_2)}$.

Proof. Since $u \in W_{loc}^{1,p}(\Omega)$, the function $f = -H(x, u, \nabla u)$ satisfies $f \in L_{loc}^{m_0}(\Omega)$ from (6.3), with $m_0 = p/q > 1$. Notice that $q < \tilde{q}$ is equivalent to $m_0 > \bar{m}$. If $m_0 > N$, then from [18, Theorem 1.2], $|\nabla u| \in L_{loc}^\infty(\Omega)$ and we get an estimate of $\| |\nabla u| \|_{L^\infty(B_1)}$ in terms of the norm $\|u\|_{W^{1,p}(B_2)}$ and $\|g\|_{L^{N+\varepsilon}(B_2)}$. Then $u \in C(\Omega)$, $f \in L_{loc}^\infty(\Omega)$, hence $u \in C^{1,\alpha}(\Omega)$ for some $\alpha \in (0, 1)$.

Next suppose that $m_0 < N$. Then from Lemma 6.3 we have $|\nabla u|^{p-1} \in L^{(p-1)m_0^*}(\Omega)$. In turn, from (6.3), $f \in L_{loc}^{m_1}(\Omega)$ with $m_1 = (p-1)m_0^*/q$. Note that $m_1/m_0 = N(p-1)/(qN-p) > 1$ since $q < \tilde{q}$. By induction, starting from m_1 , as long as $m_n < N$, we can define $m_{n+1} = (p-1)m_n^*/q$, and we find $m_n < m_{n+1}$. If $m_n < N$ for any n , then the sequence converges to $\lambda = N(q-p+1)/q$, which is impossible since $p/q < \lambda$ and $q < \tilde{q}$. Then there exists n_0 such that $m_{n_0} \geq N$. If $n_0 = N$, or if $m_0 = N$ we modify a little m_0 in order to avoid the case. Then we conclude from above. ■

Remark 6.5 *The result, which holds without any assumption on the sign of H , is sharp. Indeed for $\tilde{q} < q < p < N$, the problem $-\Delta_p u = |\nabla u|^q$ in $B(0, 1)$ with $u = 0$ on $\partial B(0, 1)$ admits the solution*

$$x \mapsto u_C(x) = C(|x|^{-\frac{p-q}{q+1-p}} - 1),$$

for suitable $C > 0$, and $u_C \in W_0^{1,p}(\Omega)$ for $\tilde{q} < q$.

Next we consider the absorption case, and for simplicity the model problem:

Theorem 6.6 *Let $0 < q < p$. Let $u \in W_{loc}^{1,p}(\Omega)$ be a nonnegative weak solution of*

$$-\Delta_p u + |\nabla u|^q = 0 \quad \text{in } \Omega.$$

Then $u \in L_{loc}^\infty(\Omega)$, and for any balls $B_1 \subset\subset B_2 \subset\subset \Omega$, $\|u\|_{L^\infty(B_1)}$ and $\|u\|_{W^{1,p}(B_1)}$ are controlled by the norm $\|u\|_{L^p(B_2)}$.

As a consequence, if $q < \tilde{q}$, then $u \in C^{1,\alpha}(\Omega)$ for some $\alpha \in (0, 1)$. In particular $\|\nabla u\|_{L^\infty(B_1)}$ is controlled by $\|u\|_{L^p(B_2)}$. If $p \geq 2$, the result is still true for any $q < p$.

Proof. Since $u \in W_{loc}^{1,p}(\Omega)$ and $-\Delta_p u \leq 0$ in Ω , from the weak Harnack inequality, for any ball $B(x_0, 3\rho) \subset \Omega$, and any $\ell > p - 1$,

$$\sup_{B(x_0, \rho)} u \leq C \left(\int_{B(x_0, 2\rho)} u^\ell \right)^{\frac{1}{\ell}}, \quad (6.4)$$

with $C = C(N, p, \ell)$, in particular for $\ell = p$. Then $u \in L_{loc}^\infty(\Omega)$. For any $\xi \in \mathcal{D}(\Omega)$, taking $u\xi^p$ as a test function, we get

$$\begin{aligned} \int_{\Omega} |\nabla u|^p \xi^p dx + \int_{\Omega} |\nabla u|^q u \xi^p dx &= -p \int_{\Omega} \xi^{p-1} u |\nabla u|^{p-2} \nabla u \cdot \nabla \xi dx \\ &\leq \frac{1}{2} \int_{\Omega} |\nabla u|^p \xi^p dx + C_p \int_{\Omega} u^p |\nabla \xi|^p dx. \end{aligned}$$

Then for any balls $B_1 \subset\subset B_2 \subset\subset \Omega$, we obtain that $\|\nabla u\|_{L^p(B_1)}$ is bounded by a constant which depends only on N, p, B and $\|u\|_{L^p(B_2)}$. The conclusions hold from Theorem 6.4.

Next assume that $p \geq 2$, and $q < p$. We still have $f = -|\nabla u|^q \in L_{loc}^{m_0}(\Omega)$, with $m_0 = p/q > 1$, since $u \in W_{loc}^{1,p}(\Omega)$. From a local version of [36, Theorem 5], there holds $\nabla u \in (W_{loc}^{\sigma, m_0(p-1)}(\Omega))^N$ for any $\sigma < 1/(p-1)$, then $u \in W_{loc}^{k, m_0(p-1)}(\Omega)$ for any $k < p'$. Since we have also $u \in L_{loc}^\infty(\Omega)$, from Gagliardo-Nirenberg inequality,

$$\|\nabla u\|_{L^{km_0(p-1)}(B_1)} \leq \|u\|_{W^{k, m_0(p-1)}(B_1)}^{\frac{1}{k}} \|u\|_{L^\infty(B_1)}^{\frac{1}{k'}}.$$

Therefore $|\nabla u| \in L^{m_0 s}(B_1)$ for any $s < p$. Then $f \in L_{loc}^{m_1}(\Omega)$, for any $m_1 < m_0 p/q = m_0^2$. By induction, for any $r \geq 1$, $f \in L_{loc}^m(\Omega)$ for any $m < m_0^r$. Thus $f \in L_{loc}^m(\Omega)$ for any $m \geq 1$. We deduce as above that $|\nabla u| \in L_{loc}^\infty(\Omega)$ from [18, Theorem 1.2], and then $u \in C^{1,\alpha}(\Omega)$ for some $\alpha \in (0, 1)$. ■

Remark 6.7 *Up to our knowledge, the regularity in the case $p < 2$ is still an open problem, even if $p > 2 - 1/N$.*

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